Real Numbers:-

There are two fundamental algebraic on the set R of all real numbers called addition and multiplication.

This operation should satisfies the following axiom.

A1 : closure for addition.

The set R is closed with respect to addition.

$a, b \in R \Rightarrow a+b \in R$.

A2 : Associative law of addition

The operation of addition in R is associative

$a+(b+c) = (a+b)+c \ \forall \ a, b, c \in R$

A3 : Identity element for addition

There exists a real number zero such that

$a+0 = 0+a \ \forall a \in R$.

A_4 : existence for (-ve) addition

Corresponding to each $a \in R$, there exist $b \in R$ such that

$$a+b = b+a = 0.$$

A_5 : cummulative

For each pair of real numbers a and b such that $a+b = b+a$.)

M_1 : closure for multification

The set R, is closed with respect to multiplication

$$a.b \in R \ \forall a, b \in R$$

M_2 : Associative law of multiplication

The operation of mutiplication in R on associative.

$$a.(b.c) = (b.c).c \ \forall a, b \in R$$

M_3 : Identity element for multiplication

There exists a real number 1 such that

$$a.1 = 1.a \ \forall a \in R$$

M_4 : existence of inverse

Corresponding to each $a \in R$, there exists $b \in R$ such that $a.b = b.a = 1$

D: Distributely of:

M_5: commutative law of multiplication for each pair of real numbers a and b such that

$$a \cdot b = b \cdot a$$

D: Distributely of multiplication over addition multiplication of distributely over addition in R. $a \cdot (b+c) = a \cdot b + a \cdot c \;\forall\; a, b, c \in R$.

NOTE:-

i) R satisfies A_1 to A_5 and M_1 to M_5, D.

$\therefore$ R is a field.

ii) Q and C are also field since they also satisfies the above axiom.

Theorem 2 uniqueness of zero

Statement :

There can exits at the most one identity for addition in R.

proof:-

If possible let there are two identities 0 and 0′ in R.

0 is the addition identity

$0+a = a+0 = a \ \forall a \in R \longrightarrow$ ①

0′ is the addition identity

$0'+a = a+0' = a \ \forall a \in R \longrightarrow$ ②

put

$a = 0'$ in equation ①

$\Rightarrow 0+0' = 0'+0 = 0' \longrightarrow$ ③

put $a=0$ in equation ②

$0'+0 = 0+0' = 0 \longrightarrow$ ④

From ③ & ④

$0 = 0'$

∴ Additine identity in R is unique.

Theorem 2:-

Uniqueness of Negative:

statement:-

Two each x in R there correspont one and only real number y such that $x+y = y+x = 0$

Proof:-

If possible, let $y_1, y_2 \in R$ be the equation of $x \in R$

When, y_1 negative element

$\Rightarrow x+y_1 = y_1+x = 0 \longrightarrow ①$

y_2 negative element

$\Rightarrow x+y_2 = y_2+x = 0$

Let

$y_1 = y_1+0$

$= y_1+(x+y_2)$

$= (y_1+x)+y_2$

$= 0+y_2$

$y_1 = y_2$

Negative of an element unique

Theorem 3 :

Translation Law of addition :-

Statement

If $x, y, z \in R$ and $x+z=y+z$.

then $x=y$

Proof :-

Let $x = x+0$

$= x+(z+(-z))$

$= (x+z)+(-z)$

$= (y+z)+(-z)$

$= y+(z+(-z))$

$= y+0$

$\boxed{x=y}$

Theorem 4 :-

For each $x \in R$

(i) $(-1)(-x) = x$

(ii) $-(x+y) = (-x)+(-y)$

Proof :-

i) Let z be the negative of $-x$

$(-x)+z = z+(-x) = 0$.

Now, $x = x+0$ (Property of 0)

$= x + [-x + z]$ (by our assumption)

$= [x + (-x)] + z$ (by associative law)

$= z$

$\boxed{x = -(-x)}$

(ii) $-(x+y) = 0 + 0 + [-(x+y)]$ (by identity law)

$= [x + (-x)] + [y + (-y)] + [-(x+y)]$ (by -ve law)

$= [x+y] + [-(x+y)] + [(-x) + (-y)]$

$= [x+y] + [-(x+y)] + [(-x) + (-y)]$

$= 0 + (-x) + (-y)$ [by -ve law]

$\boxed{-(x+y) = (-x) + (-y)}$ (by identity law)

Theorem 5:-

Uniqueness of identity for multiplication

Statement:-

There can exist at the most one identity element for multiplication in R.

Proof:

If possible there are two identity element $1, 1' \in R$

When 1 is identity

$x \cdot 1 = 1 \cdot x = x \quad x \in R \rightarrow (1)$

When $1'$ is identity

$x \cdot 1' = 1' x = x \cdot \quad x \in R \rightarrow (2)$

Suppose,

$1' \in R$ sub in eqn (1)

$\Rightarrow 1' \cdot 1 = 1 \cdot 1' = 1' \rightarrow (3)$

$1 \in R$ sub in eqn (2)

$\Rightarrow 1 \cdot 1' = 1' \cdot 1 = 1 \rightarrow (4)$

From (3) and $(4) \Rightarrow$

$1 \cdot 1' = 1'$

$1 \cdot 1' = 1$

$\boxed{1' = 1}$

Identity element for multiplication in unique.

Theorem 6 :-

uniqueness of inverse for multiplication

Statement:

To each $x \in R$; $x \neq 0$ there corresponding one and only one real number y such that $x \cdot y = y \cdot x = 1$

Proof:

It possible let there are two inverse $y_1, y_2 \in R$ for $x \in R$

y_1 is inverse

$$x y_1 = y_1 x = 1 \longrightarrow (1)$$

y_2 is inverse

$$x y_2 = y_2 x = 1 \longrightarrow (2)$$

Let $y_1 = y_1 \cdot 1$ (by identity law)

$= y_1 (x y_2)$ (by 2)

$= (y_1 x) y_2$ (by associative law)

$= (1 \cdot y_2)$ (by 1)

$= y_2$ (by identity law)

$$\boxed{y_1 = y_2}$$

Inverse of each element is uniqueness.

