UNIT - IV

Exponential function of a complex variable:

we are already familiar with the exponential function, when x is real

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \infty$$

Similarly we can define the exponential function of the complex variable $z = x + iy$

as $$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \infty \rightarrow (1)$$

Putting $x = 0$ in (1) we get

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots \infty$$

$$= 1 + \frac{iy}{1!} - \frac{y^2}{2} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \dots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} \dots\right)$$

formula 7.

$$e^{iy} = \cos y + i \sin y$$

Hence $e^z = e^{x+iy}$

$= e^x . e^{iy}$

$$e^z = e^x(\cos y + i \sin y)$$

Similarly

$$e^{-iy} = \cos y - i \sin y$$

Hence $\cos y$ is the real part of e^{iy}

and $\sin y$ is the Imaginary Part of e^{iy}

<u>circular founction of a complex variable</u>:

$$\sin\theta = e^{iy} = \cos y + i\sin y \text{ and}$$

$$e^{-iy} = \cos y - i\sin y$$

The circular function of reel angles can bee written.

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on..}$$

circular function of the complex variable z by the angle.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}.$$

with $\operatorname{cosec} z$, $\sec z$, $\cot z$. as their respective reciprocalls.

De Moivre's theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

be expressed as

$(e^{i\theta})^n = e^{in\theta}$

Note:

From the above result we have for all values of x, real (or) complex.

$$e^{ix} = \cos x + i\sin x \rightarrow ①$$

$$e^{-ix} = \cos x - i\sin x \rightarrow ②$$

Adding ① & ② we get

$$e^{ix} + e^{-ix} = 2\cos x$$

Subtracting ① & ② we get

$$e^{ix} - e^{-ix} = 2i\sin x$$

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan x = \left(\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right)\frac{1}{i}$$

$$\sec x = \frac{1}{\cos x}$$

$$\boxed{\sec x = \frac{2}{e^{ix} + e^{-ix}}}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\boxed{\operatorname{cosec} x = \frac{2i}{e^{ix} - e^{-ix}}}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\boxed{\cot x = i\,\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}}$$

Hyperbolic functions

If x be a real (or) complex, the expression i) $\frac{e^{x}-e^{-x}}{2}$ is defined as hyperbolic sin of x and it is denoted by $\sinh x$

(i.e.) $\boxed{\sinh x = \frac{e^{x}-e^{-x}}{2}}$

ii) $\frac{e^{x}+e^{-x}}{2}$ is defined as hyperbolic cosine of x and it is denoted by $\cosh x$.

(i.e) $\boxed{\cosh x = \frac{e^{x}+e^{-x}}{2}}$

Similarly we can define $\tanh x = \frac{\sinh x}{\cosh x}$

$$\tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\boxed{\coth x = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\boxed{\operatorname{sech} x = \frac{2}{e^{x}+e^{-x}}}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\boxed{\operatorname{cosech} x = \frac{2}{e^{x}-e^{-x}}}$$

Relative between hyperbolic functions and circular functions.

We know that for all (var) values of θ.

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \text{ and}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

Putting $\theta = ix$ we get

$$\sin ix = \frac{e^{i(ix)}-e^{-i(ix)}}{2i}$$

$$\sin ix = \frac{e^{-x}-e^{x}}{2i}$$

$$= -\left(\frac{e^{x}-e^{-x}}{2i}\right)$$

$$= i^2 \left(\frac{e^{x} - e^{-x}}{2i} \right)$$

$$= i \sinh x$$

$$\boxed{\sin ix = i \sinh x} - i(ix)$$

$$\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2}$$

$$\cos ix = \frac{e^{-x} + e^{x}}{2}$$

$$\cos ix = \frac{e^{x} + e^{-x}}{2}$$

$$\boxed{\cos ix = \cosh x}$$

$$\tan ix = \frac{\sin ix}{\cos ix}$$

$$\tan ix = \frac{i \sinh x}{\cosh x}$$

$$\boxed{\tan ix = i \tanh x}$$

* $\sin ix = i - \sinh x$
* $\cos ix = \cosh x$
* $\tan ix = i \tanh x$

Corollary :-

W.K.T $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

Put $\theta = ix$ we get.

$\sinh ix = e^{ix}$

$\sinh ix = i\dfrac{(e^{ix} - e^{-ix})}{2i}$

$\boxed{\sinh ix = i\sin x}$

Also, $\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$

Put $\theta = ix$ we get

$\cosh ix = \dfrac{e^{ix} + e^{-ix}}{2}$

$\cosh ix = \cos x$

$\boxed{\tanh ix = \sinh ix}$

$\tanh ix = \dfrac{\sinh ix}{\cosh ix}$

$= \dfrac{i\sin x}{\cos x}$

$\tanh ix = i\tan x$

* $\sinh ix = i\sin x$
* $\cosh ix = \cos x$
* $\tanh ix = i\tan x$

FORMULAE FOR HYPERBOLIC FUNCTIONS :-

1) $\cos^2 hx - \sin^2 hx = 1$

Proof:

$$LHS = \cosh^2 x - \sinh^2 x$$

$$= (\cos ix)^2 - \left(\frac{\sin ix}{i}\right)^2$$

$$= \cos^2 ix - \frac{\sin^2 ix}{i^2}$$

$$= \cos^2 ix + \sin^2 ix$$

$$= 1 = RHS$$

$$LHS = RHS$$

Similarly we can easily prove that following results.

2. $\operatorname{sech}^2 x + \tanh^2 x = 1$

3. $\coth^2 x - \operatorname{cosech}^2 x = 1$

4. $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \sinh y$

Proof:

$$\sinh(x+y) = \frac{1}{i} \sin(x+y)$$

$$= \left\{\sinh x = \frac{1}{i} \sin ix\right\}$$

$$= -i\left[\sin(ix+iy)\right]$$

$$= -i\left[\sin x \cos iy \pm \cos ix \cdot \sin iy\right]$$

$$= -i\left[i\sinh x \cosh y \pm i\cosh x \cdot \sinh y\right]$$

$$\sin(x \pm y) = \sinh x \cosh y \pm \cosh y \pm \cosh x \sinh y$$

(13) $\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

(14) $\sinh x - \sinh y = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

(15) $\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

(16) $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\frac{x+y}{2}$

Inverse hyperbolic functions:

If $x = \cosh y$, then we define $y = \cosh^{-1} x$ similarly if $x = \sinh y$ then $y = \sinh^{-1} x$ and so on. we will now show that $\sinh^{-1} x$, $\cosh^{-1} x$ are all single valued logarithmic function of x.

