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Unit - III

Some theorems on limit

1: Cauchy's first limit theorem

Statement :-

If $(a_n) \to l$ then $\left(\frac{a_1+a_2+\dots+a_n}{n}\right) \to l$

Proof :-
case (i)

Let $l=0$

Let $b_n = \frac{a_1+a_2+\dots+a_n}{n}$

Let $\epsilon > 0$ be given, since $(a_n) \to 0$

there exists $m \in N$ such that $|a_n| < \frac{1}{2}\epsilon$

for all $n \geq m \to (1)$

Now, let $n \geq m$

then $|b_n| = \left|\frac{a_1+a_2+\dots+a_m+a_{m+1}+\dots+a_n}{n}\right|$

$\leq \frac{|a_1|+|a_2|+\dots|a_m|}{n} + \frac{|a_{m+1}|+\dots+|a_n|}{n}$

$= \frac{k}{n} + \frac{|a_{m+1}|+\dots+|a_m|}{n}$

where $|a_1| + |a_2| + \cdots + |a_n| = k$

$\leq k/n + \left(\frac{n-m}{n}\right) \epsilon/2$

$\leq k/n + \epsilon/2$ $\left[\because \frac{n-m}{n} < 1\right] \longrightarrow (2)$

Now since $(k/n) \longrightarrow 0$, there exists $n_0 \in N$ (belongs to) such that $k < \frac{1}{2}\epsilon$ for all $n \geq n_0 \longrightarrow (3)$

Let $n_1 = \max\{m, n_0\}$

then $|b_n| < \epsilon$ for all $n \geq m$.

$\therefore (b_n) \longrightarrow 0$

Case (ii)

Since $(a_n) \longrightarrow l$,

$(a_n - l) \longrightarrow 0$

$$\therefore \frac{((a_1 - l) + (a_2 - l) + \cdots + (a_n - l))}{n} \longrightarrow 0$$

$$\therefore \left(\frac{a_1 + a_2 + \cdots + a_n - nl}{n}\right) \longrightarrow 0$$

$$\therefore \left(\frac{a_1 + a_2 + \cdots + a_n}{n} - l\right) \longrightarrow 0$$

$$\therefore \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \longrightarrow l.$$

2. Cesaro's theorem :

Statement :-

If $(a_n) \to a$ $(b_n) \to b$ then

$$\frac{(a_1 b_n + a_2 b_{n-1} + \cdots a_n b_1)}{n} \to ab$$

Proof :

Let $$c_n = \frac{a_1 b_n + \cdots + a_n b_1}{n}$$

now, put $a_n = a + r_n$, so that $(r_n) \to 0$

then $$c_n = \frac{(a+r_1) b_n + \cdots + (a+r_n) b_1}{n}$$

$$= \frac{a(b_1 + \cdots + b_n)}{n} + \frac{r_1 b_n + \cdots + r_n b_1}{n}$$

now by cauchy's first limit theorem

$$\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \to b$$

$$\left(\frac{a(b_1 + b_2 + \cdots + b_n)}{n}\right) \to ab$$

Hence it is enough if we

Prove that $\left(\frac{r_1 b_n + \cdots + r_n b_1}{n}\right) \to 0$

Now, since $(b_n) \longrightarrow b_1$

(b_n) is a bounded sequences.

$\therefore$ There exists a real number $k > 0$

Such that $|b_n| \leq k$ for all n.

$$\therefore \left| \frac{r_1 b_n + \cdots + r_n b_n}{n} \right| \leq k \left| \frac{r_1 + \cdots + r_n}{n} \right|$$

Since $(r_n) \longrightarrow 0$, $\left(\frac{r_1 + \cdots + r_n}{n} \right) \longrightarrow 0$

Hence the theorem.

$\therefore$ Any convergent sequences is a bounded sequence.

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3. Cauchy's second limit theorem.

Statement :-

Let (a_n) be a sequence of positive terms. Then $\lim_{n \to \infty} a_n^{1/n} = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$

Provided the limit on the right

hand side exists (whether finite or infinite.

