SWAMI DAYANANDA
COLLEGE OF ARTS & SCIENCE,
MANJAKKUDI-612610

DEPERTMENT OF MATHEMATICS

Integral Calculus(16SCCMM2)

Study Material

Class : I-B.Sc Mathematics

Prepared by
M.Gunanithi,
Assistant Professor,
Department of Mathematics.

CORE COURSE II

INTEGRAL CALCULUS

Objectives

1. To inculcate the basics of integration and their applications.

2. To study some applications of definite integrals.

3. To understand the concepts of Beta, Gamma functions

UNIT I

- Revision of all integral models – simple problems -

UNIT II

- Definite integrals - Integration by parts & reduction formula

UNIT III

- Geometric Application of Integration-Area under plane curves: Cartesian co-ordinates -Area of a closed curve - Examples - Areas in polar co-ordinates.

UNIT IV

- Double integrals – changing the order of Integration – Triple Integrals.

UNIT V

- Beta & Gamma functions and the relation between them – Integration using Beta & Gamma functions

TEXT BOOK(S)

1. S.Narayanan and T.K.Manicavachagom Pillai, Calculus Volume II, S.Viswanathan (Printers & Publishers) Pvt Limited, Chennai -2011.

UNIT I : Chapter 1 section 1 to 10

UNIT II : Chapter 1 section 11, 12 & 13

UNIT III : Chapter 2 section 1.1, 1.2, 1.3 & 1.4

UNIT IV : Chapter 5 section 2.1, 2.2 & 4

UNIT V : Chapter 7 section 2.1 to 2.5

REFERNECE(S)

1. Shanti Narayan, Differential & Integral Calculus.

Some Important Formula's

1. $\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq 1)$

2. $\int \frac{1}{x} dx = \ln|x|$

3. $\int e^x dx = e^x$

4. $\int a^x dx = \frac{a^x}{\ln a}$

5. $\int \sin x \, dx = -\cos x$

6. $\int \cos x \, dx = \sin x$

7. $\int \sec^2 x \, dx = \tan x$

8. $\int \csc^2 x \, dx = -\cot x$

9. $\int \sec x \tan x \, dx = \sec x$

10. $\int \csc x \cot x \, dx = -\csc x$

11. $\int \sec x \, dx = \ln|\sec x + \tan x|$

12. $\int \csc x \, dx = \ln|\csc x - \cot x|$

13. $\int \tan x \, dx = \ln|\sec x|$

14. $\int \cot x \, dx = \ln|\sin x|$

15. $\int \sinh x \, dx = \cosh x$

16. $\int \cosh x \, dx = \sinh x$

17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

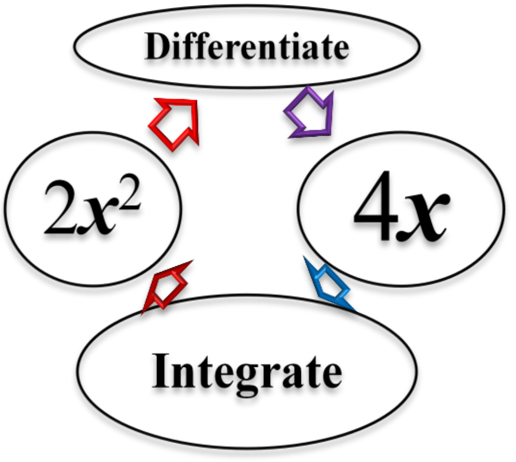
*19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

- # Introduction

In Differential Calculus, we are given functions of x and asked to obtain their derivatives. In Integral Calculus, we are given functions of x and asked what they are the derivatives of. The process of answering this question is called "integration". Integration is the reverse of differentiation.

The process of integration reverses the process of differentiation. In differentiation, if $f(x) = 2x^2$, then $f'(x) = 4x$. Thus the integral of $4x$ is $2x^2$. We can represent this process pictorially as follows:



The situation gets a bit more complicated, because there are an infinite number of functions we can differentiate to give $4x$. Here are some of those functions:

$\boldsymbol{f}(x) = 2x^2 + 7$; $\boldsymbol{g}(x) = 2x^2 - 8$; $\boldsymbol{h}(x) = 2x^2 + \frac{1}{2}$.

Write down **at least five** other functions whose derivative is 12**x**. (For example 6x^2+1,6x^2+10…)

You would have realised that all the functions have the same derivative of 12**x**, because when we differentiate the constant term we obtain zero.

Hence, when we **reverse** the process, we have no idea what the original constant term might have been.

So we include in our response an unknown constant, **c**, called the **arbitrary constant of integration.**

The **integral** of 12**x** then is $6\mathbf{x}^2 + \mathbf{c}$.

There must always be a term of the ***form dx***

$$\int 4x\, dx = 2x^2 + \boxed{c}, \quad c \in \mathbb{R}.$$

Integral sign

This term is called the **integrand**

Constant of integration

Integration Methods

1)Substitution

2) Decomposition into sum

3) Integration by Parts

4) Reduction method

Substitution Method- Procedure

•Simplify the integrand if possible.

•Look for an obvious substitution.

•Classify the integrand according to its form.

•Try again.

Sometimes, the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. Try to find some function u = g(x) in the integrand whose differential du = g’(x) dx also occurs, apart from a constant factor.

If Steps 1 and 2 have not led to the solution, we take a look at the form of the integrand f(x). If f is a rational function, we use the procedure involving partial fractions. If f(x) is a product of a power of x (or
a polynomial) and a transcendental function (a trigonometric, exponential, or logarithmic function), we try integration by parts.Algebraic manipulations (rationalizing the denominator, using trigonometric identities) may be useful in transforming the integral into an easier form.

Sometimes, two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types.It might even combine integration by parts with one or more substitutions.

Syllabus

	... INTEGRAL CALCULUS ...
unit : 1	Revision of all integral models - Simple problems.
unit : 2	Definite integrals - integration by parts and reduction formula.
unit : 3	Geometric Application of integration - Area under plane curves: Cartesian co-ordinates - Area of a closed curve - Examples: Areas in polar co-ordinates.
unit : 4	Double integrals - changing the order of integration - Triple integrals.
unit : 5	Beta and Gamma functions and the relation between them - Integration using Beta and Gamma functions.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ for all values of n except when $n = -1$

2. In the case when $n = -1$, $\frac{dx}{x} = \log x + c$.

3. $\int e^x\,dx = e^x$

4. $\int \sin x\,dx = -\cos x$

5. $\int \cos x\,dx = \sin x$

6. $\int \sec^2 x\,dx = \tan x$ $= \sec^2 - 1$

7. $\int \operatorname{cosec}^2 x\,dx = -\cot x$

8. $\int \sec x \tan x\,dx = \sec x$

9. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$

10. $\int \cosh x\,dx = \sinh x$

11. $\int \sinh x\,dx = \cosh x$

12. $\int \frac{dx}{1+x^2} = \tan^{-1} x$, Or $-\cot^{-1} x$.

13. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$, Or $-\cos^{-1} x$

14. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x$, $\log\left(x + \sqrt{x^2-1}\right)$

15. $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x$, $\log\left(x + \sqrt{x^2+1}\right)$

16. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$, Or $-\operatorname{cosec}^{-1} x$.

8|9

Exercise - 1

Integrate the following with respect to x.

1. x^{-4}

$$x^n = \frac{x^{n+1}}{n+1} + c$$

$$= \int x^{-4} dx$$

$$= \frac{x^{-4+1}}{-4+1} + c$$

$$= \frac{x^{-3}}{-3} + c$$

$$= \frac{-x^{3}}{3} + c$$

2. $x^{3/2}$

$$= \int x^{3/2} dx$$

$$= \frac{x^{3/2+1}}{3/2+1} + c$$

$$= \frac{x^{5/2}}{5/2} + c$$

3. $ax + \frac{b}{x^2}$

$$= \int ax + \frac{b}{x^2} dx$$

$$= a\int x dx + b\int x^{-2} dx$$

$$= a\left[\frac{x^{1+1}}{1+1}\right] + b\left[\frac{x^{-2+1}}{-2+1}\right] + c$$

$$= a\left[\frac{x^2}{2}\right] + b\left[\frac{x^{-1}}{-1}\right] + c$$

$$= \frac{ax^2}{2} + b\left[\frac{x^{-1}}{-1}\right] + c$$

$$= \frac{ax^2}{2} - bx^{-1} + c$$

$$= \frac{ax^2}{2} - \frac{b}{x} + c$$ //

4. $\frac{ax^2+bx+c}{x^3}$

$$\int \frac{ax^2+bx+c}{x^3}\,dx = \int \frac{ax^2}{x^3}\,dx + \int \frac{bx}{x^3}\,dx + \int \frac{c}{x^3}\,dx$$

$$= a\int \frac{1}{x}\,dx + b\int \frac{1}{x^2}\,dx + c\int x^{-3}\,dx$$

$$= a\log x + b\int x^{-2}\,dx + c\int x^{-3}\,dx$$

$$= a\log x + \frac{bx^{-1}}{-1} + \frac{cx^{-2}}{-2}$$

$$= a\log x - \frac{b}{x} - \frac{c}{2x^2}$$ //

(HW) 5. $\frac{ax^{-2}+bx^{-1}+c}{x^{-4}}$

$$\int \frac{ax^{-2}+bx^{-1}+c}{x^{-4}}\,dx = \int \frac{ax^{-2}}{x^{-4}}\,dx + \int \frac{bx^{-1}}{x^{-4}}\,dx + \int \frac{c}{x^{-4}}\,dx$$

$$= \int \frac{a}{x^{-2}}\,dx + \int \frac{b}{x^{-3}}\,dx + \int \frac{c}{x^{-4}}\,dx$$

$$= \int (ax^2 + bx^3 + cx^4)\,dx$$

$$= a\int x^2\,dx + b\int x^3\,dx + c\int x^4\,dx$$

$$= \frac{ax^3}{3} + \frac{bx^4}{4} + \frac{cx^5}{5}$$ //

6. $(x + 1/x)^2$

$$\int (x+1/x)^2\,dx = \int \left(x^2 + 2x(1/x) + (1/x)^2\right)dx$$

$$= \int x^2\,dx + \int 2\,dx + \int 1/x^2\,dx$$

$(a+b)^2 = a^2+b^2+2ab$

$$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1}$$

$$= \frac{x^3}{3} + 2x - 1/x$$ //

7) $(x^{2/5}-x^{-3/5})^2$

$$\int (x^{2/5}-x^{-3/5})^2 dx = \int \left\{ (x^{2/5})^2 - 2(x^{2/5})(x^{-3/5}) + (x^{-3/5})^2 \right\} dx$$

$$= \int (x^{4/5} - 2x^{-1/5} + x^{-6/5})\, dx$$

$$= \int x^{4/5} dx - 2\int x^{-1/5} dx + \int x^{-6/5} dx$$

$$= \frac{x^{4/5+1}}{4/5+1} - 2\frac{x^{-1/5+1}}{-1/5+1} + \frac{x^{-6/5+1}}{-6/5+1}$$ (d)

$$= \frac{x^{9/5}}{9/5} - \frac{2x^{4/5}}{4/5} + \frac{x^{-1/5}}{-1/5} \qquad = -5x^{-1/5}$$

$$= 5/9\, x^{9/5} - \frac{10}{4} x^{4/5} - 5/x^{1/5}$$ //

8 $x^2(1-x)^2$ (d)

$$\int x^2(1-x)^2 dx = \int (x^2 - x^4)\, dx$$

$x^2(1-x^2)$

$$= \int x^2 dx - \int x^4 dx$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + c$$ //

$\frac{x^3}{3} + \frac{x^5}{5} - \frac{2x^5}{5}$

9. $\frac{(x+1)^4}{x^2}$

$$\int \frac{(x+1)^4}{x^2} dx = \int \frac{(x+1)^2\,(x+1)^2}{x^2} dx$$

$$= \int \frac{(x^2+2x+1)(x^2+2x+1)}{x^2} dx$$

$$= \int \frac{(x^4+2x^3+x^2+2x^3+4x^2+2x+x^2+2x+1)}{x^2} dx$$

$$= \int \frac{(x^4+4x^3+6x^2+4x+1)}{x^2} dx$$

11. $\left(x^2 - x^{-3/5}\right)^2$

$$\int\left(x^2 - x^{-3/5}\right)^2 dx = \int\left[(x^2)^2 - 2(x^2)\left(x^{-3/5}\right) + \left(x^{-3/5}\right)^2\right] dx$$

$$= \int\left[x^4 - 2\left(x^{10-3/5}\right) + x^{-6/5}\right] dx$$

$$= \int x^4 dx - 2\int x^{7/5} dx + \int x^{-6/5} dx$$

$$= \frac{x^5}{5} - 2\frac{x^{7/5+1}}{7/5+1} + \frac{x^{-6/5+1}}{-6/5+1}$$

$$= \frac{x^5}{5} - 2\frac{x^{12/5}}{12/5} + \frac{x^{-1/5}}{-1/5}$$

$$= \frac{x^5}{5} - \cancel{2}\times 5\frac{x^{12/5}}{\cancel{12}\,6} - 5x^{-1/5}$$

$$= \frac{x^5}{5} - \frac{5x^{12/5}}{6} - 5/x^{1/5} + C$$

13. $\dfrac{3x^2+4x-5}{\sqrt{x}}$

$$\int\frac{3x^2+4x-5}{\sqrt{x}} dx = \int\frac{3x^2}{\sqrt{x}} dx + \int\frac{4x}{\sqrt{x}} dx - \int\frac{5}{\sqrt{x}} dx$$

$$= 3\int x^{2-1/2} dx + 4\int x^{1-1/2} dx - 5\int x^{-1/2} dx$$

$$= 3\int x^{3/2} dx + 4\int x^{1/2} dx - 5\int x^{-1/2} dx$$

$$= 3\frac{x^{3/2+1}}{3/2+1} + 4\frac{x^{1/2+1}}{1/2+1} - 5\frac{x^{-1/2+1}}{-1/2+1}$$

$$= 3\frac{x^{5/2}}{5/2} + 4\frac{x^{3/2}}{3/2} - 5\frac{x^{1/2}}{1/2}$$

$$= 3\times 2\frac{x^{5/2}}{5} + 4\times 2\frac{x^{3/2}}{3} - 5\times 2\frac{x^{1/2}}{1}$$

$$= 6\frac{x^{5/2}}{5} + 8\frac{x^{3/2}}{3} - 10x^{1/2} + C$$

14. $\dfrac{(x^2+4x)(2x-3)}{x^3}$

$$\int \frac{(x^2+4x)(2x-3)}{x^3}\,dx = \int \frac{2x^3-3x^2+8x^2-12x}{x^3}\,dx$$

$$= \int \frac{2x^3+5x^2-12x}{x^3}\,dx$$

$$= 2\int \frac{x^3}{x^3}\,dx + 5\int \frac{x^2}{x^3}\,dx - 12\int \frac{x}{x^3}\,dx$$

$$= 2\int 1\,dx + 5\int x^{-1}\,dx - 12\int x^{-2}\,dx$$

$$= 2x + 5\log x - 12\,\frac{x^{-2+1}}{-2+1}$$

$$= 2x + 5\log x + 12x^{-1}$$

$$= 2x + 5\log x + 12/x + C$$

17. $(\tan x - 2\cot x)^2$

$$\int (\tan x - 2\cot x)^2\,dx = \int [\tan^2 x - 2(\tan x)(2\cot x) + (2\cot x)^2$$

$$= \int (\tan^2 x - 4\tan x\cdot\cot x + 4\cot^2 x)\,dx$$

$$= \int [(\sec^2 x - 1) - 4\left(\frac{\sin x}{\cos x}\times\frac{\cos x}{\sin x}\right) + 4(\operatorname{cosec}^2 x - 1)]$$

$$= \int [\sec^2 x - 1 - 4 + 4\operatorname{cosec}^2 x - 4]\,dx$$

$$= \int (\sec^2 x - 9 + 4\operatorname{cosec}^2 x)\,dx$$

$$= \int \sec^2 x\,dx - 9\int dx + 4\int \operatorname{cosec}^2\,dx$$

$$= \tan x - 9x - 4\cot x$$

$$= \tan x - 4\cot x - 9x + C$$

18. $\dfrac{1}{\sin^2 x \cos^2 x}$

$\int \dfrac{1}{\sin^2 x \cos^2 x} dx = \int \dfrac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \dfrac{\cancel{\sin^2 x}}{\sin^2 x \cos^2 x} dx + \int \dfrac{\cancel{\cos^2 x}}{\sin^2 x \cos^2 x} dx$

$= \int \dfrac{1}{\cos^2 x} dx + \int \dfrac{1}{\sin^2 x} dx$

$\frac{1}{\sin} = \text{cosec}$

$= \int \left(\dfrac{1}{\cos x}\right)^2 dx + \int \left(\dfrac{1}{\sin x}\right)^2 dx$

$\frac{1}{\cos} = \sec$

$= \int \sec^2 x \, dx + \int \text{cosec}^2 x \, dx$

$= \tan x + (-\cot x) + c$ //

20. $\dfrac{\cos^2 x}{1-\sin x} = \int \dfrac{\cos^2 x}{1-\sin x} dx$ $a^2 - b^2 = (a+b)(a-b)$

$= \int \dfrac{(1-\sin^2 x)}{1-\sin x} dx = \int \dfrac{(1^2 - \sin^2 x)}{1-\sin x} dx = \int \dfrac{(\cancel{1-\sin x})(1+\sin x)}{\cancel{1-\sin x}} dx$

$= \int (1+\sin x) dx$

$= \int dx + \int \sin x \, dx = x - \cos x + c$ //.

19, 22

19. $\dfrac{\sin^2 x}{1+\cos x}$

$= \int \dfrac{1^2 - \cos^2 x}{1+\cos x} dx = \int \dfrac{(1-\cos x)(1+\cos x)}{(1+\cos x)} dx$

$= \int 1 - \cos x \, dx$

$= x - \sin x + c$ //

22. $\sqrt{1+\sin 2x}$ $\qquad \sin 2x = 2\sin x \cos x$

$= \int \sqrt{1+\sin 2x}\ dx$

$= \int \sqrt{(\cos^2 x + \sin^2 x) + (2\sin x \cos x)}\ dx$

$= \int \sqrt{(\cos x + \sin x)^2}\ dx$

$= \int (\cos x + \sin x)\ dx$

$= \int \cos x\, dx + \int \sin x\, dx$

$= \sin x - \cos x + C$ //

$$= \int \frac{x^4}{x^2}\,dx + 4\int \frac{x^3}{x^2}\,dx + 6\int \frac{x^2}{x^2}\,dx + 4\int \frac{x}{x^2}\,dx + \int 1/x^2\,dx$$

$1/x^2 = x^{-2}$

$= \frac{x^{-2+1}}{-2+1} = x^{-1}$

$$= \int x^2\,dx + 4\int x\,dx + 6x + 4\int 1/x\,dx + \int 1/x^2\,dx$$

$$= x^3/3 + 4\,x^2/2 + 6x + 4\log x - x^{-1}$$

$$= \frac{x^3}{3} + \frac{4x^2}{2} + 6x + 4\log x - 1/x \;//.$$

10. (HW) $\frac{(1-x^2)^2}{x}$

$$\int \frac{(1-x^2)}{x}\,dx = \int \frac{(1-2x^2+x^4)}{x}\,dx$$

$$= \int 1/x\,dx - 2\int \frac{x^{\cancel{2}}}{\cancel{x}}\,dx + \int \frac{x^{\cancel{4}\,3}}{\cancel{x}}\,dx$$

$$= \log x - 2\int x\,dx + \int x^3\,dx$$

$$= \log x - \frac{\cancel{2}x^2}{\cancel{2}} + \frac{x^4}{4}$$

$$= \log x - x^2 + \frac{x^4}{4} \;//$$

12. $\frac{x+3}{x\sqrt{x}}$

$$\int \frac{x+3}{x\sqrt{x}}\,dx = \int \frac{x+3}{x \cdot x^{1/2}}\,dx$$

$$= \int \frac{x+3}{x^{3/2}}\,dx$$

$1 + 1/2$

$\frac{2+1}{2} = 3/2$

$$= \int \frac{\cancel{x}}{\cancel{x}\cdot x^{1/2}}\,dx + \int \frac{3}{x^{3/2}}\,dx$$

$$= \int x^{-1/2}\,dx + 3\int x^{-3/2}\,dx$$

15. $\tan^2 x = \int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x + C \;//$

16. $\cot^2 x = \int \cot^2 x\,dx = \int \operatorname{cosec}^2 x - 1\,dx = -\cot x - x + C \;//$

30/9 21. $\frac{3}{\sqrt{1-x^2}} + e^x + 8.$

$$\int\left(\frac{3}{\sqrt{1-x^2}} + e^x + 8\right)dx = \int\frac{3}{\sqrt{1-x^2}}\,dx + \int e^x\,dx + \int 8\,dx$$

$\frac{\int dx}{\sqrt{1-x^2}} = \sin^{-1}x$ (or) $-\cos^{-1}x$

$$= 3\int\frac{dx}{\sqrt{1-x^2}} + \int e^x dx + 8\int dx$$

$$= 3\sin^{-1}x + e^x + 8x + C$$

(or)

$$= -3\cos^{-1}x + e^x + 8x + C$$

23. $\frac{1}{1+\sin x}\,dx$

$$\int\frac{1}{1+\sin x}\,dx = \int\left(\frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\right)dx$$

$$= \int\frac{1-\sin x}{1^2-\sin x^2}\,dx$$

$a^2-b^2 = (a+b)(a-b)$

$\frac{1}{\cos} = \sec$

$$= \int\frac{1-\sin x}{1-\sin^2 x}\,dx = \int\frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int\frac{1}{\cos^2 x}\,dx - \int\frac{\sin x}{\cos^2 x}\,dx$$

$$= \int\left(\frac{1}{\cos x}\right)^2 dx - \int\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\,dx$$

$$= \int\sec^2 x\,dx - \int\tan x\sec x\,dx$$

$$= \tan x - \sec x + C$$

24. $\frac{1}{1-\sin x}$

$$\int\frac{1}{1-\sin x}\,dx = \int\left(\frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\right)dx$$

$$= \int \frac{1+\sin x}{1^2 - \sin^2 x}\, dx \quad = \int \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}\, dx$$

$$= \int \frac{1}{\cos^2 x}\, dx + \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \left(\frac{1}{\cos x}\right)^2 dx + \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\, dx$$

$$= \int \sec^2 x\, dx + \int \tan x \sec x\, dx$$

$$= \tan x + \sec x + c$$

25 $\frac{1}{1+\cos x}$

$$\int \frac{1}{1+\cos x}\, dx$$

$$= \int \left(\frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x}\right) dx$$

$$= \int \frac{1-\cos x}{1^2 - (\cos x)^2}\, dx = \int \frac{1-\cos x}{1-\cos^2 x}\, dx$$

$$= \int \frac{1-\cos x}{\sin^2 x}\, dx = \int \frac{1}{\sin^2 x}\, dx - \frac{\cos x}{\sin^2 x}\, dx$$

$$= \int \left(\frac{1}{\sin x}\right)^2 dx - \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\, dx$$

$$= \int \operatorname{cosec}^2 x\, dx - \int \cot x \cdot \operatorname{cosec} x\, dx$$

$$= -\cot x + \operatorname{cosec} x + c$$

$\frac{1}{\sin} = \operatorname{cosec}$ $\quad \frac{\cos}{\sin} = \cot$ $\quad \frac{\sin}{\cos} = \tan.$

26. $\frac{1}{1-\cos x}$

$$= \int \frac{1}{1-\cos x}\, dx$$

$$= \int \left(\frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} \right) dx$$

$$= \int \frac{1+\cos x}{1^2-(\cos x)^2}\, dx = \int \frac{1+\cos x}{1-\cos^2 x}\, dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\, dx$$

$$= \int \frac{1}{\sin^2 x}\, dx + \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\, dx$$

$$= \int \left(\frac{1}{\sin x}\right)^2 dx + \int \cot x \cdot \operatorname{cosec} x\, dx$$

$$= \int \operatorname{cosec}^2 x\, dx + \int \cot x \cdot \operatorname{cosec} x\, dx$$

$$= -\cot x - \operatorname{cosec} x + C$$

Definite integral

$$\boxed{\int f(x)\, dx = F(x) + C}$$

$\int f(x)\, dx$ at $x = a$

$= F(a) + C$

$\int f(x)\, dx$ at $x = b$

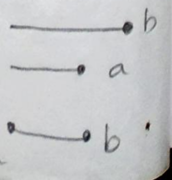
$= F(b) + C$

$$\int_a^b f(x)\, dx = \text{integral at } x = b - \text{integral at } x = a$$

$$= (F(b) + C) - (F(a) + C)$$

$$= F(b) + \cancel{C} - F(a) - \cancel{C}$$

$$= F(b) - F(a)$$



Examples.

1. $\int_1^2 (x^2 - 3x^{1/2} + 1/x^2)\,dx$

$= \left[\frac{x^3}{3} - 2x^{3/2} - 1/x\right]_1^2$

$(2)^3 \quad 2(2)^{3/2} \quad 2(1)^{3/2}$

$= \left[\frac{8}{3} - 4\sqrt{2} - 1/2\right] - \left[1/3 - 2 - 1\right]$

$= \frac{29}{6} - 4\sqrt{2}$

2. $\int_0^{\pi/6} \cos^2\frac{x}{2}\,dx$ $\qquad \boxed{\frac{1+\cos \cancel{2}\frac{x}{\cancel{2}}}{2}}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin\pi/6 = 1/2$

$= 1/2 \int_0^{\pi/6} (1+\cos x)\,dx$

$= 1/2 \left[x + \sin x\right]_0^{\pi/6}$

$= 1/2\left[(\pi/6 + \sin\pi/6) - 0\right] = 1/2(\pi/6 + 1/2)$

$= \pi/12 + 1/4.$

$\int_1^2 (x^2 - 3x^{1/2} + 1/x^2)\,dx$

$\frac{3x^{1/2+1}}{1/2+1} = \frac{3x^{3/2}}{3/2} = \frac{2}{\cancel{3}}(\cancel{3}x^{3/2}) = 2x^{3/2}$

$= \left[\frac{x^3}{3} - 2x^{3/2} - \frac{1}{x}\right]_1^2$

$\sqrt{2} = 3/2$

$x^{-2} = x^{-2+1} = x^{-1} = -1/x$

$= \left[\frac{(2)^3}{3} - 2(2)^{3/2} - 1/2\right] - \left[\frac{(1)^3}{3} - 2(1)^{3/2} - 1/1\right]$

$= \left[\frac{8}{3} - 2(2)(\sqrt{2}) - 1/2\right] - \left[1/3 - 2 - 1\right]$

$= 8/3 - 4\sqrt{2} - 1/2 - 1/3 + 3 = (8/3 - 1/2 - 1/3 + 3) - 4\sqrt{2}$

$= \frac{16 - 3 - 2 + 3(6)}{6} - 4\sqrt{2}$

$= \frac{29}{6} - 4\sqrt{2}$

20/9
1/10

Methods of Integration.

1. Substitution
2. Decomposition into a Sum
3. Integration by parts
4. Successive reduction.

Substitution

$\int f(x)\,dx$ put $x = y(t)$

$x = y(t)$

① ② $a^2 \pm x^2 = \int(ax+b)$

$\frac{dx}{dt} = y'(t) \Rightarrow dx = y'(t)\,dt$ ③ $\int f(x)^n x^{n-1}\,dx$

④ $\int f(x)^n f'(x)\,dx$

$\int f(x)\,dx = \int f(y(t))\, y'(t)\,dt$ ⑤ $\int F\int f(x)\, f(x)\,dx$

Integral of functions contains linear functions

(i) $f(ax+b)$

Put $ax + b = t$

differentiate $a(1)\,dx + 0 = dt$

$a \cdot dx = dt \Rightarrow dx = \frac{1}{a}\,dt$

$\therefore \int f(ax+b)\,dx = \int f(t) \cdot \frac{1}{a}\,dt$

$= \frac{1}{a}\int f(t)\,dt$

1. $\int (ax+b)^n\,dx$ $(n \neq -1)$

Put $t = ax+b$ $dt = a\,dx$ $dx = \frac{1}{a}\,dt$

$\int (ax+b)^n\,dx = \int t^n \frac{1}{a}\,dt = \frac{1}{a}\int t^n\,dt$

$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right] = \frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right]$

$= \frac{(ax+b)^{n+1}}{a(n+1)}$

2. $\int \frac{dx}{ax+b}\, dx$

Put $t = ax+b$ $\quad dt = a\,dx \quad dx = \frac{1}{a}\,dt$

$$\int \frac{1}{t}\frac{1}{a}\,dt = \frac{1}{a}\int \frac{dt}{t} = \frac{1}{a}\log t$$

$$= \frac{1}{a}\log(ax+b)$$

3. $\int e^{ax+b}\,dx$ $\quad ax+b = t$

$a\,dx = dt$

$dx = \frac{1}{a}\,dt$

$$= \int e^t \frac{1}{a}\,dt$$

$$= \frac{1}{a}\int e^t\,dt$$

$$= \frac{1}{a}e^t$$

$$= \frac{1}{a}e^{ax+b}$$

4. $\int \sin(ax+b)\,dx$

$t = ax+b$

$$= \frac{1}{a}\int \sin t\,dt$$

$$= \frac{1}{a}[-\cos t] = -\frac{1}{a}\cos(ax+b)$$

5. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$

6. $\int \sec^2(ax+b)\,dx = \frac{1}{a}\int \sec^2 t\cdot dt$

$$= \frac{1}{a}\tan t = \frac{1}{a}\tan(ax+b)$$

7. $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)$

8. $\int \sec(ax+b)\tan(ax+b)\,dx$

$$= \frac{1}{a}\int \sec t\tan t\,dt = \frac{1}{a}\sec t$$

$$= \frac{1}{a}\sec(ax+b)$$

9) $\int \operatorname{cosec}(ax+b)(\cot(ax+b))\,dx$

$$= -\frac{1}{a}\operatorname{cosec}(ax+b)$$

Problems

1. Find $\int \frac{x^2}{(a+bx)^3}\,dx$

Put $t = a+bx$ $\quad dt = b\,dx$

$bx = t-a$ $\quad dx = \frac{1}{b}\,dt$

$x = \frac{t-a}{b}$

now we convert our problem into new one which contain t terms only

$$\int \frac{\left(\frac{t-a}{b}\right)^2 \frac{1}{b}\,dt}{t^3}$$

$$= \int \frac{\frac{(t-a)^2}{b^2} \cdot \frac{1}{b}\,dt}{t^3}$$

$$= \frac{1}{b^3} \int \frac{(t-a)^2}{t^3}\,dt = \frac{1}{b^3} \int \frac{t^2 - 2at + a^2}{t^3}\,dx$$

$(a-b)^2$

$$= \frac{1}{b^3}\left[\int \frac{t^2}{t^3}\,dt - 2a\int \frac{t}{t^3}\,dt + a^2\int \frac{1}{t^3}\,dt\right]$$

$$= \frac{1}{b^3}\left[\int \frac{1}{t}\,dt - 2a\int \frac{1}{t^2}\,dt + a^2\int t^{-3}\,dt\right]$$

$\boxed{nx^{n-1} = x^n}$

$$= \frac{1}{b^3}\left[\log t - 2a\left(\frac{t^{-2+1}}{-2+1}\right) + a^2\left(\frac{t^{-3+1}}{-3+1}\right)\right]$$

$$= \frac{1}{b^3}\left[\log t - 2a\left(\frac{t^{-1}}{-1}\right) + a^2\left(\frac{t^{-2}}{-2}\right)\right]$$

$$= \frac{1}{b^3}\left[\log t - \frac{2a}{b}\frac{1}{t} - \frac{a^2}{2b^3}\frac{1}{t^2}\right]$$

$$= \frac{1}{b^3}\log(a+bx) + \frac{2a}{b^3(a+bx)} - \frac{a^2}{2b^3(a+bx)^2}$$

Oct 4: $\int \sin^2 3x\,dx$

(HW)

$\sin^2 A = \frac{1-\cos 2A}{2}$

Put $A = 3x$

$$\int \sin^2 3x\,dx = \int \frac{1-\cos 2(3x)}{2}\,dx$$

$$= \int \frac{1}{2}\,dx - \int \frac{\cos(6x+0)}{2}\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin(6x+0)}{6}$$

$$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right)$$

$\int \cos^3 x\,dx$

$\cos^3 A = \frac{3\cos A + \cos 3A}{4}$

~~3cos~~

$$= \int \frac{\cos 3x + 3\cos x}{4}\,dx$$

$$= \frac{1}{4}\int \cos 3x\,dx + \frac{3}{4}\int \cos x\,dx$$

$$= \frac{1}{4}\frac{\sin 3x}{3} + \frac{3}{4}\sin x$$

$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x$$

$\int \sin^4 x\,dx$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \int (\sin^2 x)^2\,dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \int \frac{(1-\cos 2x)^2}{4}\,dx$$

$$= \frac{1}{4}\int (1^2 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\left[\int dx - 2\int \cos 2x\,dx + \int \frac{1+\cos 2(2x)}{2}\,dx\right]$$

$$= \frac{1}{4}\left[x - \frac{\cancel{2}\sin 2x}{\cancel{2}} + \frac{1}{2}x + \frac{\sin 4x}{2(4)}\right]$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right]$$

$$= \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \;//.$$

Integrals of functions involving $a^2 \pm x^2$

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

Put $x = a\sin\theta$ $\qquad$ $\sin\theta = x/a$

$dx = a\cos\theta\, d\theta$ $\qquad$ $\theta = \sin^{-1}(x/a)$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-(a\sin\theta)^2}}$$

$$= \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\, d\theta \qquad \sqrt{a^2(1-\sin^2\theta)}$$

$$= \int \frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}\, d\theta$$

$$= \int \frac{a\cos\theta}{a\sqrt{\cos^2\theta}}\, d\theta = \int \frac{a\cos\theta}{a\cos\theta}\, d\theta$$

$$= \int d\theta = \theta = \sin^{-1} x/a$$

$$\int \frac{dx}{a^2+x^2}$$

Put $x = a\tan\theta$ $\qquad$ $\tan\theta = x/a$

$dx = a\sec^2\theta\, d\theta$ $\qquad$ $\theta = \tan^{-1}(x/a)$

$$\int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2+(a\tan\theta)^2}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2+a^2\tan^2\theta} = \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(\sec^2\theta)} = 1/a \int d\theta$$

$$= 1/a\ \theta$$

$$= 1/a \tan^{-1}(x/a) + c$$

$$\int \frac{dx}{\sqrt{4-9x^2}}$$

$$\left\{ \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) \right\}$$

$$\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{dx}{\sqrt{9(4/9-x^2)}}$$

$$= \int \frac{dx}{3\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\frac{4}{9}-x^2}}$$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2-x^2}} = \frac{1}{3}\sin^{-1}\left(\frac{x}{2/3}\right)$$