Example:-

Prove that $\sinh^{-1} x = \log_e (x + \sqrt{x^2+1})$

Proof:-

$\sinh x = \frac{e^x - e^{-x}}{2}$

LHS

G.T: $\sinh^{-1} x = y$

$x = \sinh y$

$x = \frac{e^y - e^{-y}}{2}$

$x = \frac{e^y - 1/e^y}{2}$

$x = \frac{\frac{e^{2y} - 1}{e^y}}{}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{e^{2y} - 1}{2e^y}$$

$$2x e^y = e^{2y} - 1$$

$$2x e^y = (e^y)^2 - 1$$

$$(e^y)^2 - 2x e^y - 1 = 0$$

$$e^y = \frac{\pm 2x \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= \frac{\cancel{2}(x \pm \sqrt{x^2+1})}{\cancel{2}}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$e^y > 0$ so, we take +ve term of e^y

$$e^y = x + \sqrt{x^2+1}$$

Take log on both sides

$$\log_e (e^y) = \log_e (x + \sqrt{x^2+1})$$

$y = \log_e (x + \sqrt{x^2+1})$

$\sin^{-1} x = \log_e (x + \sqrt{x^2+1})$

Jense Proof

2. Prove that $\tanh^{-1} x = \frac{1}{2} \log \left(\frac{1+x}{1-x}\right)$

Proof

Take $\tanh^{-1} x = y$ $\quad \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$x = \tanh y$$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$= \frac{e^y - 1/e^y}{e^y + 1/e^y}$$

$$= \frac{\dfrac{e^{2y} - 1}{e^y}}{\dfrac{e^{2y}+1}{e^y}}$$

$$= \frac{e^{2y} - 1}{e^{2y}+1}$$

$(e^{2y}+1)x = e^{2y} - 1$

$e^{2y}x + x = e^{2y} - 1 \quad = e^{2y}x - e^{2y} =$

$e^{2y} - e^{2y}x = x + 1$

$e^{2y}(1 - x) = x + 1$

$$e^{2y} = \frac{x+1}{1-x}$$

$$\log e^{2y} = \log \frac{x+1}{1-x}$$

$$2y = \log\left(\frac{x+1}{1-x}\right)$$

$$y = \frac{1}{2} \log \left(\frac{x+1}{1-x}\right)$$

$$\tanh^{-1} x = \frac{1}{2} \log \left(\frac{1+x}{1-x}\right)$$

(3) Separate in to real and imaginary Parts of $\sin(x+iy)$ $[\because \cos iy = \cosh y]$

$[\because \sin iy = i\sinh y]$

Sol

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

$$\sin(x+iy) = \sin x \cancel{+} \cos$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + \cos x\, i \sinh y$$

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

Real Part $= \sin x \cosh y$

Imaginary Part $= \cos x \sinh y$

2) $\tan(x+iy)$

Sol

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$\sin(A+B)+\sin(A-B) = 2\sin(A+B)\cos(A-B)$$

$$= \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin(x+iy+x-iy)+\sin(x+iy-x+iy)}{\cos(x+iy+x-iy)+\cos(x+iy-x+iy)}$$

$$= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos(2iy)}$$

$$= \frac{\sin 2x + \sin i(2y)}{\cos 2x + \cos i(2y)}$$

$$= \frac{\sin 2x + i\sinh(2y)}{\cos 2x + \cosh(2y)}$$

$$\tan(x+iy) = \frac{\sin 2x}{\cos 2x + \cosh y} + i\,\frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Real Part $= \dfrac{\sin 2x}{\cos 2x + \cosh y}$

Imaginary Part $= \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$

$\cosh^{-1}x = \log(x+\sqrt{x^2-1})$

$\cosh^{-1}x = y$

$x = \cosh y$

$\cosh y = \dfrac{e^x+e^{-x}}{2}$

$x = \dfrac{e^y+e^{-y}}{2}$

$= \dfrac{e^y+(1/e^y)}{2}$

$= \dfrac{\dfrac{e^{2y}+1}{e^y}}{2}$

$= \dfrac{e^{2y}+1}{2e^y}$

$\cancel{2xe^y}$

$2xe^y = e^{2y}+1$

$2xe^y = (e^y)^2+1$

$(e^y)^2 - 2xe^y + 1 = 0$

$e^y = \dfrac{\pm 2x \sqrt{(-2x)^2-4(1)(1)}}{2(1)}$

$= \dfrac{\cancel{\pm} 2x \pm \sqrt{4x^2-4}}{2}$

$= \dfrac{2x \pm \sqrt{4(x^2-1)}}{2}$

$= \dfrac{\cancel{2y} \pm 2\sqrt{x^2-1}}{2}$)

$$= \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$= \frac{\cancel{2}(x \pm \sqrt{x^2-1})}{\cancel{2}}$$

$$= x \pm \sqrt{x^2-1}$$

$$e^y = x + \sqrt{x^2-1}$$

$e^y > 0$ so, we take + ve term of e^y

$$e^y = x + \sqrt{x^2-1}$$

Take log on both sides

$$\log_e(e^y) = \log_e(x + \sqrt{x^2-1})$$

$$y = \log_e(x + \sqrt{x^2-1})$$

$$\cos^{-1}x = \log_e(x + \sqrt{x^2-1})$$

Hense Prufe.