Proof :-

case (i)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l, \text{ finite}$$

Let $\epsilon > 0$, Fl (there exists) $m \in N \ni$ (such that)

$$l - \tfrac{1}{2}\epsilon < \frac{a_{n+1}}{a_n} < l + \tfrac{1}{2}\epsilon \;\; \forall n \geq m$$

Now choose $n \geq m$

Then

$$l - \tfrac{1}{2}\epsilon < \frac{a_{m+1}}{a_m} < l + \tfrac{1}{2}\epsilon$$

$$l - \tfrac{1}{2}\epsilon < \frac{a_{m+2}}{a_{m+1}} < l + \tfrac{1}{2}\epsilon$$

$$\vdots$$

$$l - \tfrac{1}{2}\epsilon < \frac{a_n}{a_{n+1}} < l + \tfrac{1}{2}\epsilon$$

Multipling these inequalities, we get

$$\left(l - \tfrac{1}{2}\epsilon\right)^{n-m} < \frac{a_n}{a_m} < \left(l + \tfrac{1}{2}\epsilon\right)^{n-m}$$

$$\frac{\left(l - \tfrac{1}{2}\epsilon\right)^n}{\left(l - \tfrac{1}{2}\epsilon\right)^m} < \frac{a_n}{a_m} < \frac{\left(l + \tfrac{1}{2}\epsilon\right)^n}{\left(l + \tfrac{1}{2}\epsilon\right)}$$

Multipling an throughout. $\left(\because a_m\left(\frac{1}{(l-\frac{1}{2}\epsilon)^m}\right)=\ldots\right)$

$$a_m\frac{(l-\frac{1}{2}\epsilon)^n}{(l-\frac{1}{2}\epsilon)^m} < a_n < a_m\frac{(l+\frac{1}{2}\epsilon)^n}{(l+\frac{1}{2}\epsilon)^m}$$

$$k_1(l-\tfrac{1}{2}\epsilon)^n < a_n < k_2(l+\tfrac{1}{2}\epsilon)^n$$

$$k_1^{1/n}(l-\tfrac{1}{2}\epsilon) < a_n^{1/n} < k_2^{1/n}(l+\tfrac{1}{2}\epsilon) \to (1)$$

Now,

$$\left(k_1^{1/n}(l-\tfrac{1}{2}\epsilon)\right) \longrightarrow l-\tfrac{1}{2}\epsilon$$

There exists $n \in N$ such that

$\exists\, n_1 \in N \ni$:

$$(l-\tfrac{1}{2}\epsilon)-\tfrac{1}{2}\epsilon < k_1^{1/n}(l-\tfrac{1}{2}\epsilon) < (l-\tfrac{1}{2}\epsilon)+\tfrac{1}{2}\epsilon$$

Now,

$$k_2^{1/n}(l+\tfrac{1}{2}\epsilon) \longrightarrow l+\tfrac{1}{2}\epsilon \to (2)$$

Similarly, $\exists\, n_2 \in N \ni$:

$$(l+\tfrac{1}{2}\epsilon)-\tfrac{1}{2}\epsilon < k_2^{1/n}(l+\tfrac{1}{2}\epsilon) < (l+\tfrac{1}{2}\epsilon)+\tfrac{1}{2}\epsilon \longrightarrow (3)$$

Let $n_0 = \max\{m, n_1, n_2\}$

From (1), (2) & (3)

$$l-\epsilon < a_n^{1/n} < l+\epsilon$$

Hence $a_n^{1/n} \to l$

Case (ii) :-

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$$

Then $$\lim_{n\to\infty} \left(\frac{a_n}{a_{n+1}}\right) = 0$$

By known result, if $a_n \to 0$, then

$(1/a_n) \to \infty$.