Theorem 7 :-

$x \cdot 0 = 0$ for all $x \in R$

Proof :-

Now,

$$x \cdot 0 = x(0+0) \quad (\text{by } A_3)$$

$$= x \cdot 0 + x \cdot 0 \quad (D.l)$$

$$x \cdot 0 + x \cdot 0 = x \cdot 0$$

$$x \cdot 0 + x \cdot 0 = x \cdot 0 + 0 \quad (\text{by } A_3)$$

Add

$-(x \cdot 0)$ on both sides

$$-(x \cdot 0) + (x \cdot 0) + (x \cdot 0) = -(x \cdot 0) + (x \cdot 0)$$

$$0 + (x \cdot 0) = 0 \quad (\text{by } A_3)$$

$$\boxed{x \cdot 0 = 0}$$

Theorem 8 :-

If x, y be the real number such that $xy = 0$ then either $x = 0$ (or) $y = 0$.

Proof:

Given $xy = 0$

If $x = 0$ then there is nothing to prove so take $x \neq 0$.

$\therefore x^{-1} \in R$

$xy = 0$

$x^{-1}(xy) = x^{-1}0$ (by theorem 4)

$(x^{-1}x)y = 0$

$1 \cdot y = 0$ (by inverse law)

$\boxed{y = 0}$. (by identity law)

Hence Proved.

Theorem 9 :-

cancellation law of multiplication

If x, y, z be the real numbers such that $xz = yz$ and $z \neq 0$ then $x = y$.

Proof:

Now,

$(x + (-y))z = xz + (-y)z$ (D.L)

$= yz + (-y)z$ (given)

$= (y + (-y))z$ (D.L)

$= 0 \cdot z$ (-ve law)

$(x + (-y))z = 0$ (by theorem 7)

Given $z \neq 0$

$x + (-y) = 0$ (by theorem 8)

$x + (-y) = y + (-y)$ (by -ve law)

$\boxed{x = y}$ (by cancellation law)

Hence Proved.

Theorem 10:-

For all $x, y \in R$.

(i) $x(-y) = -(xy)$

(ii) $(-x)y = -(xy)$

(iii) $(-x)(-y) = xy$

Proof:-

i) $x(-y) = -xy$

Add both sides on xy

$xy + (x)(-y) = xy - xy$

$= x(y + (-y))$ (D.L)

$= x(0)$ (N.L)

$xy + x(-y) = 0$ (I.L)

Adding both sides on $-xy$

$-xy + xy + x(-y) = -xy + 0$

$0 + x(-y) = -xy$

$\boxed{x(-y) = -xy}$

(ii) $y(-x)$

$= -(yx)$

$= -xy$

$$\boxed{(-x)y = -xy}$$

iii) $(-x)(-y) = -[(-x)y]$

$= -(-xy)$

$$\boxed{(-x)(-y) = xy}$$

Definition : Difference :

The difference between two real numbers x and y is given by $x + (-y)$ and it is denoted by $x - y$.

The operation of finding the difference is called subtraction

Quotient definition:-

The quotient of a real number x and y ($y \neq 0$) by is given by xy^{-1} its denoted by x/y (or) $x \div y$.

The operation of finding quotient is called a division order in R

O_1 : law of trichotomy :

Given any two real numbers a, b one and only one of the following holds.

$a > b$, $a = b$, $a < b$

O_2 : Transitivity :

For each triple of real numbers a, b, c if $a > b$, $b > c$ then $a > c$

O_3 : Monotone property for addition

For all real numbers a, b and c.

$a > b \Rightarrow a + c > b + c$

O_4 : Monotone property for multiplication

For all real number a, b and c

$a > b$ and $c > 0 \Rightarrow ac > bc$

Note :

(i) The field R of all real numbers satisfies $O_1 - O_4$

$\therefore$ R is an ordered field

(ii) The set of all rational numbers Q is also an ordered field.

(iii) The set of all complex numbers is not ordered.

Positive Numbers:

A real number a is said to be positive if $a > 0$.

Theorem : 11 :-

For each real number a one and only one of the following holds $a < 0$, $a = 0$, $-a > 0$.

<u>Proof</u>:-

It is enough to prove

$0 > a \Rightarrow -a > 0$.

Now,

(i) $0 > a$

$\Rightarrow a + (-a) > a + (-a)$ (by O_3)

$\Rightarrow -a > 0$ (by A_3, A_4)

(ii) Take $-a > 0$

$\Rightarrow (-a) + a > 0 + a$ (by O_3)

$\Rightarrow 0 > a$

Hence the theorem.

Theorem 12:-

If a, b be positive real numbers then $a+b$ is a positive real numbers.

Proof:

Let a, b be two positive real numbers

$a > 0, b > 0$

$a > 0$

Add b on both sides

$a+b > 0+b$ (by O_3)

$a+b > b$ and $b > 0$

$\Rightarrow (a+b) > 0$ (by O_2)

$\therefore (a+b)$ is a positive real number.

Theorem 13:-

If a, b be positive real numbers then ab is a positive real number.

proof:-

Let a, b be two positive real numbers $\therefore a > 0, b > 0$

$a > 0$ (by O. law)

Multiplication b on both sides

$ab > 0b$ (by Q9)

$ab > 0$, (by $x \cdot 0 = 0$),

$\therefore ab$ is a positive real number

Definition:-

A real number 'a' is said to be less than b is $b > a$.

A real number 'a' is said to be negative if $a < 0$.

