Trigonometry

UNIT – III

Properties of arithmetic operations

i) $z_1+z_2 = z_2+z_1$ commulativity of addition

ii) $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ associativity of addition.

iii) For any complex Numbers z_1 and z_2 there is unique Numbers z. Such that $z_1+z_2=z_2$

(iv) $z_1 z_2 = z_2 z_1$ commulativity of multiplication.

(v) $(z_1 z_2)z_3 = z_1(z_2 z_3)$ associativity of multiplication.

(vi) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$ $\overline{z_1-z_2} = \overline{z_1}-\overline{z_2}$

$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ $\overline{(z_1/z_2)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

1) If $|z_1| = |z_2| = c$ PT $|z_1+z_2|^2+|z_1-z_2|^2 = 4c^2$

Sol $|z|^2 = z\bar{z}$ it follows that

$|z_1+z_2|^2+|z_1-z_2|^2$

$= (z_1+z_2)(\overline{z_1+z_2}) + (z_1-z_2)(\overline{z_1-z_2})$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 (z_1\bar{z}_2 + z_2\bar{z}_1) + z_1\bar{z}_1 + z_2\bar{z}_2 - (z_1\bar{z}_2 + z_2\bar{z}_1)$

$= |\bar{z}_1|^2 + |\bar{z}_2|^2 + |\bar{z}_1|^2 + |\bar{z}_2|^2$

$= c^2 + c^2 + c^2 + c^2$

$= 4c^2$ //

DEMOIVRE'S THEOREM

$\boxed{\text{U-III}}$

2m) If n is any integer then

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

If n is a fraction then $\cos n\theta + i\sin n\theta$

is one of the values of $(\cos\theta + i\sin\theta)^n$

NOTE

(i) $\dfrac{1}{\cos\theta + i\sin\theta} = [\cos\theta \pm i\sin\theta]^{-1}$

$= \cos\theta + i\sin\theta$

(ii) $(\cos\theta - i\sin\theta)^n = [\cos\theta + i\sin\theta^{-1}]^n$

$= (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n)\theta + i\sin(-n)\theta$

$= \cos n\theta - i\sin n\theta$

Expansions of $\cos n\theta$ and $\sin n\theta$ in powers of $\sin\theta$ and $\cos\theta$, n being a positive integer

$$(x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \cdots + nc_r x^{n-r} a^r + \cdots + a^n$$

Sol

By using Binomial Theorem

we have $(\cos\theta + i\sin\theta)^n = \cos^n\theta + nc_1 \cos^{n-1}\theta\,(i\sin\theta) + nc_2 \cos^{n-2}\theta\,(i\sin\theta)^2 + nc_3 \cos^{n-3}\theta\,(i\sin\theta)^3 + nc_4 \cos^{n-4}\theta\,(i\sin\theta)^4 + \cdots$

$= \cos^n\theta + i\,nc_1 \cos^{n-1}\theta \sin\theta + nc_2 \cos^{n-2}\theta \sin^2\theta - i\,nc_3 \cos^{n-3}\theta \sin^3\theta + \cdots$

$= \cos^n\theta + i\,nc_1 \cos^{n-1}\theta \sin\theta - nc_2 \cos^{n-2}\theta \sin^2\theta - i\,nc_3 \cos^{n-3}\theta \sin^3\theta + nc_4 \cos^{n-4}\theta \sin^4\theta + \cdots$

But

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ {By De Moivre's Theorem

There fore

$\cos n\theta + i\sin n\theta = [\cos^n\theta + i\,nc_1 \cos^{n-1}\theta \sin\theta - nc_2 \cos^{n-2}\theta \sin^2\theta - i\,nc_3 \cos^{n-3}\theta$

$$\left[i\,{}^nC_1 \cos^{n-1}\theta \sin\theta - i\,{}^nC_3 \cos^{n-3}\theta \sin^3\theta + i\,{}^nC_5 \cos^{n-5}\theta \sin^5\theta\right]$$

$$(\cos n\theta + i\sin n\theta) = [\cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta + i\{{}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin$$

Equating real and Imaginary Parts we get

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4} \sin^4\theta$$

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta + \sin^3\theta + {}^nC_5 \cos^{n-5} \sin^5\theta$$

Expansion of $\tan n\theta$:-

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$\tan n\theta = \frac{nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + nc_5 \cos^{n-5}\theta \sin^5\theta}{\cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta \ldots}$$

Dividing both ~~nu~~ numerator and denominator of bothsides we get.

$$\tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^3\theta + nc_5 \tan^5\theta \ldots}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta + \ldots}$$

formula for any numbers by angles

w.k.T

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots = (\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

Also $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1 + i\tan\theta_1)$

$\cos\theta + i\sin\theta_2 = \cos\theta_2(1 + i\tan\theta_2)$

w.k.T

$\cos(\theta_1 + \theta_2 + \ldots \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

$= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n [1 + i \tan\theta_1 + \tan\theta_2 \cdots 1 + \tan\theta_n)$

$= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n (1 + i\tan\theta_1)(1 + i\tan\theta_2) \cdots (1 + i\tan\theta_n)$

$= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n [1 + i(\tan\theta_1 + \tan\theta_2 + \tan\theta_n)$
$+ i^2(\tan\theta_1 \tan\theta_2 + \tan\theta_2 \tan\theta_3 + \cdots)$
$+ i^3(\tan\theta_1 \tan\theta_2 \tan\theta_3 + \cdots + \cdots]$

$= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n [1 - iS_1 - iS_2 + S_3 \cdots]$

where $S_1 = \tan\theta_1 + \tan\theta_2 + \cdots + \tan\theta_n$

$S_2 = \Sigma \tan\theta_1 \tan\theta_2 \cancel{\tan\theta_3} \cdots + \tan\theta_n$

$S_3 = \Sigma \tan\theta_1 \tan\theta_2 \tan\theta_3 \cdots$ etc.

Equating real and Imaginary Parts get.

$\cos(\theta_1 + \theta_2 + \cdots \theta_n) = \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n (1 - S_2 + S_4 \cdots)$

$\sin(\theta_1 + \theta_2 + \cdots \theta_n) = \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n (S_1 - S_3 + S_4 - \cdots)$

Ex: Express cos.

Dividing get

$$\tan(\theta_1+\theta_2+\dots\theta_n) = \frac{S_1 - S_3 + S_4 + \dots}{1 - S_2 + S_4 + \dots}$$

5m

(4) Express $\cos 5\theta$ interms of $\cos\theta$ $\sin^2\theta = 1-\cos^2\theta$ ~~$\cos^2\theta = 1-\sin^2\theta$~~

Solution.

$$\cos n\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta \rightarrow (1)$$

$n = 5$

$$\cos 5\theta = \cos^5\theta - 5c_2 \cos^{5-2}\theta \sin^2\theta + 5\cos^{5-4}\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta (1-\cos^2\theta) + 5\cos\theta (1-\cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta [1 - 2(\cos^2\theta) + \cos^4\theta]$$

$$= [\cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos 5\theta]$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

5m

(5) Express of $\cos 6\theta$ interms of $\cos\theta$

$$\cos 6\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta$$

$$\sin^4\theta - nc_6 \cos^{n-6}\theta \sin^6\theta$$

$$\cos 6\theta = \cos^6\theta - 6c_2 \cos^4\theta \sin^2\theta + 6c_4$$

$$\cos^2\theta \sin^4\theta - 6c_6 \cos^0\theta \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta \sin^2\theta + 15\cos^2\theta$$

$$\sin^4\theta - \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta (1-\cos^2\theta) + 15\cos^2\theta$$

$$(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$$= \cos^6\theta - 15\cos^4\theta (1-\cos^2\theta) + 15\cos^2\theta$$

$$(1 - 2\cos^2\theta + \cos^4\theta) \cdot (1 - 3\cos^2\theta$$

$+3\cos^4\theta - \cos^6\theta)$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta$