$\cosh(x+iy)$

W.K.T

$\cosh x = \cos ix$

$\cosh(x+iy) = \cos i(x+iy)$

$= \cos(ix-y)$

$= \cos ix \cos y + \sin ix \cdot \sin y$

$\cosh(x+iy) = \cosh x \cos y + i \sinh x \cdot \sin y$

Real Part $= \cosh x \cos y$

Imaginary Part $= \sinh x \cdot \sin y$

(ii) $\coth(x+iy)$

$\coth(x+iy) = \dfrac{\cosh(x+iy)}{\sinh(x+iy)}$

$$\coth(x+iy) = \frac{\cos i(x+iy)}{\frac{1}{i}\sin i(x+iy)}$$

$$= i\,\frac{\cos(ix-y)}{\sin(ix-y)}$$

$$= i\,\frac{2\cos(ix-y)\cdot\sin(ix+y)}{2\sin(ix-y)\cdot\sin(ix+y)}$$

$$= i\left[\frac{\sin[ix-y+ix+y]-\sin[ix-y-ix-y]}{\cos[ix-y-ix-y]-\cos(ix-y+ix+y)}\right]$$

$$= i\left[\frac{\sin(2ix)-\sin(-2y)}{\cos(-2y)-\cos(2ix)}\right]$$

$$= i\left[\frac{i\sinh 2x+\sin 2y}{\cos 2y-\cosh 2x}\right]$$

$$= \frac{-\sinh 2x+i\sin 2y}{\cos 2y-\cosh 2x}$$

$$\coth(x+iy) = \frac{\sinh 2x - i\sin 2y}{\cosh 2x-\cos 2y}$$

$\therefore$ Real Part $= \frac{\sinh 2x}{\cosh 2x-\cos 2y}$

imaginary Part $= \frac{-\sin 2y}{\cosh 2x-\cos 2y}$

(iii) $\operatorname{cosech}(x+iy) = \dfrac{1}{\sinh(x+iy)}$

$$= \frac{1}{\frac{1}{i}[\sin i(x+iy)]}$$

$\sin(ix) = i\sinh x$

$\sinh x = \frac{1}{i}\sin(ix)$

$$= \frac{i\,2\sin(ix+y)}{2\sin(\underset{A}{x-y})\sin(\underset{B}{ix+y})}$$

$$= \frac{i2(\sin(ix)\cdot\cos y + \cos(ix)\sin y)}{\cos(ix-y-ix-y) - \cos(ix-y+ix+y)}$$

$$= \frac{i\,2\sinh x\cos y + i2\cosh x\sin 2y}{\cos 2y - \cosh(2ix)}$$

Real Part $= \dfrac{-2\sinh x\cos y}{\cos 2y - \cosh 2x}$

imaginary part $= \dfrac{2\cosh x\sin y}{\cos 2y - \cosh 2x}$

2M

(1) separate into real and imaginary part

i) $\cos(x+iy)$

Sol

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(x+iy) = \cos x\cos iy - \sinh\sin iy$

$\cos(x+iy) = \cos x \cosh y - i\sin x \sinh y$

$\cos(x+iy) = \cos x \cosh y - i\sin x \sinh y$

Hence

Real Part $= \cos x \cosh y$

$\operatorname{cosec}(x+iy)$

Sol

$$\operatorname{cosec}(x+iy) = \frac{1}{\sin(x+iy)}$$

$$= \frac{\sin(x-iy)}{\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\sin x \cos iy - \cos x \sin iy}{\frac{1}{2}[\cos(x-iy+x+iy) - \cos(x-iy-x-iy)]}$$

$$= \frac{\sin x \cosh y - \cos x \sinh y}{\frac{1}{2}[\cos(2x) - \cos(-2iy)]}$$

$$\operatorname{cosec}(x+iy) = \frac{2\sin x \cosh y}{\cos 2x - \cosh 2y} - \frac{2i\cos x \sinh y}{\cos 2x - \cosh 2y}$$

Imaginary Part $= \dfrac{-2\cos x \sinh y}{\cos 2x - \cosh 2y}$

ii) $\sinh(x+iy)$

Sol

$$\frac{i\sinh x}{i} = \frac{\sin ix}{i}$$

$$\sinh(x+iy) = \frac{1}{i}\left(\sin i(x+iy)\right)$$

$$= (-i)\sin(ix-y)$$

$$= (-i)(\sin ix\cos y - \cos ix \sin y)$$

$$= -i(i\sinh x\cos y - \cosh x\sin y)$$

$$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$$

Real Part $= \sinh x\cos y$

Imaginary Part $= \cosh x\sin y$

iii) $\tanh(x+iy)$

Sol

$$\tan(x+iy) = \frac{\sinh(x+iy)}{\cosh(x+iy)}$$

$$= \frac{-i\sin i(x+iy)}{\cos i(x+iy)}$$

$$= \frac{-i\sin(ix-y)}{\cos(ix-y)}\,\frac{\cos(ix+y)}{\cos(ix+y)}$$

$$= \frac{2\sin(ix-y)\cdot\sin(ix+y)}{\frac{1}{i}\,2\cos(ix-y)\cdot\sin(ix+y)}$$

$$= \frac{-i \sin(x-y)\cos(ix+y)}{\cos(ix-y)\cos(ix+y)}$$

$$= \frac{-i(\sin(ix-y+ix+y)+\sin(ix-y-ix-y)}{\cos(ix-y+ix+y)+\cos(ix-y-ix-y)}$$

$$= \frac{-i(\sin i2x + \sin(-2y))}{\cos 2ix + \cos(-2y)}$$

$$= \frac{-i(i\sinh 2x - \sin 2y)}{\cosh 2x - \cos 2y}$$

$$= \frac{\sinh 2x + i\sin 2y}{\cosh 2x - \cos 2y}$$

Real Part $= \frac{\sinh 2x}{\cosh 2x - \cos 2y}$

Imaginary Part $= \frac{\sin 2y}{\cosh 2x - \cos 2y}$

11) $\operatorname{sech}(x+iy)$

Sol

$$\operatorname{sech}(x+iy) = \frac{1}{\cosh(x+iy)}$$

$$= \frac{1}{\cos i(x+iy)}$$

$$= \frac{1}{\cos(ix-y)}$$

$$= \frac{\cos(ix+y)}{\cos(ix-y)\cos(ix+y)}$$

$$= \frac{\cos ix \cos y - \sin ix \sin y}{\frac{1}{2}\cos(ix-y+ix+y) + \cos(ix-y-ix-y)}$$

