UNIT - V

Note:

D'Alembert's ratio test:-

Let Σa_n be a series of Positive terms. Then Σa_n converges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Problem:

1. Test the convergence of the series $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \dots$

Sol

$$\text{Let } a_n = \frac{1.2.3\dots n}{3.5.7\dots(2n+1)} \qquad a_{n+1} = \frac{1.2\dots(n+1)}{3.5\dots(2(n+1)+1)}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2n+3}{n+1} \qquad a_n = \frac{n/2n+1}{n+1/2(n+1)+1}$$

$$= \frac{n(2+3/n)}{n(1+1/n)} \qquad = \frac{n}{2n+1} \times \frac{2n+2+1}{n+1}$$

$$= \frac{2+3/n}{1+1/n} \qquad = \frac{n(2n+3)}{(n+1)(2n+1)}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2 > 1$$

$\therefore$ By D'Alembert's ratio test.

Σa_n is convergent.

2. Test the convergence $\Sigma \frac{n^n}{n!}$ $\qquad a_n = \frac{n^n}{n!}$ $\qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

Sol

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(n+1)n^n}{(n+1)^{n+1}} \qquad = \frac{n^n/n!}{(n+1)^{n+1}/(n+1)!}$$

$$= \frac{1}{(1+1/n)^n} \qquad = \frac{n^n / n!}{(n+1)^{n+1}/\dots}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e} < 1$$

$\therefore \sum a_n$ is divergent.

3. Test the convergence of the series $\sum \frac{2^n n!}{n^n}$

<u>Sol</u>

Let $a_n = \frac{2^n n!}{n^n}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(n+1)^{n+1}}{2(n+1)n^n}$$

$$= \frac{1}{2}\left(1+\frac{1}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = e/2 > 1$$

$\therefore$ By ratio test the series converges.

4. Test the convergence of the $\sum \frac{3^n n!}{n^n}$

As in above Problem, we find the

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = e/3 < 1$$

$\therefore$ By ratio test the series diverges.

5. Test the convergence of the series $\sum \sqrt{\frac{n}{n+1}}$ where x is any +ve real number.

<u>Sol</u>

since x is +ve the given series is a series of +ve terms.

Now, $\frac{a_n}{a_{n+1}} = \sqrt{\frac{n(n+2)}{n+1}}\,(1/x)$

$= \sqrt{\frac{(1+2/n)}{(1+1/n)}}\,(1/x)$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1/x.$

$\therefore$ By ratio test $\sum a_n$ converges if $x<1$ and diverges if $x>1$

if $x=1$

when $x=1$, $a_n = \sqrt{\frac{n}{n+1}} = \frac{1}{\sqrt{(1+1/n)}}$

$\therefore \lim_{n\to\infty} a_n = 1$

$\therefore$ The series diverges

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6. Test the convergence of the series $1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots$ where x is any +ve and real number.

Sol

Since x is a positive real number, the given series is a series of +ve terms.

Let $a_n = \frac{x^{2n-2}}{2n-2}, (n>1)$

$\therefore \frac{a_n}{a_{n+1}} = \frac{2n}{2n-2}\,(1/x^2)$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x^2}$

$\therefore$ By ratio test the series converges if $x^2<1$ and diverges if $x^2>1$

∴ The series converges if $x<1$ and diverges if $x>1$

if $x=1$ the test fails

when $x=1$, $a_n = \frac{1}{2n-2}$

By comparing with the series $\Sigma(1/n)$ we see that the series diverges.

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7. Test the convergence of the series $\Sigma \frac{n^2+1}{5^n}$

Sol

$$\frac{a_n}{a_{n+1}} = \frac{5(n^2+1)}{(n+1)^2+1} \qquad a_{n+1} = \frac{5[(n+1)^2+1]}{5^{n+1}}$$

$$= \frac{5(n^2+1)}{(n+1)^2+1}$$

$$= \frac{5(1+1/n^2)}{1+2/n+2/n^2}$$

$$\frac{5(n^2+1)}{n^2+2n+1+1} = \frac{5(n^2+1)}{n^2+2n+2}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 5$$

∴ By ratio test the series converges.