$a = 2/3$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)$$

$$\int \frac{dx}{4+9x^2} = \frac{1}{9}\int \frac{dx}{4/9+x^2}$$

$$= \frac{1}{9}\int \frac{dx}{(2/3)^2+x^2} = \frac{1}{9}\left[\frac{1}{2/3}\tan^{-1}\left(\frac{x}{2/3}\right)\right]$$

$$= \frac{1}{9}\int \frac{dx}{4/9+x^2}$$

$$= \frac{1}{9}\int \frac{dx}{(2/3)^2+x^2} = \frac{1}{9}\left[\frac{1}{2/3}\tan^{-1}\left(\frac{x}{2/3}\right)\right]$$

$$= \frac{1}{9}\left[\frac{3}{2}\tan^{-1}\left(\frac{3x}{2}\right)\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$\int d\theta = \theta$

$x/a = \cosh\theta$

$\theta = \cosh^{-1}(x/a)$

HW

1. $$\int \frac{dx}{1+7x^2}$$

$$= \frac{1}{7}\int \frac{dx}{1/7+x^2} = \frac{1}{7}\int \frac{dx}{(1/7)^2+x^2} = \frac{1}{7}\left[\frac{1}{(1/7)}\tan^{-1}\left(\frac{x}{(1/7)}\right)\right]$$

$$= \frac{1}{7}\left[7\tan^{-1}7x\right] = 7/7\tan^{-1}7x = \tan^{-1}7x$$

2. $$\int \frac{dx}{1+4x^2} = \frac{1}{4}\int \frac{dx}{1/4+x^2} = \frac{1}{4}\int \frac{dx}{(1/4)^2+x^2} = \frac{1}{4}\left[\frac{1}{(1/4)}\tan^{-1}\left(\frac{x}{(1/4)}\right)\right]$$

$$= \frac{1}{4}\left[4\tan^{-1}4x\right] = 4/4\tan^{-1}4x = \tan^{-1}4x$$

Dt: 6/2024

$$\int \frac{dx}{\sqrt{x^2+a^2}}$$

$x/a = \sinh\theta$

$\theta = \sinh^{-1}(x/a)$

$\sin\theta = \frac{e^{\theta}-e^{\theta}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{i\theta}}{2i}$

$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$

Put $x = a\sinh\theta$

$dx = a\cosh\theta\, d\theta$

$$= \int \frac{a\cosh\theta\, d\theta}{\sqrt{a^2+(a\sinh\theta)^2}} = \int \frac{a\cosh\theta\, d\theta}{\sqrt{a^2+a^2\sinh^2\theta}}$$

$\frac{d}{dx}(\sinh x) = \cosh x$

$d/dx(\cosh x) = \sinh x$

$$= \int \frac{a\cosh\theta\, d\theta}{a\sqrt{1+\sinh^2\theta}} = \int \frac{a\cosh\theta\, d\theta}{a\sqrt{\cosh^2\theta}}$$

$\boxed{\cosh^2 - \sinh^2 x = 1}$

$\cosh^2 x = 1 + \sinh^2 x$

$\boxed{\cosh^2 x - 1 = \sinh^2 x}$

$$= \int \frac{a\cosh\theta}{a\cosh\theta}\, d\theta = \int d\theta = \theta$$

$$= \sinh^{-1}(x/a)$$

$$\int \frac{dx}{\sqrt{x^2-a^2}}$$

$\theta = \cos^{-1}(x/a)$

Put $x = a\cosh\theta$ $\quad dx = a\sinh\theta\, d\theta$

$$\int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta - a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2(\cosh^2\theta-1)}} = \int \frac{a\sinh\theta\, d\theta}{a\sqrt{\sinh^2\theta}}$$

$$= \int \frac{a\sinh\theta\, d\theta}{a\sinh\theta} = \int d\theta = \theta = \cos^{-1}(x/a)$$

5m.

$$\boxed{\int \frac{dx}{a^2-x^2}} = \int \frac{dx}{(a+x)(a-x)}$$

use partial function $\frac{1}{(a+x)(a-x)} = \frac{A}{(a+x)} + \frac{B}{(a-x)}$

$$\frac{1}{(a+x)\cancel{(a-x)}} = \frac{A(a-x)+B(a+x)}{(a+x)\cancel{(a-x)}}$$

$$1 = A(a-x) + B(a+x) \longrightarrow (1)$$

Put $x = a$ in (1)

$$1 = A(a-a) + B(a+a)$$

$$1 = A(0) + 2Ba$$

$$B = \frac{1}{2a}$$

Put $x = -a$ in (1)

$$1 = A(a-(-a)) + B(a-a)$$

$$1 = 2aA$$

$$A = \frac{1}{2a}$$

$$\frac{1}{(a+x)(a-x)} = \frac{1}{2a}\left(\frac{1}{(a+x)}\right) + \frac{1}{2a}\left(\frac{1}{a-x}\right)$$

$$\int \frac{dx}{(a+x)(a-x)} = \int \frac{1}{2a} \cdot \frac{1}{(a+x)} dx + \int \frac{1}{2a} \cdot \frac{1}{(a-x)} dx$$

$$= \frac{1}{2a}\left(\int \frac{1}{a+x} dx + \int \frac{1}{a-x} dx\right)$$

$\log m - \log n$

$= \log \frac{m}{n}$

$$= \frac{1}{2a}\left(\log(a+x) - \log(a-x)\right) + c$$

$$= \frac{1}{2a} \log \frac{a+x}{a-x} + c$$

$$\int \frac{dx}{(x^2-a^2)}$$

use partial fraction:

$$\frac{1}{(x+a)(x-a)} = \frac{A}{(x+a)} + \frac{B}{(x-a)}$$

$$1 = A(x-a) + B(x+a)$$

Put $x = a$

$1 = A(a-a) + B(a+a)$

$1 = A(0) + B(2a)$

$B = 1/2a$

put $x = -a$

$1 = A(-a-a) + B(-a+a)$

$1 = -2a(A)$

$A = -1/2a$

$$\int \frac{dx}{(x+a)(x-a)} = \int 1/2a \frac{1}{(x-a)}dx - \int 1/2a \frac{1}{(x+a)}dx$$

$$= 1/2a \left[\log(x-a) - \log(x+a)\right] + c$$

$$= 1/2a \log\left|\frac{x-a}{x+a}\right| + c$$

9/10/2021

Prove that $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$

$$\frac{1}{x^2-a^2} = \frac{A}{(x+a)} + \frac{B}{(x-a)}$$

$$\frac{1}{(x+a)(x-a)} = \frac{A(x-a)+B(x+a)}{(x+a)(x-a)}$$

$$1 = A(x-a) + B(x+a) \longrightarrow (1)$$

Put $x = a$ in (1)

$1 = A(a-a) + B(a+a)$

$1 = A(0) + B(2a)$

$1 = B(2a)$

$B = 1/2a$

Put $x = -a$ in (1)

$1 = A(-a-a) + B(-a+a)$

$1 = A(-2a) + B(0)$

$1 = A(-2a)$

$A = -1/2a$

$$\therefore \frac{1}{x^2-a^2} = -\frac{1}{2a}\left(\frac{1}{x+a}\right) + \frac{1}{2a}\frac{1}{(x-a)}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$$

$\log a - \log b$
$= \log a/b$ //

$$= \frac{1}{2a}\log(x-a) - \log(x+a)$$

$$= \frac{1}{2a}\log\frac{x-a}{x+a} \quad //$$

2. $\int \frac{dx}{\sqrt{a^2-b^2x^2}}$

$$= \int\frac{dx}{\sqrt{\left((a/b)^2-x^2\right)b^2}} \qquad = \int\frac{dx}{\sqrt{\left((a/b)^2-\frac{b^2x^2}{b^2}\right)b^2}}$$

$\frac{1}{\sqrt{1-x^2}} = \sin^{-1}$

$$= \int\frac{dx}{b\sqrt{(a/b)^2-x^2}} = \frac{1}{b}\int\frac{dx}{\sqrt{(a/b)^2-x^2}}$$

$a = a/b$

$$= \frac{1}{b}\int\frac{dx}{\sqrt{(a/b)^2-x^2}} = \frac{1}{b}\sin^{-1}\left(\frac{x}{a/b}\right)$$

$$= \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) //$$

3. $\int\frac{dx}{a^2+b^2x^2}$

$$= \int\frac{dx}{b^2\left(\frac{a^2}{b^2}+x^2\right)} = \frac{1}{b^2}\int\frac{dx}{(a/b)^2+x^2}$$

$$= \frac{1}{b^2}\left[\frac{1}{(a/b)}\tan^{-1}\frac{x}{(a/b)}\right]$$

$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$$= \frac{1}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{bx}{a}\right)\right]$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)$$

4. $\int \frac{dx}{\sqrt{4+x^2}}$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a)$$

$= \int \frac{dx}{\sqrt{2^2+x^2}}$

$= \sinh^{-1}(x/2)$

5. $\int \frac{dx}{\sqrt{9+25x^2}}$

$= \int \frac{dx}{\sqrt{25(9/25+x^2)}}$

$= \int \frac{dx}{5\sqrt{(3/5)^2+x^2}} = 1/5 \int \frac{dx}{\sqrt{(3/5)^2+x^2}}$

$= 1/5 \sinh^{-1}\left(\frac{x}{3/5}\right)$

$= 1/5 \sinh^{-1}\left(\frac{5x}{3}\right)$

6. $\int \frac{dx}{\sqrt{4x^2-1}}$

$= \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a)$

$= \int \frac{dx}{\sqrt{4(x^2-1/4)}} = \int \frac{dx}{2\sqrt{x^2-(1/2)^2}}$

$= 1/2 \int \frac{dx}{\sqrt{x^2-(1/2)^2}} = 1/2 \cosh^{-1}\left(\frac{x}{1/2}\right)$

$= 1/2 \cosh^{-1}(2x)$

$\frac{1}{2a} \log \frac{a+x}{a-x}$

7. $\frac{1}{a^2-b^2x^2} = \int \frac{dx}{a^2-b^2x^2}$

$\int \frac{dx}{\left(\left(\frac{a^2}{b^2}\right)-x^2\right)b^2} = \frac{1}{b^2} \int \frac{dx}{(a/b)^2-x^2}$

$$= \frac{1}{b^2}\left[\frac{1}{2\left(\frac{a}{b}\right)} \log \frac{\left(\frac{a}{b}\right)+x}{\left(\frac{a}{b}\right)-x}\right]$$

$\frac{a}{b}+x \Rightarrow \frac{a+bx}{b}$

$\boxed{a = a/b}$ $\frac{a-bx}{b}$

$$= \frac{1}{b^2}\left[\frac{b}{2a} \log \frac{\frac{a+bx}{b}}{\frac{a-bx}{b}}\right]$$

$\boxed{\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}}$

$$= \frac{1}{2ab} \log \frac{a+bx}{a-bx}$$

8. $\frac{1}{ax^2-b^2}$

$$\int \frac{dx}{ax^2-b^2}$$

$$\int \frac{dx}{a\left(x^2-\frac{b^2}{a}\right)} = \frac{1}{a}\int \frac{dx}{x^2-\frac{b^2}{(\sqrt{a})^2}} = \frac{1}{a}\int \frac{dx}{x^2-\left(\frac{b}{\sqrt{a}}\right)^2}$$

$$= \frac{1}{a}\left[\frac{1}{2\left(\frac{b}{\sqrt{a}}\right)} \log \frac{x-\left(\frac{b}{\sqrt{a}}\right)}{x+\left(\frac{b}{\sqrt{a}}\right)}\right]$$

$a = \frac{b}{\sqrt{a}}$

$$= \frac{1}{a}\left[\frac{\sqrt{a}}{2b} \log \frac{\frac{\sqrt{a}x-b}{\sqrt{a}}}{\frac{\sqrt{a}x+b}{\sqrt{a}}}\right]$$

$a = \sqrt{a} \times \sqrt{a}$

$$= \frac{1}{2\sqrt{a}b} \cdot \log \left[\frac{\sqrt{a}x-b}{\sqrt{a}x+b}\right]$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$$

Integrals of functions of the form

$\int f(x^n)\, x^{n-1}\, dx$

Put $x^n = t$

differentiate both sides.

$n x^{n-1} dx = dt$

$x^{n-1} dx = \frac{1}{n} dt$

$\int f(x^n) x^{n-1} dx = \int f(t) \frac{dt}{n}$

$= \frac{1}{n} \int f(t) dt$

$\int x^2 \cos(x^3)\, dx$

Put $\boxed{x^3 = t}$ $\quad 3x^2 dx = dt$

$x^2 dx = \frac{dt}{3}$

$\int x^2 \cos(x^3)\, dx = \int \cos(t) \frac{dt}{3}$

$= \frac{1}{3} \int \cos(t) dt = \frac{1}{3} \sin t$

$= \frac{1}{3} \sin(x^3)$

13

$\int \frac{x^3}{\sqrt{1-x^8}} dx$

$(x^4)^2 t = (x^4)$

$= \int \frac{x^3 dx}{\sqrt{1-(x^4)^2}}$ $\qquad \int f(x^n) = x^{n-1} dx$

Put $t = x^4$, $\quad dt = 4x^3 dx$

$\frac{dt}{4} = x^3 dx$

$= \int \frac{x^3 dx}{\sqrt{1-(x^4)^3}}$

$$= \int \frac{dt}{4\sqrt{1-t^2}}$$

$$= \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{4} \sin^{-1}(t)$$

$$= \frac{1}{4} \sin^{-1}(x^4)$$

$$\int \frac{3x}{1+2x^4}\, dx$$

$\int f(x^n)\, x^{n-1}\, dx$

$$= 3 \int \frac{x}{1+2(x^2)^2}\, dx$$

Put $x^2 = t$

$2x\,dx = dt$ and $x\,dx = dt/2$

$$= 3 \int \frac{dt}{2(1+2t^2)}$$

$$= \frac{3}{2} \int \frac{dt}{2(\frac{1}{2}+t^2)}$$

$x^n = n x^{n-1}$

$$= \frac{3}{4} \int \frac{dt}{(1/\sqrt{2})^2 + t^2}$$

$(\sqrt{a})^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

$= \left(\frac{1}{\sqrt{2}}\right)^2$

$= 3/4$ ~~dt~~

$$\int \frac{dt}{a^2+t^2} = \frac{1}{a} \tan^{-1}(t/a)$$

$a = 1/\sqrt{2}$

here $a = 1/\sqrt{2}$

$\boxed{\frac{1}{2} \tan^{-1} x/a}$

$$= \frac{3}{4}\left(\frac{1}{1/\sqrt{2}} \tan^{-1}\left(\frac{t}{1/\sqrt{2}}\right)\right)$$

$$= \frac{3}{4} \sqrt{2} \tan^{-1}(\sqrt{2}t)$$

$$= \frac{3}{4}(\sqrt{2} \tan^{-1} \sqrt{2}x^2)$$

$$\int_0^2 \frac{5x+1}{x^2+4}\,dx$$

$$= \int_0^2 \frac{5x}{x^2+4}\,dx + \int_0^2 \frac{dx}{x^2+4}$$

Put $t = x^2$

$dt = 2x\,dx$

$x\,dx = \frac{dt}{2}$

$$\int \frac{5x}{x^2+4} = \frac{1}{2}\int \frac{5\,dt}{t+4}$$

$$= \frac{5}{2}\int \frac{dt}{(\sqrt{t})^2+(2)^2}$$

$$= \frac{5}{2}\left[\frac{1}{2}\tan^{-1}\left(\frac{\sqrt{t}}{2}\right)\right]$$

$$= \frac{5}{4}\tan^{-1}(x/2)$$

$$\int_0^2 \frac{5x}{x^2+4}\,dx = \left[\frac{5}{4}\tan^{-1}(x/2)\right]_0^2$$

$$= \frac{5}{4}\left[\tan^{-1}(2/2) - \tan^{-1}(0/2)\right]$$

$$= \frac{5}{4}\tan^{-1}(1)$$

$$= \frac{5}{4}\cdot\frac{\pi}{4}$$

$$= \frac{5\pi}{16}$$

$$\int_0^2 \frac{dx}{x^2+4} = \int_0^2 \frac{dx}{2^2+x^2}$$

$$= \left[\frac{1}{2}\tan^{-1}(x/2)\right]_0^2$$

$= \frac{1}{2}\left[\tan^{-1}(2/2) - \tan^{-1}(0/2)\right]$

$= \frac{1}{2}\left[\tan^{-1}(1) - 0\right]$

$= \frac{1}{2}\ \pi/4$

$= \pi/8$

$$\int_0^2 \frac{5x+1}{x^2+4}\,dx = \frac{5\pi}{16} + \frac{\pi}{8} = \frac{6\pi}{8} = \cancel{\frac{3\pi}{4}} \quad \frac{7\pi}{16}$$

$$\int \frac{x\,dx}{\sqrt{x^2+1}} \qquad x^2 = t$$

$dt = 2x\,dx$

$\frac{dt}{2} = x\,dx$

$= \int \frac{dt}{2\sqrt{t+1}}$

$= \frac{1}{2}\int \frac{dt}{\sqrt{(\sqrt{t})^2+1^2}}$

$\sqrt{t} = \cancel{\sqrt{x^2}}$

$= \frac{1}{2}\int \frac{dt}{\sqrt{1^2+(\sqrt{t})^2}}$

$\sqrt{t} = x$

$= \frac{1}{2}\sinh^{-1}\left(\frac{\sqrt{t}}{1}\right)$

$= \frac{1}{2}\sinh^{-1}(\sqrt{t})$

$= \frac{1}{2}\sinh^{-1}(x)$

(H.W)

① $\int \frac{x^2}{1-x^6}\,dx$

$= \int \frac{x^2}{1-(x^3)^2}\,dx$

Put $x^3 = t$

$3x^2\,dx = dt$

$x^2\,dx = \frac{dt}{3}$

$$\int \frac{x^2}{1-x^6}\,dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left[\frac{a+x}{a-x}\right]$$

$$\int \frac{x^2}{1-x^2}\,dx = \frac{1}{3}\,\frac{1}{2(1)}\log\left(\frac{1+t}{1-t}\right) = \frac{1}{6}\log\left(\frac{1+t}{1-t}\right)$$

$$\int \frac{x^2}{1-x^6}\,dx = \frac{1}{6}\log\left(\frac{1+x^3}{1-x^3}\right) + c$$

(2) $$\int \frac{x}{x^4+a^4}\,dx$$

$$= \int \frac{x\,dx}{a^4\left(\frac{x^4}{a^4}+1\right)} = \frac{1}{a^4}\int \frac{x\,dx}{\left(\frac{x^2}{a^2}\right)^2+1}$$

(a)

Put $t = x^2$

$dt = 2x\,dx$

$x\,dx = dt/2$

$$= \frac{1}{a^4}\int \frac{x\,dx}{\left(\frac{x^2}{a^2}\right)^2+1}$$

$$= \frac{1}{a^4}\int \frac{dt}{2\left(\frac{t^2}{a^2}\right)^2+1}$$

$$= \frac{1}{2 \cdot a^4}\int \frac{dt}{\left(\frac{t^2}{a^2}\right)^2+1}$$

$$= \frac{1}{2a^4}\tan^{-1}\left(t^2/a^2\right)$$

$$= \frac{1}{2a^4}\tan^{-1}\left(x^2/a^2\right)$$

Put $x^2 = t$

$2x\,dx = dt$

$x\,dx = dt/2$

$$= \int \frac{dt/2}{x^4+a^4}$$

$$= \frac{1}{2}\int \frac{dt}{\frac{x^4}{a^2}+a^2}$$

$$= \frac{1}{2a^2}\int \frac{dt}{t^2+a^2}$$

$$= \frac{1}{2a^2}\tan^{-1}\left(t/a^2\right)$$

$$= \frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right)$$

$$= \frac{1}{2a^2}\tan^{-1}\left(x/a\right)^2$$

20/10 Integrals of functions of the form.

$\int (f(x))^n f'(x)\, dx$

$n \neq -1,\ n = -1$

when $n \neq -1$ put $f(x) = t$

when $n \neq -1$

$f(x) = t$

$f'(x)\, dx = dt$

$\int (f(x))^n f'(x)\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1} = \frac{(f(x))^{n+1}}{n+1}$

when $n = -1$

$\int (f(x))^{-1} f'(x)\, dx = \int \frac{f'(x)\, dx}{f(x)}$

Putting $y = f(x)$ $dy = f'(x)\, dx = \int \frac{dy}{y}$

The above integral reduces to $\int \frac{dy}{y}$

$= \int \frac{dy}{y} = \log y = \log(f(x))$ //.

1) $\int \sqrt{x^2+a^2}\; x\, dx$

$= \int (x^2+a^2)^{1/2}\, x\, dx$

derivative of $x^2+a^2 = 2x \Rightarrow 2x\, dx$

Put $f(x) = t$

ie to put $x^2+a^2 = t$ $\int (f(x))^{1/2} f'(x)\, dx$.

$2x\, dx = dt$

$x\, dx = \frac{dt}{2}$

$\frac{2x\, dx}{}$ $x\, dx = \frac{dt}{2}$

$\int (x^2+a^2)^{1/2}\, x\, dx = \int t^{1/2} \frac{dt}{2}$

$= \frac{1}{2} \int t^{1/2}\, dt = \frac{1}{\cancel{2}} \left(\frac{t^{3/2}}{3/\cancel{2}} \right) = \frac{1}{3} t^{3/2}$

$= \frac{1}{3} (x^2+a^2)^{3/2}$

2. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

derivative of $\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} dx$

$\int f(x) f'(x) dx = \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

$dt = \frac{1}{\sqrt{1-x^2}} dx$ $\quad = \int \frac{dt}{t}$ $\quad \int t\, dt$ $\quad \frac{t^2}{2}$

$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t\, dt = \frac{t^2}{2} = \frac{1}{2} (\sin^{-1})^2$

imp

3. $\int \tan\theta \, d\theta$

$= \int \frac{\sin\theta}{\cos\theta} d\theta$ $\quad \cos y = -\sin y \, dy$

derivative of ~~sin θ~~ $\boxed{\cos\theta = -\sin\theta}$

Put $y = \cos\theta$

formate: $\int \frac{f'(x)}{f(x)} dx$ $\quad$ ~~cos~~

$= \int \frac{\sin\theta}{\cos\theta} d\theta = -\int \frac{dy}{y} = -\log y$

$= -\log \cos\theta$

$= \log (\cos\theta)^{-1} = \log \left(\frac{1}{\cos\theta}\right) = \log \sec\theta$

4. $\int \cot\theta \, d\theta$

$\int \cot\theta \, d\theta = \int \frac{\cos\theta}{\sin\theta} d\theta$

$\int \frac{f'(x)}{f(x)} dx$ $\quad$ put $y = \sin\theta$

$dy = \cos\theta$

$$\int \frac{\cos\theta}{\sin\theta}\, d\theta = \int \frac{dy}{y} = \log y = \log \sin\theta$$

5. $\int \sec x \, dx$

$\times$ and $\div$

$$= \int \frac{\sec x}{(\sec x + \tan x)} (\sec x + \tan x)\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

derivative of $\sec x + \tan x = \sec x \tan x + \sec^2 x$

$\int \frac{f'(x)}{f(x)} dx$ put $y = \sec x + \tan x$

$dy = \sec x \tan x + \sec^2 x$

$$= \int \frac{\sec^2 x \sec x \tan x}{\sec x + \tan x}\, dx = \int \frac{dy}{y} = \log y$$

$$= \log(\sec x + \tan x) = \log \tan\left(\pi/4 + x/2\right)$$

6. $\int \operatorname{cosec} x \, dx$

$$= \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x}$$

derivative of $\operatorname{cosec} x + \cot x = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$

$\int \frac{f'(x)}{(x)} dx$ put $y = \operatorname{cosec} x + \cot x$

$dy = -\operatorname{cosec} \cot x - \operatorname{cosec}^2 x$

$$\int \frac{-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x}\, dx = -\int dy/y = -\log y$$

$$= -\log(\operatorname{cosec} x + \cot x)$$

$$= \log \tan x/2$$

21/10

$\int F(f(x))\, f'(x)\, dx \Rightarrow$ form

Put $f(x) = y$ $\qquad f'(x)\,dx = dy$

$\int F(y)\,dy$

1. $\int x^2\sqrt{1-4x^3}\,dx$

put $1-4x^3 = y$

$-12x^2\,dx = dy$

$x^2\,dx = -dy/12$

$\int x^2\sqrt{1-4x^3}\,dx = \int \sqrt{y}\left(-dy/12\right) = -1/12 \int \sqrt{y}\,dy$

$= -1/12\left(\dfrac{y^{3/2}}{3/2}\right)$

$= \dfrac{-y^{3/2}}{18}$

2. $\int \dfrac{e^x}{e^{x/2}-1}\,dx$ $\qquad \int \dfrac{f'(x)}{f(x)}\,dx$

Put $y = e^{x/2}-1$

$y+1 = e^{x/2}$

$dy = \frac{1}{2} e^{x/2}\,dx$ $\qquad 2\,dy = e^{x/2}\,dx$

$e^{x/2}\,dx = 2\,dy$

$\int \dfrac{e^{x/2}\cdot e^{x/2}}{e^{x/2}-1}\,dx$

$= \int \dfrac{(y+1)\,2dy}{y} = \int \dfrac{2y\,dy}{y} + \int 2\,dy/y$

$= 2y + 2\log y$

$= 2(e^{x/2}-1) + 2\log(e^{x/2}-1)$

$= 2\left[e^{x/2}-1+\log(e^{x/2}-1)\right]$

$$\int \frac{dx}{(1+e^x)(1+e^{-x})}$$

$$= \int \frac{e^x}{e^x(1+e^x)(1+e^{-x})}\,dx$$

$$= \int \frac{e^x}{(e^x+e^{2x})(1+e^{-x})}\,dx$$

$$= \int \frac{e^x\,dx}{e^x+e^x e^{-x}+e^{2x}+e^{2x}e^{-x}}$$

$$\therefore (1+e^x)^2 = 1^2+2e^x+(e^x)^2$$

$$= \int \frac{e^x\,dx}{e^x+1+(e^x)^2+e^x}$$

$$= \int \frac{e^x\,dx}{(e^x)^2+2e^x+1^2}$$

$$\int F(f(x))\,f'(x)\,dx \quad \& \quad \int \frac{f'(x)}{f(x)}\,dx$$

$$= \int \frac{e^x\,dx}{(e^x+1)^2}$$

put $y = 1+e^x$

$dy = e^x\,dx$

$$\int \frac{dy}{(y)^2}$$

$\therefore$ The integral becomes

$$\int y^{-2}\,dy$$

$$\frac{1+e^x = y}{e^x.dx} = \frac{y^{-2+1}}{-2+1} = \frac{y^{-1}}{-1} = \frac{-1}{y} = \frac{-1}{1+e^x}$$

$$= \frac{y^{-2+1}}{-2+1} \Rightarrow \frac{y^{-1}}{-1} = -\frac{1}{y}$$

$$= \boxed{-\frac{1}{1+e^x}}$$

21/10

$$\int F(f(x))\,f'(x)\,dx$$

Put $f(x) = y \quad f'(x)\,dx = dy$

$$\int F(y)\,dy$$

$$\int x^2\sqrt{1-4x^3}\,dx$$

Put $1-4x^3 = y$

$-12x^2\,dx = dy$

$x^2\,dx = \frac{-dy}{12}$

$$\int x^2\sqrt{1-4x^3}\,dx = \int \sqrt{y}\left(\frac{-dy}{12}\right) = -\frac{1}{12}\int \sqrt{y}\,dy = -\frac{1}{12}\left(\frac{y^{3/2}}{3/2}\right)$$

$$= \frac{y^{3/2}}{18}$$

27/10 $\int \frac{dx}{\sin x \cos^2 x}$

$= \int \frac{(\sin^2 x + \cos^2 x)}{\sin x \cos^2 x} dx$

$= \int \frac{\sin^2 x}{\sin x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin x \cos^2 x} dx$

$= \int \frac{\sin x}{\cos^2 x} dx + \int \frac{1}{\sin x} dx$

$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx + \int \frac{1}{\sin x} dx$ — $\log \sec x$

$= \int \tan x \cdot \sec x \, dx + \int \frac{dx}{\sin x}$ — $\int \frac{1}{\operatorname{cosec}} dx = \log \tan x/2$

$= \sec x + \log \tan x/2$

$\int \frac{\tan x \, dx}{\sec x + \cos x}$

$= \int \frac{\tan x \, dx}{\frac{1}{\cos x} + \cos x} = \int \frac{\tan x \, dx}{\frac{1 + \cos^2 x}{\cos x}}$

$= \int \frac{\tan x}{1 + \cos^2 x} \cdot \cos x \, dx$ — $\frac{\frac{\sin x}{\cancel{\cos x}}}{1 + \cos^2 x} \cos x \, dx$

$= \int \frac{\frac{\sin x}{\cancel{\cos x}}}{1 + \cos^2 x} \cdot \cancel{\cos x} \, dx$

$= \int \frac{\sin x}{1 + \cos^2 x} dx$

Put $t = \cos x$

$dt = -\sin x \, dx$

$\sin x \, dx = -dt$

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = -\int \frac{dt}{1+t^2} = -\tan^{-1}(t/1) = -\tan^{-1}(t)$$

$$= -\tan^{-1}(\cos x)$$

$$\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}\,dx$$

$$d/dx\,(4\cos x + 5\sin x) = -4\sin x + 5\cos x$$

(d/dx denominator)

Putting the numerator.

$$2\cos x + 3\sin x = l(4\cos x + 5\sin x) + m(-4\sin x + 5\cos x) \longrightarrow (1)$$

we have to find l, m

from (1) we will get

$$2\cos x + 3\sin x = 4l\cos x + 5l\sin x - 4m\sin x + 5m\cos x$$

$$= (4l + 5m)\cos x + (5l - 4m)\sin x$$

$(4l+5m)\cos x = 2\cos x$

$4l+5m = 2$

$$\Rightarrow 4l + 5m = 2 \longrightarrow (2)$$

$$5l - 4m = 3 \longrightarrow (3)$$

$4l + 5m = 2$

$5l - 4m = 3$

$$(2) \Rightarrow 4l + 5m = 2$$

$$(3) \times \frac{5}{4} \Rightarrow \frac{5}{4}(5l) - 4\left(\frac{5}{4}\right)m = \frac{5}{4}(3)$$

$$4l + 5m = 2$$

$$\frac{25}{4}l - 5m = \frac{15}{4}$$

$$4l + \frac{25}{4}l = 2 + 15/4$$

$$\frac{16l + 25l}{4} = \frac{8+15}{4}$$

$41l = 23$

$$\frac{41l}{\cancel{4}} = \frac{23}{\cancel{4}}$$

$$41l = 23$$

$$l = 23/41$$

Put $l = \frac{23}{41}$ in (2)

$$4\left(\frac{23}{41}\right) + 5m = 2$$

$$92/41 + 5m = 2$$

$$5m = 2 - 92/41$$

$$5m = \frac{82-92}{41}$$

$$5m = -10/41$$

$$m = \frac{-10}{41 \times 5} = -2/41$$

$\therefore \int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$ reduces to

$$= \int \left(\frac{23}{41}\right) \frac{\cancel{4\cos x + 5\sin x}}{\cancel{4\cos x + 5\sin x}} dx + \int \left(-\frac{2}{41}\right) \frac{-4\sin x + 5\cos x}{4\cos x + 5\sin x} dx$$

$$= \frac{23}{41} \int dx + \int \left(-2/41\right) \frac{d\,(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$= \frac{23}{41} x - \frac{2}{41} \int \frac{d(4\cos x + 5\sin x)}{4\cos x + 5\sin x}$$

$$= \frac{23}{41} x - \frac{2}{41} \log(4\cos x + 5\sin x)$$

$$\int \frac{dx}{1+\tan x} = \int \frac{dx}{1+\frac{\sin x}{\cos x}} = \int \frac{dx}{\frac{\cos x+\sin x}{\cos x}} = \int \frac{\cos x}{\cos x+\sin x}\,dx$$

$$\int \frac{\cos x}{\cos x+\sin x}\,dx = \int \frac{\frac{1}{2}\,2\cos x}{\cos x+\sin x}\,dx$$

$$= \frac{1}{2}\int \frac{\cos x+\cos x}{\cos x+\sin x}\,dx$$

$$= \frac{1}{2}\int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{\cos x+\sin x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$$

$\sin x + \cos x =$

$d(\cos - \sin x)$

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{d(\sin x+\cos x)}{\sin x+\cos x}\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{d(\sin x+\cos x)}{(\sin x+\cos x)}\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log(\sin x+\cos x)$$

$$\int \frac{dx}{1+\tan x}$$

$$= \int \frac{dx}{1+\frac{\sin x}{\cos x}} = \int \frac{dx}{\frac{\cos x+\sin x}{\cos x}} = \int \frac{\cos x}{\cos x+\sin x}\,dx$$

Integration of rational algebric functions.

We proceed to integrate fractions whose numerator and denominator contain positive integral power of x with constant coefficients.

Rule (a):

If the degree of the numerator is equal to or greather than the degree of the denominator, divide the numerator by the denominator until the remainder is of the lower degree that the denominator.