The properties of the relation $>$ (greater than) are also the true for $<$ (less than) [Except that of multiplication].

Absolute value:-

If x be a real number Then its absolute value denoted by $|x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note :-

$\geq, \leq$ the relation are called

the weak inequalities.

The relation $>, <$ are called the

Strict inequalities.

Theorem 14 :-

For every $x \in \mathbb{R}$ $|x| = \max \{-x, x\}$

Proof :-

By definition.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

when $x \geq 0$

$|x| = x$ and $x \geq -x$

when $x < 0$.

$|x| = -x$ and $-x > x$

$\therefore |x| = \max \{-x, x\}$

Corollary :-

For every $x \in \mathbb{R}$, $x \leq |x|$

Proof: By above theorem

$|x| = \max \{-x, x\} \geq x$.

$|x| \geq x$, $x \leq |x|$

Theorem 15:

For every $x \in R$, $|x|^2 = x^2 = |-x|^2$

By definition

$$|x| = \begin{cases} x & \text{is } x \geq 0 \\ -x & \text{is } x < 0 \end{cases}$$

$$|x|^2 = \begin{cases} x^2 & \text{is } x \geq 0 \\ (-x)^2 & \text{is } x < 0 \end{cases}$$

$$|x|^2 = \begin{cases} x^2 & \text{is } x \geq 0 \\ x^2 & \text{is } x < 0 \end{cases}$$

$$|x^2| = x^2 \quad \forall x \longrightarrow ①$$

put $x = -x$

$$|-x|^2 = (-x)^2.$$

$$|-x|^2 = x^2 \longrightarrow ②$$

from ① and ②

$$|x|^2 = x^2 = |-x|^2$$

Theorem 16:-

For every $x \in R$ $|x| = |-x|$

Proof:

W.K.T

$|x| = \max\{-x, x\} \longrightarrow (1)$

Put $x = -x$

$|-x| = \max\{-(-x), -x\}$

$= \max\{x, -x\}$

$= |x|$ {from (1)}

$|-x| = |x|$

Theorem 17:-

For all $x, y \in R$ $|xy| = |x||y|$

Proof:

W.K.T

$|x|^2 = x^2$

$|xy|^2 = (xy)^2$

$|xy|^2 = |x|^2 |y|^2$

Take positive square root

$|xy| = |x| \cdot |y|$

Theorem 18 :-

Triangle inequality

For all real number $x, y \in R$ then

$|x+y| \leq |x| + |y|$

proof:-

case (i):

when $x+y \geq 0$

$|x+y| = x+y \longrightarrow (1)$

w.k.T

$x \leq |x|$, $y \leq |y|$ (by theorem (4))

$x+y \leq |x| + |y|$

$|x+y| \leq |x| + |y|$

Case (ii)

when $x+y < 0$

$|x+y| \leq -(x+y)$

$\Rightarrow (x+y) < 0$

$\Rightarrow -(x+y) < 0$

$\Rightarrow (-x) + (-y)) > 0$

$|x| = |-x|$ (by theorem (ii b))

$$|x+y| = |-(x+y)|$$

$$= |(-x)+(-y)|$$

$$\leq |-x| + |-y| \quad \text{(by case (i))}$$

$$= |x| + |y|$$

$$|x+y| \leq |x| + |y|$$

Theorem 1(9) :-

For all real numbers x and y

$$|x-y| \geq ||x| - |y||$$

Proof:

Let $|x| = |x-y+y|$ {∵ add and sub y}

$\leq |x-y| + y$ {by triangle inequality}

$$|x| \leq |x-y| + y$$

$$|x| - |y| \leq |x-y| \longrightarrow (1)$$

Now,

Consider

$|y| = |y-x+x|$ {∵ add and sub x}

$\leq |y-x| + |x|$ {$\because$ By triangle inequality}

$|y| \leq |y-x| + |x|$ ($\because |x| = |-x|$).

$-(|x|-|y|) \leq |x-y| \longrightarrow$ ②

W.K.T

$|x| = \max\{-x, x\}$

put $x = |x| - |y|$

$||x|-|y|| = \max\{(|x|-|y|), |x|-|y|\}$

From ① and ② $\Rightarrow \leq |x-y|$

$\therefore ||x|-|y|| \leq |x-y|$

$\therefore |x-y| \geq ||x|-|y||$

Hence Proved

Problem

1. $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$

Proof

$$|x+y|^2 = (x+y)^2 \qquad (\because |x|^2 = x^2 = |-x|^2)$$

$$|x-y|^2 = (x-y)^2$$

$$|x+y|^2 + |x-y|^2 = (x+y)^2 + (x-y)^2$$

$$= x^2+y^2+2xy+x^2+y^2-2xy$$

$$= 2x^2+2y^2$$

$$= 2(x^2+y^2)$$

$$|x+y|^2+|x-y|^2 = 2(|x|^2+|y|^2)$$

Hence Proved.

2. If x, l, ϵ be real numbers and $\epsilon > 0$ such that $|x-l| < \epsilon \Leftrightarrow l-\epsilon < x < l+\epsilon$

Proof:

$$|x-l| < \epsilon \qquad \left\{ \because |x| = \max\{-x, x\} \right.$$

$$\Leftrightarrow \max\{-(x-l), (x-l)\} < \epsilon$$

$$\Leftrightarrow \max\{-x+l, x-l\} < \epsilon$$

$$\Leftrightarrow -x+l < \epsilon \ \& \ x-l < \epsilon \quad ①$$

$$\Leftrightarrow l-x < \epsilon \ \& \ x-l < \epsilon$$

$$\Leftrightarrow l-\epsilon < x \ \& \ x < l+\epsilon$$

$\Leftrightarrow l-\epsilon < x < l+\epsilon$

$|x-l| < \epsilon \Leftrightarrow l-\epsilon < x < l+\epsilon$

Completness

Definition :-

upper bound

If for a set S of real numbers there exist a real number u such that

$$x \in S \Rightarrow x \leq u.$$

Then u is called an upper bound of S. If there exist an upper bound for a set S. the S is said to be bounded above.