$- 30\cos^4\theta + 15\cos^6\theta - 1 + 3$

$\cos^2\theta - 3\cos^4\theta + \cos^6\theta$

$$\boxed{\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}$$

5m

3. Show that $\dfrac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta$

$+ nc_5 \cos^{n-5}\theta \sin^5\theta$

$\sin 6\theta = 6c_1 \cos^{6-1}\theta \sin\theta - 6c_3 \cos^{6-3}\theta \sin^3\theta$

$+ 6c_5 \cos^{6-5}\theta \sin^5\theta$

$= 6c_1 \cos^5\theta \sin\theta - 6c_3 \cos^3\theta \sin^3\theta$

$+ 6c_5 \cos\theta \sin^5\theta$

$= 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta$

$+ 6\cos\theta \sin^5\theta$

$\dfrac{\sin 6\theta}{\sin\theta} = \dfrac{6\cos^5\theta \sin\theta}{\sin\theta} = \dfrac{20\cos^3\theta \sin^3\theta}{\sin\theta} + \dfrac{6\cos\theta \sin^5\theta}{\sin\theta}$

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta\sin^2\theta + 6\cos\theta\sin^4\theta$$

$$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$$

$$= 6\cos^5 - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-2\cos^2\theta+\cos^4\theta)$$

$$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta - 12\cos^3\theta + 6\cos^5\theta$$

$$\boxed{\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta}$$

5m

1)

$$\frac{\cos 6\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$$

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - nC_6\cos^{n-6}\theta\sin^6\theta$$

$$= \cos^6\theta - 6C_2\cos^{6-2}\theta\sin^2\theta + 6C_4\cos^{6-4}\theta\sin^4\theta - 6C_6\cos^{6-6}\theta\sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-2\cos^2\theta + \cos^4\theta)$$
$$-(1-3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta) - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$$

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

$$\frac{\cos 6\theta}{\cos\theta} = \frac{32\cos^6\theta}{\cos\theta} - \frac{48\cos^4\theta}{\cos\theta} + \frac{18\cos^2\theta}{\cos\theta} - \frac{1}{\cos\theta}$$

$$\boxed{\frac{\cos 6\theta}{\cos\theta} = 32\cos^5\theta - 48\cos^3\theta + 18\cos\theta - \frac{1}{\cos\theta}}$$

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② sin

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\cdot\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - {}^nC_7\cos^{n-7}\theta\sin^7\theta$$

$$\sin 7\theta = {}^7C_1\cos^{7-1}\theta\sin\theta - {}^7C_3\cos^{7-3}\sin^3\theta + {}^7C_5\cos^{7-5}\sin^5\theta - {}^7C_7\cos^{7-7}\theta\sin^7\theta$$

$$= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

3

$$\frac{\sin 7\theta}{\sin\theta} = \frac{7\cos^6\theta\,\sin\theta}{\sin\theta} - \frac{35\cos^4\theta\,\sin^3\theta}{\sin\theta} + \frac{21\cos^2\theta\,\sin^5\theta}{\sin\theta} - \frac{\sin^7\theta}{\sin\theta}$$

$$= 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$$

$$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35(1-2\sin^2\theta+\sin^4\theta)\sin^2\theta + 2(1-2\sin^2\theta+\sin^4\theta)\sin^4\theta - \sin^6\theta$$

$$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35 + 70\sin^2\theta - 35\sin^4\theta - 35\sin^2\theta + 2 - 4\sin^2\theta + 2\sin^4\theta\ 2\sin^4\theta - \sin^6\theta$$

$$\boxed{\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta}$$

$\frac{\cos 5\theta}{\cos\theta} = 1 - 12\sin^2\theta + 16\sin^4\theta$

Sol

$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta$

$\cos 5\theta = \cos^5\theta - 5C_2 \cos^3\theta \sin^2\theta + 5C_4 \cos^{5-4}\theta \sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$

$\frac{\cos 5\theta}{\cos\theta} = \frac{\cos^5\theta}{\cos\theta} - \frac{10\cos^3\theta \sin^2\theta}{\cos\theta} + \frac{5\cos\theta\sin^4\theta}{\cos\theta}$

$= \cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta$

$= (1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + 5\sin^4\theta$

$= (1 - 2\sin^2\theta + \sin^4\theta) - 10(1-\sin^2\theta)\sin^2\theta + 5\sin^4\theta$

$= 1 - 2\sin^2\theta + \sin^4\theta - 10\sin^2\theta + 10\sin^4\theta + 5\sin^4\theta$

$$\boxed{\frac{\cos 5\theta}{\cos\theta} = 1 - 12\sin^2\theta + 16\sin^4\theta}$$

10M

(4) $\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$

Sol

$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - nC_6\cos^{n-6}\theta\sin^6\theta + nC_8\cos^{n-8}\theta\sin^8\theta$

$n = 8$

$\cos 8\theta = \cos^8\theta - 8C_2\cos^{8-2}\theta\sin^2\theta + 8C_4\cos^{8-4}\theta\sin^4\theta$

$-8C_6 \cos^{8-6}\theta \sin^6\theta + 8C_8 \cos^{8-8}\theta \sin^8\theta$

$= \cos^8\theta - 28\cos^6\theta \sin^2\theta + 70\cos^4\theta \sin^4\theta - 28\cos^2\theta \sin^6\theta + \sin^8\theta$

$= (1-\sin^2\theta)^2(1-\sin^2\theta)^2 - 28(1-\sin^2\theta)^3\sin^2\theta + 70(1-\sin^2\theta)^2 \sin^4\theta$

$- 28(1-\sin^2\theta)\sin^6\theta + \sin^8\theta$

$= (1+\sin^4\theta - 2\sin^2\theta)(1+\sin^4\theta - 2\sin^2\theta) - 28$

$(1-\sin^6\theta + 3\sin^4\theta - 3\sin^2\theta)\sin^2\theta + 70(1+\sin^4\theta$

$-2\sin^2\theta)\sin^4\theta$

$-28(1-\sin^2\theta)\sin^6\theta + \sin^8\theta$

$= 1+\sin^4\theta - 2\sin^2\theta + \sin^4\theta + \sin^8\theta - 2\sin^6\theta - 2\sin^2\theta$

$-2\sin^6\theta + 4\sin^4\theta - 28\sin^2\theta + 28\sin^8\theta - 84\sin^6\theta$

$+84\sin^4\theta + 70\sin^4\theta + 70\sin^8\theta - 140\sin^6\theta -$

$28\sin^6\theta + 28\sin^8\theta + \sin^8\theta$

$= 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$

$$\boxed{\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta}$$

Nov

2) Expans of $\sin^n\theta$ & $\cos^n\theta$ interms of sin and cosine of multiplies of θ, n being a Positive indigen.

Let $x = \cos\theta + i\sin\theta \rightarrow (1)$

$\frac{1}{x} = x^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos\theta - i\sin\theta \rightarrow (2)$

$x^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta \rightarrow (3)$

$\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n$

$= \cos n\theta - i\sin n\theta \rightarrow (4)$

from ① & ③ we get

$$x+\frac{1}{x}=2\cos\theta \rightarrow ⑤$$

$$x-\frac{1}{x}=2i\sin\theta \rightarrow ⑥$$

from ③ & ④ we get

$$x^n+\frac{1}{x^n}=2\cos n\theta \rightarrow ⑦$$

$$x^n-\frac{1}{x^n}=2i\sin n\theta \rightarrow ⑧$$

to get expansion of $\cos^n\theta$, we have to consider

$$(2\cos\theta)^n=\left(x+\frac{1}{x}\right)^n \text{ (from ⑤)}$$

Expand RHS by using Binomial theorem and using the results ⑦ & we get the required expansion.