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8. Test the convergence of the series $(1/2+1/3)+(1/2^2+1/3^2)+(1/2^3+1/3^3)+\cdots$

Sol Let $a_n = \frac{1}{2^n} + \frac{1}{3^n}$

$$a_{n+1} = \frac{2^{n+1}+3^{n+1}}{2^{n+1}3^{n+1}}$$

$$a_n = \frac{2^n+3^n}{2^n 3^n}$$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{2^n+3^n}{2^n3^n}}{\frac{2^{n+1}+3^{n+1}}{2^{n+1}3^{n+1}}}$$

$$\frac{a_n}{a_{n+1}} = \frac{6(2^n+3^n)}{2^{n+1}+3^{n+1}}$$

$$= \frac{2\left[1+(2/3)^n\right]}{\left[1+(2/3)^{n+1}\right]}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2$$

$\therefore$ By ratio test the given series converges.

9. Test the convergence of the series $\sum \frac{x^n}{n}$

Sol

Let $a_n = \frac{x^n}{n}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n+1}{n}(1/x)$$

$$= (1+1/n)(1/x)$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1/x$$

$\therefore$ The series converges if $x<1$ and diverges if $x>1$

if $x=1$ the series becomes $\sum 1/n$ which is divergent.

. Test the convergence of the series $\sum \frac{n^p}{n!}$ $(p>0)$

Sol

Let $a_n = \frac{n^p}{n!}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n^p(n+1)}{(n+1)^p}$$

$$= \frac{n+1}{(1+1/n)^p}$$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty.$

By ratio test.

Σa_n is convergent.

11. Test the convergence of the series $\frac{1}{3}x + \frac{1}{3}\cdot\frac{2}{5}x^2 + \frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}x^3 + \cdots$

Sol

Let $a_n = \frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots(2n+1)} x^n$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2n+3}{n+1}\left(\frac{1}{x}\right)$$

$a_{n+1} = \frac{12\cdots n+1}{3\cdot 5\cdots[2(n+1)+1]} x^{n+1}$

$$= \frac{2+3/n}{1+1/n}\left(\frac{1}{x}\right)$$

$= \frac{12\cdots n+1}{3\cdot 5\cdots(2n+3)} x^{n+1}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{2}{x}$$

$\therefore$ By ratio test the series converges

if $\frac{2}{x} > 1$.

$\therefore$ The series converges if $x < 2$ and diverges if $x > 2$.

if $x = 2$ the ratio test fails

In this case $\frac{a_n}{a_{n+1}} = \frac{2n+3}{2n+2}$

$$\therefore \frac{a_n}{a_{n+1}} - 1 = \frac{1}{2n+2}$$

$$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+2} = \frac{1}{2+\frac{2}{n}}$$

$$\therefore \lim_{n\to\infty} \left(\frac{a_n}{a_{n+1}} - 1\right) = 1/2$$

$\therefore$ By Raabe's test the series diverges.

9. Test the convergence of the hyper geometric series $1 + \frac{\alpha\beta}{\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2}x^2 + \dots$

Sol

let

$$\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)+(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)n!}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(\gamma+n)(n+1)}{(\alpha+n)(\beta+n)}(1/x)$$

$$= \frac{(1+\gamma/n)(1+1/n)}{(1+\alpha/n)(1+\beta/n)}(1/x)$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1/x$$

$\therefore$ The series converges if $x<1$

and diverges if $x>1$ when $x=1$,

the ratio test fails. In this case we apply Gauss test.