$$= \frac{\cos ix \cos y - \sin ix \sin y}{\frac{1}{2}\cos 2ix + \cos(-2y)}$$

$$= \frac{2(\cosh x - i\sin x \sin y)}{\cosh 2x - \cos 2y}$$

$$= \frac{2\cosh x - 2i\sinh x \sin y}{\cosh 2x - \cos 2y}$$

Real Part

$$= \frac{2\cosh x}{\cosh 2x - \cos 2y}$$

Imaginary Part $= \dfrac{-2\sinh x \sin y}{\cosh 2x - \cos 2y}$

$\cot(x+iy)$

Sol

$$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)}$$

$$= \frac{\cos(x+iy)}{\sin(x+iy)} \times \frac{\sin(x+iy)}{\sin(x-iy)}$$

$$= \frac{\cos(x+iy)\ \sin(x-iy)}{\sin(x+iy)\ \sin(x-iy)}$$

$$= \frac{2(\sin(x+iy+x-iy) - \sin(x-iy-x+iy))}{2(\cos(x+iy-x+iy) - \cos(x+iy+x-iy))}$$

$$= \frac{\sin 2x - \sin 2iy}{\cos 2iy - \cos 2x}$$

$$= \frac{\sin 2x - \sin 2iy}{\cos 2iy - \cos 2x}$$

$$= \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$= \frac{-i\sinh 2y}{\cosh 2y - \cos 2x}$$

Real Part.

$$= \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Imaginary Part $= \dfrac{-\sin 2y}{\cosh 2y - \cos 2x}$

Separate real and imaginary parts of

$\tan^{-1}(\alpha+i\beta)$

Sol:

Let $\tan^{-1}(\alpha+i\beta) = x+iy \longrightarrow (1)$

x, y real and imaginary part

changing 'i' into '-i'

$\alpha - i\beta = \tan(x-iy) \longrightarrow (2)$

$2x = (x+iy)+(x-iy)$

$\tan 2x = \tan((x+iy)+(x-iy))$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$= \frac{\alpha+i\beta+\alpha-i\beta}{1-(\alpha+i\beta)(\alpha-i\beta)}$$

$$= \frac{2\alpha}{1-(\alpha^2+\beta^2)}$$

$$\tan 2\alpha = \frac{2\alpha}{1-\alpha^2-\beta^2}$$

$$x = \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$$

$2iy = (x+iy)-(x-iy)$

$\tan 2iy = \tan((x+iy)-(x-iy))$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$\tan 2iy = \frac{\alpha+i\beta-\alpha+i\beta}{1+(\alpha+i\beta)(\alpha-i\beta)}$$

$$= \frac{2i\beta}{1+(\alpha^2+\beta^2)}$$

$$2iy = \tan^{-1}\left(\frac{2i\beta}{1+\alpha^2+\beta^2}\right)$$

$$y = \frac{1}{2}\tan^{-1}\left(\frac{2\beta}{1+\alpha^2+\beta^2}\right)$$

4) If $\sin(A+iB) = x+iy$ Prove

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1 \text{ and } \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

Sol

$$\sin(A+iB) = x+iy$$

$$\sin A \cos iB + \cos A \sin iB = x+iy$$

$$\sin A \cosh B + i\cos A \sinh B = x+iy$$

Equating real Part and Imaginary Part $\left\{\begin{array}{l}\cos ix = \cosh x \\ \sin ix = i\sinh x\end{array}\right\}$

we get

$$x = \sin A \cosh B$$

$$y = \cos A \sinh B$$

$$x^2 = \sin^2 A \cosh^2 B$$

$$y^2 = \cos^2 A \sin^2 B$$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \frac{\sin^2 A \cosh^2 B}{\cosh^2 B} + \frac{\cos^2 A \sinh^2 B}{\sinh^2 B}$$

$$= \sin^2 A + \cos^2 A$$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \frac{\sin^2 A \cosh^2 B}{\sin^2 A} - \frac{\cos^2 A \sinh^2 B}{\cos^2 A}$$

$$= \cosh^2 B - \sinh^2 B$$

$$= 1$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

5m

2. If $u+iv = \cosh(x+iy)$ Prove that

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1 \text{ and } \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$$

G.T $u+iv = \cosh(x+iy)$

$\cos ix = \cosh x$

$= \cos i(x+iy)$

$\sin ix = i\sinh$

$= \cos(ix-y)$

$u+iv = \cos ix \cos y + \sin ix \sin y$

$= \cosh x \cos y + i\sinh x \sin y$

$u = \cosh x$

$\sinh x \sin y$

(1)

$$u^2 = \cosh^2 x \cos^2 y \rightarrow (1)$$

$$v^2 = \sinh^2 x \sin^2 y \rightarrow (2)$$

$(1) \Rightarrow$

$$\frac{u^2}{\cosh^2 x} = \cos^2 y \rightarrow (3)$$

$$(2) \Rightarrow \frac{v^2}{\sinh^2 x} = \sin^2 y \rightarrow (4)$$

$$(3)+(4) \Rightarrow \frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = \cos^2 y + \sin^2 y$$

$$\boxed{\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1} \quad \cos^2 x + \sin^2 x = 1$$

$$(1) \Rightarrow \frac{u^2}{\cos^2 y} = \cosh^2 x$$

$$(2) \Rightarrow \frac{v^2}{\sin^2 y} = \sinh^2 x$$

$(1) - (2)$

$$\Rightarrow \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = \cosh^2 x - \sinh^2 x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\boxed{\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1}$$

If $x+iy = c\cos(A-iB)$ show that

$$\frac{x^2}{c^2\cosh^2 B} + \frac{y^2}{c^2\sinh^2 B} = 1 \text{ and } \frac{x^2}{c^2\cos^2 A} - \frac{y^2}{c^2\sin^2 A} = 1$$

Sol

$c\cos(A-iB) = x+iy$

$\because \cos i\alpha = \cosh\alpha$, $\sin i\alpha = i\sinh\alpha$

$c\cos c\cos iB = c\sin A\ c\sin iB = x+iy$

$c\cos A\ c\cosh B = i c\sin A\ c\sinh B = x+iy$

Equating real and Imaginary Part we get.