To get the expansion of $\sin^n\theta$ we have to consider

$$(2i\sin\theta)^m(2\cos\theta)^n=\left(x-\frac{1}{x}\right)^m\left(x+\frac{1}{x}\right)^n$$

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⑥ Prove that $\sin^5\theta=\frac{1}{16}(\sin5\theta-5\sin3\theta+10\sin\theta)$

Sol

Let $x=\cos\theta+i\sin\theta$

$\frac{1}{x}=\cos\theta-i\sin\theta$

$x^n=\cos n\theta+i\sin n\theta$

$$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\left(x+\frac{1}{x}\right) = 2\cos\theta$$

$$\left(x-\frac{1}{x}\right) = 2i\sin\theta$$

$$\left(x^n+\frac{1}{x^n}\right) = 2\cos n\theta$$

$$\left(x^n-\frac{1}{x^n}\right) = 2i\sin n\theta$$

Binomial theorem

$$(x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \cdots$$

$$\left(x-\frac{1}{x}\right)^5 = (2i\sin\theta)^5$$

$$(2i\sin\theta)^5 = \left(x-\frac{1}{x}\right)^5$$

$$2^5 i^5 \sin^5\theta = x^5 + 5c_1 x^{5-1}\left(-\frac{1}{x}\right) + 5c_2 x^{5-2}\left(-\frac{1}{x}\right)^2 + 5c_3 x^{5-3}\left(-\frac{1}{x}\right)^3 + 5c_4 x^{5-4}\left(-\frac{1}{x}\right)^4 + 5c_5 x^{5-5}\left(-\frac{1}{x}\right)^5$$

$$= x^5 - 5x^4\frac{1}{x} + 10x^3\cdot\frac{1}{x^2} - 10x^2\frac{1}{x^3} + 5x\cdot\frac{1}{x^4} - \frac{1}{x^5}$$

$$= x^5 - 5x^3 + 10x - 10\cdot\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}$$

$$= \left(x^5-\frac{1}{x^5}\right) - 5\left(x^3-\frac{1}{x^3}\right) + 10\left(x-\frac{1}{x}\right)$$

$$= 2i\sin 5\theta - 5\,2i\sin 3\theta + 10\,2i\sin\theta$$

$\sin 5\theta = \cancel{2}i \left[\frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{\cancel{2}\cancel{5}.\ i\cancel{5}4}\right]$

$$\boxed{\sin^5\theta = \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]}$$

sm

1.) $\sin^7\theta = -\frac{1}{64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$

Sol

Let $x = \cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$(x + \frac{1}{x}) = 2\cos\theta$

$(x - \frac{1}{x}) = \cancel{+}2i\sin\theta$

$(x - \frac{1}{x})^7 = (2i\sin\theta)^7$

$(2i\sin\theta)^7 = (x - 1/x)^7$

Binomial Theorem

$(x+a)^n = x^n + nc_1 x^{n-1}a + nc_2 x^{n-2}a^2 + nc_3 x^{n-3}a^3 + nc_4 x^{n-4}a^4 + nc_5 x^{n-5}a^5 + nc_6 x^{n-6}a^6 + nc_7 x^{n-7}a^7$

$= x^7 + 7c_1 x^6(-1/x) + 7c_2 x^5(-1/x)^2 + 7c_3 x^{7-3}(-1/x)^3 + 7c_4 x^{7-4}(-1/x)^4 + 7c_5 x^2(-1/x)^5 + 7c_6 x(-1/x)^6 + 7c_7 x^0(-1/x)^7$

$= x^7 + 7x^6(-1/x) + 21x^5\frac{1}{x^2} + 35x^4(-1/x^3) + 35x^3\frac{1}{x^4} + 21x\frac{1}{x^6} - \frac{1}{x^7}$

$= x^7 + 7x^5 + 21x^3 - 35x^{2} + 35\frac{1}{x} + 21\frac{1}{x^5}$

$- \frac{1}{x^7}$

$= \left(x^7 - \frac{1}{x^7}\right) + 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right)$

$-35\left(x - \frac{1}{x}\right)$

$= 2i\sin 7\theta - 7\, 2i\sin 5\theta + 21\, 2i\sin 3\theta$

$- 35\, 2i\sin\theta$

$2^7 i^7 \sin^7\theta = 2i\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$

$\overline{\qquad 2^7 i\, i^6 \qquad}$

$$\boxed{\sin^7\theta = -\frac{1}{64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]}$$

$\sin^8\theta = \frac{1}{2^7}\left[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35\right]$

Sol

Let $x = \cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Binomial Theorem

$(x + 1/x) = 2\cos\theta$

$(x - 1/x) = 2i\sin\theta$

$(x^n + 1/x^n) = 2\cos n\theta$

$(x^n - \frac{1}{x^n}) = 2i\sin\theta$

Binomial Theorem :-

$(x+a)^n = x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \dots$

$(x - \frac{1}{x})^8 = (2i\sin\theta)^8$

$(x + \frac{1}{x})^8 = (2\cos\theta)^8$

$2^8 i^8 \sin^8\theta = x^8 + 8C_1 x^{8-1}(-1/x) + 8C_2 x^{8-2}(-1/x)^2$

$+ 8C_3 x^{8-3}(-1/x)^3 + 8C_4 x^{8-4}(-1/x)^4$

$+ 8C_5 x^{8-5}(-1/x)^5 + 8C_6 x^{8-6}(-1/x)^6$

$+ 8C_7 x^{8-7}(-1/x)^7 + 8C_8 x^{8-8}(-1/x)^8$

$= x^8 + 8x^7(-1/x) + 28x^6(1/x^2) + 56x^5(-1/x^3)$

$+ 70x^4(1/x^4) + 56x^3(-1/x^5) + 28 x^2(-1/x^6) + (-1/x)^8$

$x^8 - 8x^7(1/x) + 28x^6(1/x^2) - 56x^5(1/x^3)$

$+ 70x^4(1/x^4) - 56x^3(1/x^5) + 28x^2(1/x^6)$

$- 1/x^8$

$x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - 56(1/x^2)$

$+ 28(1/x^4) - 8(1/x^6) + 1/x^8$

$= x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - 56(1/x^2) + 28$

$(1/x^4) - 8(1/x^6) + 1/x^8$

$(x^8+1/x^8) - 8(x^6 - 1/x^6) + 28(x^4+1/x^4) - 56$

$(x^2 - 1/x^2) + 70$

$= 2\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 70$

$\cancel{+70}$

$\sin^8\theta = 70[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 70$

$\sin^8\theta = 35[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35]$

$2^8 ; 8$

$$\boxed{\sin^8\theta = 1/27[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta]}$$

Lemma 5

$\sin^9\theta = \frac{-1}{256}[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta]$

Sol

Let

$(x - 1/x) = 2i\sin\theta$

Binomial Theorem.

$(x+a)^n = x^n + n c_1 x^{n-1} a + n c_2 x^{n-2} a^2 + \ldots$

$$\left(x-\frac{1}{x}\right)^9 = (2i\sin\theta)^9$$

$$(2i\sin\theta)^9 = \left(x-\frac{1}{x}\right)^9$$

$$2^9 i^9 \sin^9\theta = x^9 + {}^9C_1\, x^{9-1}(-1/x)^1 + {}^9C_2\, x^{9-2}(-1/x)^2$$

$$+ {}^9C_3\, x^{9-3}(-1/x)^3 + {}^9C_4\, x^{9-4}(-1/x)^4 + {}^9C_5\, x^{9-5}(-1/x)^5$$

$$+ {}^9C_6\, x^{9-6}(-1/x)^6 + {}^9C_7\, x^{9-7}(-1/x)^7 + {}^9C_8\, x^{9-8}(-1/x)^8$$

$$+ {}^9C_9\, x^{9-9}(-1/x)^9$$

$$= x^9 - 9x^8(1/x) + 36x^7(1/x^2) - 84x^6(1/x^3) +$$

$$126x^5(1/x^4) - 126x^4(1/x^5)$$

$$+ 84x^3(1/x^6) - 36x^2(1/x^7) + 126(x - 1/x)$$

$$- 1/x^9$$

$$= (x^9 - 1/x^9) - 9(x^7 - 1/x^7) + 36(x^5 - 1/x^5)$$

$$- 84(x^3 - 1/x^3) + 126(x - 1/x)$$

$$= \frac{2i\sin 9\theta - 9\,2i\sin 7\theta + 36\,2i\sin 5\theta - 84\,2i\sin 3\theta + 126\sin\theta.}{2^9 i^9}$$

$$= \frac{2i[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta]}{2^{9^8} i^{9^8}}$$

$$\sin^9\theta = 1/256\,[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta]$$