(1) $\int \frac{x^2}{x+2}\,dx$

$$\begin{array}{r|l} & \quad x \\ \hline x+2 & \quad x^2 \\ & \quad x^2+2x \\ & \quad (-)\;\;(-) \\ \hline & \quad -2x \end{array} \qquad \begin{array}{r|l} & \quad x-2 \\ \hline x+2 & \quad x^2 \\ & \quad x^2-4 \\ & \quad (-)\;\;(+) \\ \hline & \quad 4 \end{array}$$

$$\frac{x^2}{x+2} = x-2+\frac{4}{x+2}$$

$$\int \frac{x^2}{x+2}\,dx = \int \left(x-2+4/_{x+2}\right) dx$$

$$= \frac{x^2}{2} - 2x + 4\log(x+2)$$

(2) $\int \frac{2+3x}{3-4x}\,dx$

$$\begin{array}{r|l} & \quad -3/4 \\ \hline 3-4x & \quad 2+3x \\ & \quad -9/4+3x \\ & \quad (+)\;\;(-) \\ \hline & \quad 2+9/4 = 17/4 \end{array}$$

$$\therefore \frac{2+3x}{3-4x} = -3/4 + 17/4\,\frac{1}{3-4x}$$

$$\int \frac{2+3x}{3-4x}\,dx = -\int \frac{3\,dx}{4} + \frac{17}{4}\int \frac{dx}{3-4x}$$

$$= \frac{3x}{4} + \frac{17}{4}\left[\frac{1}{(4)}\log(3-4x)\right]$$

Since $\int \frac{dx}{ax+b} = 1/a\log(ax+b)$

$$= -\frac{3x}{4} - \frac{17}{16}\log(3-4x)$$

$$\int \frac{x^{24}}{x^{10}+1}\,dx$$

$$x^{10}+1 \,\overline{)\, x^{24} \,}\; x^{14}$$

$x^{24} + x^{14}$

$(-) \quad (-)$

$-x^{14}$

$$x^{10}+1 \,\overline{)\, x^{24} \,}\; x^{14} - x^{4}$$

$x^{24} + x^{14}$

$(-) \quad (-)$

$-x^{14}$

$-x^{14} - x^{4}$

x^{4}

$$\frac{x^{24}}{x^{10}+1} = x^{14} - \frac{\cancel{x^4} + x^4}{x^{10}+1}$$

$$\int \frac{x^{24}}{x^{10}+1}\,dx = \int \left(x^{14} - x^{4} + \frac{x^4}{x^{10}+1} \right) dx$$

$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \int \frac{x^4\,dx}{x^{10}+1}$$

Put $x^5 = y$

$5x^4\,dx = dy$

$x^4\,dx = 1/5\,dy$

$$= \frac{x^{15}}{15} - \frac{x^5}{5} + 1/5 \int \frac{dy}{y^2+1^2}$$

$$= \frac{x^{15}}{15} - \frac{x^5}{5} + 1/5 \tan^{-1}(y)$$

$$= \frac{x^{15}}{15} - \frac{x^5}{5} + 1/5 \tan^{-1}(x^5)$$

Rule b.

Denominator is of the second degree and does not resolve into rational factor. It has been shown that.

① $\int \frac{dx}{x^2+a^2} = \frac{1}{2}\tan^{-1}(x/a)$

② $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$

③ $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$

Type ①

1) $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{x^2+2x+1+5}$

$2Ax = 2x$

$A = 1$

add & substract by A^2

$= \int \frac{dx}{(x+1)^2+2^2}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$

2) $\int \frac{dx}{4x^2-4x+2} = \frac{1}{4}\int\frac{dx}{x^2-x+2/4} = \frac{1}{4}\int\frac{dx}{x^2-x+1/4-1/4+2/4}$

$= \frac{1}{4}\frac{dx}{x^2-x\ {1/2}^2+1/4}$

$2Ax = x$

$A = -\frac{1}{2}$

$A^2 = \frac{1}{4}$

$= \frac{1}{4}\int\frac{dx}{(x-1/2)^2+{1/2}^2}$

$= \frac{1}{4}\times\frac{1}{1/2}\tan^{-1}\left[\frac{(x-1/2)}{1/2}\right]$

$= \frac{2}{4}\tan^{-1}\left(\frac{(2x-1)/2}{1/2}\right)$

$= \frac{1}{2}\tan^{-1}(2x-1)$

3 $\displaystyle\int \frac{dx}{x^2-8x-7} = \int \frac{dx}{x^2+8x+16-16-7}$

$2Ax = 8x$
$A = 4$
add & sub by
$A^2 = 16$

$$= \int \frac{dx}{(x+4)^2-23}$$

$$= \int \frac{dx}{(x+4)^2-(\sqrt{23})^2}$$

$$= \frac{1}{2\sqrt{23}} \log \frac{x+4-\sqrt{23}}{x+4+\sqrt{23}}$$

4) $\displaystyle\int \frac{dx}{3x^2-4x+5}$

$2Ax = -4/3\, x$
$A = -4/3 \cdot 1/2$
$= -2/3$
$A^2 = 4/9$

$$= \frac{1}{3}\int \frac{dx}{x^2-\frac{4}{3}x-5/3}$$

$$= \frac{1}{3}\int \frac{dx}{x^2-4/3x+4/9-4/9-5/3}$$

$$= \frac{1}{3}\int \frac{dx}{(x-2/3)^2-(4/9+5/3)}$$

$$= \frac{1}{3}\int \frac{dx}{(x-2/3)^2-\left(\frac{4+15}{9}\right)} - \frac{1}{2}\int \frac{dx}{(x-2/3)^2-19/9}$$

$$= \frac{1}{3}\int \frac{dx}{(x-2/3)^2 (\sqrt{19/9})^2}$$

$$= \frac{1}{3}\left[\frac{1}{2(\sqrt{19/9})} \log\left(\frac{x-2/3-\sqrt{19/9}}{x-2/3+\sqrt{19/9}}\right)\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{9}}{2\sqrt{19}} \log\left(\frac{x-2/3-\sqrt{19}/3}{x-2/3+\sqrt{19}/3}\right)\right]$$

$$\cancel{= \frac{1}{3}\left[\frac{3}{2\sqrt{19}} \log\left(\frac{x-2/3-\sqrt{19}/3}{x \cdot \quad 3}\right.\right.}$$

$$= \frac{1}{3}\left[\frac{3}{2\sqrt{19}} \log\left(\frac{\frac{3x-2-\sqrt{19}}{3}}{\frac{3x-2+\sqrt{19}}{3}}\right)\right]$$

$$= \frac{1}{2\sqrt{19}} \log\left(\frac{3x-2-\sqrt{19}}{3x-2+\sqrt{19}}\right) //.$$

5. $\int \frac{dx}{1+x-x^2}$

$$= \int \frac{dx}{1-(x^2-x)} = \int \frac{dx}{1-(x^2-x+\frac{1}{4}-\frac{1}{4})}$$

$2Cx = -x$

$2C = -1$

$C = -\frac{1}{2}$

$C^2 = \frac{1}{4}$

$$= \int \frac{dx}{(1+\frac{1}{4})-(x^2-x+\frac{1}{4})}$$

$$= \int \frac{dx}{\frac{5}{4}-(x-\frac{1}{2})^2}$$

$$= \int \frac{dx}{\left(\left(\frac{\sqrt{5}}{2}\right)\right)^2-(x-\frac{1}{2})^2}$$

$$= \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)} \log \frac{\frac{\sqrt{5}}{2}+\left(\frac{(2x-1)}{2}\right)}{\frac{\sqrt{5}}{2}\,(x-\frac{1}{2})}$$

$$= \frac{1}{\sqrt{5}} \log \frac{\frac{\sqrt{5}+2x-1}{2}}{\frac{\sqrt{5}-2x+1}{2}}$$

$$= \frac{1}{\sqrt{5}} \log \frac{\sqrt{5}+2x-1}{\sqrt{5}-2x-1}$$

6) $\int \frac{dx}{1-6x-9x^2}$

$$= \frac{1}{9}\int \frac{dx}{\frac{1}{9}-\frac{2}{3}x-x^2}$$

$$= \frac{1}{9}\int \frac{dx}{\frac{1}{9}-(x^2+\frac{2}{3}x)}$$

$2cx = \frac{2}{3}x$

$2c = \frac{2}{3}, \quad c = \frac{1}{3}$

$c^2 = \frac{1}{9}$

$$= \frac{1}{9}\int \frac{dx}{\frac{1}{9}-(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9})}$$

$$= \frac{1}{9}\int \frac{dx}{(\frac{1}{9}+\frac{1}{9})-(x^2+\frac{2}{3}x+\frac{1}{9})} = \frac{1}{9}\int \frac{dx}{\frac{2}{9}-(x+\frac{1}{3})^2}$$

$$= \frac{1}{9}\int \frac{dx}{(\frac{\sqrt{2}}{3})^2-(x+\frac{1}{3})^2}$$

$$= \frac{1}{9}\,\frac{1}{2(\frac{\sqrt{3}}{2})}\log \frac{\frac{\sqrt{2}}{3}+(x+\frac{1}{3})}{\frac{\sqrt{2}}{3}-(x+\frac{1}{3})}$$

$$= \frac{1}{6\sqrt{2}}\log \frac{\frac{\sqrt{2}+3x+1}{3}}{\frac{\sqrt{2}-3x-1}{3}} = \frac{1}{6\sqrt{2}}\log \frac{\sqrt{2}+3x+1}{\sqrt{2}-3x-1}$$

Rule (b)

Type : 2

$$\int \frac{lx+m}{ax^2+bx+c}\,dx$$

Numerator = A(derivative of the denominator) + B

1) $\int \frac{2x+3}{x^2+x+1} = \int \frac{lx+m}{ax^2+bx+c}$

denominator $= x^2+x+1$

$\frac{d}{dx}$(denominator) $= 2x+1$

$$2x+3 = A(2x+1)+B$$
$$= 2Ax + A + B$$

$2 = 2A$ $\qquad$ $A+B = 3$

$A = 1$ $\qquad$ $1+A = 3$

$B = 2$

$2x+3 = 1\,(2x+1)+2$

$$\int \frac{2x+3}{x^2+x+1}\,dx = \int \frac{2x+1}{x^2+x+1} + \frac{2dx}{x^2+x+1}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{x^2+x+1}$$

$2Cx = x$

$C = 1/2$

$C^2 = 1/4$

$$= \log(x^2+x+1) + 2\int \frac{dx}{x^2+x+1/4-1/4+1}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+1/2)^2+3/4}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+1/2)^2+(\sqrt{3}/2)^2}$$

$$= \log(x^2+x+1) + 2\,\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{\left(\frac{x+1/2}{2}\right)}{\sqrt{3}/2}$$

$$= \log(x^2+x+1) + 4/\sqrt{3}\,\tan^{-1}\left(\frac{\frac{2x+1}{2}}{\sqrt{3}/2}\right)$$

$$= \log(x^2+x+1) + 4/\sqrt{3}\,\tan^{-1} 2x+1/\sqrt{3}$$

2) $$\int \frac{x+4}{6x-7-x^2}\,dx$$

d/dx (denominator) $= 6-2x$

$x+4 = A(6-2x)+B$

$= 6A - 2Ax + B$

$-2Ax = x$

$A = -1/2$

$6A+B = 4$

$6(-1/2)+B = 4$

$-3+B = 4$

$B = 7$

$$\int \frac{x+4}{6x-7-x^2}\,dx = -\tfrac{1}{2}\int \frac{6-2x}{6x-7-x^2}\,dx + 7\int \frac{dx}{6x-7-x^2}$$

$$= -\tfrac{1}{2}\log(6x-7-x^2) + 7\int \frac{dx}{6x-7-x^2}$$

$$= -\tfrac{1}{2}\log(6x-7-x^2) + 7\int \frac{dx}{-(x^2-6x+9)+9-7}$$

$$= -\tfrac{1}{2}\log(6x-7-x^2) + 7\int \frac{dx}{-(x-3)^2+(\sqrt{2})^2}$$

$$= -\tfrac{1}{2}\log(6x-7-x^2) + 7\frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+x-3}{\sqrt{2}-x+3}$$

Rule - 8

Type - 2

Method of partial fraction.

If the denominator can be resolved into fractional factors of the 1st or Second degree

1) $\int \frac{dx}{x^2-a^2}$

$$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{(x+a)} + \frac{B}{(x-a)}$$

$$= \frac{A(x-a)+B(x+a)}{(x+a)(x-a)}$$

$$1 = A(x-a) + B(x+a) \longrightarrow (1)$$

Put $x=a$ in (1) we get.

$$1 = A(a-a) + B(a+a)$$

$$1 = B(2a)$$

$$B = \tfrac{1}{2a}$$

Put $x = -a$ in (1) we get

$$1 = A(-a-a) + B(-a+a)$$

$$1 = -2aA$$

$$A = -1/2a$$

$$\int \frac{dx}{x^2-a^2} = A\int \frac{dx}{(x+a)} + B\int \frac{dx}{(x-a)}$$

$$= -1/2a \int \frac{dx}{x+a} + 1/2a \int \frac{dx}{x-a}$$

$$= -1/2a \log(x+a) + 1/2 \log(x-a)$$

$$= 1/2a\left[\log(x-a) - \log(x+a)\right]$$

Ex

$$= 1/2a\left[\log \frac{x-a}{x+a}\right] //.$$

2 $\int \frac{dx}{a^2-x^2}$

$$\frac{1}{a^2-x^2} = \frac{1}{(a+x)(a-x)} = \frac{A}{(a+x)} + \frac{B}{(a-x)}$$

$$= \frac{A(a-x) + B(a+x)}{(a+x)\ (a-x)}$$

$$1 = A(a-x) + B(a+x) \longrightarrow (1)$$

Put $a = x$ in (1) we get

$$1 = A(x-x) + B(x+x)$$

$$1 = A(0) + B(2x)$$

$$B = 1/2x$$

Put $a = -x$ in (1) we get

$$1 = A(-x-x) + B(-x+x)$$

$$1 = A(-2x) + 0$$

$$A = -1/2x$$

$$\int \frac{dx}{a^2-x^2} = A\int \frac{dx}{(a+x)} + B\int \frac{dx}{(a-x)}$$

$$= -\frac{1}{2x}\int \frac{dx}{a+x} + \frac{1}{2x}\int \frac{dx}{a-x}$$

$$= -\frac{1}{2x}\log(a+x) + \frac{1}{2x}\log(a-x)$$

$$= \frac{1}{2x}\left[\log(a-x) - \log(a+x)\right]$$

$$= \frac{1}{2x}\left[\log\frac{a-x}{a+x}\right] //.$$

Ex

3) $\int \frac{x^3}{(x-1)(x-2)}\,dx$

The degree of the Numerator is higher than that of the denominator. (rule a is applied)

$$\int \frac{x^3}{(x-1)(x-2)}\,dx$$

$(x-1)(x-2) = x^2-2x-x+2$

$= x^2-3x+2$

$$\begin{array}{r|l} & x+3 \\ \hline x^2-3x+2 & x^3 \\ & x^3-3x^2+2x \\ & (-)\quad(+)\quad(-) \\ \hline & 3x^2-2x \\ & 3x^2-9x+6 \\ & (-)\quad(+)\quad(-) \\ \hline & 7x-6 \end{array}$$

$$\therefore \frac{x^3}{(x-1)(x-2)} = x+3+\frac{7x-6}{(x-1)(x-2)}$$

$$\frac{7x-6}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$7x - 6 = A(x-2) + B(x-1) \longrightarrow (1)$$

Put $x = +1$ in (1)

$$7(1) - 6 = A(1-2) + B(1-1)$$

$$1 = -A + 0$$

$$A = -1$$

Put $x = 2$ in (1)

$$7(2) - 6 = A(2-2) + B(2-1)$$

$$8 = B$$

$$B = 8$$

$$\frac{7x-6}{(x-1)(x-2)} = \frac{-1}{(x-1)} + \frac{8}{(x-2)}$$

$$\int \frac{x^3}{(x-1)(x-2)} dx = \int \left[(x+3) - \frac{1}{(x-1)} + \frac{8}{(x-2)} \right] dx$$

$$= \int (x+3)\,dx - \int \frac{dx}{(x-1)} + 8 \int \frac{dx}{(x-2)}$$

$$= \frac{x^2}{2} + 3x - \log(x-1) + 8 \log(x-3)$$

4) $\int \frac{3x+1}{(x-1)^2 (x+3)} dx$

$$\frac{3x+1}{(x-1)^2 (x+3)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)}$$

$$\frac{3x+1}{(x-1)^2 (x+3)} = \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2 (x+3)}$$

$$3x + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \longrightarrow (1)$$

Put $x=1$ in (1)

$$3(1)+1 = A(1-1)(1+3) + B(1+3) + C(1-1)^2$$

$$4 = 4B$$

$$B = 1$$

Put $x=-3$ in (1)

$$3(-3)+1 = A(-3-1)(-3+3) + B(-3+3) + C(-3-4)^2$$

$$-8 = 16C$$

$$C = -1/2$$

Put $x=0$ in (1)

$$1 = A(-1)(3) + B(3) + C(-1)^2$$

$$1 = -3A + 3 + (-1/2)$$

$$3A = 3 - 1/2 - 1$$

$$3A = \frac{6-1-2}{2}$$

$$3A = 3/2$$

$$A = 1/2$$

$$\int \frac{3x+1}{(x-1)^2(x+3)}\,dx = 1/2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} - 1/2\int \frac{dx}{x+3}$$

$$= 1/2\log(x-1) + \frac{(x-1)^{-2+1}}{-2+1} - 1/2\log(x+3)$$

$$= \frac{1}{2}\log(x-1) - \frac{1}{(x-1)} - \frac{1}{2}\log(x+3)$$

$$= \frac{1}{2}\left[\log(x-1) - \log(x+3)\right] - \frac{1}{x-1}$$

$$= 1/2\log\left(\frac{x-1}{x+3}\right) - \frac{1}{(x-1)} \quad //.$$

5) $\int \frac{2dx}{(1-x)(1+x^2)}$

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

$$\frac{2}{(1-x)(1+x^2)} = \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$$

$$2 = A(1+x^2)+(Bx+C)(1-x) \longrightarrow (1)$$

Put $x=1$ in (1)

$$2 = A(1+1)+(B(1)+C)(1-1)$$

$$2A = 2$$

$$A = 1$$

put $x=0$ in (1)

$$2 = A(1)+B(0)+C(1-0)$$

$$2 = A+C$$

$$2 = 1+C$$

$$C = 1$$

Put $x=-1$ in (1)

$$2 = A(1+1)+(B(-1))(1-(-1))$$

$$2 = 1(2)+(-B+1)(2)$$

$$2 = 2-2B+2$$

$$2B = 2$$

$$B = 1$$

$$\int \frac{2dx}{(1-x)(1+x^2)} = \int \frac{dx}{1-x} + \int \frac{x+1}{x^2+1}\,dx$$

$$= -\log(1-x) + \int \frac{x}{x^2+1}\,dx + \int \frac{dx}{x^2+1}$$

$$= -\log(1-x) + \frac{1}{2}\int \frac{dt}{t+1} + \tan^{-1}x$$

$= -\log(1-x) + \frac{1}{2}\log(t+1) + \tan^{-1}x$

$= -\log(1-x) + \frac{1}{2}\log(x^2+1) + \tan^{-1}x.$

Special Cases

Nov-8 (1) In certain cases a substitution materially shortens the work. This is especially so if some power of x, say, x^{n-1}, is a factor of the numerator and the rest of the fraction is a rational function of x^n.

1. $\int \frac{x^2\,dx}{x^6+2x^3+2}$

Put $x^3 = t$, $3x^2\,dx = dt$

$\int \frac{x^2\,dx}{(x^3)^2+2x^3+2} = \frac{1}{3}\int \frac{dt}{t^2+2t+2}$

$= \frac{1}{3}\int \frac{dt}{t^2+2t+1-1+2}$

$= \frac{1}{3}\int \frac{dt}{(t+1)^2+1^2}$ [$2At = 2t$, $A = 1$, $A^2 = 1$, add subtract by 1]

$= \frac{1}{3}\tan^{-1}(t+1)$

$= \frac{1}{3}\tan^{-1}(x^3+1)$

Eg 2. $\int \frac{dx}{x(x^3+1)}$

$= \int \frac{x^2\,dx}{x^3(x^3+1)}$ put $x^3 = t$, $\therefore 3x^2\,dx = dt$

$= \frac{1}{3}\int \frac{dt}{t(t+1)} = \frac{1}{3}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt$

$= \frac{1}{3}\left\{\log t - \log(t-1)\right\} = \frac{1}{3}\log\frac{t}{t+1}$

$= \frac{1}{3}\log\frac{x^3}{x^3+1}$

(2) In fractions in which there is no odd power of x and in which the denominator can be broken up into factors of the form $x^2 \pm a^2$ it is not necessary to resolve the denominator into linear factors. The partial fraction corresponding to each factor x^2+a^2 (or) x^2-a^2 should be obtained regarding x^2 as the variable.

2) $$\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{A}{(x^2+a^2)} + \frac{B}{(x^2+b^2)}$$

$$\frac{1}{(x^2+a^2)(x^2+b^2)}$$

$$\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2} = \frac{x^2+a^2-x^2-b^2}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{a^2-b^2}{(x^2+a^2)(x^2+b^2)}$$

$$\frac{1}{a^2-b^2}\left[\frac{1}{(x^2+b^2)} - \frac{1}{(x^2+a^2)}\right] = \frac{1}{(x^2+a^2)(x^2+b^2)}$$

$$\int \frac{dx}{(x^2+a^2)(a^2+b^2)} = \frac{1}{a^2-b^2}\left[\int \frac{dx}{x^2+b^2} - \int \frac{dx}{x^2+a^2}\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1} x/a\right]$$

Functions involving x^2+a^2

Put $x = a\tan\theta$

1) $\displaystyle\int \frac{dx}{(1+x^2)^2}$

Put $x = a\tan\theta$

$a=1$ in our problem

$\therefore x = \tan\theta,\ dx = \sec^2\theta\, d\theta$

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$$

$$= \int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \int \frac{1}{\sec^2\theta}\, d\theta = \int \cos^2\theta\, d\theta$$

$$= \tfrac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{2}\left[\frac{\sin 2\theta}{2}\right]$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta$$

$$= \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{4}\sin = \theta$$

$$= \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{4}\cdot 2\sin 2\theta$$

$$= \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{2}\sin 2\theta$$

$$= \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{2}\frac{x}{1+x^2}$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\frac{\sin\theta}{\cos\theta} \times \cos\theta \times \cos\theta$$

$$= 2(\tan\theta) \times \cos^2\theta$$

$$= 2\tan\theta\ \frac{1}{\sec^2\theta}$$

$$= 2\tan\theta\ \frac{1}{1+\tan^2\theta}$$

$$= \frac{2x}{1+x^2}$$

2) $\int \frac{x\,dx}{(x^2+2x+2)^2}$

$\int \frac{x\,dx}{(x^2+2x+1-1+2)^2} = \int \frac{x\,dx}{((x+1)^2+1^2)^2}$

Put $x+1 = \tan\theta$ $\quad dx = \sec^2\theta\,d\theta$

$x = \tan\theta - 1$

$\int \frac{x\,dx}{(x^2+2x+2)^2} = \int \frac{x\,dx}{((x+1)^2+1^2)^2}$

$= \int \frac{(\tan\theta-1)\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2}$

$= \int \frac{(\tan\theta-1)\sec^2\theta\,d\theta}{(\sec^2\theta)^2}$

$= \int \frac{(\tan\theta-1)\sec^2\theta\,d\theta}{\sec^4\theta}$

$= \int \frac{(\tan\theta-1)}{\sec^2\theta}\,d\theta = \int (\tan\theta-1)\cos^2\theta\,d\theta$

$= \int \left(\frac{\sin\theta}{\cos\theta}-1\right)\cos^2\theta\,d\theta$

$= \int (\sin\theta\cos\theta - \cos^2\theta)\,d\theta$

$= \int (\tfrac{1}{2}\sin 2\theta)d\theta - \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$

$= -\tfrac{1}{4}\cos 2\theta - \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta$

$\cos 2\theta = ?$

$\cos 2\theta = \frac{1+\cos 2\theta}{2}$

$2\cos 2\theta = 1+\cos 2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$= \frac{2}{\sec^2\theta} - 1$

$$= \frac{2}{(1+\tan^2\theta)} - 1$$

$$= \frac{2}{(1+(x+1)^2)} - 1$$

$$= \frac{2-1-(x+1)^2}{1+(x+1)^2}$$

$$= \frac{1-(x+1)^2}{(x+1)^2+1}$$

$$-\frac{1}{4}\cos 2\theta = -\frac{1}{4}\left(\frac{1-(x+1)^2}{(x+1)^2+1}\right)$$

$\sin 2\theta = ?$

$$\sin 2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{2\sin\theta}{\cos\theta} \times \cos^2\theta$$

$$= 2\tan\theta \cdot \frac{1}{\sec^2\theta}$$

$$= \frac{2(x+1)}{1+\tan^2\theta}$$

$$= \frac{2(x+1)}{1+(x+1)^2}$$

$$-\frac{1}{4}\sin 2\theta = -\frac{1}{4}\,\frac{2(x+1)}{1+(x+1)^2}$$

$$\int \frac{x\,dx}{(x^2+2x+2)^2} = \frac{1}{4}\,\frac{(x+1)^2-1}{(x+1)^2+1} - \frac{1}{2}\tan^{-1}(x+1).$$

$$-\frac{1}{4}\,\frac{2(x+1)}{1+(x+1)^2}$$

$$= \frac{x^2+2x+1-1}{4(x^2+2x+1+1)} - \frac{1}{2}\frac{(x+1)}{1+x^2+2x+1} - \frac{1}{2}\tan^{-1}(x+1)$$

$$= \frac{x^2+2x}{4(x^2+2x+2)} - \frac{(x+1)}{2(x^2+2x+2)} - \frac{1}{2}\tan^{-1}(x+1)$$

$$= \frac{x^2+2x-2(x+1)}{4(x^2+2x+2)} - \frac{1}{2}\tan^{-1}(x+1)$$

$$= \frac{x^2-2}{4(x^2+2x+2)} - \frac{1}{2}\tan^{-1}(x+1)$$

5) Integrals of the form $\int \frac{(ax^2+b)\,dx}{x^4+cx^2+1}$

$$ax^2+b = \frac{a+b}{2}(x^2+1) + \frac{a-b}{2}(x^2-1)$$

$$\int \frac{(ax^2+b)\,dx}{x^4+cx^2+1} = \frac{a+b}{2}\int \frac{x^2+1}{x^4+cx^2+1}\,dx + \frac{a-b}{2}\int \frac{x^2-1}{x^4-cx^2+1}\,dx$$

let

$$I_1 = \frac{a+b}{2}\int \frac{x^2+1}{x^4+cx^2+1}\,dx = \frac{a+b}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+c+\frac{1}{x^2}}$$

let $x - \frac{1}{x} = t$ $\quad \left(1-\left(\frac{-1}{x^2}\right)\right)dx = dt$

$$\left(1+\frac{1}{x^2}\right)dx = dt$$

let $I_2 = \frac{a-b}{2}\int \frac{x^2-1}{x^4+cx^2+1}\,dx = \frac{a-b}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+c+\frac{1}{x^2}}$

let $x + \frac{1}{x} = t$ $\quad \left(1-\frac{1}{x^2}\right)dx = dt$

Example.

$$\int \frac{ax^2+b}{x^4+cx^2+1}$$

$a=1, b=1, c=-1$

1) $$\int \frac{x^2+1}{x^4-x^2+1}\,dx$$

$$x^2+1 = \frac{(1+1)}{2}(x^2+1) + \frac{(1-1)}{2}(x^2-1)$$

$$\int \frac{x^2+1}{x^4-x^2+1}\,dx = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2-1+1/x^2}$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2-1\cdot x\frac{1}{x}+\frac{1}{x^2}}$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2-1+\frac{1}{x^2}-1+1} = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2-2+\frac{1}{x^2}+1}$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{(x-1/x)^2+1^2}$$

put $x-1/x = t$

$(1-(-1/x^2))dx = dt$

$(1+1/x^2)dx = dt$

$$= \int \frac{dt}{t^2+1^2}$$

$$= \tan^{-1}(t) = \tan^{-1}(x-1/x)$$

$$= \tan^{-1}\left(\frac{x^2-1}{x}\right) //.$$

8. Integration of irrational functions.

Many irrational expressions can be rationalised by a suitable change of variable as will be explained later on.

It has already been shown that

(1) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(2) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$, or $\log(x+\sqrt{x^2+a^2})$

(3) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$ or $\log(x+\sqrt{x^2-a^2})$.

1) $\int \sqrt{a^2-x^2}\, dx$

Put $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$\int \sqrt{a^2-x^2}\, dx = \int (\sqrt{a^2-a^2\sin^2\theta})\, a\cos\theta\, d\theta$

$= \int \sqrt{a^2(1-\sin^2\theta)}\, a\cos\theta\, d\theta$

$= \int a\sqrt{\cos^2\theta}\, a\cos\theta\, d\theta$

$= \int a^2\cos^2\theta\, d\theta = a^2\int \cos^2\theta\, d\theta$

$= a^2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$

$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$

Since $x = a\sin\theta$

$\frac{x}{a} = \sin\theta$

$\theta = \sin^{-1} x/a$

$= \frac{a^2}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right]$

$\frac{x}{a} = \sin\theta$

$(x/a)^2 = \sin^2\theta$

$= 1-\cos^2\theta$

$= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right]$

$\cos^2\theta = 1-(x/a)^2$

$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\left(\sqrt{1-\frac{x^2}{a^2}}\right)\right]$

$\cos\theta = \sqrt{1-\frac{x^2}{a^2}}$

$= \frac{a^2}{2}\left[\sin^{-1} x/a + x/a\,\sqrt{\frac{a^2-x^2}{a^2}}\right]$

$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\,\frac{\sqrt{a^2-x^2}}{a}\right]$

$= \frac{a^2}{2}\sin^{-1}(x/a) + (x/2\,\sqrt{a^2-x^2})$ //

2) $\int \sqrt{a^2+x^2}\, dx$

Put $x = a \sinh\theta$

$dx = a\cosh\theta\, d\theta$

$$\int \sqrt{a^2+x^2}\, dx = \int \left(\sqrt{a^2+a^2\sinh^2\theta}\right) a\cosh\theta\, d\theta$$

$$= \int a\sqrt{1+\sinh^2\theta}\; a\cosh\theta\, d\theta$$

$$= \int a\sqrt{\cosh^2\theta}\; a\cosh\theta\, d\theta$$

$$= a^2 \int \cosh^2\theta\, d\theta$$

$$= a^2 \int \left(\frac{1+\cosh 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2} \int (1+\cosh 2\theta)\, d\theta$$

Since $x = a\sinh\theta$

$\frac{x}{a} = \sinh\theta$

$\frac{x^2}{a^2} = \sinh^2\theta$

$= \cosh^2\theta - 1$

$\cosh^2\theta = \frac{x^2}{a^2} + 1$

$\cosh\theta = \sqrt{\frac{x^2}{a^2}+1}$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sinh\theta\cosh\theta$$

$$= \frac{a^2}{2}\sinh^{-1}(x/a) + \frac{a^2}{2}\left[\frac{x}{a}\left(\sqrt{\frac{x^2}{a^2}+1}\right)\right]$$

$$= \frac{a^2}{2}\sinh^{-1}(x/a) + \frac{a^2}{2}\left[\frac{x}{a}\,\frac{\sqrt{x^2+a^2}}{a}\right]$$

$$= \frac{a^2}{2}\sinh^{-1}(x/a) + (x/2)\sqrt{x^2+a^2}$$

3) $\int \sqrt{x^2-a^2}\, dx$

Put $x = a\cosh\theta$

$dx = a\sinh\theta\, d\theta$

$$\int \sqrt{x^2-a^2}\, dx = \int \sqrt{a^2\cosh^2\theta - a^2}\; a\sinh\theta\, d\theta$$

$$= \int a\sqrt{\cosh^2\theta - 1}\; a\sinh\theta\, d\theta$$

$$= \int a^2\sqrt{\sinh^2\theta}\; \sinh\theta\, d\theta$$

$$= \int a^2 \sinh^2\theta\, d\theta$$

$$= a^2 \int (\cosh^2\theta - 1)\, d\theta$$

$$= a^2 \int \left(\frac{\cosh 2\theta}{2} - \frac{1}{2} \right) d\theta$$

$$= \frac{a^2}{2} \int (\cosh 2\theta - 1)\, d\theta$$

$$= \frac{a^2}{2} \left[\frac{\sinh 2\theta}{2} - \theta \right] //$$

$$\int \sqrt{x^2 - a^2}\, dx = \int \sqrt{a^2\cosh^2\theta - a^2}\; a \sinh\theta\, d\theta$$

$$= \int a\sqrt{\cosh^2\theta - 1}\; a\sinh\theta\, d\theta$$

$$= \int a^2\sqrt{\sinh^2\theta}\; \sinh\theta\, d\theta$$

$$= \int a^2 \sinh^2\theta\, d\theta$$

$$= a^2 \int$$

$$= \frac{a^2}{2} \sinh\theta\cosh\theta - \frac{a^2}{2}\theta$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{x^2-a^2}}{a}\right)\left(\frac{x}{a}\right) - \frac{a^2}{2}\cosh^{-1}(x/a)$$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}(x/a)$$

$x = a\cosh\theta$

$\cosh\theta = x/a$

$\cosh^2\theta = \frac{x^2}{a^2}$

$\sinh^2\theta = \frac{x^2}{a^2} - 1$

$\sinh\theta = \sqrt{\frac{x^2 - 1}{a^2}}$

Nov 9 Case (i) Integration of the form $\frac{1}{\sqrt{ax^2+bx+c}}$

Divide the expression under the root by the numerical value of the co-efficient of x^2 and complete the square of the terms will contain x, the integral reduce to one of the forms above.