By case (i) $(1/a_n)^{1/n} \to 0 \Rightarrow a_n^{1/n} \to \infty$

Theorem :-

Statement :-

Let (a_n) be any sequences and $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = l$

If $l > 1$, then $(a_n) \to 0$

Proof :-

Let k be any real number such that

$1 < k < l$

Since $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = l$, there exists

$m \in N$ such that

$$l - \epsilon < \left|\frac{a_n}{a_{n+1}}\right| < l + \epsilon \text{ for all } n \geq$$

$k = l - \epsilon$

$\epsilon = l - k.$

choosing $\epsilon = l-k$ we obtain $\left|\frac{a_n}{a_{n+1}}\right| > k$ for all $n \geq m$ $k = l-\epsilon$

Now, fix $n \geq m$

then,

$$\left|\frac{a_m}{a_{m+1}}\right| > k, \quad \left|\frac{a_{m+1}}{a_{m+2}}\right| > k \ldots \left|\frac{a_{n-1}}{a_n}\right| > k$$

multipling the above inequalities we get

$$\left|\frac{a_m}{a_n}\right| > k^{n-m} \quad \frac{k^n}{k^m} \Rightarrow \left|\frac{a_n}{a_m}\right| < k^{m-n}$$

$$\left|\frac{a_n}{a_m}\right| < k^m (1/k)^n \Rightarrow \left|\frac{a_n}{a_m}\right| < k^m (1/k)^n$$

$$|a_n| < k^m |a_m| (1/k)^n \quad |a_m| (1/k)$$

$|a_n| < A r^n$ where $A = |a_m| k^m$ ~~is a~~

~~constant~~ and $r = 1/k$

~~Now~~

$\cancel{(k > 1 \Rightarrow 0 < r < 1)}$

$\therefore (r^n) \to 0$ (by solved)

$\therefore (a_n) \to 0$

Note :- The above theorem is true even if

Theorem :

Statement

If the sequences (a_n) and (b_n) converge to 0 and (b_n) is strictly monotonic decreasing then $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}}\right)$

provided the limit on the right hand side exist whether finite or infinite

Proof :-

Case (i)

$$\text{Let } \lim_{n\to\infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}}\right) = l, \text{ finite}$$

Let $\epsilon > 0$ be given

then there which exists $m \in N$ such that

($\exists\, m \in N \ni :$)

$$l - \epsilon < \frac{a_n - a_{n+1}}{b_n - b_{n+1}} < l + \epsilon \text{ for all } n \geq m$$

since $b_n - b_{n+1} > 0$, we get

$$(b_n - b_{n+1})(l - \epsilon) < a_n - a_{n+1} < (b_n - b_{n+1})(l + \epsilon) \text{ for all } n \geq m.$$

Let $n > p \geq m$

Then $(b_p - b_{p+1})(l-\epsilon) < a_p - a_{p+1} < (b_p - b_{p+1})(l+\epsilon)$

$(b_{p+1} - b_{p+2})(l-\epsilon) < (a_{p+1} - a_{p+2})(l+\epsilon)$

. . . .

. . . .

$(b_{n-1} - b_{p+2}$

$(b_{n-1} - b_n)(l-\epsilon) < a_{n-1} - a_n < (b_{n-1} - b_n)(l+\epsilon)$

Adding the above inequalities, we get

$(b_p - b_n)(l-\epsilon) < a_p - a_n < (b_p - b_n)(l+\epsilon)$

Taking limit as $n \to \infty$, we get.

$b_p(l-\epsilon) < a_p < b_p(l+\epsilon)$.

$l - \epsilon < \dfrac{a_p}{b_p} < l + \epsilon$ [since $(a_n), (b_n) \to 0$]

$\left| \dfrac{a_p}{b_p} - l \right| < \epsilon$ for all $p \geq m$

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$

case (ii) $\lim_{n \to \infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right) = \infty$

Let $k > 0$ be any real number. Then there exists $m \in N$ such that $\frac{a_n - a_{n+1}}{b_n - b_{n+1}} > k$ for all $n \geq m$.

$\therefore a_n - a_{n+1} > (b_n - b_{n+1}) k$ for all $n \geq m$

Let $n > p \geq m$

writting the inequalities for $n = p, p+1 \ldots n$ and adding we get.