10m

(1) Prove that $\cos^5\theta\,(\sin^4\theta) = \frac{1}{256}[\cos 9\theta + \cos 7\theta$
$- 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta]$

Sol

$$\left(x+\frac{1}{x}\right)^5 \cdot \left(x-\frac{1}{x}\right)^4 = (2\cos\theta)^5\,(2i\sin\theta)^4$$

$$(2\cos\theta)^5(2i\sin\theta)^4 = \left(x+\frac{1}{x}\right)^5 \cdot \left(x-\frac{1}{x}\right)^4$$
$$= \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^4 \cdot \left(x-\frac{1}{x}\right)^4$$
$$= \left(x+\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^4$$

$$= \left(x+\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^4$$

$$= \left(x+\frac{1}{x}\right)\Big((x^2)^4 + 4C_1(x^2)^{4-1}\left(-\frac{1}{x^2}\right)$$
$$+ 4C_2(x^2)^{4-2}\left(-\frac{1}{x^2}\right)^2 + 4C_3$$
$$(x^2)^{4-3}\left(-\frac{1}{x^2}\right)^3 + 4C_4(x^2)^{4-4}$$
$$\left(-\frac{1}{x^2}\right)^4\Big)$$

$$= \left(x+\frac{1}{x}\right)\Big[x^8 + 4x^6\left(-\frac{1}{x^2}\right) + 6x^4\left(\frac{1}{x^4}\right)$$
$$+ 4x^2\left(-\frac{1}{x^6}\right) + \left(\frac{1}{x^8}\right)\Big]$$

$$= \left(x+\frac{1}{x}\right)\left[x^8 - 4x^4 + 6 - 4\frac{1}{x^4} + \frac{1}{x^8}\right]$$

$$= x^9 - 4x^5 + 6x - 4\frac{1}{x^3} + \frac{1}{x^7} + x^7 - 4x^3$$
$$+ 6\frac{1}{x} - 4\frac{1}{x^5} + \frac{1}{x^9}$$

$$= \left(x^9 + \frac{1}{x^9}\right) + \left(x^7 + \frac{1}{x^7}\right) - 4\left(x^5 + \frac{1}{x^5}\right)$$

$-4(x^5+1/x^5)-4(x^3+1/x^3)+6(x+1/x)$

$= [2(\cos 9\theta)+2(\cos 7\theta)-4(2)\cos 5\theta-4(2)\cos 3\theta+6(2)\cos\theta]$

$= 2[\cos 9\theta+\cos 7\theta-4\cos 5\theta-4\cos 3\theta+6\cos\theta]$

$(2^5\cos^5\theta)\cdot(2^4 . i^4 . \sin^4\theta) = 2[\cos 9\theta+\cos 7\theta-4\cos 5\theta-4\cos 3\theta+6\cos\theta]$

$\cos^5\theta\ \sin^4\theta = \frac{1}{2^5 . 2^4}[2[\cos 9\theta+\cos 7\theta-4\cos 5\theta-4\cos 3\theta+6\cos\theta]]$

$= \frac{1}{256}[2(\cos 9\theta+\cos 7\theta-4\cos 5\theta-4\cos 3\theta+6\cos\theta]$

∴ Tense Prufe

P.T $\cos^5\theta\ \sin^7\theta = \frac{-1}{2^{11}}[\sin 12\theta+2\sin 10\theta-4\sin 8\theta+10\sin 6\theta+5\sin 4\theta]$

Sol

$(x+\frac{1}{x})^5(x-1/x)^7 = (2\cos\theta)^5(2i\sin\theta)^7$

$(2\cos\theta)^5(2i\sin\theta)^7 = (x+1/x)^5(x-1/x)^7$

$= (x+1/x)^5 . (x-1/x)^5(x-1/x)^2$

$= (x-1/x)^2(x^2-1/x^2)^5$

$= (x-1/x)^2[(x^2)^5+5C_1(x^2)^{5-1}(-1/x^2)+5C_2(x^2)^{5-2}(-1/x^2)^2+5C_3(x^2)^{5-3}(-1/x^2)^3+5C_4(x^2)^{5-4}(-1/x^2)^4+5C_5(x^2)^{5-5}(-1/x^2)^5]$

$= (x - 1/x)^2 \left[x^{10} - 5x^6 + 10x^2 - 10\frac{1}{x^2} + 5\frac{1}{x^6} - \frac{1}{x^{10}}\right]$

$= \left(x^2 - 2 + \frac{1}{x^2}\right) \cdot \left[x^{10} - 5x^6 + 10x^2 - 10\frac{1}{x^2} + 5\frac{1}{x^6} - \frac{1}{x^{10}}\right]$

$= x^{12} - 5x^8 + 10x^4 - 10 + 5\frac{1}{x^4} - \frac{1}{x^8} - 2x^{10}$

$+ 10x^6 + 20x^2 -$

$20\frac{1}{x^2} + 10\frac{1}{x^6} + 2\frac{1}{x^{10}} + x^2 - 5x^6$

$+ 10 - 10\frac{1}{x^4} + 5\frac{1}{x^8} - \frac{1}{x^{12}}$

$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right)$

$+ 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$

$\cos 5 \sin^7\theta = \cancel{2i}\,[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta$

$+ 10\sin 6\theta + 5\sin 4\theta$

$- 20\sin 2\theta$

2

$2^{12},\ i^{4},\ i^{2}$

$\cos^5\theta \sin^7\theta = \frac{-1}{2^{n}}\,[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta$

$+ 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta$

5 mark

(3) $\sin^4\theta \cos^2\theta = \frac{1}{2^5}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$

Sol

$$(x - 1/x)^4 \cdot (x + 1/x)^2 = \sin^4\theta \cos^2\theta$$

$$(x - 1/x)^4 \cdot (x + 1/x)^2 = (2i\sin\theta)^4 (2\cos\theta)^2$$

$$(2i\sin\theta)^4 (2\cos\theta)^2 = (x - 1/x)^4 \cdot (x + 1/x)^2$$

$$= (x - 1/x)^2 (x - 1/x)^2 \cdot (x + 1/x)^2$$

$$= (x - 1/x)^2 (x^2 - 1/x^2)^2$$

$$= (x - 1/x)^2 \left[(x^2)^2 + 2C_1 (x^2)^{2-1} (-1/x^2)^1 + 2C_2 (x^2)^{2-2} (-1/x^2)^2\right]$$

$$= (x - 1/x)^2 \left[x^4 - 2x^2 (1/x^2) + 1/x^4\right]$$

$$= (x - 1/x)^2 \left[x^4 - 2 + 1/x^2\right]$$

$$= \left[x^2 + 1/x^2 - 2\right] \left[x^4 - 2 + 1/x^4\right]$$

$$= x^6 - 2x^2 + 1/x^2 + x^2 - 2/x^2 + 1/x^6 - 2x^4 + 4 - 2/x^4$$

$$= (x^6 + 1/x^6) - 2(x^4 + 1/x^4) - (x^2 + 1/x^2) + 4$$

$$\sin^4\theta = \frac{2(\cos 6\theta) - 2(2\cos 4\theta) - (\cos 2\theta) + }{2^4 \cdot i^4 \cdot 2^2}$$

$$= \frac{2[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]}{2^{\cancel{6}5} i^4}$$

$$\boxed{\sin^4\theta \cos^2\theta = \frac{1}{2^5}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]}$$

(2) $\sin^5\theta \cos^2\theta = \frac{1}{2^6}[\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta]$