$|x-y| \leq |x| + |y|$

Proof:

Case (i)

When $x \geq 0$

$|x-y| = x-y \longrightarrow$ ①

W.K.T $x \leq |x|$, $-y \leq |y|$ [$\because$ by T-4]

Theorem 20:-

The set of positive real numbers R^+ is not bounded above.

Proof:

Let u be an upper bound of R^+

$$x \in R^+ \Rightarrow x \leq u \longrightarrow (1)$$

Now

Let ~~u<b~~

$$1 \in R^+ \Rightarrow 1 \leq u$$
$$\Rightarrow u \geq 1$$
$$\Rightarrow u > 0$$
$$\Rightarrow u+1 > 0$$
$$\Rightarrow u+1 \in R^+$$

but $u+1 > 0$

which is a contradiction in (1)

$\therefore$ our assumption is wrong

(i.e) u is upper bound in R^+ is wrong

$\Rightarrow R^+$ has no upper bound

$\Rightarrow R^+$ is not bounded above.

Hence proved.

2m Definition :

supremum :

If the set of all upper bound of a set of all real numbers has a smallest number, say W is said to be a least upper bound (or) supremum of S.

Theorem 21 :

A set cannot have more than one supremum.

Proof :

If possible let there are two supremum W & W' for the set S.

Case (i)

When W is the supremum.

$\Rightarrow W \leq W'$

$\Rightarrow W \not> W' \longrightarrow (1)$

Case (ii)

When W' is the supermum

$\Rightarrow W' \leq W$.

$\Rightarrow w' \not> w \longrightarrow (2)$

By law of trichotomy

$w = w'$

$\therefore$ the supremum of a set S is unique.

Hence Proved.

Definition :-

C : Order completness property.

Every non-empty set of a real numbers which is bounded above has supremum.

Ex :

R satisfies the above property.

R is called as a completely orders.

Note 1 :

R satisfies $A_1 - A_5$, $M_1 - M_5$, D, $O_1 - O_4$ and C this property, then R is called complete ordered field.

note 2 :-

Q does not satisfies "C"

∴ Q is not complete.

Archimedean property of real numbers :-

Statement :-

If x and y be any positive real numbers. then there exists a positive integer n such that $ny > x$ (lower bound)

Proof :-

Let us assume that the statement of the theorem be false.

∴ $ny \leq x$ for all positive integer n

$\Rightarrow x$ is an upper bound of the set

$$S = \{y, 2y, 3y, \ldots\}$$

$\Rightarrow S$ is bounded above.

By order completeness property.

S must have supremum say 's'

∴ $ny \leq s$ for all positive integer.

$\Rightarrow (n+1)y \leq s$ for all positive integer n.

$\Rightarrow (ny + y) \leq s$

$\Rightarrow ny \leq s-y$

$\Rightarrow s-y$ is also an upper bounded of s

This contradiction.

Since s is the least upper bound

$\therefore$ our assumption is wrong.

$\therefore ny > x$.

$\therefore$ Hence proved.

corollary : 1.

If x be any real number, then there exists a positive integer n such that $n > x$.

<u>Proof</u>:-

Take $y=1$ in the proof of the Archimedean property theorem.

(i.e) $ny > x$

put $y=1$

$\Rightarrow n > x$

Hence proved.

corollary : 2 :

If x be any real number any y be any positive real number then there exists a positive integer n such that $ny > x$.

Proof:-

When $x > 0$

There is nothing to prove

When $x \leq 0$

takes $n = 1$

$ny = 1 \cdot y$

$y > 0$

$y > x$

$\therefore ny > x$

Hence Proved.

Theorem : 23

Let S be a non-empty set of real number bounded above. then a real number s is the supremum of S iff the following two condition hold

(i) $x \leq s$ for all $x \in S$

(ii) For each positive real number 'ϵ' there is a real number $x \in S$, $\ni x > s - \epsilon$

Proof:-

Necessary part

Let s be the supremum of S.

$\Rightarrow$ s is also an upper bound of S

$\Rightarrow x \leq s$ for all $x \in S$

$\therefore$ First condition is true.

If ϵ is any positive real number then

$s - \epsilon < s$.

$\Rightarrow s - \epsilon$ cannot be an upper bound of S.

Since

s is the least upper bound.

$\Rightarrow x > s - \epsilon \ \forall x \in S$.

The second condiction is true.

sufficient part.

Conversely

Suppose (i) and (ii) is true

We have to prove that.

s is supremum of S.

for that.

By (i) $\Rightarrow$

$x \leq s$ for all $x \in S$.

$\Rightarrow s$ is upper bound of S.

Let $s' \in S$.

s' is any real numbers.

Note :-

Q does not satisfies "c".

∴ Q is not complete.

Archimedean property of real numbers :-

Statement :-

If x and y be any positive real numbers. then there exists a positive integer n such that $ny > x$ (lower bound).

Proof :-

Let us assume that the statement of the theorem be false.

∴ $ny \leq x$ for all positive integer n

$\Rightarrow$ x is an upper bound of the se

$$S = \{y, 2y, 3y, \ldots\}$$

$\Rightarrow$ S is bounded above.

By order completeness property.

S must have supremum say 's'

∴ $ny \leq s$ for all positive integer.

$\Rightarrow (n+1)y \leq s$ for all positive inte

$\Rightarrow (ny + y) \leq s$

$\Rightarrow ny \leq s-y$

$\Rightarrow s-y$ is also an upper bounded of s.

This contradiction.

Since s is the least upper bound

$\therefore$ our assumption is wrong.

$\therefore ny > x$.