1) $\frac{dx}{\sqrt{2-3x+x^2}}$

$$= \int \frac{dx}{\sqrt{x^2-3x+2}} = \int \frac{dx}{\sqrt{x^2-3x+\frac{9}{4}-\frac{9}{4}+2}}$$

$2ax = 3x$

$a = 3/2$

$a^2 = 9/4$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2+\left(2-9/4\right)}} = \int \frac{dx}{\sqrt{(x-3/2)^2-1/4}}$$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} x/a$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2+(1/2)^2}}$$

$$= \cosh^{-1}\left(\frac{x-\frac{3}{2}}{1/2}\right)$$

$$= \cosh^{-1}\left(\frac{2x-3/2}{1/2}\right) = \cosh^{-1}(2x-3)$$

2) $\int \frac{dx}{\sqrt{3x-x^2-2}}$

$$= \int \frac{dx}{\sqrt{-2-(x^2-3x)}} = \int \frac{dx}{\sqrt{-2-(x^2-3x+9/4-9/4)}}$$

$$= \int \frac{dx}{\sqrt{-2+9/4-(x^2-3x+9/4)}} = \int \frac{dx}{\sqrt{1/4-(x-3/2)^2}}$$

$$= \int \frac{dx}{\sqrt{(1/2)^2-\left(x-\frac{3}{2}\right)^2}} = \sin^{-1} \frac{x-3/2}{1/2}$$

$$= \sin^{-1} \frac{\frac{2x-3}{2}}{1/2} = \sin^{-1}(2x-3)$$

3) $\int \frac{dx}{\sqrt{x(3-2x)}}$

$= \int \frac{dx}{\sqrt{3x-2x^2}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{3}{2}x - x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left(x^2 - \frac{3}{2}x\right)}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right)}}$

$2ax = \frac{3}{2}x$

$2a = 3/2$

$a = 3/4$

$a^2 = 9/16$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{9}{16} - \left(x^2 - \frac{3}{2}x + \frac{9}{16}\right)}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(3/4)^2 - (x - 3/4)^2}}$

$= 1/\sqrt{2} \sin^{-1}\left(\frac{x - 3/4}{3/4}\right)$

$= 1/\sqrt{2} \sin^{-1} \frac{\frac{4x-3}{4}}{3/4} = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x-3}{3}\right)$ //.

4) $\int \frac{dx}{\sqrt{3x^2+x-2}}$

$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2 + \frac{x}{3} - \frac{2}{3}}}$

$2ax = x/3$

$2a = 1/3$

$a = 1/6$

$a^2 = 1/36$

$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2 + \frac{x}{3} + \frac{1}{36} - \frac{1}{36} - 2/3}}$

$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x+1/6)^2 - \left(\frac{1}{36} + 2/3\right)}}$

$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x+1/6)^2 - \left(\frac{25}{36}\right)}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x+1/6)^2 - (5/6)^2}}$

$= \frac{1}{\sqrt{3}} \cosh^{-1}\left(\frac{x+1/6}{5/6}\right) = \frac{1}{\sqrt{3}} \cosh^{-1} \frac{\frac{6x+1}{6}}{5/6}$

$= \frac{1}{\sqrt{3}} \cosh^{-1} \frac{6x+1}{5}$ //

Case (ii) $\dfrac{px+q}{\sqrt{ax^2+bx+c}}$ $\dfrac{d}{dx}(ax^2+bx+c) = 2ax+b$

$px+q = A(2ax+b)+B$

1) $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

$\dfrac{d}{dx}(x^2+x+1) = 2x+1$

Let $x = A(2x+1)+B \longrightarrow (1)$

Put $x = -1/2$ in (1)

$-1/2 = A(2(-1/2)+1)+B$

$-1/2 = A(0)+B$

$\boxed{B = -1/2}$

Put $x=0$ in (1)

$0 = A(0+1) - 1/2$

$0 = A - 1/2 \Rightarrow \boxed{A = 1/2}$

$x = 1/2(2x+1) - 1/2$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{2x+1}{2\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1/4-1/4+1}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\int \frac{dx}{\sqrt{(x+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$= \sqrt{x^2+x+1} - \frac{1}{2} \sinh^{-1} \frac{x+\frac{1}{2}}{\sqrt{3}/2}$

$2ax = x$
$a = 1/2$
$a^2 = 1/4$

$= \sqrt{x^2+x+1} - \frac{1}{2} \sinh^{-1} \frac{2x+1}{\sqrt{3}}$ //.

2) $\int \frac{6x+5}{\sqrt{6+x-2x^2}} dx$

$\frac{d}{dx}(6+x-2x^2) = -4x+1$

let $6x+5 = A(-4x+1) + B \longrightarrow (1)$

Put $x = 1/4$ in (1)

$6(1/4)+5 = A\left(-4(1/4)+1\right) + B$

$\frac{3}{2} + 5 = A(0) + B$

$B = \frac{13}{2}$ //

Put $x=0$ in (1)

$6(0)+5 = A(0+1) + \frac{3}{2}$

$A = 5 \cdot \frac{13}{2}$

$A = -3/2$ //

$6x+5 = \frac{-3}{2}(-4x+1) + \frac{13}{2}$

$\int \frac{6x+5}{\sqrt{6+x-2x^2}} dx = -\frac{3}{2} \int \frac{-4x+1}{\sqrt{6+x-2x^2}} dx + \frac{13}{2} \int \frac{dx}{\sqrt{6+x-2x^2}}$

$= -3\sqrt{6+x-2x^2} + \frac{13}{2} \int \frac{dx}{\sqrt{6+x-2x^2}}$

$2ax = x/2$
$a = 1/4$
$a^2 = 1/16$

$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \int \frac{dx}{\sqrt{3-\left(x^2-\frac{x}{2}\right)}}$

$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \int \frac{dx}{\sqrt{3-\left(x^2-\frac{x}{2}+\frac{1}{16}-\frac{1}{16}\right)}}$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \int \frac{dx}{\sqrt{3+\frac{1}{16} - \left(x^2 - \frac{x}{2} + \frac{1}{16}\right)}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \int \frac{dx}{\sqrt{\frac{49}{16} - (x-1/4)^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \int \frac{dx}{\sqrt{(7/4)^2 - (x-1/4)^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \sin^{-1} \frac{x-1/4}{7/4}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \sin^{-1} \frac{4x-1}{7}$$

3) $\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\, dx$

$\frac{d}{dx}(4x^2-4x-5) = 8x-4$

$3x-2 = A(8x-4)+B \longrightarrow (1)$

Put $x = 1/2$ in (1)

$3(1/2) - 2 = A(8(1/2)-4) + B$

$\frac{3}{2} - 2 = B$

$B = -1/2$

Put $x = 0$ in (1)

$3(0) - 2 = A(8(0)-4) + (-1/2)$

$-2 = -4A - 1/2$

$4A = 2 - 1/2$

$4A = 3/2$

$A = 3/8$

$$\therefore \int \frac{3x-2}{\sqrt{4x^2-4x-5}}\, dx = \frac{3}{8}\int \frac{8x-4}{\sqrt{4x^2-4x-5}}\, dx - 1/2 \int \frac{dx}{\sqrt{4x^2-4x-5}}$$

$$= \frac{3}{4}\int \frac{8x-4}{2\sqrt{4x^2-4x-5}} - 1/2 \int \frac{dx}{\sqrt{4x^2-4x-5}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - 1/2 \int \frac{dx}{\sqrt{x^2-x-5/4}}$$

$$\therefore \int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = \frac{3}{8}\int \frac{8x-4}{\sqrt{4x^2-4x-5}}\,dx$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{2(2)}\int \frac{dx}{\sqrt{x^2-x+\frac{1}{4}-\frac{1}{4}-\frac{5}{4}}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2-\frac{6}{4}}}$$

$2cx = x$

$c = \frac{1}{2}$

$c^2 = \frac{1}{4}$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2-(\frac{6}{4})^2}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{6}}{2}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\frac{\frac{2x-1}{2}}{\frac{\sqrt{6}}{2}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\frac{2x-1}{\sqrt{6}}$$

4) $$\int \left(\frac{5-x}{x-2}\right)^{1/2} dx$$

$$= \int \frac{(5-x)^{1/2}}{(x-2)} \times \frac{(5-x)^{1/2}}{(5-x)^{1/2}}\,dx$$

$$= \int \frac{5-x}{\sqrt{(x-2)}\,\sqrt{(5-x)}}\,dx$$

$$= \int \frac{5-x}{\sqrt{(x-2)(5-x)}}\,dx = \int \frac{5-x}{\sqrt{-10+2x+5x-x^2}}\,dx$$

$$= \int \frac{5-x}{\sqrt{-10+7x-x^2}}\,dx = \int \frac{5-x}{\sqrt{-10+7x-x^2}}\,dx$$

$$\int \frac{5-x}{\sqrt{-10+7x-x^2}}\,dx \qquad \frac{d}{dx}(-10+7x-x^2) = 7-2x$$

$5-x = A(7-2x)+B$

Put $x = 7/2$ in (1)

$5-\frac{7}{2} = A(7-2(7/2))+B$

$\frac{3}{2} = A(0)+B$

$B = 3/2$

Put $x=0$

$5 = 7A + 3/2$

$A = 1/2$

$$\int \frac{5-x}{\sqrt{-10+7x-x^2}} = \frac{1}{2}\int \frac{7-2x}{\sqrt{-10+7x-x^2}}\,dx + \frac{3}{2}\int \frac{dx}{\sqrt{-10+7x-x^2}}$$

$$= \int \frac{7-2x}{2\sqrt{-10+7x-x^2}}\,dx + 3/2 \int \frac{dx}{\sqrt{-10-(x^2-7x)}}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\int \frac{dx}{\sqrt{-10-(x^2-7x+\frac{49}{4}-\frac{49}{4})}}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\int \frac{dx}{\sqrt{\frac{49}{4}-10-(x^2-7x+\frac{49}{4})}}$$

$2cx = 7x$
$c = 7/2$
$c^2 = 49/4$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\int \frac{dx}{\sqrt{\frac{9}{4}-(x-7/2)^2}}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\int \frac{dx}{\sqrt{(3/2)^2-(x-7/2)^2}}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\int \frac{dx}{\sqrt{(3/2)^2-(x-7/2)^2}}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\sin^{-1}\frac{x-7/2}{3/2}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\sin^{-1}\frac{\frac{2x-7}{2}}{3/2}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\sin^{-1}\left(\frac{2x-7}{3}\right)$$

Case (iii)

integrate of $\sqrt{ax^2+bx+c}$ and $(px+q)\sqrt{ax^2+bx+c}$

It has already been shown that.

(1) $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1} x/a$

(2) $\int\sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\sinh^{-1}(x/a)$

(3) $\int\sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\cosh^{-1}(x/a)$

The integration of $\sqrt{ax^2+bx+c}$ can be accomplished by reducing ax^2+bx+c to the form $a\{(x+\alpha)^2 \pm \beta^2\}$

1) $\int\sqrt{x^2+2x+10}\,dx$

$= \int\sqrt{x^2+2x+1-1+10}\,dx$

$= \int\sqrt{(x+1)^2+9}\,dx = \int\sqrt{(x+1)^2+3^2}\,dx$

$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+3^2} + \frac{9}{2}\sinh^{-1}\left(\frac{x+1}{3}\right)$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x+10} + \frac{9}{2}\sinh^{-1}\left(\frac{x+1}{3}\right)$

$= \int\sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}$

2) $\int\sqrt{1+x-2x^2}\,dx$

$= \sqrt{2}\int\sqrt{\frac{1}{2}+\frac{x}{2}-x^2}\,dx$

$2cx = x/2$

$c = 1/4$

$c^2 = 1/16$

$= \sqrt{2}\int\sqrt{\frac{1}{2}-\left(x^2-\frac{x}{2}\right)}\,dx$

$= \sqrt{2}\int\sqrt{\frac{1}{2}-\left(x^2-\frac{x}{2}+\frac{1}{16}-\frac{1}{16}\right)}\,dx$

$= \sqrt{2}\int\sqrt{\frac{1}{2}+\frac{1}{16}-(x-1/4)^2}\,dx$

$= \sqrt{2}\int\sqrt{9/16-(x-1/4)^2}\,dx = \sqrt{2}\int\sqrt{(3/4)^2-(x-1/4)^2}\,dx$

$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}x/a$

$= \sqrt{2}\left[\frac{1}{2}(x-1/4)\sqrt{\frac{9}{10}(x-1/4)^2} + \frac{1}{2}\left(\frac{9}{10}\right)\sin^{-1}\left(\frac{x-1/4}{3/4}\right)\right]$

$= \sqrt{2}\left[\frac{1}{2}\left(\frac{4x-1}{4}\right)\sqrt{\frac{9}{10}-\left(x^2-\frac{x}{2}+\frac{1}{10}\right)} + \frac{9}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)\right]$

$= \sqrt{2}\left[\frac{1}{2}\,\frac{4x-1}{4}\sqrt{\frac{1}{2}-\left(x^2-\frac{x}{2}\right)} + \frac{9}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)\right]$

$= \frac{1}{2}\,\frac{4x-1}{4}\sqrt{1+x-2x^2} + \frac{9\sqrt{2}}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)$

$= \frac{4x-1}{8}\sqrt{1+x-2x^2} + \frac{9\sqrt{2}}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)$

3) $\int (3x-2)\sqrt{x^2+x+1}\,dx$ $\qquad \frac{d}{dx}(x^2+x+1) = 2x+1$

let $3x-2 = A(2x+1) + B \longrightarrow (1)$

Put $x = -1/2$ in (1)

$$3(-1/2) - 2 = A\left(2(-1/2)+1\right) + B$$

$$\frac{-3}{2} - 2 = A(0) + B$$

$$\boxed{B = -7/2}$$

Put $x = 0$ in (1)

$$3(0) - 2 = A(0+1) - 7/2$$

$$-2 = A - 7/2$$

$$A = -2 + 7/2 = 3/2$$

$$\boxed{A = 3/2}$$

$$\int (3x-2)\sqrt{x^2+x+1}\,dx = \int \left(\frac{3}{2}(2x+1)\ 7/2\right)\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx - 7/2\int \sqrt{x^2+x+1}\,dx$$

$$\left\{ \begin{aligned} \frac{d}{dx}(x^2+x+1)^{3/2} &= \frac{3}{2}(x^2+x+1)^{3/2-1}(2x+1) \\ &= 3/2\,(x^2+x+1)^{1/2}(2x+1) \\ &= 3/2\,\sqrt{x^2+x+1}\,(2x+1) \end{aligned} \right\}$$

$$= (x^2+x+1)^{3/2} - 7/2\int \sqrt{x^2+x+1/4-1/4+1}\,dx$$

$$= (x^2+x+1)^{3/2} - 7/2\int \sqrt{(x+1/2)^2 + 3/4}\,dx$$

$$= (x^2+x+1)^{3/2} - 7/2\int \sqrt{(x+1/2)^2 + (\sqrt{3}/2)^2}\,dx$$

$$= (x^2+x+1)^{3/2} - 7/2\left[1/2(x+1/2)\sqrt{x^2+x+1} + 1/2\ 3/4\sinh^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)\right]$$

$$= (x^2+x+1)^{3/2} - 7/4\,(x+1/2)\sqrt{x^2+x+1} - \frac{7}{4}\,\frac{3}{4}\,\sinh^{-1}\frac{x+1/2}{\sqrt{3}/2}$$

$$= (x^2+x+1)^{3/2} - 7/4\left(\frac{2x+1}{2}\right)\sqrt{x^2+x+1} - \frac{21}{16}\sinh^{-1}\frac{\frac{2x+1}{2}}{\sqrt{3}/2}$$

$$= (x^2+x+1)^{3/2} - 7/8\,(2x+1)\sqrt{x^2+x+1} - \frac{21}{16}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

case (iv)

In Some cases it is more convenient to proceed as below.

An algebraical expression involving only one irrational quantity $\sqrt{ax+b}$ can be rationalised by the substitution $ax+b=t^2$ as in the following Examples and then its integral can be found by the methods already studied.

case (iv) functions involving $\sqrt{ax+b}$ put $ax+b=t^2$

1) $\int \frac{x^2}{\sqrt{x+5}}\,dx$

put $x+5=t^2$ $\quad dx = 2t\,dt$

$$\int \frac{x^2}{\sqrt{x+5}}\,dx = \int \frac{(t^2-5)^2\,2t\,dt}{t}$$

$$= 2\int (t^2-5)^2\,dt$$

$$= 2\int (t^4-10t^2+25)\,dt$$

$$= 2\left[\frac{t^5}{5} - 10\frac{t^3}{3} + 25t\right]$$

$$= \frac{2t}{15}\left[3t^4 - 50t^2 + 375\right]$$

$$= \frac{2\sqrt{x+5}}{15}\left[3(x+5)^2 - 50(x+5) + 375\right]$$

2) $\int \frac{\sqrt{x}}{1+x}\,dx$

Put $x = t^2$

$dx = 2t\,dt$

$$= (x^2+x+1)^{3/2} - 7/4\,(x+1/2)\sqrt{x^2+x+1} - \frac{7}{4}\,\frac{3}{4}\,\sinh^{-1}\frac{x+1/2}{\sqrt{3}/2}$$

$$= (x^2+x+1)^{3/2} - 7/4\left(\frac{2x+1}{2}\right)\sqrt{x^2+x+1} - \frac{21}{16}\sinh^{-1}\frac{\frac{2x+1}{2}}{\sqrt{3}/2}$$

$$= (x^2+x+1)^{3/2} - 7/8\,(2x+1)\sqrt{x^2+x+1} - \frac{21}{16}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Case (iv)

In Some cases it is more convenient to proceed as below.

An algebraical expression involving only one irrational quantity $\sqrt{ax+b}$ can be rationalised by the Substitution $ax+b=t^2$ as in the following Examples and then its integral can be found by the methods already Studied.

case (iv) functions involving $\sqrt{ax+b}$ put $ax+b=t^2$

1) $\displaystyle\int \frac{x^2}{\sqrt{x+5}}\,dx$

Put $x+5=t^2$ $\quad dx = 2t\,dt$

$$\int \frac{x^2}{\sqrt{x+5}}\,dx = \int \frac{(t^2-5)^2\,2t\,dt}{t}$$

$$= 2\int (t^2-5)^2\,dt$$

$$= 2\int (t^4-10t^2+25)\,dt$$

$$= 2\left[\frac{t^5}{5} - 10\frac{t^3}{3} + 25t\right]$$

$$= \frac{2t}{15}\left[3t^4 - 50t^2 + 375\right]$$

$$= \frac{2\sqrt{x+5}}{15}\left[3(x+5)^2 - 50(x+5) + 375\right]$$

2) $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$

Put $x=t^2$

$dx = 2t\,dt$

$$\int \frac{\sqrt{x}}{1+x}\,dx = \int \frac{t \cdot 2t\,dt}{1+t^2}$$

$$= 2\int \frac{t^2}{1+t^2}\,dt = 2\int \frac{t^2+1-1}{t^2+1}\,dt$$

$$= 2\int \left(1-\frac{1}{t^2+1}\right)dt = 2\left[\int dt - \int \frac{dt}{1+t^2}\right]$$

$$= 2\left[t - \tan^{-1} t\right]$$

3) $\displaystyle\int \frac{dx}{(1+x)^{3/2} + (1+x)^{3/2}}$ $\qquad 1+x = t^2$, $dx = 2t\,dt$

$$\therefore \int \frac{dx}{(1+x)^{3/2}+(1+x)^{1/2}} = \int \frac{2t\,dt}{(t^2)^{3/2} + (t^2)^{1/2}}$$

$$= \int \frac{2t\,dt}{t^3+t} = 2\int \frac{t\,dt}{t(t^2+1)}$$

$$= 2\int \frac{dt}{t^2+1} = 2\tan^{-1} t = 2\tan^{-1}\sqrt{1+x}$$

4) $\displaystyle\int \frac{dx}{1+(x+a)^{1/3}}$ $\qquad x+a = t^3$, $dx = 3t^2\,dt$

$$= \int \frac{3t^2\,dt}{1+(t^3)^{1/3}} = \int \frac{3t^2\,dt}{1+t}$$

$$= \int \frac{3t^2\,dt}{1+t} = \frac{3}{2}t^2 - 3t + 3\log(1+t)$$

$$= \frac{3}{2}(x+a)^{2/3} - 3(x+a)^{1/3} + 3\log\left(1+(x+a)^{1/3}\right)$$

Case (v)

Any Expression of the form

$$\frac{1}{(x-k)\sqrt{ax^2+bx+c}}$$

can be integrated by the Substitution $x-k = 1/t$ the expression is thereby reduced to the form $\dfrac{1}{\sqrt{Ax^2+B+C}}$ which has been already Considered.

case (v)

1) $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Here we have $x - \frac{1}{k} = x+1 = 1/t$

Put $x+1 = 1/t$ $\quad dx = \frac{-1}{t^2} dt$

$$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \int \frac{\frac{-dt}{t^2}}{1/t\,(1/t^2 - 1/t + 1)^{1/2}}$$

$$= -\int \frac{dt\,(1/t)}{\sqrt{1/t^2 - 1/t + 1}} = -\int \frac{dt\,(1/t)}{\sqrt{\frac{1-t-t^2}{t^2}}}$$

$$= -\int \frac{dt\,(1/t)}{1/t\,(\sqrt{1-t+t^2})} = -\int \frac{dt}{\sqrt{1-t+t^2}}$$

$2at = t$

$a = 1/2$

$a^2 = 1/4$

$$= -\int \frac{dt}{\sqrt{t^2-t+1}} = -\int \frac{dt}{\sqrt{t^2-t+1/4-1/4+1}}$$

$$= -\int \frac{dt}{\sqrt{(t-1/2)^2+3/4}} = -\int \frac{dt}{\sqrt{(t-1/2)^2+(\sqrt{3}/2)^2}}$$

$$= -\sinh^{-1} \frac{t-1/2}{\sqrt{3}/2}$$

$t = 1/(x+1)$

$$= -\sinh^{-1} \frac{1-x}{2(x+1)(\sqrt{3}/2)}$$

$$t - 1/2 = \frac{1}{x+1} - 1/2$$

$$= \frac{2-(x+1)}{2(x+1)}$$

$$= \frac{1-x}{2(x+1)}$$

$$= -\sinh^{-1} \frac{1-x}{\sqrt{2}(1+x)} \;//$$

2) $\int \frac{dx}{(3+x)\sqrt{x}}$

$3+x = 1/t \quad x = 1/t - 3$

$dx = -1/t^2\, dt$

$$= \int \frac{-1/t^2\, dt}{1/t\,(1/t-3)^{1/2}} = \int \frac{-1/t\, dt}{(1/t-3)^{1/2}}$$

$$= \int \frac{-1/t\, dt}{1/t\left(\frac{t^2}{t}-3t^2\right)^{1/2}}$$

$$= \int \frac{-1/t\, dt}{1/t\sqrt{(t-3t^2)}} = -\int \frac{dt}{\sqrt{t-3t^2}}$$

$$= -1/\sqrt{3} \int \frac{dt}{\sqrt{\frac{t}{3}-t^2}} = \frac{-1}{\sqrt{3}} \int \frac{dt}{\sqrt{\frac{1}{36}-1/36+\frac{t}{3}}}$$

$$= -1/\sqrt{3} \int \frac{dt}{\sqrt{(1/6)^2-(t-1/6)^2}}$$

$$= \frac{-1}{\sqrt{3}} \sin^{-1}\left(\frac{t-1/6}{1/6}\right) = -\frac{1}{\sqrt{3}} \sin^{-1}(6t-1)$$

$$= \frac{-1}{\sqrt{3}} \sin^{-1}\left(6\left(\frac{1}{3+x}\right)-1\right)$$

$$= -1/\sqrt{3} \sin^{-1}\left(\frac{6-(3+x)}{3+x}\right)$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3-x}{3+x}\right) //.$$

Case (vi)

Integration of $\dfrac{1}{(Ax^2+B)\sqrt{Cx^2+D}}$

To integrate this we have to put $x = \frac{1}{t}$ or $\dfrac{Cx^2+D}{Ax^2+B} = t^2$

These Substitutions will faciliate integration.

Case (vii)

To evaluate $\displaystyle\int \frac{dx}{(ax^2+bx+c)\sqrt{Ax^2+Bx+C}}$ put $\dfrac{Ax^2+Bx+C}{ax^2+bx+c} = t^2$

Example $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $\dfrac{1-x^2}{1+x^2} = t^2$ $\quad x = \frac{1}{t}$

put $x = 1/t$ $\quad dx = \dfrac{-1}{t^2}\,dt$ $\quad t = \frac{1}{x}$ $\quad t^2 = \frac{1}{x^2}$

$$\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \int \frac{\frac{-dt}{t^2}}{\left(1+\frac{1}{t^2}\right)\left(1-\frac{1}{t^2}\right)^{1/2}}$$

$$= -\int \frac{\frac{dt}{t^2}}{\left(\frac{(t^2+1)}{t^2}\right)\left(\frac{t^2-1}{t^2}\right)^{1/2}} = -\int \frac{\frac{dt}{t^2}}{\left(\frac{(t^2+1)}{t^2}\right)\left(\frac{(t^2-1)^{1/2}}{t}\right)}$$

$$= -\int \frac{dt}{t^2} \times \frac{t^3}{(t^2+1)(t^2-1)^{1/2}} = \int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

$$= -\int \frac{u\,du}{(u^2+2)\sqrt{u^2}}$$

Put $t^2-1 = u^2$

$2t\,dt = 2u\,du$

$t\,dt = u\,du$

$t^2-1 = u^2$

$t^2-1+2 = u^2+2$

$t^2+1 = u^2+2$

$$= -\int \frac{u\,du}{(u^2+2)\,u}$$

$$= -\int \frac{du}{u^2+(\sqrt{2})^2}$$

$$= -\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}\right] = \frac{-1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{t^2-1}}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{\frac{1}{x^2}-1}}{\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{1-x^2}{2x^2}}$$

$$= \frac{-1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{1-x^2}}{x\sqrt{2}}$$

Case (viii)

Many algebraical functions which involve the square root of a quadratic expression can be rationalised by a trigonometrical substitution and their integration is often thereby simplified.

If an expression involves the rational quantity $\sqrt{a^2-x^2}$ or $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$ and no other radical. we can put $x = a\sin\theta$, or $a\tan\theta$ or $a\sec\theta$ respectively in the above cases and we shall get rid of the square root

Example.

1. $\displaystyle\int \frac{x^3+1}{\sqrt{1-x^2}}$ $\quad x = a\sin\theta$

put $x = \sin\theta$, $dx = \cos\theta$

$$= \int \frac{\sin^3\theta + 1}{\cos\theta} \cos\theta \, d\theta$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$= \int (\sin^3\theta + 1)\, d\theta$$

$$= \int \left(\frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4} + 1 \right) d\theta$$

$$= -\frac{3}{4}\cos\theta + \frac{\cos 3\theta}{4\times 3} + \theta$$

$$= \theta - \frac{3}{4}\cos\theta + \frac{1}{12}(\cos 3\theta)$$

$= \theta - \frac{3}{4}\cos\theta + \frac{1}{12}(4\cos^3\theta - 3\cos\theta)$

$= \theta - \frac{3}{4}\cos\theta - \frac{\cancel{3}}{\cancel{12}_{4}}\cos\theta + \frac{1}{3}\cos^3\theta$

$\cos 3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$

$4\cos^3\theta = 3\cos\theta + \cos 3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$= \theta - \frac{4}{4}\cos\theta + \frac{1}{3}\cos^3\theta$

$= \sin^{-1}x - \sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3$

$= \sin^{-1}x - \sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}$

$x = \sin\theta$

$x^2 = \sin^2\theta$

$1-x^2 = \cos^2\theta$

$\cos\theta = \sqrt{1-x^2}$

2) $\int \frac{dx}{(a^2+x^2)^{3/2}}$

Put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$= \int \frac{a\sec^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^{3/2}} = \int \frac{a\sec^2\theta\, d\theta}{(a^2)^{3/2}(1+\tan^2\theta)^{3/2}}$

$= \int \frac{a\sec^2\theta\, d\theta}{a^3(\sec^2\theta)^{3/2}} = \int \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$

$= \frac{1}{a^2}\int \frac{d\theta}{\sec\theta} = \frac{1}{a^2}\int \cos\theta\, d\theta = 1/a^2 \sin\theta$

$\frac{x}{\sqrt{a^2+x^2}} = \frac{a\tan\theta}{\sqrt{a^2+a^2\tan^2\theta}} = \frac{a\tan\theta}{a\sqrt{1+\tan^2\theta}}$

$= \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

$= 1/a^2 \sin\theta = \frac{x}{a^2\sqrt{a^2+x^2}}$

Example 3

$\int \frac{dx}{x^2\sqrt{4+x^2}}$

Put $x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta.$

$$= \int \frac{2\sec^2\theta \, d\theta}{4\tan^2\theta\sqrt{4+4\tan^2\theta}}$$

$$= \int \frac{\cos\theta}{4\sin^2\theta} d\theta = -\frac{1}{4} \cdot \frac{1}{\sin\theta} = -\frac{\sqrt{x^2+4}}{4x}$$

Example 4

$$\int \frac{dx}{x^3\sqrt{x^2-9}}$$

Put $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta \, d\theta$

$$= \int \frac{3\sec\theta\tan\theta \, d\theta}{27\sec^3\theta\sqrt{9\sec^2\theta - 9}} = \int \frac{d\theta}{27\sec^2\theta}$$

$$= \frac{1}{27}\int \cos^2\theta \, d\theta$$

$$= \frac{\theta}{54} + \frac{1}{108}\sin 2\theta = \frac{\theta}{54} + \frac{\sin\theta\cos\theta}{54}$$

$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{18}\frac{\sqrt{x^2-9}}{x^2}$$

Case ix

$$\sqrt{(x-\alpha)(\beta-x)}, \quad \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}, \quad \left(\frac{x-\alpha}{\beta-x}\right)^{1/2}$$

where $\beta > \alpha$

Substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$

Example 1

$$\int \sqrt{(x-3)(7-x)} \, dx \qquad \cos^2\theta + \sin^2\theta = 1$$

$\alpha = 3 \quad \beta = 7$

Substitution

$x = 3\cos^2\theta + 7\sin^2\theta$

$$dx = \left(3(-2\cos\theta\sin\theta) + 7(2\sin\theta\cos\theta)\right) d\theta$$
$$= (-6\cos\theta\sin\theta + 14\sin\theta\cos\theta)\, d\theta$$
$$= 8\cos\theta\sin\theta\, d\theta$$

$$x-3 = 3\cos^2\theta + 7\sin^2\theta - 3(\sin^2\theta + \cos^2\theta)$$
$$= \cancel{3\cos^2\theta} + 7\sin^2\theta - 3\sin^2\theta - \cancel{3\cos^2\theta}$$
$$= 4\sin^2\theta$$

$$7-x = \cancel{7\sin^2\theta} + 7\cos^2\theta - 3\cos^2\theta - \cancel{7\sin^2\theta}$$
$$= 4\cos^2\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{\sin 2\theta}{2}$

$$\int \sqrt{(x-3)(7-x)}\, dx = \int \sqrt{4\sin^2\theta \cdot 4\cos^2\theta} \cdot 8\cos\theta\sin\theta\, d\theta$$
$$= \int \sqrt{4^2\sin^2\theta\cos^2\theta}\; 8\cos\theta\sin\theta\, d\theta$$
$$= \int (4\sin\theta\cos\theta)(8\cos\theta\sin\theta)\, d\theta$$
$$= 32\int (\sin\theta\cos\theta)(\sin\theta\cos\theta)\, d\theta$$
$$= 32\int \frac{\sin 2\theta}{2} \frac{\sin 2\theta}{2}\, d\theta$$
$$= \frac{32}{4}\int \sin^2 2\theta\, d\theta$$
$$= 8\int \frac{1-\cos 2(2\theta)}{2}\, d\theta$$
$$= 4\int (1-\cos 4\theta)\, d\theta$$
$$= 4\left[\theta - \left(\frac{\sin 4\theta}{4}\right)\right]$$
$$= 4\theta - \sin 4\theta \quad 2(2\theta)$$
$$= 4\theta - 2\sin 2\theta\cos 2\theta$$

$\cos^2 A = \frac{1+\cos 2A}{2}$

$$= 4\theta - 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1)$$

$2\cos^2 A = 1 + \cos 2A$

$$= 4\theta - 4\sin\theta\cos\theta\,(2\cos^2\theta - 1)$$

$x - 3 = 4\sin^2\theta$ $\quad 7 - x = 4\cos^2\theta$

$\frac{x-3}{4} = \sin^2\theta$ $\quad \frac{7-x}{4} = \cos^2\theta$

$\Rightarrow \sin\theta = \sqrt{\frac{x-3}{4}}$

$\theta = \sin^{-1}\left(\sqrt{\frac{x-3}{4}}\right)$ $\quad 4\sin\theta\cos\theta = \sqrt{(x-3)(7-x)}$

$= 4\sin^{-1}\left(\sqrt{\frac{x-3}{4}}\right) - \left(\sqrt{(x-3)(7-x)}\right)\left(\frac{7-x}{2} - 1\right)$

$= 4\sin^{-1}\left(\frac{\sqrt{x-3}}{2}\right) - \left(\sqrt{(x-3)(7-x)}\right)\left(\frac{5-x}{2}\right)$.

2) $\int\left(\frac{5-x}{x-2}\right)^{1/2} dx$ H.w.