<u>Sol</u>

$$(x - 1/x)^5 \cdot (x + 1/x)^2 = (2i\sin\theta)^5 \cdot (2\cos\theta)^2$$

$$(2i\sin\theta)^5 \cdot (2\cos\theta)^2 = (x - 1/x)^5 \cdot (x + 1/x)^2$$

$$= (x - 1/x)^3 \cdot (1 - 1/x)^2 (1 + 1/x)^2$$

$$= (x - 1/x)^3 \cdot (x^2 - 1/x^2)^2$$

$$= (x - 1/x)^3 \left[(x^2)^2 + 2c_1 (x^2)^{2-1} (-1/x^2)^1 + 2c_2 (x^2)^{2-2} (-1/x^2)^2\right]$$

$$= (x - 1/x)^3 \left[x^4 + 2x^2(-1/x^2) + 1/x^4\right]$$

$$= \left[x^3 - 3x^2(1/x) + 3x(1/x^2) - (1/x^3)\right]$$

$$\left[x^4 - 2 + 1/x^4\right]$$

$$= \left[x^3 - 3x + \frac{3}{x} - 1/x^3\right]\left[x^4 - 2 + 1/x^4\right]$$

$$= x^7 - 2x^3 + 1/x - 3x^5 - 6x - 3/x^3 + 3x^3 - 6/x + 3/x^5 - x$$

$$+ 2/x^3 - 1/x^7$$

$$= \left(x^7 - 1/x^7\right) - 3\left(x^5 - 1/x^5\right) + \left(x^3 - 1/x^3\right) + 5\left(x - 1/x\right)$$

$$= 2i\left[\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta\right]$$

$$\sin^5\theta\cos^2\theta = \frac{2i\left[\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta\right]}{2^7\, i^5}$$

$$\boxed{\sin^5\theta\,\cos^2\theta = \frac{1}{2^6}\left[\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta\right]}$$

---

(3) $\cos^4\theta\,\sin^3\theta = \frac{-1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$

Sol

$$(x + 1/x)^4 (x - 1/x)^3 = (2\cos\theta)^4 (2i\sin\theta)^3$$

$$(2\cos\theta)^4 (2i\sin\theta)^3 = (x + 1/x)^4 . (x - 1/x)^3$$

$$= (x + 1/x)(x + 1/x)^3 . (x - 1/x)^3$$

$$= (x + 1/x) . \left(x^2 - 1/x^2\right)^3$$

$$= (x + 1/x)\left[(x^2)^3 + {}^3C_1 (x^2)^{3-1}\left(-1/x^2\right) + {}^3C_2 (x^2)^{3-2}\left(-1/x^2\right)^2 + {}^3C_3 (x^2)^{3-3}\left(-1/x^2\right)^3\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 + 3x^4\left(-\frac{1}{x^2}\right) + 3x^2\left(\frac{1}{x^4}\right) - \frac{1}{x^6}\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$$

$$= (2i\sin 7\theta) + (2i\sin 5\theta) - 3(2i\sin 3\theta) - 3(\sin\theta$$

$$\cos^4\theta \sin^3\theta = \frac{2i\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]}{2^4 2^3 i^3}$$

$$= -\frac{1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$$\boxed{\cos^4\theta \sin^3\theta = -\frac{1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]}$$

Expansion of $\sin\theta$ and $\cos\theta$ in ascending power of θ.

Sol

w.k.t

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta \dots$$

$$= n\cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{2\cdot 3\cdot 1}\cos^{n-3}\theta \sin^3\theta \dots \quad \to (1)$$

Put $n\theta = \alpha$ in (1) we get $n = \alpha/\theta$

$$\sin\alpha = \frac{\alpha}{\theta}\cos^{n-1}\theta \sin\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)}{1\cdot 2\cdot 3}\cos^{n-3}\theta \sin^3\theta + \dots$$

$$\sin\alpha = \alpha\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{1\cdot 2\cdot 3}\cos^{n-3}\theta\,\frac{\sin^3\theta}{\theta^3} + \dots$$

$\theta \to 0$ and $n \to \alpha$ such that $n\theta = \alpha$ is

we know that as $\theta \to 0$ $\frac{\sin\theta}{\theta} \to 1$

$\cos\theta \to 1$ and therefore every power the quantities tends to unity

$$\therefore \sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \dots$$

we know that

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta$$

$$= \cos^n\theta - \frac{n(n-1)}{1\cdot 2}\cos^{n-2}\theta\,\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cos^{n-4}\theta\,\sin^4\theta - \cdots$$

Put $n\theta = \alpha$ we get

$$\cos\alpha = \cos^n\theta - \frac{\left(\frac{\alpha}{\theta}\right)\left(\frac{\alpha}{\theta}-1\right)}{1\cdot 2}\cos^{n-2}\theta\,\sin^2\theta + \cdots$$

$$\frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)}{1\cdot 2\cdot 3\cdot 4} - \cos^{n-4}\theta\,\sin\ldots$$

$$\cos\alpha = \cos^n\theta - \frac{\alpha(\alpha-\theta)}{1\cdot 2}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \cdots$$

$$\frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{1\cdot 2\cdot 3\cdot 4}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 \cdots$$

Let $\theta \to 0$ and $n \to \infty$ then $n\theta = \alpha$.

We know that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ and

$\lim_{\theta\to 0}\cos\theta = 1$ and there fore every Power of these quantities tends to unity.

Hense (1) becomes

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots$$

Note: α should be in radian

2.5m Expansion of $\tan\theta$ in Assending powers of θ

Sol

we know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)}{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right)}$$

$$= \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)\left[1 - \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} \cdots\right)\right]^{-1}$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} \cdots\right)\left[1 + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots\right) \left(+\frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots\right)\right]$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} \cdots\right)\left(1 + \frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots\right)$$

$$= \theta + \frac{\theta^3}{2} - \frac{\theta^5}{24} + \frac{\theta^5}{4} - \frac{\theta^3}{6} - \frac{\theta^5}{12} + \frac{\theta^5}{120} +$$

the term continuing θ^6 and so on

$$= \theta + \theta^3\left(\tfrac{1}{2} - \tfrac{1}{6}\right) + \theta^5\left(-\tfrac{1}{24} + \tfrac{1}{4} - \tfrac{1}{12} + \tfrac{1}{120}\right) + \theta^6(\cdots)$$

$$= \theta + \frac{2}{6}\theta^3 + \frac{16}{120}\theta^5 + \cdots$$

$$= \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$$

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⑥ $\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$

<u>Sol</u>

Let $(x - 1/x) = 2i\sin\theta$

Binomial Therem::

$$(x+a)^n = x^n + nC_1 x^{n-1}a + nC_2 x^{n-2} x^{n-2} a^2 + \cdots$$

$$(x - 1/x)^6 = (2i\sin\theta)^6$$

$$(2i\sin\theta)^6 = (x - 1/x)^6$$

$$2^6 i^6 \sin^6\theta = x^6 + 6C_1 x^{6-1}(-1/x) + 6C_2 x^{6-2}(-1/x)^2$$
$$+ 6C_3 x^{6-3}(-1/x)^3 + 6C_4 x^{6-4}(-1/x)^4 + 6C_5 x^{6-5}(-1/x)^5$$
$$+ 6C_5 x^{6-6}(-1/x)^6$$

$$= x^6 - 6x^5(1/x) + 15x^4(1/x^2) - 20x^3(1/x^3) + 15x^2(1/x^4) - 6x(1/x^6) + 1/x^6$$

$$= (x^6 + 1/x^6) - 6(x^4 - 1/x^4) + 15(x^2 + 1/x^2)$$

$$= \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 20$$

$$\sin^6\theta = \frac{\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 20}{2^6 i^6}$$

$$\boxed{\sin^6 = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)}$$

V:15M 1. $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$ Prove that, the angle θ is $1°58'$ nearly

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$= \theta\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots\right]$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{\sin\theta}{\theta} = \frac{5045}{5046}$$ (which is nearly $=1$ ∞ here θ must be very small)

$$= 1 - \frac{1}{5046}$$ ∴ There fore omitting θ^4 and higher powers.)