$x = c\cos A\ \cosh B$

$y = c\sin A\ c\sinh B$

$x^2 = c^2\cos^2 A\ c^2\cosh^2 B$

$y^2 = c^2\sin^2 A\ c^2\sin^2 B$

$$\frac{x^2}{c^2\cosh^2 B} + \frac{y^2}{c^2\sinh^2 B} = \frac{c^2[\cos^2 A\ c^2\cosh}{c^2\cosh^2 B}$$

$$\frac{\cancel{c^2}[\sin^2 A\ \cancel{c^2}\sinh^2}{\cancel{c^2}\sinh^2 B}$$

$= c^2\cos^2 A + c^2\sin^2 A$

$= c^2[\cos^2 A + \sin^2 A]$

$= 1$

$$\frac{x^2}{} + \frac{y^2}{\sinh^2 B}$$

$$\frac{c^2[\cos^2 A \cosh^2 B]}{c^2 \cosh^2 B}$$

$$\frac{c^2[\sin^2 A \sinh^2 B]}{c^2 \sinh^2 B}$$

$$= \cos^2 A - \sin^2 A$$

$$= 1$$

$$\frac{x^2}{c^2\cos^2 A} - \frac{y^2}{c^2\sin^2 A} = 1$$

If $\cos^{-1}(\alpha+\beta) = \theta + i\phi$ show that $\alpha^2 \operatorname{sech}^2\phi + \beta^2 \operatorname{cosech}^2 \phi = 1$

Sol

$$\cos^{-1}(\alpha + i\beta) = \theta + i\phi$$

$$\alpha + i\beta = \cos(\theta + i\phi)$$

$$= \cos\theta \cos i\phi - \sin\theta \sin i\phi$$

$$\alpha + i\beta = \cos\theta \cosh\phi - i \sin\theta \sinh\phi$$

$$\alpha = \cos\theta \cosh\phi \rightarrow (1)$$

$$\beta = \sin\theta \sinh\phi \rightarrow (2)$$

$$(1)^2 \Rightarrow \alpha^2 = \cos^2\theta \cosh^2\phi$$

$$(2)^2 \Rightarrow \beta^2 = \sin^2\theta \sinh^2\phi$$

$$\therefore \frac{\alpha^2}{\cosh^2\phi} = \cos^2\theta \rightarrow (3)$$

$$\frac{\beta^2}{\sinh^2\phi} = \sin^2\theta \rightarrow (4)$$

$$(3)+(4) \Rightarrow \frac{\alpha^2}{\cosh^2\phi} + \frac{\beta^2}{\sinh^2\phi} = \cos^2\theta + \sin^2\theta$$

$$\frac{\alpha^2}{\cosh^2\phi} + \frac{\beta^2}{\sinh^2\phi} = 1$$

$$\alpha^2 \operatorname{sech}^2\phi + \beta^2 \operatorname{cosech}^2\phi = 1$$

If $x+iy=\cos(u+iv)$ where x, y, u, v are real P.T $(1+x)^2+y^2=(\cosh v+\cos u)^2$ and $(1-x^2)+y^2$

$=(\cosh v-\cos u)^2$

Sol

G.T

$x+iy=\cos(u+iv)$

$x+iy=\cos u\cos(iv)-i\sin u\sin hv$

~~$x+iy=\cos u\cos iv-\sin u\sin(iv)$~~

$x+iy=\cos u\cosh v-i\sin u\sinh v$

Equating real and imaginary Part we

get,

$x=\cos u\cosh u$

$y=-\sin u\sinh u$

Now $(1+x)^2+y^2=(1+\cos^2 u\cosh^2 v)^2+\sin^2 u\sinh^2 v$

$(1+x^2)+y^2=\cancel{1}+\cos^2 u\cosh^2 v+\sin^2 u\sinh^2 v+2\cos u\cosh v$

$(1+x)^2+y^2=1+\cos^2 u\cosh^2 v+(1-\cos^2 u)(\cosh^2 v-1)+2\cos u\cosh v$

$(1+x)^2+y^2=\cancel{1}+\cancel{\cos^2 u\cosh^2 v}+\cosh^2 v-\cancel{1}-\cancel{\cos^2 u\cosh^2 v}+\cos^2 u$

$+2\cos u\cosh v$

$$\left\{ \begin{array}{l} \sin^2\theta + \cos^2\theta = 1 \\ \sin^2\theta = 1 - \cos^2\theta \\ \cosh^2 x - 1 = \sinh^2 x \end{array} \right.$$

$$= \cosh^2 v + \cos^2 u + 2\cos u \cosh v$$

$$\boxed{(1+x)^2 + y^2 = (\cosh v + \cos u)^2}$$

$$(1-x)^2 + y^2 = (1 - \cos u \cosh v)^2 + \sin^2 u \sinh^2 v$$

$$= 1 - 2\cos u \cosh v + \cos^2 u \cosh^2 v + \sin^2 u \sinh^2 v$$

$$= 1 - 2\cos u \cosh v + \cos^2 u \cosh^2 v + (1 - \cos^2 u)(\cosh^2 v - 1)$$

$$= 1 - 2\cos u \cosh v + \cos^2 u \cosh^2 v - 1 - \cos^2 u \cosh^2 v + \cos^2 u$$

$$(1-x)^2 + y^2 = \cos^2 u + \cosh^2 v - 2\cos u \cosh v$$

$$(1-x)^2 + y^2 = (\cosh v - \cos u)^2$$

If $\tan(\theta + i\phi) = \sin(x+iy)$ P.T $\coth y \sinh 2\phi = \cot x \sin 2\theta$