Problem:

1) Test the convergence of $\sum \frac{1}{(\log n)^n}$

Sol:-

Let $a_n = \frac{1}{(\log n)^n}$ $\quad a_n^{1/n} = \sqrt[n]{a_n} = \left[\frac{1}{(\log n)^n}\right]^{1/n}$

$\sqrt[n]{a_n} = \frac{1}{\log n}$

$\therefore \sqrt[n]{a_n} = \frac{1}{\log n}$

$\therefore \lim_{n \to \infty} \sqrt[n]{a_n} = 0 < 1$

$\therefore$ By Cauchy's root test $\sum \frac{1}{(\log n)^n}$ converges

2) Test the convergence of $\sum \left(1+\frac{1}{n}\right)^{-n^2}$

Sol

Let $a_n = (1+1/n)^{-n^2}$ $\quad \sqrt[n]{a_n} = \left[(1+1/n)^{-n^2}\right]^{1/n}$

$\therefore \sqrt[n]{a_n} = (1+1/n)^{-n}$ $\quad = \left[(1+1/n)^{-n}\right]$

$\therefore \lim_{n \to \infty} \sqrt[n]{a_n} = 1/e$ $\quad = (1+1/\infty)^{-\infty}$

< 1 $\quad = 1/e$

$= 1/e$

$\therefore$ By Cauchy's root test the series converges.

P.T the series $\sum e^{-\sqrt{n}} x^n$ converges if $0 < x < 1$ and diverges if $x > 1$

Sol

Let $a_n = e^{-\sqrt{n}} x^n$

$a_n^{1/n} = (e^{-\sqrt{n}} x^n)^{1/n}$

$= (e^{-\sqrt{n}})^{1/n} x$

$= (e^{-\sqrt{n}})^{1/\sqrt{n}\sqrt{n}}$

$= (e^{-\sqrt{\infty}})^{1/\infty}$

$\lim_{n \to \infty} = e^{-\sqrt{\infty}} x$

$= x$

$\therefore a_n^{1/n} = e^{-1/\sqrt{n}} x$

$\therefore \lim_{n \to \infty} a_n^{1/n} = x$

$\therefore$ By cauchy's root test the given series convergest $0 < x < 1$ and diverges if $x > 1$

4) Test the convergence of $\sum \frac{n^3 + a}{2^n + a}$

Sol

Let $a_n = \frac{n^3 + a}{2^n + a}$ and $b_n = \frac{n^3}{2^n}$

$$\frac{a_n}{b_n} = \left(\frac{n^3 + a}{2^n + a}\right) \Big/ \left(\frac{n^3}{2^n}\right)$$

$$= \left(\frac{n^3 + a}{n^3}\right)\left(\frac{2^n}{2^n + a}\right)$$

$$= \left(1 + a/n^3\right)\left(\frac{1}{1 + a/2^n}\right)$$

$$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = 1$$

$\therefore$ By comparison test, the given series is convergent or divergent according as $\sum \frac{n^2}{2^n}$ is convergent or divergent.

$$\text{Now, } \sqrt[n]{b_n} = \left(\frac{n^3}{2^n}\right)^{1/n} = \frac{n^{3/n}}{2}$$

$$\text{Also, } \lim_{n\to\infty} n^{3/n} = 1$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{b_n} = 1/2 < 1$$

$\therefore \sum b_n$ is convergent

$\therefore \sum a_n$ is convergent.

5) Test the convergent of $\sum \frac{1}{n\log n}$

Sol:-

By Cauchy's test, $\sum \frac{1}{n\log n}$ converges or diverges with the series

$$\sum \frac{2^n}{2^n \log 2^n} = \sum \frac{1}{n\log 2} = \frac{1}{\log 2} \sum 1/n$$

Now, the series $\sum 1/n$ diverges.

$\therefore$ The given series diverges.

6) Test the Convergence of the series $\sum \frac{1}{n(\log n)^p}$

Sol:-

The given series converges or diverges with the series

$$\sum \frac{2^n}{2^n(\log n)^p} = \sum \frac{1}{(\log n)^p}$$

diverges if $P \le 1$

$\therefore$ The given series converges if $P > 1$

and diverges if $P \le 1$.

Test the convergence of the series

$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$

Sol:-

We have $a_n^{1/n} = \begin{cases} \left(\frac{1}{3^{n/2}}\right)^{1/n} & \text{if } n \text{ is even} \\ \left(\frac{1}{2^{(n+1)/2}}\right)^{1/n} & \text{if } n \text{ is odd} \end{cases}$

$a_n^{1/n} = \begin{cases} \frac{1}{\sqrt{3}} & \text{if } n \text{ is even} \\ \frac{1}{2^{\frac{1}{2}(1+1/n)}} & \text{if } n \text{ is odd.} \end{cases}$

Now the sequence

$\left(\frac{1}{2^{\frac{1}{2}(1+1/n)}}\right)$ Converges to $\left(\frac{1}{\sqrt{2}}\right)$ as $n \to \infty$

$\therefore \frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}}$ are the only limit points of the given sequence $\limsup a_n^{1/n} = \frac{1}{\sqrt{2}} < 1$

$\therefore$ By Cauchy's root test the given series is convergent.