$\therefore$ Hence proved.

corollary : 1.

If x be any real number, then there exists a positive integer n such that $n > x$.

<u>Proof</u>:-

Take $y=1$ in the proof of the Archimedean property theorem.

(i.e) $ny > x$

put $y=1$

$\Rightarrow n > x$

Hence proved.

corollary : 2 :

If x be any real number any y be any positive real number then there exist a positive integer n such that $ny > x$.

Proof:-

when $x > 0$

There is nothing to prove

when $x \leq 0$

takes $n = 1$

$ny = 1 \cdot y$

$y > 0$

$y > x$

$\therefore ny > x$

Hence Proved.

Theorem : 23

Let S be a non-empty set of real number bounded above. Then a real number s is the supremum of S iff the following two condition hold:

(i) $x \leq s$ for all $x \in S$

(ii) For each positive real number 'ϵ'

there is a real number $x \in S$, $\ni x > s - \epsilon$

Proof:-

Necessary part

Let s be the supremum of S.

$\Rightarrow$ s is also all upper bound of S.

$\Rightarrow x \leq s$ for all $x \in S$

$\therefore$ First condition is true.

If ϵ is any positive real number

Then $s - \epsilon < s$.

$\Rightarrow s - \epsilon$ cannot be an upper bound of S.

Since

s is the least upper bound.

$\Rightarrow x > s - \epsilon \ \forall x \in S$.

The second condiction is true.

sufficient part.

conversely

Suppose (i) and (ii) is true

we have to prove that.

s is supremum of S.

for that.

By (i) $\Rightarrow$

$x \leq s$ for all $x \in S$.

$\Rightarrow s$ is upper bound of S.

Let $s' \in S$.

s' is any real numbers.

But

$s' < s$.

$\Rightarrow s - s' > 0$

Let

$s - s' = \epsilon \longrightarrow (1)$

By (2) $\Rightarrow$

there exists a real number $x \in S$ such that.

$x > s - \epsilon$.

$x > s - (s - s')$

$x > s - s + s'$

$x > s'$

$\therefore s'$ is not upper bound of S.

$\Rightarrow s$ is the upper bound of S and Any number less than s is not an upper bound of S.

$\Rightarrow S$ is bounded above.

By ordered completeness property

$\Rightarrow s$ is the supremum of S.

Hence Proved.

Definition :

Lower bound :

If for a set S of real numbers there a real number v such that

$x \in S \Rightarrow x \geq v$.

Then v is called lower bound of S.

If there exists a lower bound of the set.

Then S is said to be bounded below.

Ex :-

The set of positive real numbers is bounded below $\{0, 1, 2, \ldots\}$

Definition :-

Infimum (or) Greatest lower bound.

In the set of all lower bounds of a set S of real numbers has a greatest number say t. then t is said to be greatest lower bound (or) an infimum of S (written inf S).

Theorem : 2.4

Any non-empty set of real numbers which is bounded below has infimum.

Proof:-

Let S be the non-empty set of real numbers and.

Let v be the lower bounds of S

Let us denote by T the set of negative of number of S.

$$T = \{-x, x \in S\}$$

We show that T is bounded above for that.

Let $y = -x$ for some $x \in S \to (1)$

Then $x \geq v$

Since.

v is the lower bound of S.

$\therefore -y \geq v$ (by (1))

$y \leq -v \; \forall y \in T$

T is bounded above set.

$-v$ is being an upper bound.

By completeness property T has supremum say b

$\Rightarrow$ It can be show t is the infimum of S.

Let if w be any lower bound of s.

Then

$-t \geq w$.

$\Rightarrow w$ be the lower bound of s.

$\Rightarrow -w$ be the upper bounded of T

$\Rightarrow t \leq -w$

$\Rightarrow -t \geq w$.

$\therefore$ non-empty set of real numbers which is bounded has infimum.

Hence Proved

Theorem : 2.5

Let S be non empty set of real number bounded below. A real number t is the infimum of S. iff the following condition holds.

$\Leftrightarrow$

(i) $x \geq t$ for all $x \in S$.

(ii) For each positive real number ϵ there is a real number $x \in S$ such that $x < t + \epsilon$

Proof:

Necessary part.

Let t be the infimum of S.

$\Rightarrow$ t is also all lower bound of S.

$\Rightarrow$ $x \geq t$ for all $x \in S$.

$\therefore$ First condition is true.

If ϵ is any positive real number

Then.

$t + \epsilon > t$.

$t + \epsilon > t$.

$\Rightarrow t + \epsilon$ cannot be a lower bound of S.

Since, t is the greatest lower bound.

$\Rightarrow x < t + \epsilon$ for sum $x \in S$.

The second condition is true.

conversely

Suppose (i) and (ii) is true

we have to prove that.

t is infimum of S.

for that

By (1) $\Rightarrow$

$x \geq t'$ for all $x \in S$.

$\Rightarrow t$ is lower bounds of S.

Let

$t' \in S$.

t' is any real number

But $t' > t$.

$\Rightarrow t' - t > 0$

Let

$t' - t = \epsilon \longrightarrow (1)$

By (2) $\Rightarrow$

There exist a real number $x \in S$ such that.

$x < t + \epsilon$

$x < t + (t' - t)$

$x < t + t' - t$

$x < t'$

$\therefore t'$ is not lower bound of S

$\Rightarrow t$ is lower bound of S

$\Rightarrow$ t is lower bound of S and any number greater that t in not a lower bound of S.

$\Rightarrow$ S is bounded below.

By ordered completeness property.

$\Rightarrow$ t is the infimum of S.

Hence proved.

Subset of R.

i) Natural numbers :-

The set N of Natural Number is the smallest subset of R. $N \subset R$.

having the following property.

i) $1 \in N$

ii) $m \in N \Rightarrow m+1 \in N$.