Sol Give, $\tan(\theta + i\phi) = \sin(x+iy)$

$$\frac{2\sin(\theta+i\phi)\cos(\theta-i\phi)}{2\cos(\theta+i\phi)\cos(\theta-i\phi)} = \sin x \cos iy + \cos x \sin iy$$

$$= \frac{\sin(\theta+i\phi+\theta-i\phi)+\sin(\theta+i\phi-\theta+i\phi)}{\cos(\theta+i\phi+\theta-i\phi)+\cos(\theta+i\phi-\theta-i\phi)}$$

$$= \sin x \cosh y + i\cos x \sinh y$$

$$\frac{\sin 2\theta + \sin i2\theta}{\cos i2\theta + \cos 2\theta} = \sin x \cosh y + i\cos x \sinh y$$

$$\frac{\sin 2\theta + i\sinh 2\theta}{\cosh 2\theta + \cos 2\theta} = \sin x \cosh y + i\cos 5x \sinh y$$

Equating real and imaginary Part

$$\sin x \cosh y = \frac{\sin 2\theta}{\cosh 2\theta + \cos 2\theta} \rightarrow (1)$$

$$\cos x \sinh y = \frac{\sinh e\phi}{\cosh 2\phi + \cos 2\theta} \rightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\sin x \cosh y}{\cos x \sinh y} = \frac{\sin 2\theta}{\sin 2\phi}$$

$$\Rightarrow \tan x \coth y = \frac{\sin 2\theta}{\sinh 2\phi}$$

$$\sinh 2\phi \cdot \coth y = \cot x \cdot \sin 2\theta$$

If $\tan(A+iB) = (x+iy)$, P.T $x^2+y^2+2x\cot 2A = 1$ and $x^2+y^2-2y\coth 2B+1=0$

Sol

Given, $\tan(A+iB) = x+iy$

Changing i into $-i$ we get

$\tan(A-iB) = x-iy$

now, $2A = (A+iB) + (A-iB)$

$\tan 2A = \tan[(A+iB)+(A-iB)]$

$$\tan 2A = \frac{\tan(A+iB) + \tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$\tan 2A = \frac{2x}{1-(x^2+y^2)}$$

$$= \frac{2x}{1+(-x^2-y^2)}$$

$$1-x^2-y^2 = \frac{2x}{\tan 2A}$$

$= 2x\cot 2A$

$1-x^2-y^2 = 2x\cot 2A$

$\boxed{x^2+y^2+2x\cot 2A = 1}$

$$\tan 2B = \frac{\tan(A+iB) - \tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$= \frac{x+iy - x+iy}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+(x^2+y^2)}$$

$$\therefore \tan 2B = \frac{2iy}{1+(x^2+y^2)}$$

$$1+x^2+y^2 = \frac{2iy}{\tan 2B}$$

$$1+x^2+y^2 = 2iy \cot 2B$$

$$x^2+y^2-2iy\cot 2B + 1 = 0$$

$$\boxed{x^2+y^2-2y\coth 2B+1=0}$$

$$y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$$

If $\sin(\theta+i\phi) = \cos\alpha + i\sin\alpha$ P.T $\cos^2\theta = \pm \sin\alpha$

Sol

$\sin(\theta+i\phi) = \cos\alpha + i\sin\alpha$

$\sin\theta \cosh\phi + i\cos\theta \sinh\phi = \cos\alpha + i\sin\alpha$

$\sin\theta \cosh\phi + i\cos\theta \sinh\phi = \cos\alpha + i\sin\alpha$

Equating real and Imaginary Part

$\cos\alpha = \sin\theta \cosh\phi \rightarrow (1) \Rightarrow \cosh\phi = \dfrac{\cos\alpha}{\sin\theta}$

$\sin\alpha = \cos\theta \sinh\phi \rightarrow (2) \Rightarrow \sinh\phi = \dfrac{\sin\alpha}{\cos\theta}$

W.K.T

$\cosh^2\phi - \sinh^2\phi = 1$

Sub (1) and (2)

$$\frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta} = 1$$

$$\frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta} = 1 \left[\cos^2\theta\right.$$

$$(1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha(1-\cos^2\theta) = \frac{(1-\cos^2\theta)}{\cos^2\theta}$$

$$\cos^2\theta - \cancel{\sin^2\alpha\cos^2\theta} - \sin^2\alpha + \cancel{\sin^2\alpha\cos^2\theta}$$
$$= \cos^2\theta - \cos^4\theta$$

$$\cancel{\cos^2\theta} - \sin^2\alpha = \cancel{\cos^2\theta} - \cos^4\theta$$

$$\sin^2\alpha = \cos^4\theta$$

$$\cos^2\theta = \pm\sin\alpha$$

$$\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha \quad \text{s.t. } 2\theta = n\pi + \pi/2 + \alpha$$

and $e^{2\phi} = \pm\cot\alpha/2$

<u>Sol</u>

Given that $\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$ $\ldots\ldots \to (1)$

changing 'i' into $(-i)$ we get.

$$\tan(\theta-i\phi) = \tan\alpha - i\sec\alpha \ldots\ldots \to (2)$$

$$2\theta = (\theta+i\phi) + (\theta-i\phi)$$

$$\tan 2\theta = \tan\Big[\overbrace{(\theta+i\phi) + (\theta-i\phi)}^{A+B}\Big]$$

$$= \frac{\tan(\theta+i\phi) + \tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{\tan\alpha + i\sec\alpha + \tan\alpha - i\sec\alpha}{1 - \{(\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)\}}$$