ALTERNATING SERIES :-

Definition :-

A series whose terms are alternatively positive and negative is called an alternating series.

Thus an altering series is of the form $a_1 - a_2 + a_3 - a_4 + \dots = \Sigma(-1)^{n+1} a_n$ where $a_n > 0$ for all n.

Theorem :- (Leibnitz's test)

Let $\Sigma(-1)^{n+1} a_n$ be an alternating series whose terms a_n satisfy the following conditions (i) (a_n) is a monotonic decreasing sequence.

ii) $\lim_{n \to \infty} a_n = 0$

Then the given alternating series converges.

Proof :-

Let (S_n) denote the sequence of partial sums of the given series.

then $S_{2n} = a_1 - a_2 + a_3 - a_4 + \dots + a_{2n-1} - a_{2n}$

$S_{2n+2} = a_1 - a_2 + a_3 - a_4 + \dots + a_{2n-1} + a_{2n+1} - a_{2n+2}$

$S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2}$

$\therefore S_{2n+2} - S_{2n} = (a_{2n+1} - a_{2n+2}) \geq 0$ (by (i))

$S_{2n+2} - S_{2n} \geq 0$

$\therefore S_{2n+2} \geq S_{2n}$.

$\therefore (S_{2n})$ is a monotonic increasing sequence.

Also $S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) \ldots (a_{2n-2} - a_{2n-1}) - a_{2n}$

$\leq a_1$ by (i)

$\therefore (S_{2n})$ is bounded above.

$\therefore (S_{2n})$ is a Convergent sequence.

Let $(S_{2n}) \longrightarrow S$

Now, $S_{2n+1} = S_{2n} + a_{2n+1}$

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$$

$$= S + 0 = S. \text{ (by (ii))}$$

$$\therefore (S_{2n+1}) \longrightarrow S$$

Thus the subsequences (S_{2n}) and (S_{2n+1}) Converges to the same limits.

$\therefore (S_n) \rightarrow S$

$\therefore$ the given Series Converges.

Solved Problems:-

1. S.T the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

<u>Sol</u>

The given series is $\sum (-1)^{n+1} a_n$ where $a_n = 1/n$ cleare $a_n > a_{n+1}$ $\forall n$ and hence (a_n) is monotonic decreasing.

$a_n = 1/n$, $a_{n+1} = 1/(n+1)$

$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-n-1}{(n+1)n} = \frac{-1}{n(n+1)}$

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 1/n = 0$

$a_{n+1} - a_n \le 0$

$a_{n+1} \le a_n$

$\therefore$ By Libinitz's test the gn series converges.

$\lim\limits_{n\to\infty} \frac{(-1)}{n} =$

2. S.T the series $\sum \frac{(-1)^{n+1}}{\log(n+1)}$ converges.

<u>Sol</u>

Let $a_n = \frac{1}{\log(n+1)}$

clearly $(a_n) \to 0$ as $n \to \infty$

Also $\frac{1}{\log n} > \frac{1}{\log(n+1)}$ $\forall n \ge 2$.

$\therefore$ By Leibnitz's test the gn series converges.

3. S.T Series $\sum (-1)^{n+1} \frac{n}{3n-2}$ Oscillates

$a_n = \frac{n}{3n-2}$, $a_{n+1} = \frac{n+1}{3(n+1)-2} = \frac{n+1}{3n+3-2} = \frac{n+1}{3n+1}$

$\frac{n}{n(3-2/n)}$

<u>Sol</u> Let $a_n = \frac{n}{3n-2}$

clearly $a_n > a_{n+1}$ $\forall n$.

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{3n-2} = 1/3$

$\therefore$ The gn series oscillates.