The condition i) and (ii) are requires to principle of finite induction

The natural number satisfy the following conditions.

i). A_1, A_2, A_5

(ii) M_1, M_2, M_3, M_5

(iii) D.

(2) Integers:

The set of all integers in the smallest subset of R.

i.e) $Z \subset R$ having the following property

(i) $Z \supset N$

(ii) Z contain an identity element for addition.

(iii) Z contains the negative of each of its element.

The condition i) and (ii) and (iii) satisfy the following condition

(i) $A_1 - A_5$

(ii) $M_1 - M_3, M_5$

(iii) D.

(3) Rational numbers: (Q)

The set of all rational numbers is the smallest subset of having the following properties.

(i) $Q \supset N$

(ii) Q is a field.

Note:

Q does not satisfy completness property.

Theorem : 26.
There is no rational numbers whose square is 2.

Proof:-
If possible. Let x be a rational no's whose square is 2.

Take $x = p/q$, p and q are integers with $q \neq 0$.

$$x^2 = 2$$

$$(p/q)^2 = 2 \longrightarrow (1)$$

Let g be the greatest common devision p and q.

$$p = gm, \quad q = gn$$

Then m and n are co-prime.

From (1) $\Rightarrow$

$$\left(\frac{gm}{gn}\right)^2 = 2.$$

$$(m/n)^2 = 2$$

$$\frac{m^2}{n^2} = 2$$

$$m^2 = 2n^2 \longrightarrow (2)$$

$\therefore m^2$ is even.

m is also even.

$m = 2u$ (say) $\longrightarrow$ (3)

From (2)

$(2u)^2 = 2n^2$

$4u^2 = \cancel{4n^2}\ 2n^2$

$n^2 = 4u^2/2$

$\Rightarrow n^2 = 2u^2$

$\therefore n^2$ is even

n is also even

$n = 2v \longrightarrow$ (4)

where u and v are integers.

From (3) and (4) $\Rightarrow$

2 is the common divisor of m and n.

This $\Rightarrow$

$\angle$ to the choice of m & n.

$\therefore$ m and n are co-prime.

$(p/q)^n \neq 2$

There is no rational no. whose square is 2.

Theorem : 27.

The set of rational numbers is not complete ordered field.

proof:-

It is enough to show that there exist a non-empty subset of q, which is bounded above and has no supremum.

Define $S = \{x, x \in q^{-1} \text{ and } 0 < x^2 < 2\}$

$1 \in S \Rightarrow S$ is non empty.

since

S is the upper bound of S.

It is bounded above.

We shall show that S does not have a least upper bound.

Let x be any rational number.

case (i)

$x \leq 0$

since all the element of S are positive x is not an upper bound of S.

$\Rightarrow x$ is not supremum.

case (ii)

when $x>0$ and $0<x^2<2$

Let $y = \dfrac{4+3x}{3+2x} \longrightarrow ①$

$$y^2-2 = \left(\frac{4+3x}{3+2x}\right)^2 - 2$$

$$= \left(\frac{(4+3x)^2 - 2(3+2x)^2}{(3+2x)^2}\right)$$

$$= \frac{(16+9x^2+24x) - 2(9+4x^2+12x)}{(3+2x)^2}$$

$$= \frac{16+9x^2+24x-18-8x^2-24x}{(3+2x)^2}$$

$$y^2-2 = \frac{x^2-2}{(3+2x)} \longrightarrow (2)$$

From ①

$$y-x = \frac{4+3x}{3+2x} - x$$

$$= \frac{4+3\cancel{x}-3\cancel{x}-2x^2}{3+2x}$$

$$= 4-2x^2/3+2x$$

$$y-x = \frac{2(2-x^2)}{3+2x} \longrightarrow ③$$

by (1) $\Rightarrow$

$x > 0 \Rightarrow y > 0$

$\Rightarrow 0 < y$

$\Rightarrow 0 < y^2 \longrightarrow (4)$

$x^2 < 2 \Rightarrow x^2 - 2 < 0$

$\Rightarrow y^2 - 2 < 0$ (by 2)

$\Rightarrow y^2 < 2$

$y > 0$ and $0 < y^2 < 2$ (from 4 & 5)

$\Rightarrow y \in S$

Now,

$x^2 - 2 < 0$

multiply by (-1)

$-x^2 + 2 > 0$

$\therefore 2 - x^2 > 0$

$\Rightarrow y - x > 0$ (by 3)

$\Rightarrow y > x$

x is not an upper bound of S.

$\Rightarrow x$ is not supremum.

case (iii)

$x > 0$ and $x^2 = 2$.

This is not possible. since is no ration number whose square is 2.

Case (iv)

When $x>0$, $x^2>2$

$x>0 \Rightarrow y>0$ (by (1))

$\Rightarrow 0<y$

$\Rightarrow 0<y^2$

$x^2>2 \Rightarrow x^2-2>0$

$\Rightarrow y^2-2>0$ (by (2))

$\Rightarrow y^2>2$

$y>0$ and $y^2>2 \longrightarrow (6)$

Now,

$x^2-2>0$

multiply by $(-)$

$-x^2+2<0$

$2-x^2<0$

$\Rightarrow y-x<0$ (by (3))

$\Rightarrow y<x \longrightarrow (7)$

Let $p\in S$.

$\Rightarrow p>0$ and $0<p^2<2$

by (6) $\Rightarrow 2<y^2$

$\Rightarrow 0<p^2<2<y^2$ and $y<x$ (by 7)

$\Rightarrow 0 < p < \sqrt{2} < y < x$

x and y are both upper bound of S

But x is not least upper bound of S.

we are verified all the possibility

$\therefore x$ is not the supremum of S.

$\Rightarrow S$ is not complete.

$\therefore$ The set of rational number is not ordered field.

Hence proved.

Finite set (Definition)

A set S is said to be finite.