3) $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ where $\beta > \alpha$

Put $x = \alpha\sin^2\theta + \beta\cos^2\theta$

$dx = (2\alpha\sin\theta\cos\theta - 2\beta\cos\theta\sin\theta)\,d\theta$

$= 2(\alpha-\beta)\sin\theta\cos\theta\,d\theta$

$x - \alpha = \alpha\sin^2\theta + \beta\cos^2\theta - \alpha\sin^2\theta - \alpha\cos^2\theta$

$= (\beta-\alpha)\cos^2\theta$

$\beta - x = (\beta-\alpha)\sin^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \int \frac{2(\alpha-\beta)\sin\theta\cos\theta\,d\theta}{\sqrt{(\beta-\alpha)\cos^2\theta\,(\beta-\alpha)\sin^2\theta}}$

$= \int \frac{-2(\beta-\alpha)\sin\theta\cos\theta\,d\theta}{(\beta-\alpha)\cos\theta\sin\theta}$

$= -2\int d\theta = -2\sin^{-1}\left(\frac{\beta-x}{\beta-\alpha}\right)^{1/2}$

$= -2\theta$

Case (X)

Sometimes rationalisation of the denominator may integration

Example : 1

$$\int \frac{dx}{x+\sqrt{x^2-1}} = \int \left(x-\sqrt{x^2-1}\right) dx$$

2nd $\int \frac{dx}{\sqrt{x}-\sqrt{1+x}}$

$$= \frac{1}{2}x^2 - \int \sqrt{x^2-1}\, dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\cosh^{-1}x$$

$$\int \frac{dx}{a+b\cos x}$$

Put $t = \tan\frac{x}{2}$

$$dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$$

$$dt = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) dx$$

$$dx = \frac{2dt}{1+t^2}, \quad \cancel{\cos x} = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{dx}{a+b\cos x} = \int \frac{2\, dt}{(1+t^2)\left[a+b\left(\frac{1-t^2}{1+t^2}\right)\right]}$$

$$= \int \frac{2dt}{a(1+t^2)+b(1-t^2)}$$

$$= \int \frac{2dt}{a+at^2+b-bt^2} = \int \frac{2dt}{(a+b)+(a-b)t^2}$$

case (i) $a > b$

$$I = \frac{2}{a-b}\int \frac{dt}{\left(\frac{a+b}{a-b}\right)+t^2} \quad \left(\left(\frac{a+b}{a-b}\right)^2\right)^{1/2}$$

$$= \frac{2}{(a-b)}\left[\frac{1}{\left(\frac{a+b}{a-b}\right)^{1/2}}\tan^{-1}\frac{t}{\left(\frac{a+b}{a-b}\right)^{1/2}}\right]$$

$$= \frac{2}{(a-b)^{1/2}(a+b)^{1/2}}\left[\tan^{-1} t\left(\frac{a-b}{a+b}\right)^{1/2}\right]$$

$$= \frac{2}{\sqrt{(a-b)(a+b)}}\left[\tan^{-1}\left(\tan\frac{x}{2}\left(\frac{a-b}{a+b}\right)^{1/2}\right)\right]$$

Case (ii) $(a<b)$

$$I = 2\int\frac{dt}{(a+b)-(b-a)t^2} = \frac{2}{(b-a)}\int\frac{dt}{\frac{b+a}{b-a}-t^2}$$

$$= \frac{2}{(b-a)}\left[\frac{1}{2\left(\frac{b+a}{b-a}\right)^{1/2}}\log\frac{t+\left(\frac{b+a}{b-a}\right)^{1/2}}{\left(\frac{a+b}{b-a}\right)^{1/2}-t}\right]$$

$$= \frac{1}{(b+a)^{1/2}(b-a)^{1/2}}\log\left(\frac{\left(\sqrt{b-a}\,t+\sqrt{b+a}\right)/\sqrt{b-a}}{-\sqrt{b-a}\,t+\sqrt{b+a}/\sqrt{b-a}}\right)$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left(\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{-\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}\right)$$

Imp formulas.

(*) $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2 x/2}$ (*) $\sin x = \dfrac{2\tan x/2}{1+\tan^2 x/2}$

10. $\int\dfrac{dx}{5+4\cos x}$ (x) imp Sum

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2} = t$

$$\sec^2\frac{x}{2}\cdot\frac{dx}{2} = dt$$

$$\left(1+\tan^2\frac{x}{2}\right)\cdot\frac{dx}{2} = dt$$

$$(1+t^2)\cdot\frac{dx}{2} = dt$$

$$dx = \frac{2dt}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\int\frac{dx}{5+4\cos x} = \int\frac{2dt/1+t^2}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int\frac{2dt/1+t^2}{\frac{5(1+t^2)+4(1-t^2)}{(1+t^2)}}$$

$$= \int\frac{2dt}{5+5t^2+4-4t^2}$$

$$= 2\int\frac{dt}{t^2+9}$$

$$= 2\int\frac{dt}{t^2+3^2}$$

$$= 2\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)+c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan x}{3}\right)+c$$

11. $\int\frac{dx}{4+5\sin x}$

$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$$

Put $\tan\frac{x}{2} = t$

$$\sec^2\frac{x}{2}\cdot\frac{dx}{2} = dt$$

$$\left(1+\tan^2\frac{x}{2}\right)\cdot\frac{dx}{2} = dt$$

$$(1+t^2)\frac{dx}{2} = dt$$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\int \frac{dx}{4+5\sin x} = \int \frac{2dt/1+t^2}{4+5\left(\frac{2t}{1+t^2}\right)}$$

$$= 2\int \frac{dt/1+t^2}{\frac{4(1+t^2)+5(2t)}{(1+t^2)}}$$

$$= 2\int \frac{dt}{4+4t^2+10t}$$

$$= \frac{2}{4}\int \frac{dt}{t^2+\frac{5t}{2}+1} = \frac{2}{4}\int \frac{dt}{t^2-\frac{5}{2}t+\frac{25}{16}-\frac{25}{16}+1}$$

$$= \frac{2}{4}\int \frac{dt}{(t+5/4)^2-(3/4)^2}$$

$$= \frac{1}{2}\left[\frac{2}{3}\log\left(\frac{4t+5-3}{4t+5+3}\right)\right]+c$$

$$= \frac{1}{3}\log\left(\frac{4t+2}{4t+8}\right)+c$$

$$= \frac{1}{3}\log\left(\frac{2t+1}{2t+4}\right)+c$$

$$= \frac{1}{3}\log\left[\frac{2\tan x/2+1}{2\tan x/2+4}\right]+c$$

Ex:1

Evaluate $\int_0^{\pi} \frac{dx}{5+4\cos x}$

putting $t = \tan\frac{x}{2}$, the integral reduces to

$$\int_0^{\infty} \frac{2dt}{9+t^2} = \frac{2}{3}\left\{\tan^{-1}\left(\frac{t}{3}\right)\right\}_0^{\infty} = \pi/3$$

(The limits of the definite integral must be changed when the variable x is changed to t. when $x=0$, $t=0$, and $x=\pi$, $t\to\infty$

Ex. 2

Evaluate $\int \frac{dx}{a\cos x + b\sin x + c}$

let $a = r\cos\alpha$ and $b = r\sin\alpha$.

The auxillary constants r and α are thus given by $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\frac{b}{a}$.

Hence the integral becomes

$$\int \frac{dx}{r\cos(x-a)+c} = \int \frac{dy}{r\cos y + c} \cdot \text{ where } y = x - a.$$

This reduces to the tyne considered.

Ex. 3

$$\int_0^{\pi/2} \frac{dx}{9\cos x + 12\sin x}$$

putting $t = \tan\frac{x}{2}$ and noting that $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$ the integral reduces to $2/3 \int_0^1 \frac{dt}{3+8t-3t^2}$ as the limits for t change to 0 to 1 when x takes the values 0 and $\pi/2$. Hence the integral is

$$\frac{2}{3}\int_0^1 \frac{dt}{(3-t)(3t+1)} = \frac{1}{15}\int_0^1 \left\{\frac{3}{3t+1} + \frac{1}{3-t}\right\}dt$$

$$= \frac{1}{15}\left\{\log\frac{3t+1}{3-t}\right\}_0^1 = 1/15\left(\log 2 - \log 1/3\right) = \log 6/15$$

$$\sec^2\frac{x}{2}\cdot\frac{dx}{2} = dt$$

$$\left(1+\tan^2\frac{x}{2}\right)\frac{dx}{2} = dt$$

$$(1+t^2)\frac{dx}{2} = dt$$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int\frac{dx}{9\cos x+12\sin x} = \int\frac{2dt/1+t^2}{9\left(\frac{1-t^2}{1+t^2}\right)+12\left(\frac{2t}{1+t^2}\right)}$$

$$= \int\frac{2dt}{9-9t^2+24t} = 2/9\int\frac{dt}{1-t^2-8/3\,t}$$

$$= \frac{2}{9}\int\frac{dt}{1+\frac{8}{3}t+\frac{16}{9}-\frac{16}{9}-t^2}$$

$$= \frac{2}{9}\int\frac{dt}{\left(\frac{25}{9}\right)-(t-4/5)^2}$$

$$= 2/9\left[\frac{5}{2\times5}\log\left(\frac{5-3t-4}{5-3t+4}\right)\right]+c$$

$$= 1/15\,\log\left(\frac{5t+1}{9-3t}\right)+c$$

$$= 1/15\,\log\left[\frac{3\tan x/2+1}{9-3\tan x/2}\right]+c$$

10) Evaluate

$$\int\frac{dx}{a^2\cos^2x+b^2\sin^2x}$$

Multiplying numerator and denominator by $\sec^2x$, the integral reduces to $\displaystyle\int\frac{\sec^2x\,dx}{a^2+b^2\tan^2x}$

$$= \int\frac{dt}{a^2+b^2t^2} \quad \text{on putting } \tan x = t,\ \sec^2x\,dx = dt$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)$$

16/11/2021

unit-2

PROPERTIES OF DEFINITE INTEGERLS

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. This is obvious from the definition of a definite integral.

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is some value of x between a and b.

Let $\int f(x)\,dx = F(x)$

Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

The $R.H.S = F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$. Hence the result.

3. $\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx$, If $f(x)$ is an even function of x.

If $f(x)$ is even. $f(x) = f(-x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \text{ by (2)}$$

$$= \int_{-a}^{0} f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_a^0 f(y)\,dy + \int_0^a f(x)\,dx$$ (by putting $y = x$ in the first integral)

$$= \int_0^a f(y)\,dy + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

as in a definite integral we can replace the variable y by x.

4. $f(x)$ is odd, $f(x) = -f(-x)$

If $F(x)$ is an odd function of x, $\int_{-a}^{a} f(x)\,dx = 0$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{-a}^{0} f(-x)dx + \int_{0}^{a} f(x)dx$$

$$= +\int_{a}^{0} f(y)dy + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= 0$$

(5.) $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$

In $\int_{0}^{a} f(a-x)dx$, put $a-x=y$

$$R.H.S = -\int_{a}^{0} f(y)dy = \int_{0}^{a} f(y)dy = \int_{0}^{a} f(x)dx.$$

This result is very useful in evaluating many integrals.

Examples.

Ex : 1

Prove that $\int_{0}^{\pi/2} \sin^n x\, dx = \int_{0}^{\pi/2} \cos^n x\, dx$

Let $f(x) = \sin^n x$. Here $a = \pi/2$

$$\therefore f(a-x) = \sin^n\left(\frac{\pi}{2} - x\right) = \cos^n x.$$

Example : 2

$$\int_{0}^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\, dx = \pi/4$$

Let I be the value of this integral and $f(x)$ denote the intrest $\dfrac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}$

$$\therefore I = \int_{0}^{\pi/2} f(x)dx.$$

$$f(a-x) = \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} \quad \text{as } a = \frac{\pi}{2} \text{ here.}$$

Also $I = \int_0^{\pi/2} f(a-x)\,dx$

Adding (1) and (2)

$$2I = \int_0^{\pi/2} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\,dx$$

$$= \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2$$

Hence $I = \pi/4$.

Ex 3. $\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log 2$

Let $f(\theta) = \log(1+\tan\theta)$. Here $a = \pi/4$.

$\therefore f\left(\frac{\pi}{4}-\theta\right) = \log\left\{1+\tan\left(\pi/4-\theta\right)\right\}$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

$$= \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\pi/4\,\tan\theta}\right\} = \log\frac{2}{1+\tan\theta}$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

and $$I = \int_0^{\pi/4} \log\frac{2}{1+\tan\theta}\,d\theta$$

Adding, $$2I = \int_0^{\pi/4} \log 2\,d\theta = \log 2\,[\theta]_0^{\pi/4}$$

$$= \pi/4 \log 2.$$

Hence the result.

Ex: 4.

$$\int_0^x \theta \sin^3\theta \, d\theta = \frac{2\pi}{3}.$$

$f(\theta) = \theta \sin^3\theta$. Hence $a = \pi$

$\therefore f(a-\theta) = (\pi - \theta)\sin^3\theta$

Hence $I = \int_0^\pi \theta \sin^3\theta \, d\theta$ and $I = \int_0^x (\pi - \theta)\sin^3\theta \, d\theta$

Adding $2I = \pi \int_0^x \sin^3\theta \, d\theta$

$= \pi \int \sin^2\theta \, (-dy)$ putting $\cos\theta = y$; $-\sin\theta \, d\theta = dy$

$= -\pi \int_1^{-1} (1-y^2)\, dy = -\pi \left[y - \frac{y^3}{3} \right]_1^{-1}$

$= -\pi \left[-1 + \frac{1}{3} - 1 + \frac{1}{3} \right] = \frac{4\pi}{3}$

Hence $I = \frac{2\pi}{3}$

Ex. 5

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x) \text{ and}$$

$= 0$ if $f(2a-x) = -f(x)$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx.$$

In the second integral, put $2a - x = y$, $dx = -dy$

when $x = a$, $y = a$, and $x = 2a$, $y = 0$.

Hence $\int_a^{2a} f(x)\,dx = -\int_a^a f(2a-y)\,dy = \int_0^a f(2a-y)\,dy$

$= \int_0^a f(2a-x)\,dx.$

$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ from (1).

If $f(2a-x) = f(x)$, $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

If $f(2a-x) = -f(x)$, $2\int_0^a f(x)\,dx = 0$

Cor $\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx$.

Ex. 6 Evaluate $I = \int_0^{\pi/2} \log \sin x\,dx$.

$$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \log \cos x\,dx$$

Hence $$2I = \int_0^{\pi/2} \log \sin x\,dx + \int_0^{\pi/2} \log \cos x\,dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\,dx$$

$$= \int_0^{\pi/2} (\log \sin 2x - \log 2)\,dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \frac{\pi}{2}\log 2.$$

put $2x = z$; $dx = \frac{1}{2}dz$; then

$$\int_0^{\pi/2} \log \sin 2x\,dx = \frac{1}{2}\int_0^{\pi} \log \sin z\,dz = \frac{1}{2}\int_0^{\pi} \log \sin x\,dx$$

$$= \frac{1}{2} \times 2\int_0^{\pi/2} \log \sin x\,dx$$

$$= \int_0^{\pi/2} \log \sin x\,dx$$

Thus, $2I = I - \frac{\pi}{2}\log 2$.

ie, $I = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$

Integration by parts.

If u and v are functions of x,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$ by the product rule,

Integrating both side with respect to x:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\therefore uv = \int u\,dv + \int v\,du$$

Hence $\int u\,dv = uv - \int v\,du$

Note: The Success of this method depends on the Product choice of u and v; the auxiliary integral $\int u\,dv$ must be easier integrate then the given integral.

Examples.

Ex (1) $\int x e^{x}\,dx$

writing $dv = e^{x}$ and $u = x$, $V = \int e^{x}dx = e^{x}$

$$\therefore \int x e^{2}dx = \int x\,dx(e^{x}) = \int u\,dv = uv - \int v\,du$$

$$= xe^{x} - \int e^{x}dx = xe^{x} - e^{x}$$

$$= e^{x}(x-1)$$

Ex (2) $x \sin 2x\,dx$ $\qquad -\frac{\cos 2x}{2}$

Here $dv = \sin 2x\,dx \qquad V = \int \sin 2x\,dx = -\frac{\cos 2x}{2}$

$u = x,\ du = dx$

$$\int x \sin 2x\,dx = \int x\,d\left(-\frac{\cos 2x}{2}\right)$$

$$= uv - \int v\,du$$

$$= x\left(-\frac{\cos 2x}{2}\right) - \int \frac{-\cos 2x}{2}\,dx$$

$$= \frac{-x\cos 2x}{2} + \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{-x\cos 2x}{2} + \frac{1}{2}\,\frac{\sin 2x}{2} + c$$

$$\int x \sin x \, dx = \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} + c$$

3) $\int x^n \log x \, dx$ $\quad \log x = \frac{1}{x} dx$

$u = \log x \qquad dv = x^n$

$du = \frac{1}{x} dx \qquad v = \frac{x^{n+1}}{n+1}$

$$\int x^n \log x \, dx = \log x\left(\frac{x^{n+1}}{n+1}\right) - \int \frac{x^{n+1}}{n+1}\,\frac{1}{x}\, dx$$

$$= \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\int x^{n+1} x^{-1} dx$$

$$= \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\int x^n dx$$

$\log x$

$\frac{1}{x} dx$

$$= \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\left[\frac{x^{n+1}}{n+1}\right] + c$$

4) $\int \sin^{-1} x \, dx$

$u = \sin^{-1}(x) \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$uv - \int v\,du$

$$\int \sin^{-1} x \, dx = \sin^{-1}(x)(x) - \int x \frac{1}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1}(x) + \int \frac{-x}{\sqrt{1-x^2}} dx$$

$$= x\sin^{-1}(x) + \cancel{\tfrac{1}{2}} \cdot \cancel{2}\sqrt{1-x^2}$$

$$= x\sin^{-1}(x) + \sqrt{1-x^2} + c$$

5) $\int \tan^{-1}(x)\,dx$.

$u = \tan^{-1}(x)$ $\qquad dv = dx$

$\qquad\qquad\qquad\qquad v = x$

$du = \frac{1}{1+x^2}\,dx$

$$\int \tan^{-1}(x)\,dx = \cancel{x\tan^{-1}}(x) - \cancel{\tfrac{1}{2}\int \frac{2x}{1+x^2}dx}$$

$$= x\tan^{-1}(x) - \int x\frac{1}{1+x^2}\,dx$$

$$= x\tan^{-1}(x) - \tfrac{1}{2}\,\log(1+x^2) \cancel{\frac{2x}{1+x^2}dx} + C$$

18/11/21

6. $\int x^2 \tan^{-1}(x)\,dx$

$u = \tan^{-1}(x) \qquad dv = x^2\,dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = \frac{x^3}{3}$

$$\int x^2\tan^{-1}(x)\,dx = \frac{x^3}{3}\tan^{-1}(x) - \frac{1}{3}\int x^3\cdot\frac{1}{1+x^2}\,dx$$

$$= \frac{x^3\tan^{-1}(x)}{3} - \tfrac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{1}{3}\left[x^3\tan^{-1}(x) - \int x\,dx + \frac{1}{2}\int\frac{2x}{1+x^2}\,dx\right]$$

$$= \frac{1}{3}\left[x^3\tan^{-1}(x) - \frac{x^2}{2} + \frac{1}{2}\log(1+x^2)\right] + C$$

7) $\int (\log x)^2\,dx$

$u = (\log x)^2 \qquad dv = dx$

$du = 2\log x\cdot\frac{1}{x}\,dx \qquad v = x$

$$\int(\log x)^2 dx = x(\log x)^2 - 2\int x\log x\cdot\frac{1}{x}\,dx$$

$$= x(\log x)^2 - 2\int \log x\,dx$$

$u = \log x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$= x(\log x)^2 - 2\left[x\log x - \int x \frac{1}{x}\,dx\right]$

$= x(\log x)^2 - 2\left[x\log x - x\right] + c$

$= x\left[(\log x)^2 - 2\log x + 2\right] + c$

8) $\int \sqrt{a^2+x^2}\,dx$

$u = \sqrt{a^2+x^2}$ $\quad dv = dx \quad v = x$

Q.N

$du = \frac{+1}{2\sqrt{a^2+x^2}} \cdot 2x\,dx$

$du = \frac{x}{\sqrt{a^2+x^2}}\,dx$

$\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$

$= x\sqrt{a^2+x^2} - \int \frac{x^2+a^2-a^2}{\sqrt{a^2+x^2}}\,dx$

$= x\sqrt{a^2+x^2} - \int \frac{x^2-a^2}{\sqrt{a^2+x^2}}\,dx + x^2\int \frac{1}{\sqrt{a^2+x^2}}\,dx$

$\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} - \int \sqrt{x^2+a^2}\,dx + a^2\log\left(x+\sqrt{x^2+a^2}\right.$

$2\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)$

$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x\cdot\sqrt{a^2+x^2}\right) + c$

9) $\int \frac{x+\sin x}{1+\cos x}\,dx$

$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$

$I_1 = \int \frac{x}{1+\cos x}\,dx \quad , \quad I_2 = \int \frac{\sin x}{1+\cos x}\,dx$

$I_1 = \frac{1}{2}\int \frac{1\,x}{2\cos^2 x/2}\,dx$ $\qquad 1+\cos x = 2\cos^2 x/2$

$\qquad \cos x = 2\cos^2 x/2 - 1$

$= \frac{1}{2}\int x \sec^2 \frac{x}{2}\,dx$ $\qquad 1/\cos = \sec$

$u = x, \quad dv = \sec^2 \frac{x}{2}\,dx$ $\qquad \sec^2 \frac{x}{2} = 2\tan x/2$

$du = dx, \quad v = 2\tan\frac{x}{2}$

$= \frac{1}{2}\left[2x\tan\frac{x}{2} - 2\int \tan\frac{x}{2}\,dx\right]$ $\qquad \tan x/2 \Rightarrow \log \sec x/2$

$= \frac{1}{2}\left[2x\tan\frac{x}{2} - 2\cdot 2\log\left(\sec\frac{x}{2}\right)\right] + C_1$

$I_1 = x\tan\frac{x}{2} - 2\log\left(\sec x/2\right) + C_1$

$I_2 = \int \frac{\sin x}{1+\cos x}\,dx$ $\qquad d(1+\cos)$

$\qquad d(1) + d\cos$

$= -\int \frac{d(1+\cos x)}{1+\cos x}\,dx$

$\qquad \int \frac{1}{x} = \log x$

$I_2 = -\log(1+\cos x) + C_2$ $\qquad \int \frac{d(x)}{x}$

$I = I_1 + I_2$

$= x\tan\frac{x}{2} - 2\log\left(\sec\frac{x}{2}\right) - \log(1+\cos x) + C$

where $C = C_1 + C_2$ //.

10) $\displaystyle\int e^x \frac{(x+1)}{(x+2)^2}\,dx$ $\qquad 2-1=1$

$$= \int e^x \frac{(x+2-1)}{(x+2)^2}\,dx$$

$$= \int \left[\frac{e^x (x+2)}{(x+2)^{2}} - \frac{e^x}{(x+2)^2} \right) dx$$

$$= \int \frac{e^x}{x+2}\,dx - \int \frac{e^x}{(x+2)^2}\,dx$$

$$\int \frac{e^x}{x+2}\,dx$$

$u = \frac{1}{x+2}$, $dv = e^x\,dx$

$du = \frac{-1}{(x+2)^2}\,dx$, $v = e^x$

$$= \frac{e^x}{x+2} + \int \frac{e^x}{(x+2)^2}\,dx - \int \frac{e^x}{(x+2)^2}\,dx$$

$$= \frac{e^x}{x+2} + c$$

11) $\displaystyle\int e^x(\sin x + \cos x)\,dx$

imp

$$\int e^x(\sin x + \cos x)\,dx = \int e^x \sin x\,dx + \int e^x \cos x\,dx$$

$$\int e^x \sin x\,dx$$

$u = \sin x$, $dv = e^x\,dx$

$du = \cos x\,dx$, $v = e^x$

$$= e^x \sin x - \int e^x \cos x\,dx + \int e^x \cos x\,dx$$

$$= e^x \sin x + c$$

Reduction formulae.

$I_n = \int x^n e^{ax} dx$, where n is a positive integer

Here $dv = e^{ax} dx$, i.e, $v = \int e^{ax} dx = \frac{e^{ax}}{a}$ and $u = x^n$

$\therefore I_n = \int x^n d\left(\frac{e^{ax}}{a}\right) = \frac{e^{ax}}{x} x^n - \frac{n}{a}\int e^{ax} x^{n-1} dx$

$= \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}$.

$I_n\ \int x^n \cos ax\, dx$ (n is positive integer)

$I_n = \int x^n \cos ax\, dx = \int x^n d\left(\frac{\sin ax}{a}\right)$ [Here $u = x^n$ and $v = \frac{\sin ax}{a}$]

$\int u dv = uv - \int v du$

$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} \sin ax\, dx$

$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} d\left(\frac{-\cos ax}{a}\right)$

$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2}\int x^{n-2} \cos ax\, dx$.

$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2} I_{n-2}$,

The ultimate integral is either $\int x \cos ax\, dx$ or $\int \cos ax\, dx$ as n is odd or even.

(i) $\int x \cos ax\, dx = \int x d\left(\frac{\sin ax}{a}\right) = \frac{x \sin ax}{a} - \frac{1}{a}\int \sin ax\, dx$

$= \frac{x \sin ax}{a} + \frac{1}{a^2} \cos ax$

(ii) $\int \cos ax\, dx = \frac{\sin ax}{a}$.

1) $I_n\ \int x^n e^{ax} dx$ where n is a positive integer.

$dv = e^{ax} dx$ $\quad u = x^n$

$v = \frac{e^{ax}}{a}$. $\quad du = n x^{n-1}$

$I_n = uv - \int v du$

$$= \frac{x^n e^{ax}}{a} - \int \frac{e^{ax}}{a} n x^{n-1} dx$$

$$= \frac{e^{ax}}{a} x^n - \frac{n}{a} \int e^{ax} x^{n-1} dx$$

$$= \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}$$

The auxillary integral is of the same type as the given integral, but with index n reduce by 1 such a formula is called a reduction formula.

2) $I_n = \int x^n \cos ax \, dx$ (n is a positive integer)

$I_n = \int x^n \cos ax \, dx$

$u = x^n$ $\quad$ $dv = \cos ax \, dx$

$du = nx^{n-1}$ $\quad$ $v = \frac{\sin ax}{a}$

$$I_n = uv - \int v \, du = \frac{x^n \sin ax}{a} - \int \frac{\sin ax}{a} n x^{n-1} dx$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax \, dx$$

$du = x^{n-1}$ $\quad$ $dv = \sin ax$

$du = (n-1) x^{n-2} dx$ $\quad$ $v = -\frac{\cos ax}{a}.$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} d\left(-\frac{\cos ax}{a}\right)$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a} \left[x^{n-1} \frac{(-\cos ax)}{a} - \int -\frac{\cos ax}{a} (n-1) x^{n-2} dx \right]$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1} \cos ax - \frac{n}{a} \int \frac{\cos ax}{a} (n-1) x^{n-2} \, dx.$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2} \int x^{n-2} \cos ax \, dx$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2} I_{n-2}$$

The ultimate integral is either $\int x \cos ax \, dx$ or $\int \cos ax \, dx$ according as n is odd or even

(i) $\int x \cos ax$

$u = x \qquad dv = \cos ax$

$du = dx \qquad v = \frac{\sin ax}{a}$

$$\int x \cos ax = \frac{x \sin ax}{a} - \int \frac{\sin ax}{a} dx$$

$$= \frac{x \sin ax}{a} - \frac{1}{a} \int \sin ax \, dx.$$

$$= \frac{x \sin ax}{a} - \frac{1}{a}\left(-\frac{\cos ax}{a}\right)$$

$$= \frac{x \sin ax}{a} + \frac{1}{a^2} \cos ax$$

ii) $\int \cos ax \, dx = \frac{\sin ax}{a}$

3) I_n $\int \sin^n x \, dx$ (n being positive integer)

$u = \sin^{n-1} x$

$du = (n-1) \sin^{n-2} x \cos x \, dx$

$dv = \sin x \, dx$

$v = -\cos x$

$$I_n = \int \sin^{n-1} x \; \sin x \, dx$$

$$= uv - \int v \, du$$

$$= -\sin^{n-1} x \cos x \int (-\cos x)(n-1) \sin^{n-2} x \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int (\sin^{n-2} x - \sin^n x) \, dx$$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$

$I_n = \sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = \sin^{n-1}x\cos x + (n-1)I_{n-2}$

$\therefore nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$

$$I_n = -\frac{\sin^{n-1}x\cos x}{A} + \frac{(n-1)}{n}I_{n-2}$$

The ultimate integral $\begin{cases}\text{is } \int \sin x\,dx & \text{if } n \text{ is odd} \\ \text{is } \int dx & \text{if } n \text{ is even.}\end{cases}$

imp

$\sin \pi/2 = 1$
$\cos \pi/2 = 0$
$\sin 0 = 0$
$\cos 0 = 1$

corollary

$$\int_0^{\pi/2} \sin^n x\,dx.$$

$$= \left[\frac{-\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}dx$$

(0) (0) I_{n-2} (0)

$$= \left[-\left(\frac{\sin^{n-1}(\pi/2)\cos(\pi/2)}{n} - \frac{\sin^{n-1}(0)\cos(0)}{n}\right)\right]$$

$I_m = \left[-\frac{\sin^{m-1}\cos}{m}\right.$

$= \frac{m-1}{m}I_{m-2}$

$= \frac{n-2-1}{n-2}I_{n-2-2}$

$$+\frac{n-1}{n}\,\frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4}x\,dx$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\,\frac{n-7}{n-6}$$

If n is even the ultimate integral is

$$\int_0^{\pi/2} dx = (x)_0^{\pi/2} = \pi/2$$

If n is odd The ultimate integral is

$$\int_0^{\pi/2} \sin x\,dx = (-\cos x)_0^{\pi/2} = 1$$

$$\int_0^{\pi/2} \sin^n x = \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{1}{2}\,\frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n}\,\frac{n-3}{n+2}\cdots\frac{2}{3} & \text{if } n \text{ is odd} \end{cases}$$

Reduction formulae:

1) $\int \sin^n x\,dx$
2. $\int \cos^n x\,dx$
3. $\int \tan^n x\,dx$
4. $\int \cot^n x\,dx$
5. $\int \sec^n x\,dx$
6. $\int \mathrm{cosec}^n x\,dx$
7. $\int e^{ax}\cos bx\,dx$
8. $\int e^{ax}\sin bx\,dx$
9. $\int \sin^m x\cos^n x$
10. $\int x^n e^{ax}\,dx$
11. $\int x^m(\log x)^n\,dx$
12. $\int x^n \sin mx\,dx$
13. $\int x^n \cos mx\,dx$.

	0	30	45	60	90
sin	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

1. $\int \sin^n x\,dx$

Consider:

$$I_n = \int \sin^n x\,dx$$

$$= \int \sin^{n-1} x\,\sin x\,dx$$

$$= \int \sin^{n-1} x\,d(-\cos x)$$

$$= \sin^{n-1} x\cos x - \int (-\cos x)(n-1)\sin^{n-2} x\cos x\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$$

$$I_n + nI_n - I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$$

$$n I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$$

$$\therefore I_n = \frac{-1}{n} \sin^{n-1}x \cos x + \frac{n-1}{n} I_{n-2}$$

2) $\int \cos^n x \, dx$

consider:

$$I_n = \int \cos^n x \, dx$$

u dv

$$= \int \cos^{n-1}x \, \cancel{dx} \cos x \, dx$$

$$= \int \cos^{n-1}x \, d(\sin x)$$

$u = \cos^{n-1}x$, $dv = \cos x \, dx$
$du = (n-1) \cos^{n-2} \sin x \, dx$
$dv = \cos x \, dx$
$v = \sin x$

$$= \cos^{n-1}x \sin x - \int \sin x \cdot (n-1) \cos^{n-2}x (-\sin x) \, dx$$

$$= \cos^{n-1}x \sin x + (n-1) \int \cos^{n-2}x \sin^2 x \, dx$$

$$= \cos^{n-1}x \sin x + (n-1) \int \cos^{n-2}x (1 - \cos^2 x) dx$$

$$= \cos^{n-1}x \sin x + (n-1) \left[\int \cos^{n-2}x \, dx - \int \cos^n x \, dx \right]$$

$$= \cos^{n-1}x \sin x + (n-1) \left[I_{n-2} - I_n \right]$$

$$= \cos^{n-1}x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \cos^{n-1}x \sin x + (n-1) I_{n-2}$$

$$n I_n = \cos^{n-1}x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{1}{n} \cos^{n-1}x \sin x + \frac{(n-1)}{n} I_{n-2}$$

	diff	int
sin	cos	−cos
cos +	−sin	sin

3) $\int \tan^n x \, dx$

consider:

$$I_n = \int \tan^n x \, dx$$

$$= \int \tan^{n-2} x \tan^2 x \, dx$$

$$= \int \tan^{n-2} x (-1 + \sec^2 x) \, dx$$

$$= -\int \tan^{n-2} x \, dx + \int \tan^{n-2} x \sec^2 x \, dx$$

$$= -\int \tan^{n-2} x \, dx + \int \tan^{n-2} x \, d(\tan x)$$

$$I_n = -I_{n-2} + \frac{\tan^{n-1} x}{n-1}$$

4) $\int \cot^n x \, dx$

consider.

$$I_n = \int \cos^n x \, dx$$

$$= \int \cos^{n-2} x \cot^2 x \, dx$$

$$= \int \cos^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\int \cot^{n-2} x \, dx + \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx$$

$$= -\int \cot^{n-2} x \, dx - \int \cot^{n-2} x \, d(\cot x)$$

$$I_n = -I_{n-2} - \frac{\cot^{n-1}}{n-1}$$

5) $\int \sec^n x \, dx.$

consider:

$$I_n = \int \sec^n x \, dx$$

$$= \int \sec^{n-2} x \sec^2 x \, dx$$

$$= \int \sec^{n-2} x \, d(\tan x)$$

$= \sec^{n-2}x \tan x - \int \tan x\,(n-2)\sec^{n-3}x \sec x \tan x\,dx$

$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \tan^2 x\,dx$

$= \sec^{n-2}x \tan x - (n-2)\left[\cancel{\int \sec^n x\,dx}\ \int \sec^{n-2}x(\sec^2 x - 1)\,dx\right.$

$= \sec^{n-2}x \tan x - (n-2)\left[\int \sec^n x\,dx - \int \sec^{n-2}x\,dx\right]$

$= \sec^{n-2}x \tan x - (n-2)\left[I_n - I_{n-2}\right]$

$= \sec^{n-2}x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$I_n + (n-2) I_n = \tan x \sec^{n-2}x + (n-2) I_{n-2}$

$n I_n - I_n = \tan x \sec^{n-2}x + (n-2) I_{n-2}$

$I_n (n-1) = \tan x \sec^{n-2}x + (n-2) I_{n-2}$

$I_n = \frac{1}{n-1}\tan x \sec^{n-2}x + \frac{(n-2)}{(n-1)} I_{n-2}$

6. $\int \operatorname{cosec}^n x\,dx$

consider.