$$\tan 2\theta = \frac{2\tan\alpha}{1-(\tan^2\alpha - i^2\sec^2\alpha)}$$

$$= \frac{2\tan\alpha}{1-\{\tan^2\alpha + (1+\tan^2\alpha)\}}$$

$$= \frac{2\tan\alpha}{\cancel{1} - \tan^2\alpha - \cancel{1} - \tan^2\alpha}$$

$$= \frac{-\cancel{2}\tan\alpha}{\cancel{2}\tan^2\alpha}$$

$$= -\frac{1}{\tan\alpha}$$

$$\tan 2\theta = -\cot\alpha$$

$\{\because \tan\theta = \tan\alpha$ then $\theta = n\pi + \alpha$

$$\tan 2\theta = \tan(\pi/2 + \alpha)$$

$$\boxed{2\theta = n\pi + \pi/2 + \alpha}$$

$$2i\phi = (\theta + i\phi) - (\theta - i\phi)$$

$$\tan(2i\phi) = \tan[\underbrace{(\theta + i\phi) - (\theta - i\phi)}_{A-B}]$$

$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$= \frac{\cancel{\tan\alpha} + i\sec\alpha - \cancel{\tan\alpha} + i\sec\alpha}{1 + (\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$$

$$= \frac{2i\sec\alpha}{1 + (\tan^2\alpha - i^2\sec^2\alpha)}$$

$$= \frac{2i\sec\alpha}{1 + \tan^2\alpha + \sec^2\alpha}.$$

$(1 + \tan^2\alpha = \sec^2\alpha)$

$$= \frac{\cancel{2}i\sec\alpha}{\cancel{2}\sec^2\alpha}$$

$$= \frac{i}{\sec\alpha} \quad [\because \tan 2i\phi = i\cdot\tanh 2\phi]$$

$\tan(2i\phi) = i\cos\alpha$

$\cancel{i}\tanh 2\phi = \cancel{i}\cos\alpha.$

$\tanh 2\phi = \cos\alpha.$

$$\frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \cos\alpha$$

By componendo and (dividendo) we get

$$\frac{(e^{2\phi} - \cancel{e^{-2\phi}}) + (e^{2\phi} + \cancel{e^{-2\phi}})}{(\cancel{e^{2\phi}} + e^{-2\phi}) - (\cancel{e^{2\phi}} - e^{-2\phi})} =$$

$= \dfrac{1+\cos\alpha}{1-\cos\alpha}$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\therefore \cos^2\theta/2 = 1 + \cos 2(\theta/2)$

$\dfrac{\cancel{1}+\ldots 2\phi}{\cancel{1}-\ldots 2\phi} = \dfrac{\cancel{2}\cos^2\alpha/2}{\cancel{2}\sin^2\alpha/2}$

$2\cos^2\theta/2 = 1+\cos\theta$

$e^{2\phi} \cdot e^{2\phi} = \cot^2\alpha/2$

$1-\cos\alpha = 2\sin^2\alpha/2$

$(e^{2\phi})^{\cancel{2}} = \cot^{\cancel{2}}\alpha/2$

$\boxed{e^{2\phi} = \pm\cot\alpha/2}$

4) If $\cos(\alpha+i\beta) = \cos\theta + i\sin\theta$

P.T $\sin^2\alpha = \pm\sin\theta$

Sol

$$\cos(\alpha+i\beta) = \cos\theta + i\sin\theta$$

$$\cos\alpha \cos i\beta - \sin\alpha \sin i\beta = \cos\theta + i\sin\theta$$

$$\cos\alpha \cosh\beta - i\sin\alpha \sinh\beta = \cos\theta + i\sin\theta$$

Equating real and Imaginary Part we get,

$$\cos\theta = \cos\alpha \cosh\beta$$

$$\sin\theta = -\sin\alpha \sinh\beta$$

$$\cosh\beta = \frac{\cos\theta}{\cos\alpha}$$

$$\sinh\beta = \frac{-\sin\theta}{\sin\alpha}$$

W.K.T

$$\cosh^2\beta - \sinh^2\beta = 1$$

$$\frac{\cos^2\theta}{\cos^2\alpha} - \frac{\sin^2\theta}{\sin^2\alpha} = 1$$

$$\cos^2\theta \cdot \sin^2\alpha - \sin^2\theta \cos^2\alpha = \cos^2\alpha \sin^2\alpha$$

$$(1-\sin^2\theta) \cdot \sin^2\alpha - \sin^2\theta (1-\sin^2\alpha)$$

$\sin^2\alpha - \sin^2\theta \sin^2\alpha - \sin^2\theta + \sin\theta \sin^2\alpha$

$= \sin^2\alpha - 2\sin^4\alpha$

$\sin^2\alpha - \sin^2\theta = \sin^2\alpha - \sin^4\alpha$

$-\sin^2\theta = -\sin^4\alpha$

$\sin\theta = \pm \sin^2\alpha$

If $\tan(\theta + i\phi) = \sin(x + iy)$

PT $\sin 2\theta + \tanh y = \tan x \sinh 2\phi$

Sol

Given $\tan(\theta + i\phi) = \sin(x+iy)$

$$\frac{\sin(\theta+i\phi)}{\cos(\theta+i\phi)} = \sin x \cos iy + \cos x \sin iy$$

$$\frac{2\sin(\theta+i\phi)\cos(\theta-i\phi)}{2\cos(\theta+i\phi)\cos(\theta-i\phi)} = \sin x \cosh y + i\cos x \sinh y$$

$$\frac{\sin[(\theta+i\phi)+(\theta-i\phi)] + \sin[(\theta+i\phi)-(\theta-i\phi)]}{\cos[\cos + (\theta+i\phi)+(\theta-i\phi)] + [\cos (\theta+i\phi)-(\theta+i\phi)]} = \sin x \cosh y + i\cos x \sinh y$$

$$\frac{\sin 2\theta + \sin 2i\phi}{\cos 2\theta + \cos 2i\phi} = \sin x \cosh y + i \cos x \sinh y$$

$$\frac{\sin 2\theta + i\sinh 2\phi}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y + i \cos x \sinh y$$

$$\frac{\sin 2\theta + i\sinh 2\phi}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y + i \cos x \sinh y$$

Equating real and Imaginary part,

we get.