If either it is empty or for some natural number n, there exist a one-to-one maping from the set.

$\{1, 2, \ldots n\}$ on to this set S.

If a set S is not finite its said to be infinite.

Example :-

(i) ϕ is a finite set

(ii) $\{e, \phi, \sqrt{2}\}$ is a finite set.

(iii) The set of all natural number is infinite set.

Enumerable :

A set S is said to be enumerable if there exist a one-to-one maping from the set N of all natural number on to the sets S.

Countable :-

A set S is said to be countable if it is either finite or enumerable.

If a set is not countable, then its said to be uncountable.

Example :-

(i) The set N of all Natural number is enumerable. The identity maping is the require one-too-one maping.

(i) The empty set is countable.

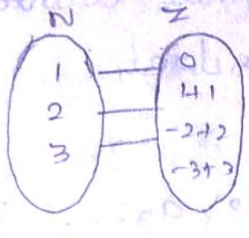
(ii) The set Z of the integer is countable.

Note :-

The maping $f: N \rightarrow Z$ define by.

$$f(n) = \begin{cases} \left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ -(n/2) & \text{if } n \text{ is even} \end{cases}$$

is the required one-to-one maping.



RESULTS:-

(i) Every subset of a finite is finite

(ii) Every super set of a infinite set is infinite

(iii) The intersection of every non-empty family of finite set is finite.

(iv) The union of every non empty family of infinite set is infinite.

Theorem : 28 : (X)

Every subset of a countable set is countable.

Proof!

Let A be a countable set and B be subset of A.

If B is finite. Then there is nothing to prove.

Without loss of generality assume that A is an infinite countable set and B its infinite subset.

Let,

$$A = \{a_1, a_2, a_3 \ldots\}$$

Then each element of B is an $a_i \in A$ for some index i.

Let n_1 be the smallest index for which $a_{n_1} \in A$ and as well as $a_{n_1} \in B$.

consider $A \sim \{a_{n_1}\}$

Let n_2 be the smallest index for which a_{n_2} belongs to B as well as to $A \sim \{a_n\}$.

consider $A \sim \{a_{n_1}, a_{n_2}\}$

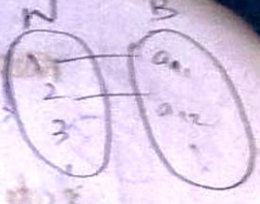
Let n_3 be the smallest index for which a_{n_3} belongs to B well as to $A \sim \{a_{n_1}, a_{n_2}\}$

Consider

$$A \sim \{a_{n_1}, a_{n_2}, a_{n_3}\}$$

Proceeding in this way we can obtain.

$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$



Then $k \longrightarrow a_{n_k}$ is the 1-1

maping from N on to B.

$\Rightarrow$ B is countable.

Theorem 29 :-

Every super set of an uncountable set is uncountable. Set is true

Proof:

Let A be an uncountable set.

Let $B \supset A$

If B is countable, then the set A must also be countable

(for it is subset of the countable set B)

$\therefore$ since A is given to be uncountable

It follows that B must also be U.C.

$\therefore$ Every super set of an U.C set is U.C

Theorem 20:-

If $A_1, A_2, A_3 \ldots$ are countable sets. then is $\bigcup_{n=1}^{\infty} A_n$ is countable

(or) Show that the infinite union of countable sets in countable.

proof:

Let

$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, a_{23} \ldots\}$

$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\}$

$\vdots$

$A_n = \{a_{n1}, a_{n2}, a_{n3} \ldots\}$

Here,

a_{ij} = j^{th} element in the ith set.

Define height of $a_{ij} = i+j$

$\therefore$ Height of $a_{11} = 2$ and this is the only element of height 2.

Similarly the height of a_{12} and a_{21} $= 3$ and there are the only element of height 3.

proceeding similarly there are $(m-1)$ element with height m.

Also height of each element is unique.

using these heights we can arrange these elements as $a_{11}, a_{12}, a_{13}, \ldots$

This process is shown below.

$$\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

Thus all the elements are counted as in the figure according to their heights.

Hence $\bigcup_{n=1}^{\infty} A_n$ is countable.

Theorem 31:-

The set $N \times N$ is countable

Proof:-

We may arrange the elements of $N \times N$ and ~~cannot~~ count them as in the following diagram.

$$\begin{array}{llll} (1,1) \to & (1,2) & (1,3) \to & (1,4) \ldots \\ (2,1) & (2,2) & (2,3) & (2,4) \ldots \\ (3,1) & (3,2) & (3,3) & (3,4) \ldots \\ (4,1) & (4,2) & (4,3) & (4,4) \ldots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

The counting power is shown by the arrows

Hence $N \times N$ is countable.

Corollary :-

i) The set of all positive rational numbers is countable.

Proof:-

Let A be the set of all positive rational numbers.

$A = \{ p/q : p$ and q are positive integers prime to each other $\}$.

Define $B = \{(p,q) : (p,q)$ ~~positive~~ and p, q are prime to each other.

Then there is a 1-1 correspondence between the elements of A and B.

$\Rightarrow$ A is countable iff B is countable.

Since B is a subset of the countable set $N \times N$, B is countable.

$\Rightarrow$ A is also a countable.

Corollary:-

(2) the set of all negative rational numbers is countable.

Proof:

Let A be the set of all positive rational numbers and C be the set of all negative rational numbers.

Then there is a 1-1 correspondence between the elements of A and B.

$\Rightarrow$ A is countable iff B is countable.

Since B is a subset of the countable set $N \times N$, B is countable.

$\Rightarrow$ A is also a countable.

corollary

(2) the set of all negative rational numbers is countable.

Proof:

Let A be the set of all positive rational numbers and c be the set of all negative rational numbers.

Then there is a 1-1 correspondence between the elements of A and c.