$$1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} = 1 - \frac{1}{5046}.$$

$$\cancel{1} - \frac{\theta^2}{3!} = \cancel{1} - \frac{1}{5046}$$ $\cancel{1} - \frac{\theta^2}{3!} - 1 = \frac{1}{5046}$

$$\frac{\theta^2}{3!} = \frac{1}{5046}$$ $\frac{\theta^2}{3!} = \frac{1}{}$

$$\frac{\theta^2}{6} = \frac{1}{5046}$$

$$\theta^2 = \frac{6}{5046}$$

841

$$\theta^2 = \frac{1}{841}$$

$$\theta = \frac{1}{29} \text{ radians}.$$

$$\theta = \frac{1}{2a} \times \frac{180}{\pi}$$

$$= 1^\circ 58'$$

2. Find $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

Sol

$$= \lim_{x \to 0} \frac{x - \sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{x - \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\left[x - x + \frac{x^3}{3!} - \frac{x^5}{5!} \cdots\right]}{x^3}$$

$$= \lim \frac{\left[\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\cancel{x^3}\left[\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right]}{\cancel{x^3}}$$

Hence $x \to 0$ omitting x^2 and higher powers.

$$= \frac{1}{3!}$$

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \frac{1}{3!} = 1/6$$

3. Evaluate $\lim_{\theta \to 0} \frac{\tan\theta - \sin\theta}{\theta^3}$

5M

Sol

$$= \lim_{\theta \to 0} \frac{\left[\left(\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots\right) - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)\right]}{\theta^3}$$

$$= \lim_{\theta \to 0} \frac{\left[\left(\cancel{\theta} + \frac{\theta^3}{3} + \frac{2\theta^5}{15} - \cancel{\theta} + \frac{\theta^3}{3!} - \frac{\theta^5}{5!}\right) + \text{higher Powers of } \theta\right]}{\theta^3}$$

$$= \lim_{\theta \to 0} \frac{\left[\theta^3\left(\frac{1}{3} + \frac{2\theta^2}{15} + \frac{1}{3!} - \frac{\theta^2}{5!}\right) + \text{higher Powers of } \theta\right]}{\theta^3}$$

$$= \lim_{\theta \to 0} \left[\frac{1}{3} + \frac{2\theta^2}{15} + \frac{1}{3!} - \frac{\theta^2}{5!}\right) + \text{higher Powers of } \theta\Big]$$

$$= \frac{1}{3} + \frac{1}{6}$$

$$= \frac{2+1}{6}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

H.W

4. Find θ approximately to the nearest minute of $\cos\theta = \frac{1681}{1682}$ 10m

Sol

$$\cos\theta = \frac{1681}{1682}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\cos\theta = \frac{1681}{1682}$$

(which is nearly = 1 x hents to must be very small)

There fore omitting θ^6 and higher power

$$= 1 - \frac{1}{1682}$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} = 1 - \frac{1}{1682}$$

$$= 1 - \frac{\theta^2}{2!} = 1 - \frac{1}{1682}$$

$$\Rightarrow \frac{\theta^2}{2!} = \frac{1}{1682}$$

$$\theta^2 = \frac{2}{1682}$$

$$\theta^2 = \frac{1}{841}$$

$$\theta = \frac{1}{29.13} \text{ radians}$$

$$\theta = \frac{1}{29.13} \times \frac{180}{\pi}$$

$$\theta = 1^\circ 58' 0.84''$$

Solve approximately $\sin(\pi/6 + \theta) = 0.51$

$\frac{300}{180/6}$

Sol

$$\sin(\pi/6 + \theta) = 0.51$$

$$\sin(\pi/6 + \theta) = \sin\pi/6 \cdot \cos\theta + \cos\pi/6 \sin\theta$$

$$= 1/2 \cdot \cos\theta + \sqrt{3}/2 \sin\theta$$

$$= \boxed{1/2\left(1 - \frac{\theta^2}{2!} + \cdots\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \cdots\right)}$$

which is nearly = 1/2 have θ must be very small omitting θ^2 and higher powers.

$$\sin(\pi/6 + \theta) = 1/2 + \frac{\sqrt{3}}{2}\theta$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51$$

$$\frac{\sqrt{3}}{2}\theta = 0.51 - \frac{1}{2}$$

$$= \frac{1.02 - 1}{2}$$

$$= \frac{0.02}{2}$$

$\frac{0.51 \times 2}{1.02}$

1.02
− 1.02
0.0

1.02
− 1.00
0.02

$$\frac{\sqrt{3}}{2}\theta = 0.01$$

$$\frac{\sqrt{3}}{2}\theta = \frac{1}{100}$$

$$\theta = \frac{1}{\cancel{100}_{50}} \times \frac{\cancel{2}}{\sqrt{3}}$$

$$\theta = \frac{1}{50\sqrt{3}}$$

$$= \frac{\sqrt{3}}{50 \times 3}$$

$$\theta = \frac{\sqrt{3}}{150} \text{ radian}$$

$$= \frac{\sqrt{3}}{150} \times \frac{180}{\pi}$$

$$= 0.1155 \times 57.325$$

$$= 0^\circ 39^\circ 43' 5''$$

$$\boxed{\theta = 39^\circ 43'}$$

6. Evaluate

$$\lim_{\theta \to 0} \left(\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} \right)$$

Sol

$$\lim_{\theta \to 0} \left(\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta} + 1} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\frac{\sin\theta + 1 - \cos\theta}{\cancel{\cos\theta}}}{\frac{\sin\theta - 1 + \cos\theta}{\cancel{\cos\theta}}} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\sin\theta + 1 - \cos\theta}{\sin\theta - 1 + \cos\theta} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right) + 1 - \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right)}{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right) - 1 + \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right)} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cancel{1} - \cancel{1} + \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)}{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cancel{1} + \cancel{1} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right)} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\cancel{\theta}\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \frac{\theta}{2!} - \frac{\theta^3}{4!} + \cdots\right)}{\cancel{\theta}\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \frac{\theta^2}{2!} + \frac{\theta^3}{4!} \cdots\right)} \right)$$

$$= (1/1)$$

$$\lim_{\theta \to 0} \left(\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} \right) = 1$$

---

5 mark

Evaluate

7. $$\lim_{\theta \to 0} \frac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \cos n\theta)}$$

$$= \lim_{\theta \to 0} \left(\frac{\left(n\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right) - \left(n\theta - \frac{n^3\theta^3}{3!} + \frac{n^5\theta^5}{5!} \cdots\right)\right)}{\left(\theta\left[\left(1 - \theta^2/2! + \theta^4/4! \cdots\right) - \left(n - \frac{n^2\theta^2}{2!} + \frac{n^4\theta^4}{4!} \cdots\right)\right]\right)} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\left(\cancel{n}\theta - \frac{n\theta^3}{3!} + \frac{n\theta^5}{5!} - \cancel{n}\theta + \frac{n^3\theta^3}{3!} - \frac{n^5\theta^5}{5!} + \cdots\right)}{\left(\theta\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - n + \frac{n^2\theta^2}{2!} - \frac{n^4\theta^4}{4!} \cdots\right)\right.} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\left(-\frac{n\theta^3}{3!} + \frac{n\theta^5}{5!} + \frac{n^3\theta^3}{3!} - \frac{n^5\theta^5}{5!} + \cdots\right)}{\theta\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - n + \frac{n^2\theta^2}{2!} - \frac{n^4\theta^4}{4!} + \cdots\right)} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{(n^3 - n)\theta^3/3! + (n - n^5)\theta^5/5! + \cdots}{\theta (n^2-1)\theta^3/2! + (1-n^4)\theta^4/4! \cdots} \right)$$

$$= \lim_{\theta \to 0} \left[\frac{\cancel{\theta^3}\left[\left(\frac{n^3-n}{3!}\right) + (n-n^5)\frac{\theta^2}{5!} + \cdots\right]}{\cancel{\theta^3}\left[\frac{(n^2-1)}{2} + (1-n^4)\frac{\theta^5}{4} \cdots\right]} \right]$$

$$= \frac{\frac{n^3-n}{3!}}{\frac{n^2-1}{2!}}$$

$$= \frac{n^3-n}{\cancel{6}\,3} \frac{\times\cancel{2}}{n^2-1}$$

$$= \frac{(n^3-n)}{3(n^2-1)}$$

$$= \frac{n(n^2-1)}{3(n^2-1)}$$

$$= n/3$$

10 m)