$I_n = \int \operatorname{cosec}^n x\,dx$

$= \int \operatorname{cosec}^{n-2}x\ \boxed{\operatorname{cosec}^2 x\,dx}$

$= \int \operatorname{cosec}^{n-2}x\,d(-\cot x)$

$= -\cot x \operatorname{cosec}^{n-2}x - \int (-\cot x)(n-2)\operatorname{cosec}^{n-2}x\,(-\operatorname{cosec}\cot x)\,dx$

$= -\cot x \operatorname{cosec}^{n-2}x - (n-2)\int \operatorname{cosec}^{n-2}x \cot^2 x\,dx$

$= -\cot x \operatorname{cosec}^{n-2}x - (n-2)\int \operatorname{cosec}^{n-2}x(\operatorname{cosec}^2 x - 1)\,dx$

$= -\cot x \operatorname{cosec}^{n-2}x - (n-2)\left[\int \operatorname{cosec}^n x\,dx - \int \operatorname{cosec}^{n-2}x\,dx\right]$

$= -\cot x \operatorname{cosec}^{n-2}x - (n-2)\left[I_n - I_{n-2}\right]$

$= -\cot x \operatorname{cosec}^{n-2}x - (n-2) I_n + (n-2) I_{n-2}$

$$nI_n + (n-2)I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}$$

$$nI_n - I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}$$

$$I_n(n-1) = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{-1}{n-1} \cot x \operatorname{cosec}^{n-2} x + \frac{n-2}{n-1} I_{n-2}.$$

7) $\int e^{ax} \cosh x \, dx$

$$I_n = \int e^{ax} \cosh x \, dx$$

$$= \int e^{ax} \, d\left(\frac{\sinh x}{b}\right)$$

$$= e^{ax} \frac{\sinh x}{b} - \int \frac{\sinh x}{b} \, a e^{ax} \, dx$$

$$= \frac{e^{ax} \sinh x}{b} - \frac{a}{b} \int e^{ax} \sinh x \, dx$$

$$= \frac{e^{ax} \sinh x}{b} - \frac{a}{b} \int e^{ax} \, d\left(-\frac{\cosh x}{b}\right)$$

$$= \frac{e^{ax} \sinh x}{b} - \frac{a}{b}\left[-\frac{e^{ax} \cosh x}{b} - \int \left(-\frac{\cosh x}{b}\right) a e^{ax} \, dx\right]$$

$$= \frac{e^{ax} \sin hx}{b} + \frac{a}{b^2} e^{ax} \cosh x - \frac{a^2}{b^2} \int e^{ax} \cosh x \, dx$$

$$= \frac{e^{ax} \sinh x}{b} + \frac{a}{b^2} e^{ax} \cos hx - \frac{a^2}{b^2} I_n$$

$$I_n\left(1 + \frac{a^2}{b^2}\right) = \frac{be^{ax} \sinh x + ae^{ax} \cosh x}{b^2}$$

$$I_n\left(\frac{a^2+b^2}{b^2}\right) = \frac{be^{ax}\left[a \cosh x + b \sin hx\right]}{b^2}$$

$$\therefore I_n = \frac{e^{ax}}{a^2+b^2}\left[a \cosh x + b \sinh x\right]$$

8) $\int e^{ax} \sinh bx \, dx$

consider:

$$I_n = \int e^{ax} \sinh bx \, dx$$

$$= \int e^{ax} d\left(-\frac{\cosh bx}{b}\right)$$

$$= \frac{-e^{ax}\cosh bx}{b} - \int -\frac{\cosh bx}{b} \, a e^{ax} \, dx$$

$$= \frac{-e^{ax}\cosh bx}{b} + \frac{a}{b}\int e^{ax}\cosh bx \, dx$$

$$= \frac{-e^{ax}\cosh bx}{b} + \frac{a}{b}\int e^{ax} d\left(\frac{\sinh bx}{b}\right)$$

$$= \frac{-e^{ax}\cosh bx}{b} + \frac{a}{b}\left[\frac{e^{ax}\sinh bx}{b} - \int \frac{\sinh bx}{b} a e^{ax} dx\right]$$

$$= \frac{-e^{ax}\cosh bx}{b} + \frac{a}{b^2} e^{ax}\sinh bx - \frac{a^2}{b^2}\int e^{ax}\sinh bx \, dx$$

$$= \frac{-e^{ax}\cosh bx}{b} + \frac{a}{b^2} e^{ax}\sinh bx - \frac{a^2}{b^2} I_n$$

$$I_n\left(1+\frac{a^2}{b^2}\right) = \frac{-e^{ax}\cosh bx}{b} + \frac{a e^{ax}\sinh bx}{b^2}$$

$$I_n\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}\left[a \sinh bx - b\cosh bx\right]}{b^2}$$

$$I_n = \frac{e^{ax}}{a^2+b^2}\left[a \sinh bx - b \cosh bx\right]$$

9) $\int \sin^m x \cos^n x \, dx$

Reducing 'n'

$$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \sin^m x \cos^{n-1} x \cos x \, dx$$

$$= \int \sin^m x \cos^{n-1} x \, d(\sin x)$$

$$= \int \cos^{n-1} x \, d\left(\frac{\sin^{m+1} x}{m+1}\right)$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} - \int \frac{\sin^{m+1} x}{m+1} (n-1) \cos^{n-2} x (-\sin x)\, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \,(\overset{\sin^2 x}{\cancel{1-\cos^2 x}})\, dx$$

$$= \frac{\sin^{m+1} \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x (1-\cos^2 x)\, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, dx - \frac{n-1}{m+1} \int \sin^m x \cos^n x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} dx - \frac{n-1}{m+1} I_{m,n}$$

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^n x \cos^{n-2} x \, dx$$

$$I_{m,n}\left(1 + \frac{n-1}{m+1}\right) = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n}\left(\frac{m+1+n-1}{m+1}\right) = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n}\left(\frac{m+n}{m+1}\right) = \frac{\sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2}}{m+1}$$

$$I_{m,n} = \frac{1}{m+n}\left[\sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2}\right]$$

(ii)

Reducing 'm'

$$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \sin^{m-1} x \cos^n x \sin x \, dx$$

$$= \int \sin^{m-1} x \cos^n x \, d(-\cos x)$$

formula. $\int -y^n dy = \frac{-y^{n+1}}{n+1}$

$$= \int \sin^{m-1} x \, d\left(-\frac{\cos^{n+1} x}{n+1} \right)$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \int \frac{\cos^{n+1} x}{n+1} (m-1) \sin^{m-2} x \cos x \, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \cos^n x \sin^{m-2} x \cos^2 x \, dx$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \cos^n x \sin^{m-2} x \, (1 - \sin^2 x) dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \cos^n x \sin^{m-2} - \cos^n x \sin^m x \, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{n,m-2} - \frac{m-1}{n+1} I_{m,n}.$$

$$I_{mn} + \frac{m-1}{m+1} I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n}$$

$$\left(\frac{n+1+m-1}{n+1}\right) I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n}$$

$$I_{m,n} \left(\frac{m+n}{n+1} \right) = \frac{-\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}}{n+1}$$

$$I_{m,n} = +\frac{1}{m+n} \left[-\sin^{m-1} x \cos^{n+1} + (m-1) I_{m-2,n} \right]$$

11) $\cancel{\int x^n \sin mx}\, dx$.

$\int x^m (\log x)^n dx$

$I_n = \int x^m (\log x)^n dx$

$= \int (\log x)^n d\left(\frac{x^{m+1}}{m+1}\right)$

$= (\log x)^m \frac{x^{m+1}}{m+1} - \int \frac{x^{m+1}}{m+1} d((\log x)^n)$

$= \frac{x^{m+1}(\log x)^n}{m+1} - \int \frac{x^{m+1}}{m+1} n(\log x)^{n-1} \frac{1}{x} dx$

$= \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx$

$I_n = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1} I_{m,n-2}$

12) $\int x^n \overset{e^{ax}}{\cancel{\sin mx}}\, dx$

$I_n = \int x^n \overset{e^{ax}}{\cancel{\sin mx}}\, dx$

$= \int x^n d\left(\frac{e^{ax}}{a}\right)$

$= \frac{x^n e^{ax}}{a} - \int \frac{e^{ax}}{a} d(x^n)$

$= \frac{x^n e^{ax}}{a} - \int \frac{e^{ax}}{a} n(x^{n-1}) dx$

$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int e^{ax} x^{n-1} dx$

$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$.

12) $\int x^n \sin mx \, dx$

$$I_n = \int x^n \sin mx \, dx$$

$$= \int x^n d\left(-\frac{\cos mx}{m}\right)$$

$$= -\frac{x^n \cos mx}{m} + \int \frac{\cos mx}{m} d(x^n)$$

$$= -\frac{x^n \cos mx}{m} + \int \frac{\cos mx}{m} n x^{n-1} dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\int \cos mx \, x^{n-1} dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\int x^{n-1} d\left(\frac{\sin mx}{m}\right)$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left[\frac{x^{n-1}\sin mx}{m} - \int \frac{\sin mx}{m} d(x^{n-1})\right]$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left[\frac{x^{n-1}\sin mx}{m} - \int \frac{\sin mx}{m}(n-1) x^{n-2} dx\right]$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left[\frac{x^{n-1}\sin mx}{m} - \frac{(n-1)}{m}\int x^{n-2} \sin mx \, dx\right]$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left[\frac{x^{n-1}\sin mx}{m} - \frac{(n-1)}{m} I_{n-2}\right]$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m^2}\left[x^{n-1}\sin mx - (n-1) I_{n-2}\right]$$

13. $\int x^n \cos mx \, dx$

$$I_n = \int x^n \cos mx \, dx$$

$$= \int x^n d\left(\frac{\sin mx}{m}\right)$$

$$= \frac{x^n \sin mx}{m} - \int \frac{\sin mx}{m} d(x^n)$$

$$= \frac{x^n \sin mx}{m} - \int \frac{\sin mx}{m} n x^{n-1} dx$$

$$= \frac{x^n \sin mx}{m} - \frac{n}{m} \int \sin mx \, x^{n-1} dx$$

$$= \frac{x^n \sin mx}{m} - \frac{n}{m} \int x^{n-1} d\left(\frac{-\cos mx}{m}\right)$$

$$= \frac{x^n \sin mx}{m} - \frac{n}{m} \left[-\frac{x^{n-1} \cos mx}{m} + \int \frac{\cos mx}{m} \frac{d}{dx}(x^{n-1}) dx \right]$$

$$= \frac{x^n \sin mx}{m} - \frac{n}{m} \left[-\frac{x^{n-1} \cos mx}{m} + \int \frac{\cos mx}{m} (n-1) x^{n-2} dx \right]$$

$$= \frac{x^n \sin mx}{m} - \frac{n}{m} \left[-\frac{x^{n-1} \cos mx}{m} + \frac{(n-1)}{m} \int \cos mx \, x^{n-2} dx \right]$$

$$= \frac{x^n \sin mx}{m} - \frac{n}{m} \left[-\frac{x^{n-1} \cos mx}{m} + \frac{(n-1)}{m} I_{n-2} \right]$$

$$I_n = \frac{x^n \sin mx}{m} - \frac{n}{m^2} \left[-x^{n-1} \cos mx + (n-1) I_{n-2} \right]$$

25 November 2021

$$I_n = \int \sin^n x \, dx$$

$$n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$$

corollary

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= -\left[\frac{\sin^{n-1}\pi/2 \cos \pi/2}{n} - \frac{\sin 0 \cos 0}{n}\right] + \frac{n-1}{n}$$

$$= \frac{n-1}{n}\left[\frac{n-2-1}{n-2} I_{n-2-2}\right]$$

$$= \frac{n-1}{n}\left[\frac{n-3}{n-2} I_{n-4}\right]$$

$$= \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \int_0^{\pi/2} \sin x & n \text{ is odd} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \int_0^{\pi/2} dx & \text{if } n \text{ is even.} \end{cases}$$

$$= \int_0^{\pi/2} \sin x = -\cos x \int_0^{\pi/2} = -[\cos \pi/2 \;\cos 0]$$

$$= -[0-1] = 1$$

$$= \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2 - 0 = \pi/2$$

Formula $I_n = \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{2}{3} & \text{If } n \text{ is odd.} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{1}{2}\,\pi/2 & \text{If } n \text{ is even.} \end{cases}$

1) $\int_0^{\pi/2} \sin^7 x\,dx$ $\quad \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{}$ $\quad \frac{n-1}{n}$

n is odd $\quad n = 7$

$$I_7 = \int_0^{\pi/2} \sin^7 x\,dx = \left(\frac{7-1}{7}\right)\left(\frac{7-3}{7-2}\right)\left(\frac{7-5}{7-4}\right)\left(\frac{7-7}{7-6}\right)$$

$$= (6/7)(4/5)(2/3) = 16/35 \,//.$$

2) $\int_0^{\pi/2} \sin^6 x \, dx$

$n = 6$

n is even

$$I_6 = \int_0^{\pi/2} \sin^6 x \, dx = \left(\frac{6-1}{6}\right)\left(\frac{6-3}{6-2}\right)\left(\frac{6-5}{6-4}\right) \pi/2$$

$$= (5/6)(3/4)(1/2)\,\pi/2$$

$$= \frac{5\pi}{32} //.$$

$\sin^m x \cos^{n-1} x \, dx$,

$\sin^n x \, dx$

If $\int \cos^n x \, dx$ Show that

$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$.

$\int \cos^n x \, dx$

$I_n = \int \cos^n x \, dx$

$= \int \cos^{n-1} x \cos x \, dx$

$= I_n \; uv - \int v \, du$

$u = \cos^{n-1} x \quad dv = \cos x \, dx$

$du = (n-1) \cos^{n-2} x \sin x \, dx$

$dv = \cos x \, dx$

$v = \sin x$

$= \cos^{n-1} x \sin x + \int (\sin x)(n-1) \cos^{n-2} x \sin x \, dx$

$= \cos^{n-1} x \sin x (n-1) \int \cos^{n-2} x \sin^2 x \, dx.$

$I_n = \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx$

$= \cos^{n-1} x \sin x + (n-1) \left[\int \cos^{n-2} x \, dx - \cos^2 x \cos^{n-2} x \, dx \right]$

$= \cos^{n-1} x \sin x + (n-1) \left[I_{n-2} - I_n \right]$

$I_n = \cos^{n-1} \sin x + (n-1) I_{n-2} - (n-1) I_n$

$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} = I_n = \frac{\cos^{n-1} x \, \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$

$\int_0^{\pi/2} \cos^n x \, dx$

$= \left[\cos^{n-1} x \, \sin x \right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} dx.$

$= \frac{(n-1)}{n} \int_0^{\pi/2} \cos n-2 \, dx$

$$= \frac{(n-1)}{n} \frac{n-2-1}{n-2} \int_0^{\pi/2} \cos^{n-4} dx$$

$$= \frac{(n-1)}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos x\,dx \text{ if n is odd}$$

$$\int_0^{\pi/2} 2\,dx \text{ if n is even.}$$

Corallary

$$\int_0^{\pi/2} \cos^n x\,dx$$

$$I_n = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}, \frac{n-3}{n-2}, \frac{n-5}{n-4} \cdots I_0 \text{; where } I_0 = \pi/2 \text{ n is even} \\ \frac{n-1}{n}, \frac{n-3}{n-2}, \frac{n-5}{n-4} \cdots I_1 \text{, where } I_1 = 1 \text{ n is odd.} \end{cases}$$

Qm

3) $\int_0^{\pi/2} \cos^8 x\,dx$

$n = 8$ (even)

$$I_n = \frac{n-1}{n}, \frac{n-3}{n-2}, \frac{n-5}{n-4} \cdots \pi/2$$

$$= \frac{8-1}{8}, \frac{8-3}{8-2}, \frac{8-5}{8-4} \frac{8-7}{8-6} \cdots \pi/2$$

$$= 7/8, 5/6 \cdot 3/4 \cdot 1/2 \cdot \pi/2$$

$$I_8 = \frac{35\pi}{256}$$

4) $\int_0^{\pi/2} \cos^5 x\,dx$ $(n=5) \Rightarrow$ odd

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2}, \frac{n-5}{n-4} \cdots 1$$

$$= \frac{5-1}{5} \cdot \frac{5-3}{5-2} \cdots 1 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$I_5 = 8/15$$

5 $\int \sin^6 x \cos^3 x \, dx$

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx$$
$$= \int \sin^6 x (1 - \sin^2 x) \, d(\sin x)$$
$$= \int (\sin^6 x - \sin^5 x) \, d(\sin x)$$
$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + c$$

6 $\int \sin^9 x \cos^5 x \, dx$

$$\int \sin^9 x \cos^5 x \, dx = \int \sin^9 x \cos^4 x \cos x \, dx$$
$$= \int \sin^9 x (1 - \sin^2 x)^2 \, d(\sin x)$$
$(a-b)^2$

$(\cos^2)$
$$= \int (\sin^9 x (1 + \sin^4 x - 2\sin^2 x)) \, d \sin x$$
$$= \int (\sin^9 x + \sin^{13} x - 2\sin^2 x) \, d \sin x$$

(doubt)
$$= \frac{\sin^9 x}{9} + \frac{\sin^{14} x}{14} - \frac{\sin^{12} x}{6} + c$$

7) $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$

x

10mark
$$\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \int_0^{\pi/2} \sin^6 x \cos^4 x \cos x \, dx$$
$$= \int_0^{\pi/2} \sin^6 x (1 - \sin^2 x)^2 \, d(\sin x)$$
$$= \int_0^{\pi/2} (\sin^6 x (1 + \sin^4 x - 2\sin^2 x)) \, d(\sin x)$$
$$= \int_0^{\pi/2} \sin^6 x + \sin^{10} x - 2\sin^8 x) \, d(\sin x)$$
$$= \left[\frac{\sin^7 x}{7} + \frac{\sin^{11} x}{11} - \frac{2\sin^9 x}{9} \right]_0^{\pi/2} + c$$
$$= \frac{1}{7} + \frac{1}{11} - \frac{2}{9} = \frac{1}{7} - \frac{2}{9} + \frac{1}{11}$$
$$= \frac{9-14}{63} + \frac{1}{11} = \frac{-5}{63} + \frac{1}{11} = \frac{-55+63}{693}$$
$$= \frac{8}{693}$$

ex : 3 $\int \sin^5 x \, dx$

Put $y = \cos x$;

$dy = -\sin x \, dx$

$$\int \sin^5 x \, dx = -\int \sin^4 x \, dy = -\int (1-y^2)^2 dy$$

$$= -\int (1 - 2y^2 + y^4) dy$$

$$= -y + \frac{2y^3}{3} - \frac{y^5}{5} = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}$$

Exam 4 Evaluate $\int_0^1 x(1-x^2)^{1/2} dx$

x	0	1
θ	0	$\pi/2$

$\xrightarrow{\sin x}$

Put $x = \sin\theta$; $dx = \cos\theta \, d\theta$

when $x = 0$, $\theta = 0$, and $x = 1$, $\theta = \pi/2$

The integral becomes

$$\int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta = \int \cos^2\theta \, d(-\cos\theta)$$

$$= \left[-\frac{\cos^3\theta}{3} \right]_0^{\pi/2} = 1/3$$

$\int \cos^7 x \, dx$

$$= \int \cos^6 x \cos x \, dx = \int (1-y^2)^3 dy$$

$$= \int (1 - 3y^2 + 3y^4 - y^6) dy$$

$$= y - y^3 + \frac{3y^5}{5} - \frac{y^7}{7}$$

$$= \sin x - \sin^3 x + \frac{3\sin^5 x}{5} - \frac{\sin^7 x}{7}$$

$\int_0^{\pi/2} \cos^8 x \, dx$

$\int_0^{\pi/2} \sin^6 x \, dx$

Formula

$$I_{m,1} = \int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}$$

$$I_{1,n} = \int \sin x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n+1}$$

For examples.

1) $\int \sin^6 x \cos^3 x \, dx$

Put $y = \sin x$,

$dy = \cos x \, dx$

$$\int \sin^6 x \cos^3 x \, dx = \int y^6 (1-y^2)\, dy = \frac{y^7}{7} - \frac{y^9}{9}$$

$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9}.$$

2) $\int \sin^9 x \cos^5 x \, dx$

$\sin x = y$

$\cos x \, dx = dy$

$$\int \sin^9 x \cos^5 x \, dx = \int y^9 (1 - 2y^2 + y^4)\, dy$$

$$= \frac{y^{10}}{10} - \frac{y^{12}}{12} + \frac{y^{14}}{14}$$

$$= \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{6} + \frac{\sin^{14} x}{14}.$$

Case (ii)

$$I_{m,0} = \int \sin^m x \, dx \quad \left[\begin{matrix} (n = + \text{ inte}) \\ n < m \end{matrix}\right]$$

corollary

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx \quad (m, n \text{ being } + \text{ integer})$$

$$= \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n} \cdot \int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx$$

$$= \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx \quad \text{as the first term.}$$

(vanishes all limits)

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \int_0^{\pi/2} \sin^m x \cos^{n-4} x \, dx$$

$$\frac{n-1}{m+n}\;\frac{n-3}{m+n-2}\;\frac{n-5}{m+n-4}\;\dots\dots\; I_{m,1} \text{ or } I_{m,0}$$

according as n is odd or even.

i) If n is odd, $I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x \, dx$

$$= \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}$$

when n is odd,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\,\dots\dots\frac{2}{m+3}\,\frac{1}{m+1}$$

ii) If n is ~~odd~~ even,

$$I_{m,0} = \int_0^{\pi/2} \sin^m x \, dx = \frac{m-1}{m}\cdot\frac{m-3}{n-2}\dots\frac{1}{2}\,\frac{\pi}{2}$$

when m is even

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \frac{n-1}{m+n}\;\frac{n-3}{m+n-2}\;\dots\;\frac{1}{m+1}\;\frac{m-1}{m}\;\frac{m-3}{n-2}\;\dots\frac{1}{2}\,\frac{\pi}{2}$$

Ex :1

$$\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$$

$$= \frac{4}{11}\cdot\frac{2}{9}\cdot\frac{1}{7}$$

$$= \frac{18}{693}.$$

$$\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$$

$$= \frac{5}{10}\cdot\frac{3}{8}\cdot\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi}{512}$$

Eg:1 $\int \tan^4 x\, dx$

$$= \frac{\tan^3 x}{3} - \int \tan^2 x\, dx \quad \text{Put } n=4$$

$$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\, dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x$$

Eg:2 $\int_0^{\pi/4} \tan^3 x\, dx$

$$= \left[\frac{\tan^2 x}{2}\right]_0^{\pi/2} - \int_0^{\pi/4} \tan x\, dx \quad \text{put } n=3$$

$$= \tfrac{1}{2} + \left[\log \cos x\right]_0^{\pi/4} = \tfrac{1}{2} + \log \frac{1}{\sqrt{2}}$$

$$= \tfrac{1}{2}(1 - \log 2)$$

3. $\int \sec^3 x\, dx$

$$= \int \sec x\, d(\tan x)$$

$$= \sec x \tan x - \int \tan^2 x \sec x\, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx$$

$$= \sec x \tan x - I + \log(\sec x + \tan x)$$

$$\therefore 2I = \sec x \tan x + \log(\sec x + \tan x)$$

$\int \sec^6 x\, dx$

$$= \int \sec^4 x\, d(\tan x) = \int (1+t^2)^2\, dt \quad [t = \tan x]$$

$$= \int (1 + 2t^2 + t^4)\, dt = t + \frac{2t^3}{3} + \frac{t^5}{5}$$

$$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5}$$

$\int cosec^4 x\,dx$

$= -\int cosec^2 x\, d(\cot x)$ where $y = \cot x$

$= -\int (1+y^2)\,dy$

$= -y - \frac{y^3}{3} = -\cot x - \frac{\cot^3 x}{3}$

$\int cosec^5 x\,dx$

$n = 5$

$\int cosec^5 x\,dx = \frac{-cosec^3 x \cot x}{4} + \frac{3}{4}\int cosec^3 x\,dx$

$= \frac{-cosec^3 x \cot x}{4} - 3/8\ cosec\ x \cot x$

$-3/8 \log(cosec x + \cot x)$

$I_{m,n} = \int x^m (\log x)^n dx$ n is +ve

otherwise evaluate $\int x^4 (\log x)^3 dx$

$I_{m,n} = \int (\log x)^n d\left(\frac{x^{m+1}}{m+1}\right)$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$

The ultimate integral is $I_{m,0} = \int x^m dx = \frac{x^{m+1}}{m+1}$.

$\int (\log x)^3 x^4 dx = \int (\log x)^3 d\left(\frac{x^5}{5}\right)$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}(\log x)^2 x^4 dx$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int (\log x)^2 d\left(\frac{x^5}{5}\right)$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\int x^4(\log x)\,dx$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\left\{\frac{x^5}{5}\log x - \frac{x^5}{25}\right\}$$

$$= x^5\left\{\frac{1}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 + \frac{6}{125}\log x - \frac{6}{625}\right\}$$

10 mark

If $\int_0^{\pi/2} \cos^m x \cos nx\,dx = f(m,n)$, prove that $f(m,n) = \frac{m}{m+n} f(m-1, n-1)$. Hence prove that $f(n,m) = \frac{\pi}{2^{n+1}}$

$$f(m,n) = \int \cos^m x \cos nx\,dx$$

$$= \int \cos^m x\, d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin x \sin nx\,dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\left[\cos(n-1)x - \cos nx \cos x\right]dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx - \frac{m}{n}\int \cos^m x \cos nx\,dx$$

$$f(m,n) = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx - \frac{m}{n} f(m,n)$$

$$\left(1 + \frac{m}{n}\right) f(m,n) = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx$$

$$\left(\frac{m+n}{n}\right) f(m,n) = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx$$

Hence $f(m,n) = \int_0^{\pi/2} \cos^m x \cos nx\,dx$

$$f(m,n) = \frac{1}{m+n}\left\{\left(\cos^m x \sin nx\right)_0^{\pi/2} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x\,dx\right.$$

$$= \frac{m}{m+n} f(m-1, n-1)$$

Putting $m = n$

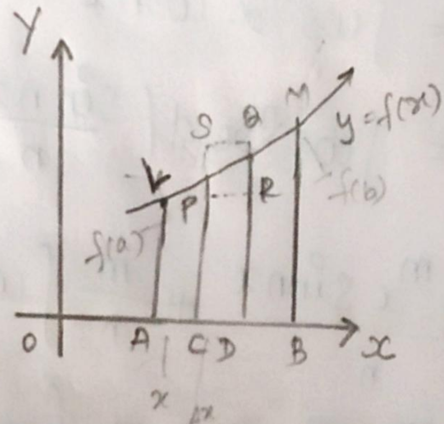
$$f(n,n) = \frac{1}{2} f(n-1, n-1) = \frac{1}{2^2} f(n-2, n-2)$$

$= \frac{1}{2^n} f(0,0)$ by repeated application of the same fomula

$$= \frac{1}{2^n} \int_0^{\pi/2} dx = \frac{\pi}{2^{n+1}}$$

Geomentrical Application of integration.

Unit - 3



Rectangle $CPRD = CP \cdot CD = y \cdot \Delta x$

rectangle $CSQD = DQ \cdot CD = (y + \Delta y)\Delta x$

$y = f(x)$ $\quad LPQM$

$OA = a$ $\quad OB = b$

$AL = f(a)$ $\quad BM = f(b)$

A = Area bounded by arc LP, ordinates AL, CP & the portion AC of x axis x

Area $ALQD = A + \Delta A$

CP

Area $CPQD >$ Area $CPRD$

$\Delta A > y\Delta x$

Area $CPQD <$ ~~$y\Delta x$~~ Area $CSQD$

$\Delta A < (y+\Delta y)\Delta x$ ← $= y\Delta x < \Delta A < (y+\Delta y)\Delta x$

$y < \frac{\Delta A}{\Delta x} < y + \Delta y$

Apply the limit when $\Delta x \to 0$

$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$ $\quad \Delta x \to 0$, $\Delta y \to 0$ $\quad \theta y + \Delta y \to y$

$y < \frac{dA}{dx} < y$

$\Rightarrow \frac{dA}{dx} = y.$

Apply inte

$A = \int y\,dx + c$

$A = \int f(x)\,dx + c$

let us denote $\int f(x)\,dx = F(x)$

$A = F(x) + c$

when $x = a$, $A = 0$ As $A = ALPC$

$= 0 = F(a) + c \longrightarrow$ (1) by definition

when $x = b$, ~~$A =$~~

$A =$ Area $ALMP$ by definition

$\therefore ALMP = F(b) + c$

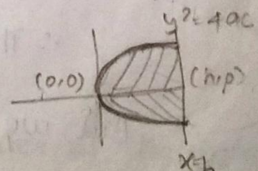
$= F(b) + c - (F(a) + c)$

$= F(b) - F(a)$

$= [f(x)]_a^b = \int_a^b f(x)\,dx.$

1) find the area bounded by the curve $y^2 = 4ax$, the x axis and the ordinate $x = h$.

2mrk repeated



The curve is the parabola which, we know passes through the orgin.

The limits for the area are 0 and h

$\sqrt{4ax}$

$y^2 = 4ax$ $\quad y = f(x)$ $\quad \int_0^h \sqrt{4ax}\,dx$

$y = 2\sqrt{ax}$ $\quad = \sqrt{4a}\int_0^h \sqrt{x}\,dx$

$$2\int_0^h 2\sqrt{ax}\,dx = \overset{2}{\cancel{4}}\sqrt{a}\int_0^h x^{1/2}\,dx$$ $\boxed{h^{3/2} = h^1\sqrt{h}}$

$$= \overset{2}{\cancel{4}}\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^h$$

$\frac{1+1/2}{\frac{2+1}{2}} = 3/2$

$$= \frac{\overset{2}{\cancel{4}}\sqrt{a}\,(2)}{3}\,(h^{3/2} - 0)$$

$\frac{8\sqrt{a}\,h^3\sqrt{h}}{3}$

$$= \frac{\overset{4}{\cancel{8}}\sqrt{a}}{3}\,h^{3/2}$$

$h \cdot h^{1/2}$ $\sqrt{h}$

$$= \frac{4h\sqrt{a}\sqrt{h}}{3}$$

Required area $= 2\int_0^h 2\sqrt{ax}\,dx = \frac{8h\sqrt{ah}}{3}$ sq. units

2) Find the area bounded by one arch of the curve $y = \sin ax$ and the x axis

2m repeated

To find area between one arch & axis

axes $\Rightarrow y = 0$ $\quad y = \sin ax$

$\Rightarrow \sin ax = 0$

$ax = \sin^{-1}(0)$

$ax = 0, n\pi$

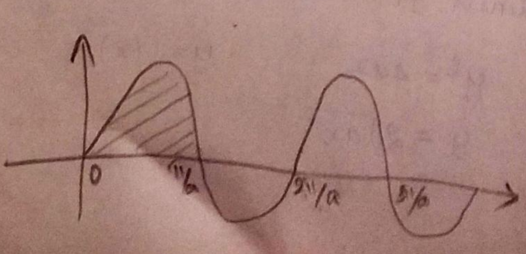
$x = 0, \frac{n\pi}{a}$

$\therefore$ The curve cuts the x axis at 0 and $n\pi/a$

But we have to find the area bounded by one curve

req limi $x = 0, n\pi/a$ $\therefore n = 1$

$\therefore$ limit of $x = 0, \pi/a$



Required Area

$$= \int_0^{\pi/a} y\,dx = \int_0^{\pi/a} \sin ax\,dx = \left[\frac{-\cos ax}{+a}\right]_0^{\pi/a}$$

$$= \frac{1}{a}\left[-\cos a(\pi/a) - (-\cos a(0))\right]$$

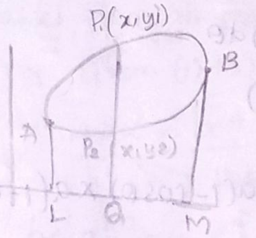
$\cos\pi = -1$
$\cos 0 = 1$

$$= 1/a\,[-(-1) - (-1)]$$

$$= 1/a\,(2)$$

$$= 2/a \text{ sq. units.}$$

04/12/2021 Area of a closed curve.



let AL, BM be the tangents to the closed curve Parallel to y axis.

Let an intermediate ordinate meet the curve in two points P_1, P_2 Let $y_1 > y_2$.

let $OL = a$ $OM = b$

Area of $LAP_1BM = \int_b^a y_1\,dx$

Area of $LAP_2BM = \int_b^a y_2\,dx$

By Subtraction we get the area of the closed curve be $\boxed{\int_a^b (y_1 - y_2)\,dx}$ closed circle

① Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ and its base.

5 mark
repeated

Area bounded by one arch. As a point P describe one arch the parameter θ varies from 0 to 2π

$$\text{Required Area} = \int_0^{2\pi} y\,dx$$

$$= \int_0^{2\pi} y \frac{dx}{d\theta}\, d\theta$$

$$x = a(\theta - \sin\theta)$$

$$dx = a(1-\cos\theta)\,d\theta \Rightarrow \frac{dx}{d\theta} = a(1-\cos\theta)$$

$$y = a(1-\cos\theta)$$

diff sin = cos

$\int \sin = -\cos$

$$\text{Required Area} = \int_0^{2\pi} a(1-\cos\theta) \times a(1-\cos\theta)\,d\theta$$

$$= a^2 \int_0^{2\pi} (1-\cos\theta)^2\, d\theta$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$= a^2 \int_0^{2\pi} (1^2 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= a^2 \int_0^{2\pi} \left(\frac{2 - 4\cos\theta + 1 + \cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2} \int_0^{2\pi} (3 - 4\cos\theta + \cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\Big[[6\pi - 4(0) + 0] - [0]\Big]$$

$$= \frac{a^2}{2}(6\pi)$$

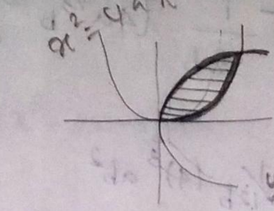
$$= 3\pi a^2 \text{ sq. units.}$$

$\sin \pi/2 = 1$
$\sin \pi = 0$
$\cos \pi = -1$

$\sin \pi = 0$
$\cos \pi = -1$
$\cos 0 = 1$

(2) Find the area lies between the parabolas $y^2 = 4ax$, $x^2 = 4by$.