7. S.T the following series converges $\frac{1}{2^3} - \frac{1}{3^3}(1+2)$

$+\frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \dots$

Sol

Let $a_n = \frac{1+2+3+\dots+n}{(n+1)^3}$

$= \frac{n(n+1)}{2(n+1)^3} = \frac{n}{2\cdot(n+1)^2}$

Clearly $a_n > a_{n+1}$ $\forall n$. $\{a_n\}$ is monotonic decreasing sequence

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2(n+1)^2}$

$= \lim \frac{1}{2n[1+1/n]^2} = 0$

$\therefore$ By Leibnitz test the given series converge.

Cauchy's root test.

Statement:

Let Σa_n be a series of +ve terms.

Then Σa_n is convergent if $\lim_{n\to\infty} a_n^{1/n} < 1$

and divergent if $\lim_{n\to\infty} a_n^{1/n} > 1$

Proof:

Case (i)

Let $\lim_{n\to\infty} (a_n)^{1/n} = l < 1$

choose $\varepsilon > 0$ such that $l+\varepsilon < 1$. Then there exists $m \in N$ such that $(a_n^{1/n}) < l+\varepsilon$ for all $n \geq m$.

$\therefore a_n < (l+\varepsilon)^n$ for all $n \geq m$

Since $l+\varepsilon < 1$, $\Sigma (l+\varepsilon)^n$ is convergent

By comparison test Σa_n is convergent.

case (ii)

$$\text{Let } \lim_{n \to \infty} (a_n^{1/n}) = l > 1$$

choose $\varepsilon > 0$ such that $l-\varepsilon > 1$. Then there exists $m \in N$ such that $(a_n^{1/n}) > l-\varepsilon$ for all $n \geq m$.

$\therefore a_n > (l-\varepsilon)^n$ for all $n \geq m$.

Since $l-\varepsilon > 1$, $\Sigma (l-\varepsilon)^n$ is divergent

By comparison test Σa_n is divergent.

ABSOLUTE CONVERGENCE:

Definition:

A series $\sum a_n$ is said to be absolutely convergent if the series $\sum |a_n|$ is convergent.

Theorem:

Any absolutely convergent series is convergent.

Proof:-

Let $\sum a_n$ be a absolutely convergent

$\therefore \sum |a_n|$ is convergent.

Let $s_n = a_1 + a_2 + \cdots + a_n$ & $t_n = |a_1| + |a_2| + \cdots + |a_n|$,

By hypothesis (t_n) is convergent and hence is a Cauchy sequence.

Hence, given $\varepsilon > 0$, there exists $n_1 \in N$ such that $|t_n - t_m| < \varepsilon$ for all $n, m \geq n_1$ → (1)

now, let.

then $|s_n - s_m| = |a_{n+1} + a_{n+2} + \cdots + a_m|$

$\leq |a_{m+1}| + |a_{n+2}| + \cdots + |a_m|$

$= |t_n - t_m|$.

$< \varepsilon \quad \forall n, m \geq n_1$ [by (1)]

$\therefore (s_n)$ is a Cauchy sequence

$\therefore \Sigma a_n$ is a convergent series.

Theorem:-

In an absolutely convergent series, the series formed by its +ve terms alone is convergent and the series formed by its -ve terms alone is convergent and convergent and conversely.

Proof:-

Let Σa_n be the given absolutely convergent series.

we define $p_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n < 0 \end{cases}$ SIVA

(i.e) p_n is a +ve term of the given ~~series~~ series and q_n is the modulus of a -ve term.

$\therefore \Sigma p_n$ is the series formed with the +ve terms of the given series and the moduli Σa_n is the series formed with the moduli of the -ve terms of the given series.

clearly, $p_n \leq |a_n|$ and $a_n \leq |a_n| \forall n$.