8) Determinate a, b, c Search that $\lim_{\theta \to 0} \dfrac{\theta(a + b\cos\theta) - c\sin\theta}{\theta^5} =$

$$= \lim_{\theta \to 0} \frac{\theta\left(a + b\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) - c\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)\right)}{\theta^5}$$

$$= \lim_{\theta \to 0} \frac{a\theta + b\theta\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) + (\cancel{\theta} - c\theta + \frac{c\theta^3}{3!} \cdots}{\theta^5}$$

$$= \lim_{\theta \to 0} \frac{a\theta + (b\theta - \frac{b\theta^3}{2!} + \frac{b\theta^5}{4!}) + (-c\theta + \frac{c\theta^3}{3!} + \frac{c\theta^5}{5!})}{\theta^5}$$

$$= \lim_{\theta \to 0} \frac{(a+b-c)\theta + (\frac{-b}{2!} + \frac{c}{3!})\theta^3 + (\frac{b}{4!} - \frac{c}{5!})\cancel{\theta^5}}{\cancel{\theta^5}} + \text{higher power}$$

$$= \lim_{\theta \to 0} \left[\frac{a+b-c}{\theta^4} + \frac{(\frac{-b}{2!} + \frac{c}{3!})}{\theta^2} + \frac{(\frac{b}{4!} - \frac{c}{5!})}{\cancel{\theta^4}} + \text{higher power of } \theta\right]$$

$$= \lim_{\theta \to 0} \left[\frac{a+b-c}{\theta^4} + \frac{(\frac{-b}{2!} + \frac{c}{3!})}{\theta^2} + (\frac{b}{4!} - \frac{c}{5!}) + \text{higher power of } \theta\right]$$

$$= \lim_{\theta \to 0} \left[\frac{a+b-c}{\theta^4}\right] + \frac{(\frac{-b}{2!} + \frac{c}{3!})}{\theta^2} + \left(\frac{b}{4!} - \frac{c}{5!}\right] \to ①$$

Given

$$\lim_{\theta \to 0} \frac{\theta(a + b\cos\theta) - c\sin\theta}{\theta^5} = 1$$

$$\lim_{\theta \to 0} \frac{a+b-c}{\theta^4} + \frac{\frac{-b}{2!} + \frac{c}{3!}}{\theta^2} + \frac{b}{4!} - \frac{c}{5!} = 1$$

This is possible if

$$a+b-c = 0 \to ②$$

$$\frac{-b}{2!} + \frac{c}{3!} = 0 \to ③$$

$$\frac{b}{4!} - \frac{c}{5!} = 1 \to ④$$

$$② \Rightarrow \frac{-b}{2} + \frac{c}{6} = 0$$

$$-3b + c = 0 \to ⑤$$

$$③ \Rightarrow \frac{b}{24} - \frac{c}{120} = 1$$

$$5b - c = 120 \to ⑥$$

2. $\lim_{x \to \pi/2} (\sec x - \tan x)$

Sol

Take $x = \pi/2 + \theta$

as $\theta \to 0$

$$\lim_{x \to \pi/2} (\sec x - \tan x) = \lim_{\theta \to 0} \left[(\sec(\pi/2+\theta) - \tan(\pi/2+\theta) \right]$$

$$= \lim_{\theta \to 0} \left[\frac{1}{\cos(\pi/2+\theta)} + \cot\theta \right]$$

$$= \lim_{\theta \to 0} \left[\frac{1}{-\sin\theta} + \frac{\cos\theta}{\sin\theta} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{-1+\cos\theta}{\sin\theta} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{-1+1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}}{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{\cancel{\theta^2}\left(-\frac{1}{2!} + \frac{\theta^2}{4!} \dots\right)}{\cancel{\theta}\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} \dots\right)} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{\theta\left(-\frac{1}{2!} + \frac{\theta^2}{4!} \dots\right)}{1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} \dots)} \right]$$

$$\times = \left[\frac{0\left(-1-\frac{1}{2!}\right) + \frac{0^2}{4!} \dots)}{\left(1 - \frac{0^2}{3!} + \frac{0^4}{5!} \dots\right)} \right]$$

$$= 0/1$$

$$= 0$$

$$\lim_{x \to \pi/2} (\sec x - \tan x) = 0$$

$$\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$$

Sol:-

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \lim_{x\to 0}\left[\frac{1}{x^2} - (\sin x)^{-2}\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x^2} - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\cdots\right)^{-2}\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x^2} - x^{-2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\cdots\right)^{-2}\right]$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$$

$$x = \frac{x^2}{6} - \frac{x^4}{120}$$

$$= \lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x^2}\left(1 + 2\left(\frac{x^2}{6} - \frac{x^4}{120}\right)\right)\right]$$

$$= \lim_{x\to 0}\left[\cancel{\frac{1}{x^2}} - \cancel{\frac{1}{x^2}} - \frac{2}{x^2}\left(\frac{x^2}{6} - \frac{x^4}{120}\right) + \cdots\right]$$

$$= \lim_{x\to 0}\left[\frac{-2}{x^2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)\right]$$

$$\Rightarrow \lim_{x\to 0} \frac{-2}{x^2} \cdot x^2\left(\frac{1}{6} - \frac{x^2}{120}\right)$$

$$= -\frac{2}{6} + 0 = -\frac{1}{3}$$

$$\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] = -\frac{1}{3}$$

8th sum continue.

(5) => $-3b+c=0$

$$5b-c=120$$

$$2b=120$$

$$\boxed{b=60}$$

$b=60$ in (5)

$$-3(60)+c=0$$

$$-180+c=0$$

$$\boxed{c=180}$$

$b=60$, $c=180$ in (2)

$$a+b-c=0$$

$$a+60-180=0$$

$$a-120=0$$

$$\boxed{a=120}$$

5m (2) Evaluate:

$$\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$$

Sol

$$\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$$

$$= \lim_{x\to 0} \frac{\left(x+\frac{x^3}{3}+\frac{2x^5}{15}\right) - \left(x-\frac{x^3}{3!}+\dots\right)}{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\right)^3}$$

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right) + \left(-x + \frac{x^3}{3!} - \frac{x^5}{5!}\right) + \cdots}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)^3}$$

$$= \lim_{x \to 0} \left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right) + \left(-x + \frac{x^3}{3!} - \frac{x^5}{5!}\right) +$$

$$= \lim_{x \to 0} \frac{x^3\left(\frac{1}{3!} + \frac{1}{3!}\right) + x^5\left(\frac{2}{15} - \frac{1}{5!}\right) + \cdots}{x^3\left(1 - \frac{x^2}{3!} + \frac{x^5}{5!} \cdots\right)^3}$$

$$= \lim_{x \to 0} \frac{x^3\left(\frac{1}{3!} + \frac{1}{3!}\right) + x^5\left(\frac{2}{15} - \frac{1}{5!}\right) + \cdots}{x^3\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)^3}$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{3} + \frac{1}{3!}\right) + x^2\left(\frac{2}{15} - \frac{1}{5!}\right) + \cdots}{x^3\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots\right)^3}$$

$$= \frac{\frac{1}{3} + \frac{1}{3!}}{1}$$

$$\left(x\left[1 - \frac{x^2}{3!} + \frac{x^5}{5!} \cdots\right]\right)^3$$

$$= 1/3 + 1/6$$

$$x^3\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right]^3$$

$$= \frac{2+1}{6}$$

$$= 3/6$$

$$= 1/2.$$

Q1) Evaluate $\lim_{x\to 0} \frac{\sin x - \tan x}{x^2}$

Sol

$= \lim_{x\to 0} \frac{\sin x - \tan x}{x^2}$

$= \lim_{x\to 0} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) - \left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right.$

$= \lim_{x\to 0} x^3\left(-\frac{1}{3} - \frac{1}{3!}\right) + x^5\left(\frac{1}{5!} - \frac{2}{15}\right)$