$a_p - a_n > k (b_p - b_n)$

Taking limit as $n \to \infty$, we get $a_p \geq k b_p$

$\therefore \frac{a_p}{b_p} \geq k$ for all $p \geq m$

$\therefore \left(\frac{a_n}{b_n} \right)$ diverges to ∞

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① Show that $\lim_{n \to \infty} \frac{1}{n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) = 0$

Sol Let $a_n = \frac{1}{n}$

we know that $(a_n) \to 0$. Hence by

cauchy's first limit theorem. we get

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right) \to 0$$

$$\left(\frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right) \to 0$$

2. show that $\lim_{n\to\infty} n^{1/n} = 1$

Sol Let $a_n = n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$$

By cauchy's second limit theorem we get. $\lim_{n\to\infty} n^{1/n} = 1$

Prove that $\frac{1}{n}\left[(n+1)(n+2)+\cdots+(n+n)\right]^{1/n} \to 4/e$

Sol

Let $a_n = \frac{1}{n}\left[(n+1)(n+2)+\cdots+(n+n)\right]^{1/n}$

$$= \left[\frac{(n+1)(n+2)+\cdots+(n+n)}{n^n}\right]^{1/n}$$

$$= \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

Let $b_n = \dfrac{(1+1/n)(1+2/n)\cdots(1+n/n)}{n^n}$

so that $a_n = b_n^{1/n}$.

Now.

$$\frac{b_{n+1}}{b_n} = \frac{\dfrac{(1+1/_{n+1})(1+2/_{n+1})\cdots(1+\frac{n+1}{n+1})}{(n+1)^{n+1}}}{\dfrac{(1+1/n)(1+2/n)\cdots(1+n/n)}{n^n}}$$

$$= (2n+1)(2n+2)\frac{n^n}{(n+1)^{n+2}}$$

$$= \frac{2(2n+1)}{n+1}\cdot\frac{n^n}{(n+1)^n}$$

$$= 2\left(\frac{2+1/n}{1+1/n}\right)\frac{1}{(1+1/n)^n}$$

$n \to \infty$ $\therefore \left(\dfrac{b_{n+1}}{b_n}\right) \to \dfrac{4}{e}$

By theorem 3.24 we get $(b_n^{1/n}) \to 4/e$

$(a_n) \to 4/e$

4) Prove that $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$

Sol

Let $a_n = \dfrac{x^n}{n!}$ $a_{n+1} = \dfrac{x^{n+1}}{(n+1)!}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{x^n}{n!}\cdot\frac{(n+1)!}{x^{n+1}} = \frac{n+1}{x}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty$$

$\therefore (a_n) \to 0$ (by theorem 3-25)

5) Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Sol

Let $a_n = \frac{n!}{n^n}$

$$\frac{n!}{n^n} \times \frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{(n+1)} \times \cancel{n!}}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n!}{n^n} \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1+1/n\right)^n$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left(1+1/n\right)^n$$

$$= e$$

$$> 1$$

$\therefore (a_n) \to 0$

By ~~note~~ known result.

Sub sequence :-

Let a_n be a sequence.

Let n_k be a strictly increasing sequence of natural numbers. then (a_{n_k}) is called a subsequence of (a_n)

Theorem :

If a sequence a_n converges to l, then every subsequence (a_{n_k}) of (a_n) also converges to l

Proof :

Let $\varepsilon > 0$, since $(a_n) \to l$ & there exists $m \in N$ such that $|a_n - l| < \varepsilon$.

for all $n \geq m \longrightarrow ①$

Now choose $n_{k_0} \geq m$

Then $k \geq k_0 \Rightarrow n_k \geq n_{k_0}$

$\left\{ \because n_k \text{ is a monotonic increasing} \right\}$

$\Rightarrow n_k \geq m$

$\Rightarrow |a_{n_k} - l| < \varepsilon \longrightarrow ①$

Thus $|a_{n_k} - l| < \epsilon$ for all $k \geq k_0$

$\therefore a_{n_k} \to l$

Theorem:

If the sub sequences (a_{2n-1}) and (a_{2n}) of a sequences a_n converge to the same limit l then (a_n) also converges to l

Proof:

Let $\epsilon > 0$, since $(a_{2n-1}) \to l$ converges there exists $n_1 \in N$. such that

$|a_{2n-1} - l| < \epsilon$ for all

$2n-1 \geq n_1$

IIIly

Let $\epsilon > 0$, since $(a_{2n}) \to l$ there exists $n_2 \in N$ such that

$|a_{2n} - l| < \epsilon$ for all $2n \geq n_2$

Let $m = \max\{n_1, n_2\}$

clearly $|a_n - l| < \epsilon$ for all $n \geq m$

$\therefore a_n \to l$

Theorem

Statement:

Every bounded sequence as a converges Sub sequence.