Since the given series is absolutely convergent $\Sigma |a_n|$ is a convergent series

of +ve terms. Hence by comparison

test $\sum p_n$ and $\sum q_n$ ~~be~~ are convergent.

conversely, $\sum p_n$ and $\sum q_n$ converges to p and q respectively we claim that $\sum a_n$ is absolutely convergent.

we have $|a_n| = p_n + q_n$

$$\sum |a_n| = \sum (p_n + q_n)$$

$$= \sum p_n + \sum q_n$$

$$= p + q$$

$\therefore \sum a_n$ is absolutely convergent

Theorem:

Statement:

If $\sum a_n$ is an absolutely convergent series. and (b_n) is a bounded sequences. then the series $\sum a_n b_n$ is an absolutely convergent series.

Proof:

Since (b_n) is a bounded sequences, there exists a real number $k > 0$ such that.

$|b_n| \leq k$ for all n.

$\therefore |a_n b_n| = |a_n||b_n| \leq k|a_n| \ \forall n$.

Since $\sum a_n$ is absolutely convergent.

$\sum |a_n|$ is convergent.

$\therefore \sum |a_n|$ is convergent.

$\therefore$ By comparison test $\sum |a_i b_i|$ is convergent

$\sum a_n b_n$ is absolutely convergent

Problems:

1. Test for convergence of the series $\sum \frac{(-1)^n}{n^p}$

sol

Case (i)

Let $p > 1$

Then $\sum \left|\frac{(-1)^n}{n^p}\right| = \sum \frac{1}{n^p}$ is convergent.

$\therefore$ the gn series is absolutely convergent and hence convergent.

Case (ii)

Let $0 < p \leq 1$.

then $\left(\frac{1}{n^p}\right)$ is a monotonic decreasing sequence converging to 0.

$\therefore$ By Lebinitz's test the gn series convergent.

In this case the convergence is not absolute since $\sum \frac{1}{n^p}$ diverges when $0 < p \leq 1$

Case (iii)

Let $p < 0$.

Then the series reduces which finetely.

$-1+1-1 \dots$

case (iv)

Then the sequence $(1/n^p)$ is unbounded.
Hence the given series oscillates infinitely.

S.T the series $\sum (-1)^n [\sqrt{(n^2+1)} - n]$ is conditionally convergent.

Let

$$a_n = \sqrt{n^2+1} - n$$

$$= \sqrt{n^2+1} - n \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$= \frac{(\sqrt{n^2+1})^2 - n^2}{\sqrt{n^2+1}+n}$$

$$a_n = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$a_{n+1} = \frac{1}{\sqrt{(n+1)^2+1}+(n+1)}$$

$$a_{n+1} - a_n = \frac{1}{\sqrt{(n+1)^2+1}+(n+1)} - \frac{1}{\sqrt{n^2+1}+n}$$

$$= \frac{\sqrt{n^2+1}+n - \sqrt{(n+1)^2+1} - (n+1)}{\left[\sqrt{(n+1)^2+1}\,(n+1)\right]\left[\sqrt{n^2+1}+n\right]} < 0$$

$$a_{n+1} - a_n < 0$$

$$a_{n+1} < a_n$$

Clearly (a_n) is a montonic decreasing sequence converging to 0

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0$$

∴ By limits Lebinitz test the given series converges.

now, we prove that

$\sum (-1)^n (\sqrt{n^2+1} - n)$ is divergent

$|(-1)^n [\sqrt{n^2+1} - n]| = a_n = \frac{1}{\sqrt{n^2+1}+n}$

$b_n = \frac{1}{n}$

$\frac{a_n}{b_n} = \frac{n}{\sqrt{n^2+1}+n}$

$= \frac{n}{\sqrt{n^2(1+1/n^2)}+n}$

$= \frac{n}{n(\sqrt{1+1/n^2}+1)}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}+1}$

$= \frac{1}{\sqrt{1+\frac{1}{\infty}}+1}$

$= \frac{1}{1+1}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2} < 1$

∴ By comparison test Σa_n is divergent

∴ The given series is not a absolutely convergent

∴ The given series is conditionally convergent

4. Test the convergence of $\sum \frac{(-1)^n \sin n\alpha}{n^3}$

Sol

We have $\left|\frac{(-1)^n \sin n\alpha}{n^3}\right| \leq 1/n^3$ (since $|\sin\theta| =$

$\therefore$ By comparison test the series is absolutely convergent.