(X) 10 mark imp



$$y^2 = 4ax \longrightarrow (1)$$

$$x^2 = 4by \longrightarrow (2)$$

$$y = \frac{x^2}{4b}$$

$$y^2 = \frac{x^4}{16b^2} \longrightarrow (3)$$

are passes through origin point of intersection.

(from (1), (3))

$$4ax = \frac{x^4}{16b^2}$$

$$64ab^2x = x^4$$

$$x^4 - 64ab^2x = 0 \Rightarrow x(x^3 - 64ab^2) = 0$$

$$x = 0 \text{ (or) } x^3 - 64ab^2 = 0$$

$$\Rightarrow x^3 = 64ab^2$$

$$x = (64)^{1/3} a^{1/3} b^{2/3}$$

$$x = 4a^{1/3}b^{2/3}$$

limits are $x = 0$ & $x = 4a^{1/3}b^{2/3}$

Required area

$$= \int_0^{4a^{1/3}b^{2/3}} (y_1 - y_2)\,dx$$

$y^2 = 4am$

$y = 2\sqrt{a}\sqrt{m}$

where $y_1 = 2\sqrt{a}\sqrt{x}$ & $y_2 = \frac{x^2}{4b}$

$x^2 = 4by$

$x^2/4b = y$

$$\therefore \text{Required Area} = \int_0^{4a^{1/3}b^{2/3}} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right) dx$$

$2\sqrt{a}\int\left(\sqrt{x} - \frac{x^2}{4b}\right)dx$

$$= \left[2a^{1/2}\frac{x^{3/2}}{3/2} - \frac{1}{4b}\frac{x^3}{3}\right]_0^{4a^{1/3}b^{2/3}}$$

$\frac{x^{n+1}}{n}$

$$= \left(\frac{4}{3} a^{1/2} \frac{(4a^{1/3} b^{2/3})^{3/2}}{4 \times 4^{1/2}} - \frac{1}{12b} (4a^{1/3} b^{2/3})^3 \right)$$

$$= \frac{4}{3} a^{1/2} 4^{3/2} a^{1/2} b - \frac{1}{12b} (4)^3 ab^2$$

S=12
10m=17

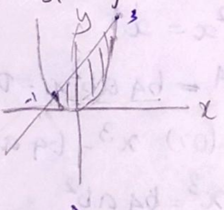
$$= \frac{4a}{3} (4)^1 (4)^{1/2} b - \frac{1}{12} 64\, ab$$

$$= \frac{4a(4)(2)b}{3} - \frac{1}{3}(16ab)$$

$$= \frac{32ab - 16ab}{3} = \frac{16ab}{3} \text{ sq. units}$$

(3) find the area enclosed between the parabola $y = x^2$ and the straight line $2x - y + 3 = 0$

10 mark
Repeated



$$y = x^2 \longrightarrow (1)$$

$$2x - y + 3 = 0 \Rightarrow y = 2x + 3 \longrightarrow (2)$$

$$x^2 = 2x + 3$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3, \text{ or } x = -1$$

intersection points are $x = 3$, $x = -1$

limits are $x = 3$, $x = -1$

$= \cancel{(-27 - (2 + 1/3))}$ $y_1 = x^2$

$y_2 = 2x + 3$

The

Area = $\int_{-1} (y_1 - y_2)\,dx$

$= \int_{-1}^{3} (x^2 - 2x - 3)\,dx$

$= \left[\frac{x^3}{3} - \frac{2x^2}{2} - 3x\right]_{-1}^{3}$

$= \left[\frac{27}{3} - 9 - 9\right] - \left[-\frac{1}{3} + 1 - 3\right]$

$= (-9) + \left[\frac{1}{3} - 2\right]$

$= -11 + \frac{1}{3}$

$\Rightarrow -\frac{32}{3} \Rightarrow \frac{32}{3}$ //

Required Area bounded by the coordinates -1 & 3

$A = \int_{-1}^{3} (y_1 - y_2)\,dx = \int_{-1}^{3} ((2x+3) - x^2)\,dx$

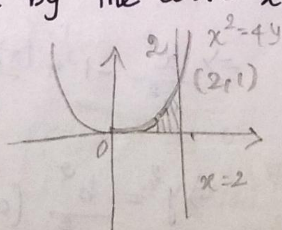
$= \left[\frac{2x^2}{2} + 3x - \frac{x^3}{3}\right]_{-1}^{3}$

$= \left[\left(3^2 + 3(3) - \frac{(3^3)}{3}\right) - \left((-1)^2 + 3(-1) - \frac{(-1)^3}{3}\right)\right]$

$= \left[9 + \frac{5}{3}\right]$

$= \frac{32}{3}$ Sq. units

(*) find the Area bounded by the curve $x^2 = 4y$, x-axis, $x = 2$



$y = \frac{1}{4}x^2$

$y = 0$

$x^2 = 4y$ $y = x^2/4$

$x = 2$

x^2 $2^2 = 4y \Rightarrow 4 = 4y$

$y = 1$

∴ (2,1) is the point of intersection.

Area enclosed by curve $x^2 = 4y$, x axis, $x = 2$

Required Area $= \int_{0}^{2} y\,dx$

$$= \int_0^2 \frac{x^2}{4}\,dx = \int_0^2 \frac{1}{4}x^2\,dx$$

$$= \frac{1}{4}\int_0^2 x^2\,dx$$

$$= \frac{1}{4}\left[x^3/3\right]_0^2$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

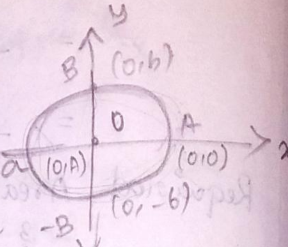
$$= \frac{1}{4}\left[\left(\frac{2^3}{3} - \frac{0}{3}\right)\right]$$

$$= \frac{1}{4}\left(8/3\right)$$

Required Area = $2/3$ sq. units.

10 mrk

2) find Area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Since the given ellipse is Symmentrical about both x-axis and y-axis

∴ required Area enclosed by ellipse = 4 × Area of OAB

Now,

$$\text{Area of } OAB = \int_0^a y\,dx$$

Now Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \underbrace{\frac{x^2}{a^2} - 1}_{LCM} = \frac{-y^2}{b^2} = 1 \qquad \frac{y^2}{b^2} = \frac{a^2-x^2}{a^2} = 1$$

$$\Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$= y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Since OAB lies in first quadranant.

$$\therefore y = \frac{b}{a}\sqrt{a^2 - x^2}$$

required area Now, Area of OAB = $\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$

$$\boxed{\text{Since, } \int \sqrt{a^2-x^2}\,dx = \left[\frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]_0^a}$$

$\Rightarrow$ Area of OAB $= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= \frac{b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2} + a^2\sin^{-1}(a/a)\right] - \left[0 + a^2\sin^{-1}(0)\right]$

$= \frac{b}{a}\left[\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}(1)\right]$

$\sin^{-1}0 = \pi/2$

$\sin\pi/2 = 1$

$= \frac{b}{a}\times\frac{a^2}{2}\sin^{-1}(1) = \frac{ab}{2}\times\frac{\pi}{2} = \frac{ab\pi}{4}$

required area of ellipse.

ans $\Rightarrow$ Area of OAB $= \frac{\pi ab}{4}$

ans $\Rightarrow$ Area enclosed by ellipse $= 4\times\frac{\pi ab}{4} = \pi ab$ sq.units

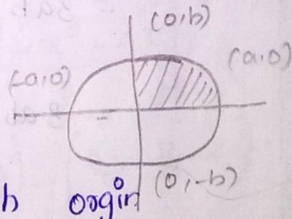
~~Hence option (a) is correct.~~

3) find the whole area of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$

5mark

$(x/a)^{2/3} + (y/b)^{2/3} = 1$

$x = 0,\ y \neq 0$



$\therefore$ Curve is not passes through origin

Put $x = -x$ $\quad y = -y$

$\left(\frac{-x}{a}\right)^{2/3} + (-y/b)^{2/3} = 1 = (x/a)^{2/3} + (y/b)^{2/3}$

curve is symmetric

$x = 0$

$\Rightarrow (0) + (y/b)^{2/3} = 1$

$y^{2/3} = b^{2/3}$

$y^2 = b^2$

$y = \pm b$

Points $(0,b),\ (0-b)$

$y = 0$

$\Rightarrow (x/a)^{2/3} + 0 = 1$

$x^{2/3} = a^{2/3}$

$x^2 = a^2$

$x = \pm a$

$(a,0),\ (-a,0)$

$(x/a)^{2/3} + (y/b)^{2/3} = 1$

put $x = a\cos^3\theta$, $y = b\sin^3\theta$

$dx = a(3\cos^2\theta(-\sin\theta))\,d\theta$

$= (-3a\cos^2\theta\sin\theta)\,d\theta$

x	0	a
θ	$\pi/2$	0

If $x = 0$ $a\cos^3\theta = 0$

$\cos\theta = 0$

$\theta = \pi/2$

If $x = a$

$a\cos^3\theta = a$

$\cos^2\theta = 1$

$\cos\theta = 1$

$\theta = 0$

$\int_0^a y\,dx$

$= \int_{\pi/2}^{0} (b\sin^3\theta)(-3a\cos^2\theta\sin\theta)\,d\theta$ formula padi $(-) = (+)$

$= 3ab\int_0^{\pi/2} (\sin^4\theta\cos^2\theta)\,d\theta$

$= 3ab\int_0^{\pi/2} \sin^4\theta(1 - \sin^2\theta)\,d\theta$

$= 3ab\left(\int_0^{\pi/2} \sin^4\theta\,d\theta - \int_0^{\pi/2} \sin^6\theta\,d\theta\right)$

$= 3ab\left[\left(\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\pi/2\right) - \left(\frac{6-1}{6}\cdot\frac{6-3}{6-2}\cdot\frac{6-5}{6-4}\cdot\pi/2\right)\right]$

$= \cancel{3ab\left[(3/4\cdot 1/2\cdot\pi/2)\cdot\pi/4\right]}$

$= \cancel{3ab\left(\frac{3\pi}{16}\cdot\frac{\pi}{4}\right) = 3ab\,5\pi/16}$

$= 3ab\left[\left(\frac{3}{4}\cdot 1/2\cdot\pi/2\right) - \left(5/6\cdot 3/4\cdot 1/2\cdot\pi/2\right)\right]$

$= 3ab\left[\frac{3\pi}{16} - \frac{5\pi}{32}\right]$

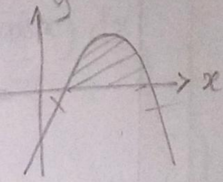
$= 3ab\left(\frac{6\pi - 5\pi}{32}\right) = \frac{3ab\pi}{32}$

Required Area $= 4 \times \int_0^a y\,dx$

$= 4\left(\frac{3ab\pi}{32}\right)$

$= \frac{3ab\pi}{8}$ sq. units

4) find the erea enclosed between.



$y = 5x - x^2 - 4$ & x axis

$y = -(x^2 - 5x + 4)$

$= -(x-4)(x-1)$

$= (x-4)(1-x)$

$y=0$ at $x=4$, $x=1$, limits are $(1,4)$

Required Area $= \int_1^4 y\,dx$

$= \int_1^4 (5x - x^2 - 4)\,dx$

$= \left[\frac{5x^2}{2} - \frac{x^3}{3} - 4x\right]_1^4$

$= \left(\frac{5(16)}{2} - \frac{(4)^3}{3} - 4(4)\right) - \left(5/2 - 1/3 - 4\right)$

$= \left(40 - \frac{64}{3} - 16\right) - \left(\frac{15-2-24}{6}\right)$

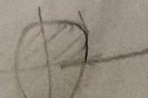
$= \left(\frac{120 - 64 - 16}{3}\right) - \left(\frac{15-26}{6}\right)$

$= 8/3 + 11/6 = \frac{16+11}{6} = 27/6 = 9/2$

polar →

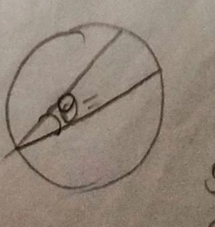
5) find the area of the circle $r = 2a\cos\theta$ lies between 0 to $\pi/2$

2mark



$= 1/2 \int_\alpha r^2 d\theta$

$= 2 \int_0^{\pi/2} 1/2\, r^2 d\theta$



$\sin\pi = 0$

$\cos\pi = 1$

$r = 2a\cos\theta$ $\;= 2\int_0^{\pi/2} \frac{1}{2} 4a^2 \cos^2\theta \, d\theta = \frac{2}{2} 4a \int_0^{\pi/2} \cos^2\theta \, d\theta$

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$= \frac{4a^2}{2} \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= 2a^2 \int_0^{\pi/2} (1+\cos 2\theta) \, d\theta$$

$\int \cos 2\theta = \frac{\sin 2\theta}{2}$ $\quad 2a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$

$$= 2a^2 \left[\left[\pi/2 + \frac{\sin 2(\pi/2)}{2} \right] - 0 \right]$$

$\sin \pi/2 = 0$

$$= 2a^2 (\pi/2 + 0)$$

$$= 2a^2 (\pi/2)$$

$$= \pi a^2 \text{ sq. units //.}$$

If $\int_0^{\pi/2} \cos^m x \cos nx \, dx = f(m,n)$ prove that $f(m,n)$

$$= \frac{m}{m+n} f(m-1, n-1)$$

unit-2

$$f(m,n) = \int_0^{\pi/2} \cos^m x \cos nx \, dx$$

$$= \left. \frac{\cos^m x \sin nx}{n} \right|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin nx}{n} (m\cos^{m-1} x)(-\sin x) \, dx$$

$u = \cos^m x$

$du = m\cos^{m-1} (-\sin x \, dx)$

$dv = \cos nx \, dx$

$v = \frac{\sin nx}{n}$

$$= \left(\frac{\cos^m \pi/2 \, \sin n\pi/2}{n} - \frac{\cos^m (0) \sin n(0)}{n} \right) + \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \, \sin nx \, \sin x \, dx$$

$$= 0 + \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \left(\cos(nx - x) + \cos nx \cos x \right) dx$$

$$= \frac{m}{n} \left[\int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x - \int_0^{\pi/2} \cos^{m-1} x \cos nx \cos x \, dx \right]$$

$$= \frac{m}{n} \left[f(m-1, n-1) - \int_0^{\pi/2} \cos^m x \cos nx \, dx \right]$$

$$f(m,n) = \frac{m}{n} f(m-1, n-1) - \frac{m}{n} f(m,n)$$

$$\left(1 + \frac{m}{n}\right) f(m,n) = \frac{m}{n} f(m-1, n-1)$$

$$\left(\frac{m+n}{n}\right) f(m,n) = \frac{m}{n} f(m-1, n-1)$$

$$f(m,n) = \frac{m}{n} \times \left(\frac{n}{m+n}\right) f(m-1, n-1)$$

$$\Rightarrow f(m,n) = \frac{m}{m+n} f(m-1, n-1)$$

eg. Find the area of the ellipse $x^2 + 4y^2 - 6x + 8y + 9 = 0$

Repeated 5mark

writing this as a quadratic in y.

$$\underset{a}{4y^2} + \underset{b}{8y} + \underset{c}{(x^2 - 6x + 9)} = 0$$

If y_1 and y_2 be the roots

$-b/a$ $\quad y_1 + y_2 = -8/4 = -2$ $\qquad y_1 y_2 = \dfrac{x^2 - 6x + 9}{4}$ $\quad c/a$

Sum of roots $\qquad$ p. of roots

$$(y_1 + y_2)^2 = y_1^2 + y_2^2 + 2y_1 y_2$$

$$(y_1 - y_2)^2 = y_1^2 + y_2^2 - 2y_1 y_2 \qquad \int^b (y_1 - y_2)\, dx.$$

$$(y_1 - y_2)^2 - (y_1 + y_2)^2 = -4y_1 y_2$$

$$(y_1 - y_2)^2 = (y_1 + y_2)^2 - 4y_1 y_2$$

$$\Rightarrow [(y_1 - y_2)] = \sqrt{(y_1 + y_2)^2 - 4y_1 y_2} \qquad \sqrt{-(x-5)(x-1)}$$

$$= \sqrt{4 - (x^2 - 6x + 9)} = \sqrt{6x - x^2 - 5} = \sqrt{(1-x)(x-5)}$$

$y_1 - y_2 = 0$ when $x = 1$, and $x = 5$

$\therefore$ limits are $x = 1, x = 5$

Put $a = x$ $\quad \sin^2\theta + b\cos^2\theta$

$$= \sin^2\theta + 5\cos^2\theta$$

$$dx = (2\sin\theta\cos\theta - 10\cos\theta\sin\theta)\, d\theta$$

$$= -8\sin\theta\cos\theta\, d\theta$$

$$x = \sin^2\theta + 5\cos^2\theta \Rightarrow 1 - \cos^2\theta + 5\cos^2\theta$$

$$x = 1 + 4\cos^2\theta$$

x	1	5
θ	$\cos^{-1}\left(\sqrt{\frac{1-1}{4}}\right) = \cos^{-1}(0) = \pi/2$	$\cos^{-1}\left(\sqrt{\frac{5-1}{4}}\right) = \cos^{-1}(1) = 0$

$$\Rightarrow \frac{x-1}{4} = \cos^2\theta$$

$$\cos\theta = \sqrt{\frac{x-1}{4}}$$

$$\theta = \cos^{-1}\left(\sqrt{\frac{x-1}{4}}\right)$$

$$1 - x = \sin^2\theta + \cos^2\theta - \sin^2\theta - 5\cos^2\theta$$

$$= -4\cos^2\theta$$

$$x-5 = \sin^2\theta + 5\cos^2\theta - 5\sin^2\theta - 5\cos^2\theta$$
$$= -4\sin^2\theta$$

$\therefore$ Area of the ellipse $= \int_1^5 (y_1 - y_2)\,dx$

$$= \int_{\pi/2}^{0} \sqrt{(1-x)(x-5)}\,dx$$

$$= -\int_0^{\pi/2} \sqrt{(-4\cos^2\theta)(-4\sin^2\theta)}\,(-8\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} 4\cos\theta\sin\theta\,(8\sin\theta\cos\theta)\,d\theta$$

$$= 32\int_\theta^{\pi/2} \sin\theta\cos\theta \cdot \sin\theta\cos\theta\,d\theta.$$

$$= 32\int_0^{\pi/2} \sin^2\theta\cos^2\theta$$

$$= 32\int_0^{\pi/2} \sin^2\theta(1-\sin^2\theta)\,d\theta$$

$$= 32\left[\int_0^{\pi/2} \sin^2\theta\,d\theta - \int_0^{\pi/2} \sin^4\theta\right]d\theta$$

$$= 32\left[\left(\frac{2-1}{2}\,\frac{\pi}{2}\right) - \left(\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right)\right]$$

$$= 32\left[\frac{\pi}{4} - \left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)\right] = \cancel{8\pi}$$

$$= 32\left[\frac{\pi}{4} - \frac{3\pi}{16}\right]$$

$$= \overset{2}{\cancel{32}}\left[\frac{4\pi - 3\pi}{\cancel{16}}\right]$$

$$= 2\pi \text{ sq. units.}$$

Find the area of loop of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$

$$y^2 = x^2\left(\frac{a+x}{a-x}\right)$$

To find Area of loop

If x is $> a$ y is imaginary

$\therefore$ The curve does not exist

If $y = 0 \Rightarrow x = 0$ and $a + x = 0$ $x = 0, -a$

If $x = a$; $y = I\ \infty$

$\therefore$ line $x = a$ assimptod

If $y = -y$ the curve does not change

$\therefore$ The x-axis is Symmetric

$$\text{Required Area} = 2\int_0^{-a} y\,dx$$

$$= 2\int_0^{-a} x\left(\frac{a+x}{a-x}\right)^{1/2} dx$$

Put $x = a\cos 2\theta$

$a + x = a + a\cos 2\theta = a(1+\cos 2\theta)$

$a + x = 2a\cos^2\theta$

$a - x = a - a\cos 2\theta = a(1 - \cos 2\theta)$

$a - x = 2a\sin^2\theta$

$dx = -2a\sin 2\theta\, d\theta$

$x = 0$

$\Rightarrow \cos 2\theta = 0$

$\cos 2\theta = \cos \pi/2$

$\theta = \pi/4$

$x = -a$

$\Rightarrow \cos 2\theta = -1$

$\cos 2\theta = \cos \pi$

$\theta = \pi/2$

$$A = \int_{\pi/4}^{\pi/2} (a\cos 2\theta)\left(\frac{\cos\theta}{\sin\theta}\right)(-2a\sin 2\theta)\,d\theta$$

$$= -4a^2 \int_{\pi/4}^{\pi/2} \sin 2\theta \cdot \cos 2\theta \cdot \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= -4a^2 \int_{\pi/4}^{\pi/2} 2\sin\theta\cos\theta \cdot \cos 2\theta\, \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} (\cos^2\theta \cos 2\theta)\, d\theta = -8a^2 \int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2} \cdot \cos 2\theta\, d\theta$$

$$= -4a^2 \int_{\pi/4}^{\pi/2} (\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= -4a^2 \int_{\pi/4}^{\pi/2} \left(\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= -2a^2 \int_{\pi/4}^{\pi/2} \left[2\cos 2\theta + 1 + \cos 4\theta\right] d\theta$$

$$= -2a^2 \left[\frac{2\sin 2\theta}{2} + \theta + \frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2}$$

$$= -2a^2 \left[(0 + \pi/2 + 0) - (1 + \pi/4 + 0)\right]$$

$$= -2a^2 \left[\pi/2 - 1 - \pi/4\right] = -2a^2\left[\pi/4 - 1\right]$$

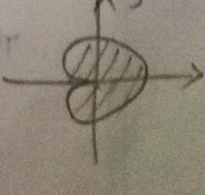
$$= -\frac{a^2\pi}{2} + 2a^2$$

$$A = \frac{a^2(4-\pi)}{2} \text{ sq. units.}$$

Find the area of cardioid $r = a(1+\cos\theta)$.

Equation of cardioid $r = a(1+\cos\theta)$

Area by the curve



This curve is symmetrical about the ~~fin~~ initial line

∴ The area bounded the limit of $0-\pi$

∴ Required Area $= 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2 d\theta$$

$$= a^2\int_0^{\pi}(1+\cos^2\theta+2\cos\theta)d\theta$$

$$= a^2\int_0^{\pi}\left(1+\frac{1+\cos2\theta}{2}+2\cos\theta\right)d\theta$$

$$= a^2\int_0^{\pi}\left(3/2+\frac{\cos2\theta}{2}+2\cos\theta\right)d\theta$$

$$= a^2\left[3/2\,\theta+\frac{\cos2\theta}{2}+2\sin\theta\right]_0^{\pi}$$

$$= a^2\left[3\pi/2-0\right]$$

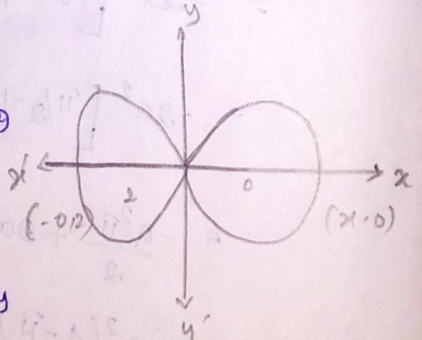
$$A = \frac{3\pi a^2}{2} \text{ Sq. units.}$$

Find the entire area of luminscate of Bernoulli $r^2=a^2\cos2\theta$

Equation of luminiscate of Bernoulli $r^2=a^2\cos2\theta$

To find

Area by the curve



Required area bounded by two loops.

This curve is symmentrical about the initial line.

$$\therefore A = 4\int_a^b \frac{1}{2} r^2 d\theta$$

The limits of them area is $0-\pi/4$

$$= 2\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$$

$$= 2\int_0^{\pi/4} a^2\cos2\theta\, d\theta$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta \, d\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= 2a^2 \left[1/2 - 0 \right]$$

$$A = a^2 \text{ sq. units}$$

17/12/21

Unit - 4
Multiple Integral

Double integral

$$\int_a^b f(x)\,dx$$

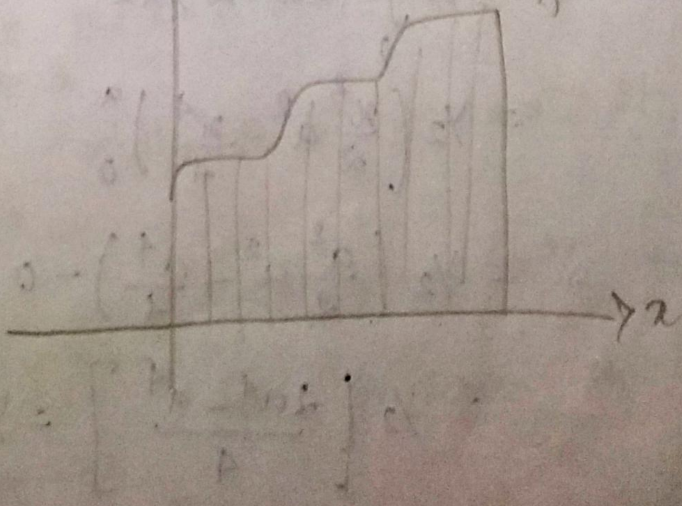
$$\int_a^b f(x)\,dx = \text{Sum of the areas of rectangles}$$

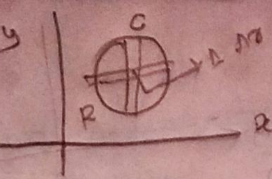
$$= \sum f(x_i)\,(x_i - x_{i-1})$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

F - anti derivative at f.

i.e. $\frac{d}{dx} f(x) = f(x)$





Area of sub regions $= \Delta A_1, \Delta A_2, \Delta A_3 \cdots \Delta A_r$

$\Delta A_1, \Delta A_2 \ldots$

1) Evaluate $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$

$$\iint xy\,dx\,dy$$

If we keep x as a constant

limits of 0 to a

y varies from 0 to $\sqrt{a^2-x^2} \Rightarrow x^2+y^2=a^2$

$$y=\sqrt{a^2-x^2}$$

To cover the total area x should varies from 0 to a

$x=0$ $\qquad$ $x=a$

$y=\sqrt{a^2-0}=a$ $\qquad$ $y=\sqrt{a^2-a^2}=0$

x	0	a
y	a	0

$$\iint xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_0^a \left[x y^2/2\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\frac{x(a^2-x^2)}{2} - 0\right) dx$$

$$= \frac{1}{2}\int_0^a (xa^2 - x^3)\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}a^2 - \frac{x^4}{4}\right)_0^a$$

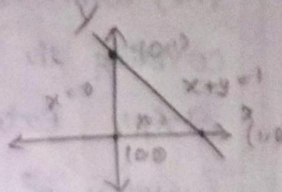
$$= \frac{1}{2}\left(\frac{a^2}{2}a^2 - \frac{a^4}{4}\right) - 0$$

$$= \frac{1}{2}\left[\frac{2a^4 - a^4}{4}\right] = \frac{1}{2}\left[\frac{a^4}{4}\right]$$

$$= \frac{a^4}{8} \text{ sq. units.}$$

Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which $x, y \geq 0$ & $x+y \leq 1$

The region is the triangle formed by the lines



$$\iint (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3} \right) dx$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{(1-x)^3}{3} \right) dx$$

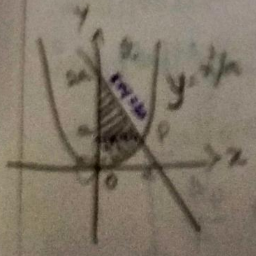
$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\left(-\frac{(1-x)^4}{4} \right) \right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{3}(0) \right] - \left[0 - 0 + \frac{1}{3}\left(-\frac{(1)^4}{4} \right) \right]$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12} = \frac{2}{12} = \frac{1}{6}.$$

① Change the order of integration in the integral $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ & evaluate it.



y varies from x^2/a to $2a-x$ (ie)

y lies between the curves

$y = \frac{x^2}{a}$ & $y = 2a - x$, $x+y = 2a$

varies from 0 to a.

In changing the order of the integration, we integrate 1st with respect to x keeping y constant (i.e) with elementary strips parallel to x axis. In covering the same, region of above the end of the strips extend to the line $x+y=2a$ & to the curve $y=\frac{x^2}{a}$ Hence we divide the region into 2 parts by the line $y=a$ which passes through p.

Hence for one region x varies from 0 to $\sqrt{ay}$ (ie, $y=\frac{x^2}{a} \Rightarrow x^2=ay \Rightarrow x=\sqrt{ay}$) & for the other region x varies from 0 to $2a-y$

$(y=2a-x \Rightarrow x=2a-y)$

In the 1st region y varies from 0 to a & ~~integer~~ in the 2nd region y varies from a to 2a.

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$$

$$= \int_0^a \left[\frac{yx^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} \left[\frac{x^2y}{2}\right]_0^{2a-y} dy$$

$$= \int_0^a \frac{ay^2}{2}\,dy + \int_a^{2a} \frac{y(2a-y)^2}{2}\,dy$$

$$= a/2 \left[\frac{y^3}{3}\right]_0^a + \int_a^{2a} \frac{y(4a^2-4ay+y^2)}{2}\,dy$$

$$= \frac{a^4}{6} + \frac{1}{2}\int_0^{2a} 4a^2y - 4ay^2 + y^3\,)\,dy$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[\frac{4a^2y^2}{2} - \frac{4ay^3}{3} + \frac{y^4}{4}\right]^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\left[\frac{4a^2(4a^2)}{2} - \frac{4a(8a^3)}{3} + \frac{16a^4}{4}\right] - \left[\frac{4a^2(a^2)}{2} - \frac{4aa^3}{3} + \frac{a^4}{4}\right]\right)$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\left[8a^4 - \frac{32a^4}{3} + 4a^4\right] - \left[2a^4 - \frac{4}{3}a^4 + \frac{a^4}{4}\right]\right)$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(10a^4 - \frac{28}{3}a^4 - \frac{a^4}{4}\right)$$

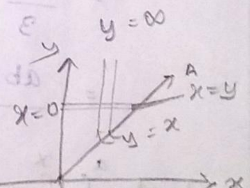
$$= \frac{a^4}{6} + \frac{1}{2}\left(\frac{120a^4 - 112a^4 - 3a^4}{12}\right)$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\frac{5a^4}{12}\right)$$

$$= \frac{a^4}{6} + \frac{5}{24}a^4 = \frac{4a^4 + 5a^4}{24} = \frac{9a^4}{24} = \frac{3a^4}{8} \,//.$$

(2) By changing the order of integration Evaluate

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$$



let
$$I = \int_0^\infty dx \int_x^\infty \frac{e^{-y}}{y}\, dy$$

Integrate w.r.t y from x to ∞ & then into w.r.t x from 0 to ∞

let OA be the straight line $y=x$ Region of integration is R above OA. change the order of integration Keep y constant. x varies from 0 to y then allow y to vary from 0 to ∞ to curve

$$I = \int_0^\infty \frac{e^{-y}}{y}\, dy \int_0^y dx$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, dy\, [x]_0^y = \int_0^\alpha \frac{e^{-y}}{y}\,(y)\, dy$$

$$= \int_0^\alpha e^{-y}\, dy = (-e^{y})_0^\alpha = (-e^{-\alpha} + e^0)$$

$$= -0 + 1 = 1 \,//.$$

(2) Evaluate

$$\int_0^a \int_0^b (x^2+y^2)\, dx\, dy$$

$$= \int_0^a \left[\frac{x^3}{3} + xy^2\right]_0^b dy$$

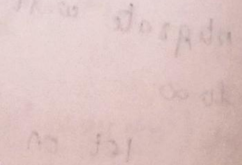
$$= \int_0^a \left(\frac{b^3}{3} + by^2\right) dy$$

$$= \left[\frac{b^3 y}{3} + \frac{by^3}{3}\right]_0^a$$

$$= \frac{b^3 a}{3} + \frac{ba^3}{3}$$

$$= \frac{ab(a^2+b^2)}{3} \,//$$

(3) Evaluate $\int_0^a \int_0^x (x^2+y^2)\, dy\, dx$



$$= \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^x dx$$

$$= \int_0^a \left(x^3 + \frac{x^3}{3} - 0\right) dx$$

$$= \left[\frac{x^4}{4} + \frac{1}{3}\left(\frac{x^4}{4}\right)\right]_0^a$$

$$= \frac{a^4}{4} + \frac{a^4}{12} = \frac{3a^4 + a^4}{12}$$

$$= \frac{a^4}{3} \,//.$$

Evaluate $\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2\,dr\,d\theta$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta \qquad \frac{n-1}{n}$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{8\cos^3\theta}{3} - 0\right) d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta = 8/3 \times 2\int_{0}^{\pi/2} \cos^3\theta\, d\theta.$$

$$= 8/3 \times 2 \times \left(\frac{3-1}{3}\right)$$

$$= \frac{8}{3} \times 2 \times \frac{2}{3} = \frac{32}{9} \;//.$$

Evaluate $\int_{0}^{\pi}\int_{0}^{a(1+\cos\theta)} r^2 \sin\theta\, dr\, d\theta$

$$I = \int_0^{\pi} \left[\frac{r^3}{3}\sin\theta\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} \left[\frac{a^3(1+\cos\theta)^3}{3}\sin\theta - 0\right] d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} (1+\cos\theta)^3 \sin\theta\, d\theta$$

Put $x = 1+\cos\theta$

$dx = -\sin\theta\, d\theta$

$-dx = \sin\theta\, d\theta$

θ	0	π
x	$1+\cos 0 = 1+1 = 2$	$1+\cos\pi = 1-1 = 0$

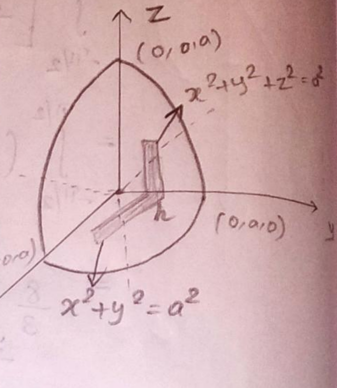
Therefore the integral becomes

$$I = \frac{a^3}{3}\int_2^0 x^3(-dx)$$

$$= \frac{a^3}{3}\int_0^2 x^3\,dx = \frac{a^3}{3}\left[\frac{x^4}{4}\right]_0^2$$

$$= \frac{a^3}{3}\left[\frac{2^4-0}{4}\right] = \frac{a^3}{3}\left(\frac{16}{4}\right) = \frac{4a^3}{3} \;//.$$

Evaluate $\iiint (xyz)\, dx\, dy\, dz$ taken through the Positive octant (8 parts) of the sphere $x^2+y^2+z^2=a^2$.