Proof:

Let (a_n) be a bounded sequence

Let (a_{n_k}) be a monotonic Sub sequence of $\{a_n\}$. Since (a_n) is bounded (a_{n_k}) is also bounded

$\therefore (a_{n_k})$ is a bounded monotonic sequence hence converges.

$\therefore (a_{n_k})$ is a Convergent Sub sequence of (a_n)

Definition

Peak Point:

Let a_n be a sequence a natural Number m is called a peak point of the sequence a_n. If $a_n < a_m$

for all $n > m$.

Theorem

Statement:

Every sequence a_n has a monotonic sub sequence.

Proof:-

Case (i)

(a_n) has a infinite number of Peak Points

Let the Peak Point be $n_1 < n_2 < \cdots < n_k < \cdots$.

Then,

$$a_{n_1} > a_{n_2} > \cdots > a_{n_k} > \cdots$$

$\therefore a_{n_k}$ is a monotonic decreasing sub sequence of (a_n)

$\therefore a_{n_k}$ is a monotonic decreasing sub sequence of a_n

Case (ii)

a_n has only a finite number of peak points or no peak points.

Choose a natural number n_1 such that there is no peak point $\geq n$.

Since n_1 is a not peak point of a_n there exists $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$ again since n_2 is a not peak point, there exists $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$ Reapeating the process we get a monotonic increasing subsequence a_{n_k} of a_n.

[∴ note $a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \cdots$] a_{n_k} a_n

Limit Points:

Definition:

Let (a_n) be a sequence of real numbers "a" is called a limit point or a cluster point of the sequence

(a_n) if given $\epsilon>0$, there exists infinite number of terms of the sequence in $(a-\epsilon, a+\epsilon)$. If the sequence (a_n) is not bounded above then ∞ is a limit point of the sequence. If (a_n) is not bounded below then $-\infty$ is a limit point of the sequence.

Theorem - 1

Statement:

Let (a_n) be a sequence. A real number "a" is a limit point of (a_n) if there exists a subsequence (a_{n_k}) of (a_n) converging to "a"

Proof:

Suppose there exists a subsequence (a_{n_k}) of (a_n) converging to a.

Let $\epsilon>0$ be given. Then there exists $k_0 \in N$ such that

$a_{n_k} \in (a-\epsilon, a+\epsilon)$ for all $k \geq k_0$

$\therefore (a-\epsilon, a+\epsilon)$ contains infinitely many terms of the sequence (a_n).

$\therefore a$ is a limit point of the sequence (a_n)

Controversly suppose "a" is a limit point of (a_n)

Then for each $\epsilon > 0$ the interval $(a-\epsilon, a+\epsilon)$ contains infinitely many terms of the sequence. In particular we can find $n_1 \in N$ such that

$a_{n_1} \in (a-1, a+1)$

Also, we can find $n_2 > n_1$ such that

$a_{n_2} \in (a-1/2, a+1/2)$

Proceeding like this we can find natural numbers $n_1 < n_2 < n_3 \ldots$ such that

$a_{n_k} \in (a-1/k, a+1/k)$

Clearly (a_{n_k}) is a subsequence

of (a_n) and $|a_{n_k} - a| < 1/k$.

For any $\epsilon > 0$ $|a_{n_k} - a| < \epsilon$ if $k > 1/\epsilon$

$\therefore (a_{n_k}) \longrightarrow a$

Theorem :-

Statement :

Every bounded sequence has at least one limit point

Proof :-

Let (a_n) be a bounded sequence. Then there exists a convergent subsequence (a_{n_k}) of (a_n) converging to l (say) (by theorem 3.31)

Hence l is a limit point of (a_n)

Theorem - 3

Statement :

A sequence (a_n) converges to l iff (a_n) is bounded and l is the only limit point of the sequence.