$\Rightarrow$ c is countable iff A is countable of A and c.

$\Rightarrow$ c is countable since A is countable.

corollary:

(3) The set of all rational number is countable.

Proof:

$Q = A \cup C \cup \{0\}$

Where,

A = The set of all positive rational

C = The set of all negative rational

Here, A, C and $\{0\}$ are countable sets

We know that the infinite union of countable sets is countable.

corollary :

(4) The set of all rational number in (0, 1) is countable.

<u>proof</u>:

The set of all rational numbers on (0, 1) is a subset of $\mathbb{Q}$ which is countable.

$\therefore$ the set of all rational numbers in (0, 1) is countable.

Theorem : 32.

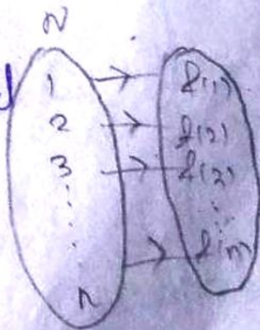
The set [0, 1] is uncountable.

<u>proof</u>:

Suppose that [0, 1] is countable. then there exits a 1-1 mapping f from N onto [0, 1].

$\therefore [0,1] = \{f(1), f(2), f(3), f(4), \ldots f(n) \ldots\}$

N



Express each $f(n)$ as a decimal

$f(1) = 0 . a_{11}, a_{21}, a_{31} \ldots$

$f(2) = 0 . a_{12}, a_{22} a_{32} \ldots$

$f(3) = 0 . a_{13} \quad a_{23} \quad a_{33} \ldots$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$f(n) = 0 . a_{1n} \quad a_{2n} \quad a_{3n} \ldots$

$\vdots \qquad \vdots \qquad \vdots$

Here $a_{ij} \in \{0, 1, 2, \ldots, 9\}$

Define for each $n \in \mathbb{N}$, a +ve integer b_n as follows

$b_n = 1$ if $a_{nn} \neq 1$

$= 2$ if $a_{nn} = 1$

$\therefore b_n \neq a_{nn}$ for each n.

Take $y = 0.b_1 b_2 b_3 \ldots$

$\Rightarrow y \in [0, 1]$.

Now, y differs from $f(1)$ in the first decimal place because $b_1 \neq a_{11}$

y differs from $f(2)$ in the second decimal place because $b_2 \neq a_{22}$

In general y differs from $f(n)$ on the n^{th} decimal place.

$\therefore y \notin \{f(1), f(2), f(3) \ldots f(n) \ldots\}$

$\Rightarrow f$ is not one to one

Which is a contradiction

$\therefore [0, 1]$ is uncountable.

Corollary : 1.

The set of all real numbers is uncountable.

proof:-

We know that

$[0,1] \subset R$.

Also, $[0,1]$ is uncountable.

The set of an an uncountable set must be uncountable.

$\Rightarrow R$ must be uncountable.

Corollary : 2.

The set of all irrational numbers is uncountable.

Let S be the set of all irrational number then $R = Q \cup S$.

Here R is uncountable and Q is countable

$\Rightarrow S$ must be uncountable since if S is countable then R will be countable.

[Infinite union of countable sets is countable].

Theorem : 3.3.

Let P_n be the set of polynomial function f of degree n defined by the relations of the form.

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

Where n is a fixed non-negative integer the co-efficients $a_0, a_1, a_2 \dots a_n$ are all integers with $a_0 \neq 0$. the set P_n is countable.

Proof:-

Let us prove the theorem by induction on n.

When $n=0$.

The set of polynomials with degree 0 is 1-1 correspondence with $Z-\{0\}$.

since $Z-\{0\}$ is countable, the set of all polynomials with degree 0 is countable.

Let us assume that the set P_k is countable for some fixed non-negative integer k.

Define.

$S_m = \{ f : mx^{k+1} + g, g \in P_k \}$

$S_{-m} = \{ f ; -mx^{k+1} + g, g \in P_k \}$ for each $m \in Z^+$

Here S_m and S_{-m} are in 1-1 correspondence with P_k.

$\Rightarrow S_m$ and S_{-m} are countable.

Since P_k is countable.

$\Rightarrow T_m = S_m \cup S_{-m}$ is countable.

$\Rightarrow \bigcup_{m=1}^{\infty} T_m$ is countable.

Since infinite union of uncountabl set is countable.

since q

$\Rightarrow P_{k+1} = \bigcup_{m+1}^{\infty} T_m$ is countable.

Hence the theorem.

Algebraic Number:-

A real number is said to be algebraic if it is the root of some polynomial equation with rational co-efficients.

Theorem : 34

The set of algebraic numbers is countable.

Proof:-

Let n be any fixed positive integer. Then Q_n is countable (by theorem 33)

Take $Q_n = \{f_{n_1}, f_{n_2}, f_{n_3} \dots f_{nk} \dots\}$

Where

f_{nk} = Polynomial of degree n with rational co-efficients.

Let A_{nk} be the real roots of $f_{nk} = 0$.

$\Rightarrow A_{nk}$ is countable.

$\Rightarrow A_n = \bigcup_{k=1}^{\infty} A_{nk}$ is countable

Since.

The infinite union of countable sets in countable.

$\Rightarrow A = \bigcup_{n=1}^{\infty} A_n$ is countable (since the infinite union of countable sets in countable

⇒ The set of algebraic numbers is Countable.

Transcendental number:

A real number is said to be transcendental if it is not algebraic.

Theorem 35:-

The set of transcendental numbers is uncountable.

Proof:

Let T be the set of transcendent-al numbers and A be the set of algebraic numbers.

Then $R = A \cup T$

Here R is uncountable and A is Countable.

If T is Countable then R must be Countable as it was the union of Countable sets.

But R is uncountable

⇒ T must be uncountable.

The set of trancendental number uncountable.