$$\frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y \to (1)$$

$$\frac{\sinh 2\theta}{\cos 2\theta + \cosh 2\phi} = \cos x \sinh y \to (2)$$

$$\frac{\sin 2\theta}{\sinh 2\phi} = \frac{\sin x \cosh y}{\cos x \sinh y}$$

$$\frac{\sin 2\theta}{\sinh 2\phi} = \tan x \cdot \coth y$$

$$\sin 2\theta + \tanh y = \tan x \sinh 2\phi$$

If $\tan x/2 = [\tanh x/2]$ P.T $\cos x \cdot \cosh x = 1$

G.T.

$$\tan x/2 = \tanh x/2$$

$$\tan x^2/2 = \tanh^2 x/2$$

$$1/\tan^2 x/2 = \frac{1}{\tan^2 h x/2}$$

Componed and divided to

$$\frac{1+\tan^2 x/2}{1-\tan^2 x/2} = \frac{1+\tan^2 h x/2}{1-\tan^2 h x/2}$$

$$\frac{1+\tan^2 h x/2}{1-\tan^2 x/2} = 1/\cos h x$$

$$\frac{1+\tan^2 x/2}{1-\tan^2 x/2} = \frac{1}{\cos x}$$

$$\frac{1}{\cos x} = \cosh x$$

$$1 = \cos x \cosh x$$

If $\tanh u/2 = \tan \theta/2$ P.T $u = \log \tan [\pi/4 + \theta/2]$.

7. If $\tanh u/2 = \tan\theta/2$ P.T $u = \log \cdot \tan(\pi/4 + \theta/2)$.

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{\tan\theta/2}{1} \qquad \tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

$$\frac{+1}{-1}$$

$$\frac{e^{u/2} - e^{-u/2} + e^{u/2} + e^{-u/2}}{e^{u/2} + e^{-u/2} - e^{u/2} + e^{-u/2}} = \frac{\tan\theta/2 + 1}{\tan\theta/2 - 1}$$

$$\frac{2e^{u/2}}{2e^{-u/2}} = \tan(\pi/4 + \theta/2)$$

$$e^{u/2} \cdot e^{u/2} = \tan(\pi/4 + \theta/2)$$

$$e^{2u/2} = \tan(\pi/4 + \theta/2)$$

$$e^{u} = \tan(\pi/4 + \theta/2)$$

$$u = \log \tan(\pi/4 + \theta/2).$$

If $(\alpha+i\beta) = \sinh^{-1}(1+i)$ P.T $\tan\beta \tanh\alpha$

Sol

$$\alpha+i\beta = \sinh^{-1}(1+i)$$

$$\sinh(\alpha+i\beta) = 1+i$$

$$\cancel{1+i = 1/i}$$

$$\left\{\begin{array}{l} \sin i = \alpha+i\beta \\ \because \sin ix = i\sinh x \\ \sinh x = 1/i \sin ix \end{array}\right.$$

$$(1+i) = \frac{1}{i}\sin(\alpha+i\beta)$$

$$= -i(\sin i(\alpha+i\beta))$$

$$= -i(\sin i\alpha-\beta)$$

$$= -i\,[i\sinh\alpha\cos\beta - \cosh\alpha\sin\beta]$$

$$= \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$$

equating real and imaginary Part

$$1+i = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$$

$$1 = \sinh\alpha\cos\beta \rightarrow (1)$$

$$1 = \cosh\alpha\sin\beta \rightarrow (2)$$

$$\frac{(2)}{(1)}$$

1) $\log(3)$

Sol

$a = 3,\ b = 0$

$\log(a+ib) = \log_e \sqrt{a^2+b^2} + i\tan^{-1}(b/a)$

$= \log_e \sqrt{(3)^2+0} + i\tan^{-1}(0/3)$

$= \log_e 3 + i\tan^{-1}(0)$

$= \log_e 3 + i(\cancel{\pi/2})0$

Real Part $= \log_e 3$. i imarginary part $= \cancel{\pi}$

$\log(-2)$

$a = -2 \quad b = 0$

$\log(a+ib) = \log_e \sqrt{a^2+b^2} + i\tan^{-1}(b/a)$

$= \log_e \sqrt{4+0} + i\tan^{-1}(0/4)$

$= \log_e 2 + i\tan^{-1}(0)$

$= \log_e 2 + i(\pi/6)0$

Real Part $= \log_e 2$ $\quad$ i imaginary part $= \pi$

$\log(i)$

$a = 0 \quad b = 1$

$\log(a+ib) = \log_e \sqrt{a^2+b^2} + i\tan^{-1}(b/a)$

$= \log_e \sqrt{0+1^2} + i\tan^{-1}(1/0)$

$= \log_e 1 + i\tan^{-1}(\infty)$

$= 0 + i\pi/2$

$= i\pi/2$

Real part $= 0$ imaginary part $= \pi/2$

8\. $\log(-i)$

$a = 0 \quad b = -1$

$\log(a+ib) = \log_e \sqrt{a^2+b^2} + i\tan^{-1}(b/a)$

$= \log_e \sqrt{0+1} + i\tan^{-1}(-1/0)$

$= \log_e 1 + i\tan^{-1}(-\infty)$

$= 0 + i(-\pi)$

$= i\pi$

real Part $= 0$ imaginary Part $= \pi$

S.T $\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}(b/a)$ and hence evaluate $\cos\left[\log\left(\frac{a+ib}{a-ib}\right)\right]$

Sol

$$\log(a+ib) = \tfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}(b/a)$$

$$\log(a-ib) = \tfrac{1}{2}\log(a^2+b^2) - i\tan^{-1}(b/a)$$

$$\log\left(\frac{a+ib}{a-ib}\right) = \log(a+ib) - \log(a-ib)$$

$$= \tfrac{1}{2}\left[\log(a^2+b^2) + i\tan^{-1}(b/a)\right] - \left[\tfrac{1}{2}\log(a^2+b^2) - i\tan^{-1}(b/a)\right]$$

$$= \tfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}(b/a) - \tfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}(b/a)$$

$$= 2i\tan^{-1}(b/a$$

$$\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}(b/a)$$