Proof :-

Let $(a_n) \to l$. Then (a_n) is bounded.

Also, l is a limit point of the sequence (a_n). Then there exists a subsequence (a_{nk}) of (a_n) such that $(a_{nk}) \to l_1$ now. Since $(a_n) \to l$, we have $(a_{nk}) \to l$.

(a_n), $\epsilon > 0$ $\exists n_0 \in N$

$|a_n - l| < \epsilon$

$a_n \to l$.

$\therefore l = l_1$

Thus l is the only limit point of the sequence.

Controversly Suppose l is the only limit point of (a_n). Suppose (a_n) does not converge to l

Then there exists at least one $\epsilon > 0$ such that infinitely many terms of the sequence lie outside $(l - \epsilon, l + \epsilon)$. Hence we can find a subsequence (a_{nk}) of (a_n) such that $a_{nk} \notin (l - \epsilon, l + \epsilon)$ for all K.

(Since (a_n) is a bounded sequence. (a_{nk}) is also a bounded sequence

Hence (a_{n_k}) has also a limit pt by theorem 3.33 say l' and $l' \neq l$

$\therefore (a_n)$ has two limit points l and l' which is contradiction.

Hence $(a_n) \to l$

Cauchy Sequences.

bn → bounded
$\exists$ (k) = (|bn| < k

Definition :-

A sequence (a_n) is said to be a Cauchy sequence if given $\epsilon > 0$, there exists $n_0 \in N$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq n_0$.

Note :-

In the above definition the condition $|a_n - a_m| < \epsilon$ for all $n, m \geq n_0$ can be written in the following equivalent ~~from~~ form namely $|a_{n+p} - a_n| < \epsilon$ for all $n \geq n_0$ and for all positive integers p.

Theorem 1:

Statement:

Any convergent sequency is a cauchy sequence.

Proof:

Let $(a_n) \to l$. Then given $\epsilon > 0$ there exists $n_0 \in N$ such that $|a_n - l| < \frac{1}{2}\epsilon$ for all $n \geq n_0$

$$\therefore |a_n - a_m| = |a_n - l + l - a_m|$$
$$\leq |a_n - l| + |l - a_m|$$
$$< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon < \epsilon$$
$$\forall n, m \geq n_0$$

Theorem 2:

Statement:

Any ~~conver~~ cauchy sequence is a bounded sequence.

Proof:

Let (a_n) be a cauchy sequence. Let $\epsilon > 0$ be given then there exists

$n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$

for all $m, n \geq n_0$. $\Rightarrow |a_n| < |a_m| + \epsilon$

$\therefore |a_n| < |a_{n_0}| + \epsilon,$

for $n \geq n_0$

Now,

let $k = \max\{|a_1|, |a_2| \ldots + |a_n|. + \epsilon\}$

then $|a_n| \leq k$ for all n.

Hence (a_n) is a bounded sequence.

Theorem: 3

Statement:-

Let (a_n) be a Cauchy sequence. If (a_n) has a subsequence (a_n) converges to l, then $(a_n) \to l$

Proof:-

Let $\epsilon > 0$ be given, then there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/2$,

Since $a_{n_k} \to l$ there exists $k_0 \in \mathbb{N}$ such that $|a_n - l| < \epsilon/2$

choose n_k such that $n_k \geq n_k$ and n_0

Then $|a_n - l| = |a_n - a_{n_k} + a_{n_k} - l| \leq$

$\leq |a_n - a_{n_k}| + |a_{n_k} - l|$

$\leq \epsilon/2 + \epsilon/2$

$\leq \epsilon$.

$(a_n) \to l$.

Cauchy's general Principle of convergent.

Statement :-

A sequence (a_n) in are R is convergent if only iff it is a Cauchy's sequence.

Proof :-

We know that any convergent sequence is a Cauchy's sequence.

Conversly, let a_n be cauchy's sequence in R. There

(a_n) is bounded sequence.

$\therefore$ There exists a Subsequence a_{n_k} of a_n such that a_{n_k} converges to l

$\therefore \; a_{\cancel{n_k}} \to l$

$\therefore \; a_n \to l$