$\lim_{x\to 0} \frac{\sin x - \tan x}{x^2} = 0$

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(2) Evaluate $\lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

Sol

$= \lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

$= \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$

$= \lim_{x\to 0} \left(\frac{1 - \cos x}{\sin x} * \frac{1}{x}\right)$

$= \lim_{x\to 0} \left[\frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)}\right] \times \frac{1}{x}$

$$= \lim_{x \to 0} \left[\frac{1 - 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots\right)} \right] \times \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2!} + \frac{x^4}{4!} \cdots}{x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}\left(\frac{1}{2!} + \frac{x^2}{4!} \cdots\right)}{\cancel{x^2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots\right)}$$

$$= \frac{1}{2!}$$

$$= \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x} = \frac{1}{2}$$

---

7) $\lim_{x \to 0} \frac{\tan 2x - 2\sin x}{x^3}$ $\qquad \tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5$

Sol

$$\lim_{x \to 0} \frac{\tan 2x - 2\sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(2x + \frac{2^3 x^3}{3} + \frac{2^5 x^5 \cdot 2}{15} + \cdots\right) - 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)}{x^3}$$

5m Show that the error involve in replaysing $\frac{1}{3}(8\sin\theta/2 - \sin\theta)$ by θ its approximatly $\frac{\theta^5}{480}$ θ is small.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$$

$$\theta/2 - \frac{(\theta/2)^3}{3!} + \frac{(\theta/2)^3}{5!} + .$$

Sol

$$= \frac{1}{3}\left(8\left(\theta/2 - \frac{(\theta/2)^3}{3!}\right.\right.$$

Sol

$= \frac{1}{3}(8\sin\theta/_2 - \sin\theta)$

$= 8/_3 \sin\theta/_2 - 1/_3 \sin\theta$

$= 8/_3\left(\theta/_2 - \frac{(\theta/2)^3}{3!} + \frac{(\theta/2)^5}{5!} - \right) - 1/_3\left(\theta - \frac{\theta^3}{3!}\right.$

$\left. + \frac{\theta^5}{5!} \cdots\right)$

$= 8/_3\left(\theta/_2 - \frac{\theta^3}{8\cdot 3!} + \frac{\theta^5}{32\cdot 5!} \cdots\right) - 1/_3\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$

$= \frac{8\theta}{6} - \frac{8\theta^3}{8\cdot 3\cdot 3!} + \frac{8\theta^5}{3\cdot 32\cdot 5!} \cdots) - \left(\theta/_3 - \frac{\theta^3}{3\cdot 3!} + \frac{\theta^5}{3\cdot 5!}\right.$

$= \left(\frac{4\theta}{3} - \frac{\theta^3}{3\cdot 3!} + \frac{\theta^5}{12\, 5!} - \frac{\theta}{3!} + \frac{\theta^3}{3\cdot 3!} - \frac{\theta^5}{3\cdot 5!}\right) +$

higher Powers of θ

$= \left(\frac{4\theta}{3} + \frac{\theta^5}{12\cdot 5!} - \frac{\theta}{3} - \frac{\theta^5}{3\cdot 5!}\right) +$ highers Powers of θ

$= \theta/_3(4-1) + \frac{\theta^5}{5!}\left(1/_{12} - 1/_3\right) +$ higher Powers of θ

$= \frac{\theta}{3}(3) + \frac{\theta^5}{5!}\left(\frac{1-4}{12}\right)$

$= \frac{\theta}{3}(3) + \frac{\theta^5}{5!}\left(-3/_{12}\right)$

$= \frac{\theta}{3}(3) + \frac{\theta^5}{5!}\left(-1/_4\right)$

$= \theta - \theta^5/_{480}$

$\lim_{x\to 0}$ hense the error in taking $\frac{1}{3}\left(\frac{8\sin\theta/2}{\theta}\right.$

$\left.-\sin\theta\right)$ θ is $\frac{\theta^5}{480}$

$$\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$$

<u>Sol</u>

$$\frac{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right] - x + \frac{x^3}{6}}{x^5}$$

$$= \frac{\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \cancel{x} + \frac{x^3}{6}}{x^5}$$

$$= \frac{\frac{x^5}{5!}}{x^5}$$

$$= 1/x^5 \left[\frac{x^5}{5!}\right]$$

$$= 1/5!$$

$$= 1/120$$

5M H.W.

If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$ find the value of θ approximately

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

$$= \theta\left[1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \cdots\right]$$

$$\frac{\tan\theta}{\theta} = 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \cdots$$

$$\frac{\tan\theta}{\theta} = \frac{2524}{2523}$$

$$= 1 + \frac{1}{2523}$$ (which is nearly = 1 & hence θ must be very small)

$$1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} = 1 + \frac{1}{2523}$$ ∴ omitting $2\theta^4$ and higher powers)

$$\frac{\theta^2}{3} + \frac{2\theta^4}{15} = \frac{1}{2523}$$

$$\frac{\theta^2}{3} = \frac{1}{2523}$$

$$\theta^2 = \frac{3}{2523}$$

$$\theta^2 = \frac{1}{841}$$

$$\theta = \frac{1}{29} \text{ radians.}$$

$$\theta = \frac{1}{29} \times \frac{180}{\pi}$$

$$\theta = 1°58'$$

2. If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$ S.T θ is nearly equal to 3.

$$\sin\theta = \left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right]$$

$$= \theta\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots\right]$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!}$$

$$\frac{\sin\theta}{\theta} = \frac{2165}{2166}$$

$$= 1 - \frac{1}{2166}$$

which is nearly $= 1$ & hence θ must be very small.

$\therefore$ omitting θ^4 and higher powers.

$$1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} = 1 - \frac{1}{2166}$$

$$\frac{\theta^2}{3!} = \frac{1}{2166}$$

$$\frac{\theta^2}{6} = \frac{1}{2166}$$

$$\theta^2 = \frac{6}{2166}$$

$$\theta^2 = \frac{1}{361}$$

19×19
171
19
361

$$\theta = \frac{1}{19} \text{ radians}$$

$$\theta = \frac{1}{19} \times \frac{180}{\pi}$$

$$\theta = \frac{180}{19\pi}$$

$$\theta = 3^\circ 0'$$

3. 5m

$$\lim_{x \to 0} \frac{5\sin x - \sin 5x}{x(\cos x - \cos 5x)}$$

<u>Sol</u>

$$\lim_{x \to 0} \frac{5\sin x - \sin 5x}{x(\cos x - \cos 5x)}$$

$$= \lim_{x \to 0} \left[\frac{5\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) - \left(5x - \frac{5^3 x^3}{3!} + 5^5 x^5\right)}{x\left(\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right) - \left(1 - \frac{5^2 x^2}{2!} + \frac{5^4 x^4}{4!}\right)\right)} \right]$$

$$= \lim_{x \to 0} \left[\frac{\cancel{5x} - \frac{5x^3}{3!} + \frac{5x^5}{5!} - \cancel{5x} + \frac{5^3 x^3}{3!} + \frac{5^5 x^5}{5!}}{x\left[\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cancel{1} + \frac{5^2 x^2}{2!} - \frac{5^4 x^4}{4!}\right]} \right]$$

$$= \lim_{x \to 0} \left[\frac{(5^3 - 5)\, x^3/3! + (5^5 - 5)\, x^5/5!}{x (5^2 - 1)\, x^2/2! - (5^4 - 1)\, x^4/4!} \right]$$

$$= \lim_{x \to 0} \left[\frac{x^3 \left(\frac{5^3 - 5}{3!}\right) + (5^5 - 5)\frac{x^5}{5!}}{x^3 \left(\frac{5^2 - 1}{2!}\right) - \frac{(5^4 - 1)x^4}{4!}} \right]$$

$$= \frac{5^3 - 5}{3!} \times \frac{2}{5^2 - 1}$$

$$= \frac{5(\cancel{5^2 - 1})}{3!} \times \frac{2}{\cancel{5^2 - 1}}$$

$$= \frac{5}{\cancel{6}\,3} \times \cancel{2}$$

$$= \frac{5}{3}$$