To cover the whole Positive octant of the Sphere $x^2+y^2+z^2=a^2$.

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

(Since $x^2+y^2+z^2=a^2$

$z^2=a^2-x^2-y^2$

$z=\sqrt{a^2-x^2-y^2}$)

y varies from 0 to $\sqrt{a^2-x^2}$

(Since $x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\sqrt{a^2-x^2}$)

x varies from 0 to a

$\therefore$ Required integral

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} (xyz)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{xyz^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(xy \frac{(a^2-x^2-y^2)}{2}\right) dy\, dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} (xya^2 - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2} \int_0^a \left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2x(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4} - 0\right] dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^4x}{2} - \frac{a^2x^3}{2} - \frac{a^2x^3}{2} - \frac{x^5}{2} - \frac{x(a^4-2a^2x^2+x^4)}{4}\right) dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^4x}{2} - \frac{a^3x^3}{2} - \frac{a^2x^3}{2} - \frac{x^5}{2} - \frac{a^4x}{4} + \frac{2a^2x^3}{4} - \frac{x^5}{4}\right) dx$$

$$= \frac{1}{2}\left[\frac{a^4x^2}{4} - \frac{a^2x^4}{8} - \frac{a^2x^4}{8} + \frac{x^6}{12} - \frac{a^4x^2}{8} + \frac{2a^2x^4}{16} - \frac{x^6}{24}\right]_0^a$$

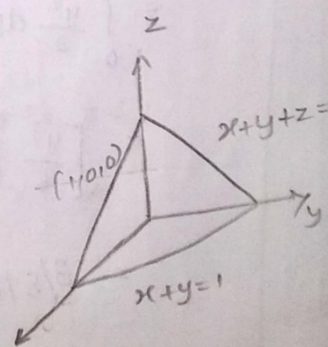
$$= \frac{1}{2}\left[\frac{a^6}{4} - \frac{a^6}{8} - \cancel{\frac{a^6}{12}} + \cancel{\frac{a^6}{12}} - \cancel{\frac{a^6}{8}} + \cancel{\frac{a^6}{8}} - \frac{a^6}{24}\right]$$

$$= \frac{a^6}{2}\left[\frac{1}{4} - \frac{1}{8} - \frac{2}{12} - \frac{1}{24}\right]$$

$$= \frac{a^6}{2}\left[\frac{6-3-1}{24}\right] = \frac{a^6}{24} \;//$$

(X) Evaluate $\iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ where V is the region bounded by $x=0,\ y=0,\ z=0,\ x+y+z=1$

(or) back pg



$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dz\,dy\,dx}{(1+x+y+z)^3}$$

$$I = \int_0^1\int_0^{1-x}\left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right] dy\,dx$$

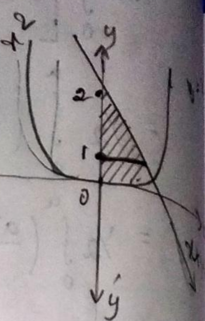
$$= -\frac{1}{2}\int_0^1\left[\frac{1}{4}y + \left(\frac{1}{(x+y+1)}\right)_0^{1-x}\right] dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right] dx = -\frac{1}{2}\left[-\frac{(1-x)^2}{8} + \frac{1}{2}x - \log(1+x)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{8} + \frac{1}{2}\log(2)\right] = \frac{1}{16}(8\log 2 - 5)$$

Change the order of integration & Evaluate

(eg) $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$



$y = x^2$ and $y = 2 - x$ are boundaries

Change the order of integration

$\Rightarrow$ limits of integration

x lies b/w 0 to $\sqrt{y}$ and y lies b/w 0 to 1

x lies b/w 0 to $2-y$ and y lies b/w 1 to 2

$$I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$$

$$= \int_0^1 \int_0^{\sqrt{y}} xy \, dy \, dx + \int_1^2 \int_0^{2-y} xy \, dx \, dy$$

$$= \int_0^1 y \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} y \, dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \tfrac{1}{2} \int_1^2 (2-y)^2 y \, dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 + \tfrac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= 3/8 \,//$$

(or)

(or) ① $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz \, dy \, dx}{(x+y+z+1)^3}$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^3 \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+1-x-y-1)^{-2}}{-2} - \frac{(x+y+2)^{-2}}{-2} \right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{(x+y+1)^{-2}}{2} - \frac{1}{8} \right) dy\, dx$$

$$= \int_0^1 \left[\frac{(x+y+1)^{-1}}{-2} - \frac{y}{8} \right]_0^{1-x} = \int_0^1 \left[\left(\frac{(2)^{-1}}{-2} - \frac{(1-x)}{8} \right) - \left(\frac{(x+1)^{-1}}{-2} - 0 \right) \right] dx$$

$$= \int_0^1 \left(-\frac{1}{4} - \frac{1-x}{8} + \frac{1}{2(x+1)} \right) dx$$

$$= \int_0^1 \left(\frac{1}{2(x+1)} + \frac{x-1}{8} - \frac{1}{4} \right) dx$$

$$= \left[\frac{1}{2} \log (x+1) + \frac{1}{8} \left(\frac{x^2}{2} - x \right) - \frac{x}{4} \right]_0^1$$

$$= \left[\frac{1}{2} \log 2 + \frac{1}{8} \left(\frac{1}{2} - 1 \right) - \frac{1}{4} \right] - \left[\frac{\log (1)}{2} + 0 + 0 \right]$$

$$= \frac{\log 2}{2} - \frac{1}{16} - \frac{1}{4} = \frac{\log 2}{2} - \frac{5}{16} \,//$$

9) $\iiint_R (x+y+z)\, dx\, dy\, dz$

where R is $1 \le x \le 2,\ 2 \le y \le 3,\ 1 \le z \le 3$.

$$\int_1^3 \int_2^3 \int_1^2 (x+y+z)\, dx\, dy\, dz$$

$$= \int_1^3 \int_2^3 \left(\frac{x^2}{2} - yx + zx \right)_1^2 dy\, dz$$

$$= \int_1^3 \int_2^3 \left[\left(\frac{4}{2} - 2y + 2z \right) - \left(\frac{1}{2} - y + z \right) \right] dy\, dz$$

$$= \int_1^3 \int_2^3 \left(2 - 2y + 2z - \frac{1}{2} + y - z \right) dy\, dz$$

$$= \int_1^3 \int_2^3 \left(\frac{3}{2} - y + z \right) dy\, dz$$

$$= \int_1^3 \left(\frac{3}{2} y - \frac{y^2}{2} + zy \right)_2^3 dz$$

$$= \int_1^3 \left[(9/2 - 9/2 + 3z) - (6/2 - \tfrac{4}{2} + 2z) \right] dz$$

$$= \int_1^3 (3z - 3 + 2 - 2z)\, dz$$

$$= \int_1^3 (z-1)\, dx \quad = \left[\frac{z^2}{2} - z \right]_1^3$$

$$= (9/2 - 3) - (1/2 - 1) \quad = 3/2 + 1/2 \quad = 4/2 = 2$$

3. $$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left(\frac{r^3}{3} \sin\theta \right)_0^a d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left(\frac{a^3}{3} \sin\theta \right) d\theta \, d\phi$$

$$= \int_0^{2\pi} \left[\frac{-a^3}{3} \cos\theta \right]_0^{\pi/4} d\phi$$

$$= \int_0^{2\pi} \frac{-a^3}{3} \left[\cos \pi/4 - \cos 0 \right] d\phi$$

$$= \int_0^{2\pi} \frac{-a^3}{3} \left[1/\sqrt{2} - 1 \right] d\phi$$

$$= \frac{-a^3}{3} \left[\frac{1-\sqrt{2}}{\sqrt{2}} \right] \int_0^{2\pi} d\phi$$

$$= \frac{(\sqrt{2}-1)\, a^3}{3\sqrt{2}} \left[\phi \right]_0^{2\pi}$$

$$= \frac{2\pi a^3 (\sqrt{2}-1)}{3\sqrt{2}}$$

$$= \frac{\pi a^3 (a - \sqrt{2})}{3}$$

4) $\iiint (x+y+z)\, dx\, dy\, dz$

where R is $x=0,\ x=1,$ & $y=0,\ y=1$ & $z=0, z=1$

$$\iiint_R (x+y+z)\, dx\, dy\, dz = \int_0^1\int_0^1\int_0^1 (x+y+z)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 \left(\frac{x^2}{2} + yx + zx\right)_0^1 dy\, dx$$

$$= \int_0^1\int_0^1 (1/2 + x + y)\, dy\, dz$$

$$= \int_0^1 \left(y/2 + zy + \frac{y^2}{2}\right)_0^1 dz$$

$$= \int_0^1 (1/2 + z + 1/2)\, dz$$

$$= \int_0^1 (1+z)\, dz = \left[z + \frac{z^2}{2}\right]_0^1$$

$$= 1 + 1/2 = 3/2.$$

5) $\int_1^2\int_1^{\log y}\int_1^{ex} \log z\, dz\, dx\, dy$

$$= \int_1^e\int_1^{\log y} (z\log z - 1)_1^{ex}\, dx\, dy$$

$$= \int_1^e\int_1^{\log y} \left[(e^x \log e^x - 1) - (\log(1) - 1)\right] dx\, dy$$

$$= \int_1^e\int_1^{\log y} x e^x\, dx\, dy$$

$$= \int_1^e \left[e^x[x-1]\right]_1^{\log y} dx\, dy$$

$$= \int_1^e \left\{e^{\log y}[\log y - 1]\right\} - \left\{e - (1-1)\right\} dy$$

$$= \int_1^e y[\log y - 1]\, dy$$

$$= \int_1^e (y \log x - y)\, dy$$

$u = \log y - 1$, $dv = y\, dy$

$du = \frac{1}{y}\, dy$, $v = \frac{y^2}{2}$

$$= \left[(\log y - 1) \frac{y^2}{2} \right]_1^e - \int_1^e \frac{y^2}{2} \cdot \frac{1}{y}\, dy$$

$$= \left[(\log y - 1) \frac{y^2}{2} \right]_1^e - \frac{1}{2} \int_1^e y\, dy$$

$$= (0 + \tfrac{1}{2}) - \tfrac{1}{2} \left(\frac{y^2}{2} \right)_1^2$$

$$= \tfrac{1}{2} - \tfrac{1}{2} \left[\frac{e^2}{2} - \tfrac{1}{2} \right]$$

$$= \tfrac{1}{2} + \tfrac{1}{4} - \frac{e^2}{4}$$

$$= \tfrac{5}{4} - \tfrac{e^2}{4} = \frac{3 - e^2}{4}$$

b) $$\int_0^a \int_0^x \int_0^{y+x} e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_0^a \int_0^x \int_0^{y+x} e^x e^y e^z\, dz\, dy\, dx$$

$$= \int_0^a \int_0^x e^x e^y (e^z)_0^{x+y}\, dy\, dx$$

$$= \int_0^a \int_0^x e^x e^y \left[e^{x+y} - 1 \right] dy\, dx$$

$$= \int_0^a \int_0^x (e^{2x} e^{2y} - e^x \cancel{e}^y)\, dy\, dx$$

$$= \int_0^a \left[e^2 \frac{e^{2y}}{2} - e^x e^y \right]_0^x dx$$

$$= \int_0^a \left(e^{2x}\, \frac{e^{2x}}{2} - e^x e^x\right) - \left(\frac{e^{2x}}{2} - e^x\right) dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2a}}{4} + e^x\right]_0^a$$

$$= \left[\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \left(\frac{1-6+8}{8}\right)\right.$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - 3/8$$

7. $\iiint (x+y+z)\, dz\, dy\, dz$

where v is $x+y+z=a$, $x=0$, $y=0$, $z=0$

$$= \int_0^a \int_0^{a-z} \int_0^{a-y-z} (x+y+z)\, dx\, dy\, dz$$

$$= \int_0^a \int_0^{a-z} \left(\frac{x^2}{2} + yx + zx\right)_0^{a-y-z} dy\, dz$$

$$= \int_0^a \int_0^{a-z} \left[\frac{(a-y-z)^2}{2} + y(a-y-z) + z(a-y-z)\right] dy\, dz$$

$$= \int_0^a \int_0^{a-z} \left[\frac{(a-y-z)^2}{2} + ay - ay^2 - zy + az - zy - z^2\right] dy\, dz$$

$$= \int_0^a \left[-\frac{(a-y-z)^3}{6} + \frac{ay^2}{2} - \frac{y^3}{3} - \frac{zy^2}{2} + ayz - \frac{zy^2}{2} - yz^2\right]_0^{a-z} dz$$

$$= \int_0^a \left[-\frac{(a-y-z)^3}{6} + y^3(a/2 - z) - \frac{y^3}{3} + (az - z^2)y\right]_0^{a-z} dz$$

$$= \int_0^a \left[0 + (a-z)^2 (a/2 - z) - \left(\frac{a-z}{6}\right)^3 - (az - z^2)(a-z)\right] dz$$

$$= \int_0^a (a^2 + z^2 - 2az)(a/2 - z) - \left(\frac{a^3 - z^3 - 3a^2 z + 3az^2}{6}\right) - (a^2 z - az^2 - az^2 + z^3)\, dz$$

$$= \int_0^a \Big[\frac{a^3}{2} - a^2 z + \frac{az^2}{2} - z^3 - \frac{2a^2 z}{2} + 2az^2 + \frac{z^3 + 3a^2 z - 3az^2 - a^3}{6}$$
$$+ a^2 z - az^2 - az^2 + z^3 \Big] dz$$

$$= \frac{1}{6} \int_0^a (2z^3 + 6a^2 z - 6az^2 - 2a^3 + 3a^2 - 6a^2 z + 3az^2)\, dz$$

$$= \frac{1}{6} \int_0^a (2z^3 - 3az^2 + a^3)\, dz$$

$$= \frac{1}{6} \left[\frac{2z^4}{4} - \frac{3az^3}{3} + a^3 z \right]_0^a$$

$$= \frac{1}{6} \left[\frac{2a^4}{4} - \frac{3aa^3}{3} + a^3 a \right]$$

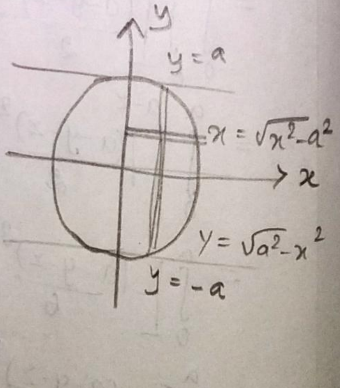
$$= \frac{1}{6} \left[\frac{a^4}{2} - a^4 + a^4 \right] \quad = a^4/12$$

Evaluate $\iiint xyz\, dx\, dy\, dz$ Taken through be positive octant

(4) change the order of integration in $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} x\, dx\, dy$ and hence evaluate.

Given

$$\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} x\, dx\, dy$$



$x = 0$, $x = \sqrt{a^2-y^2}$, $y = -a$, $y = a$

$x = 0$, $x^2 = a^2 - y^2$

$x^2 + y^2 = a^2$

By changing the order of integration. using Vertical strip on the region R x Varies from 0 to a, y Varies from $\sqrt{a^2-x^2}$ to $-\sqrt{a^2-x^2}$

$$\therefore \int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} x\,dx\,dy = \int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x\,dy\,dx$$

$$= \int_{0}^{a} x\,(y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dx$$

$$= \int_{0}^{a} x\left(2\sqrt{a^2-x^2}\right)dx$$

$$= 2\int_{0}^{a}\sqrt{a^2-x^2}\;x\,dx$$

$$= -\int_{0}^{a}\sqrt{a^2-x^2}\;x\,dx$$

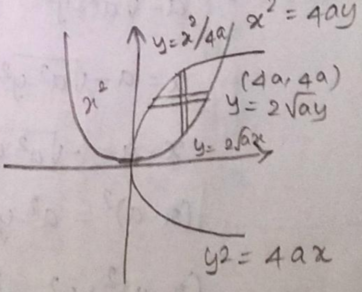
$$= -\int_{0}^{a}\sqrt{a^2-x^2}\;d\,(a^2-x^2)$$

$$= -\left[(a^2-x^2)^{3/2}\cdot 2/3\right]_{0}^{a}$$

$$= -\left[0-(a^2)^{3/2}\cdot 2/3\right] = \frac{2a^3}{3}$$

5. chg the order of inte and evaluate

$$\int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$$



$y = \dfrac{x^2}{4a}$, $y = 2\sqrt{ax}$

$4ay = x^2$, $y^2 = 4ax$

$x = 0$, $x = 4a$

By changing the order of integration using strips on R. x varies from $\dfrac{y^2}{4a}$ to $2\sqrt{ay}$ and y varies from 0 to $4a$.

$$\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx = \int_{0}^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a} y\left[\frac{x^2}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \cancel{\int_0^{4a} y\left(x^2/2\right)}^{2}$$

$$= \int_0^{4a} y\left[\frac{4ay}{2} - \frac{y^4}{32a^2}\right] dy$$

$$= \int_0^{4a} \left[2ay^2 - \frac{y^5}{32a^2}\right]_0^{4a}$$

$$= \left[\frac{124a^4}{3} - \frac{1024a^6}{192a^4}\right]$$

$$= \frac{128a^4}{3} - \frac{64a^4}{3} = \frac{64a^4}{3}.$$

6) chg the ord of inte $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$ and hence evaluate y.

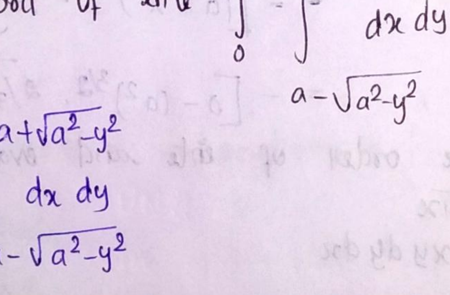
$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$$

$x = a-\sqrt{a^2-y^2}$, $x = a+\sqrt{a^2-y^2}$

$x-a = -\sqrt{a^2-y^2}$, $x-a = \sqrt{a^2-y^2}$

$(x-a)^2 = a^2-y^2$, $(x-a)^2 = a^2-y^2$

$(x-a)^2+y^2 = a^2$, $(x-a)^2+y^2 = a^2$



By changing the order of rectangle chose vertical strip on R. y varies from 0 to $\sqrt{a^2-(x-a)^2}$ and x varies from 0 to 2a

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\, dy = \int_0^{2a} \int_0^{\sqrt{a^2-(x-a)^2}} dy\, dx$$

$$= \int_0^{2a} (y)_0^{\sqrt{a^2-(x-a)^2}} dx$$

$$= \int_0^{2a} \sqrt{a^2-(x-a)^2}\, dx$$

Put $t=(x-a)$

$dt = dx$

x	0	$2a$
t	$-a$	a

$$= \int_{-a}^{a} \sqrt{a^2-t^2}\, dt$$

$$= \left[\frac{t}{2}\sqrt{a^2-t^2} + \frac{a^2}{2}\sin^{-1}\frac{t}{a}\right]_{-a}^{a}$$

$$= \frac{a^2}{2}\sin^{-1}(1) - \frac{a^2}{2}\sin^{-1}(-1)$$

$\because \sin^{-1}(-1) = -\sin^{-1}(1)$ ||

$$= a^2/2\, \sin^{-1}(1) + a^2/2\, \sin^{-1}(1)$$

$$= \frac{a^2 \cdot \pi}{2} = \frac{\pi a^2}{2} \;||$$

EXAMPLES.

Change the order of inte and then evaluate $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dx\, dy$

~~$y = x^2$ and $y = 2-x$ are boundaries.~~

$$I = \int_0^a (y)_0^{\sqrt{a^2-x^2}} dx = \int_0^a \sqrt{a^2-x^2}\, dx$$

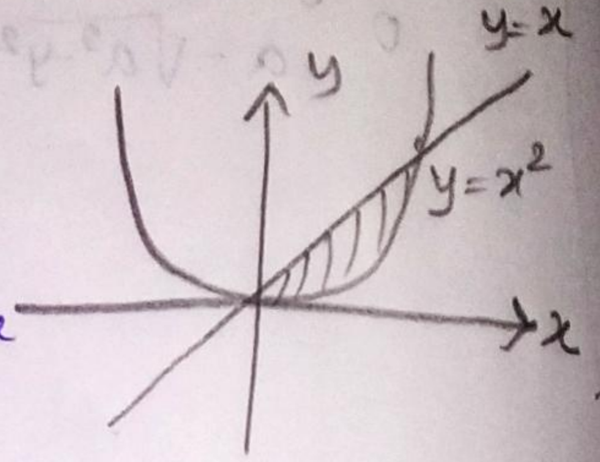
$$= \left[\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}\right]_0^a$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a^2}{4} \;||$$

Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$.

draw the curves $y = x^2$ and $y = x$ to understand the region of integration



$$I = \iint xy(x+y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + x\frac{y^3}{3} \right)_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^4}{3} \right) dx$$

$$= \left[\frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24} \right]_0^1$$

$$= 1/10 - 1/14 + 1/15 - 1/24 = 3/56$$

28/12/2021

UNIT - 5

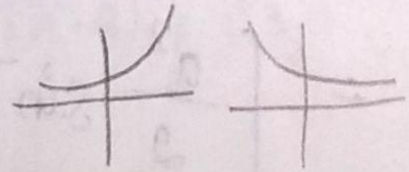
Beta, GAMMA functions.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \quad m>0, n>0$$

gamma Beta oda copy

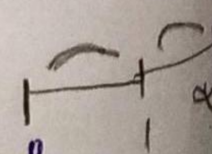
$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx \quad n>0$$

Convergence of $\Gamma(n)$



$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$ This integral exists if $n>0$.

$$\Gamma(n) = \int_0^1 x^{n-1} e^{-x}\,dx + \int_1^\alpha x^{n-1} e^{-x}\,dx.$$



the integral is $\lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{n-1} e^{-x}\,dx$ if the limit exists when x is small $\to e^{-x}$ will becomes very less.

The integral behaves like x^n & limit exists if $n>0$

Second integral $\int^{\infty} x^{n-1} e^{-x}\, dx$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \frac{x^r}{r!} \dots > \frac{x^r}{r!} \quad (r \text{ is any } + \text{integer})$$

$$\Rightarrow e^x > \frac{x^r}{r!} > \frac{x^{n+1}}{r!} \quad \text{where } n+1 < r$$

$x^2 < x^3 \quad 2<3$

$x^{n+1} < x^r$

$$e^{-x} < \frac{r!}{x^{n+1}}$$

$$x^{n+1} e^{-x} < r!$$

$$\frac{x^{n+1}}{x^2} e^{-x} < \frac{r!}{x^2}$$

$\int_1^{\infty} e^{-x} x^{n-1} dx$ doesnot exceed a constant multiple of $\int_1^{\infty} \frac{dx}{x^2}$ which converges.

$\therefore \Gamma(n)$ converges for $n>0$ //.

Recurrence formula

$$\Gamma(n+1) = n\Gamma(n) \quad \text{if } n>0$$

Proof

$$\Gamma(n+1) = \int_0^{\infty} x^{n+1-1} e^{-x} dx = \int_0^{\infty} x^n e^{-x} dx \quad \left(\begin{matrix} n+1>0 \\ n>-1 \end{matrix}\right)$$

Integration by parts $u = x^n \quad dv = e^{-x} dx$

$du = nx^{n-1} dx \quad v = -e^{-x}$

gamma

$$\therefore \Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx = \left[-x^n e^{-x}\right]_0^{\infty} - n\int_0^{\infty} -e^{-x} x^{n-1} dx.$$

$\lim_{x\to 0} e^{-x} x^n = 0 \quad \text{if } n>0$

$\therefore \lim_{x\to\infty} e^{-x} x^n = \lim_{x\to\infty} \frac{x^n}{e^x} = 0$

$\because \Gamma(n+1) = n \int_0^\infty e^{-x} x^{n-1} dx$

formula $\Gamma(n+1) = n\Gamma(n)$ if $n>0$

Note this recurrence formula is true only when $n>0$

Corollary : 1

$\Gamma(n+1) = n!$

Proof

from the recurrence formula we knows that $\Gamma(n+1) = n\Gamma(n)$

$= n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\Gamma(n-2)$

$= n(n-1)(n-2)\ldots 1\,\Gamma(1)$

$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x} dx = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty$

$= -\left[e^{-x}\right]_0^\infty = [0-1] = 1$

$\Gamma(1) = 1$

$\therefore \Gamma(n+1) = n(n-1)(n-2)\ldots 3\,2\,1$

$= 1\times 2\times 3\ldots\ldots\times n$

$= n!$

Corollary : 2

$\Gamma(n+a) = (n+a-1)(n+a-2)\ldots a\Gamma(a)$

when n is a positive integer

Properties of Beta function.

(i) $\beta(m,n) = \beta(n,m)$

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx.$

Put $x = 1-y$
$dx = -dy$
$1-x = y$

x	0	1
$y=1-x$	1	0

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy = \beta(n,m)$$

ii) $\beta(m,n)$ can be expressed as a definite integral with $0, \alpha$ as limits

$0 = \frac{y}{1+y} \Rightarrow y = 0$

Proof.

In $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ put $x = y/_{1+y}$

$y = y+1 \Rightarrow 1 = \frac{y}{1+y}$

when $x = 0$ then $y = 0$
$x = 1$ $y = \alpha$

x	0	1
y	0	α

$dx = \frac{dy}{(1+y)^2}$ $1-x = \frac{1-y}{1+y} = \frac{1}{1+y}$

$$\beta(m,n) = \int_0^\alpha \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}\right)$$

$$= \int_0^\alpha \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Properties of Beta function.

(iii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \, dx$ → property.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$\sin^2\theta = 0$
$\theta = 0$

Put $x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta \, d\theta$

$\sin^2\theta = 1$
$\sin\theta = 1$
$\theta = \pi/2$

x	0	1
θ	0	$\pi/2$

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \cos^{2n-2+1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$B(m,n) = 2I_{2m-1,\,2n-1}$$

$$I_{2m-1,\,2n-1} = \tfrac{1}{2} B(m,n)$$

$$I_{m,n} = \tfrac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$I_{m,n} = \tfrac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Relation b/w Beta & Gamma function.

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof:

$$\Gamma(m) = \int_0^{\infty} x^{m-1} e^{-x}\, dx$$

Put $x = t^2$ we have

$dx = 2t\,dt$

x	0	∞
t	0	∞

$$\Gamma(m) = \int_0^{\infty} (t^2)^{m-1} e^{-t^2}\, 2t\, dt$$

So we can take

$$\Gamma(m) = 2\int_0^{\infty} x^{2m-1} e^{-x^2}\, dx$$

Similarly we can have

$$\Gamma(n) = 2\int_0^{\infty} y^{2n-1} e^{-y^2}\, dy$$

$$\Gamma(m)\Gamma(n) = \left(2\int_0^{\infty} x^{2m-1} e^{-x^2}\, dx\right) \times \left(2\int_0^{\infty} y^{2n-1} e^{-y^2}\, dy\right)$$

$$= 4\int_0^{\infty}\int_0^{\infty} x^{2m-1} y^{2n-1} e^{-x^2-y^2}\, dx\, dy.$$

Putting $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

x, y vary from 0 to ∞

(Cartesian coordinates are in 1st quadrant)

we are transforming cartesian into polar.

by taking r vary from 0 to ∞ $\quad B(m,n)$

θ vary from 0 to $\pi/2$ $\quad \Gamma(n)$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^{\infty}\int_0^{\pi/2} (r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1} e^{-r^2}\, r\,d\theta\,dr$$

$$= 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r^{2n+2m-1} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta\,dr$$

$$= 4\int_0^{\infty} e^{-r^2} r^{2n+2m-1}\,dr \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Now. $\displaystyle\int_0^{\infty} e^{-r^2} r^{2m+2n-1}\,dr = \frac{1}{2}\int_0^{\infty} t^{m+n-1} e^{-t}\,dt$

$$= \int_0^{\infty} e^{-t}\, t^{1/2(2m+2n-1)} \left(\frac{dt}{2t^{1/2}}\right)$$

by putting $r^2 = t$

$2r\,dr = dt$

$r = t^{1/2}$

$$= \frac{1}{2}\Gamma(m+n)$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = I_{2m-1,\,2n-1}$$

$$= \frac{1}{2}B(m,n)$$

$$\therefore \Gamma(m)\Gamma(n) = 4\cdot\frac{1}{2}\Gamma(m+n)\,\frac{1}{2}B(m,n)$$

$$\Gamma(m+n)\,B(m,n)$$

$$\therefore B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Hence proved//

Corollary (i)

$\Gamma(1/2) = \sqrt{\pi}$

Proof

We know that $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

Put $m = 1/2$ $n = 1/2$

$$B(1/2, 1/2) = \frac{\Gamma(1/2)\,,\Gamma(1/2)}{\Gamma(1)}$$

$$B(1/2, 1/2) = 2\int_0^{\pi/2} \sin^{2(1/2-1)}\theta \, \cos^{2(1/2-1)}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^0\theta \, \cos^0\theta \, d\theta = 2\int_0^{\pi/2} d\theta = 2(\pi/2) = \pi$$

$\Gamma(1) = 1$

$$= \frac{\Gamma(1/2)\cdot\Gamma(1/2)}{1} = \pi$$

$$= \Gamma(1/2)^2 = \pi$$

Hence $\Gamma(1/2) = \sqrt{\pi}$

Corollary (ii)

In $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

Put $m = 1-n$, then

$$\frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(n+1-n)} = B(n, 1-n)$$

$$= \frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(1)} = B(n, 1-n)$$

$$= \frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(1/2)\,\Gamma(1/2)} = \int_0^{\infty} x^{n-1}(1+x)^{-1}\,dx$$

$$\cancel{= \int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}}$$

~~If we put n=1/2~~

$$= \int_0^\infty x^{n-1}(1-x)^{1-n-1}\,dx = \int_0^\infty \frac{x^{n-1}}{(1-x)^n}\,dx$$

$$= \int_0^\infty \frac{x^{n-1}}{1+x+x^3}\,dx$$

$$= \int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$$

If we put $n = 1/2$

$$(\Gamma(1/2))^2 = \frac{\pi}{\sin \pi/2} = \frac{\pi}{1} = \pi$$

hence $\Gamma(1/2) = \sqrt{\pi}$

corollary (iii)

the result in corollary (ii)

$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = 1/2\, B(m,n)$ is of the expressed in the following form.

Putting $2m = p$ & $2n = q$

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1} = 1/2\, B\left(\frac{p}{2}, \frac{q}{2}\right)$$

$$= 1/2 \frac{\Gamma(p/2)\,\Gamma(q/2)}{\Gamma\left(\frac{p+q}{2}\right)} \longrightarrow (1)$$

If we put $q = 1$ in (1) we get

$$\int_0^{\pi/2} \sin^{p-1}\theta\, d\theta = \frac{1/2\, \Gamma(p/2)\,\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)}$$

If we put $p = q$ in (1) we get

$$\int_0^{\pi/2} \sin^{p-1}\theta\, d\theta = 1/2 \frac{(\Gamma(p/2))^2}{\Gamma(p)}$$

We know that $2\sin\theta\cos\theta = \sin 2\theta$

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta$$

$$(\sin\theta\cos\theta)^{p-1} = \frac{1}{2^{p-1}}\sin^{p-1}2\theta\,d\theta$$

$$2\frac{1}{2^{p-1}}\int_0^{\pi/2}\sin^{p-1}2\theta\,d\theta = \left[\Gamma(p/2)\right]^2$$

Put $2\theta = \varphi$ $\quad 2d\theta = d\varphi$, we get

$$\frac{1}{2^{p-1}}\int_0^{\pi}\sin^{p-1}\varphi\frac{d\varphi}{2} = \frac{(\Gamma(p/2))^2}{2\Gamma(p)}$$

θ	0	$\pi/2$
φ	0	$2(\pi/2)$

$$\Rightarrow \frac{2}{2^{p-1}}\int_0^{\pi/2}\sin^{p-1}\varphi\,d\varphi = \frac{(\Gamma(p/2))^2}{\Gamma(p)}$$

$$\frac{2}{2^{p-1}}\left[\frac{1}{2}\frac{\Gamma(p/2)\,\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)}\right] = \frac{\{\Gamma(p/2)\}^2}{\Gamma(p)}$$

$$= \frac{\Gamma(1/2)}{2^{p-1}\,\Gamma\left(\frac{p+1}{2}\right)} = \frac{\Gamma(p/2)}{\Gamma(p)}$$

$$\Gamma(p/2)\,\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^{p-1}}\Gamma(p) \longrightarrow (3)$$

Put $p = 2n$, we have

$$\Gamma(n)\,\Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n) \longrightarrow (4)$$

Put $n = 1/4$

$$\Gamma(1/4)\,\Gamma(1/4+1/2) = \frac{\sqrt{\pi}}{2^{2(1/4)-1}}\Gamma(2(1/4))$$

$$\Gamma(1/4)\,\Gamma(3/4) = \frac{\sqrt{\pi}}{2^{-1/2}}\Gamma(1/2)$$

$$= \sqrt{\pi}\,2^{1/2}\,\Gamma(1/2)$$

$$= \sqrt{2}\,\sqrt{\pi}\,\sqrt{\pi}$$

$$= \sqrt{2}\,\pi \,//$$

1) $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$

Put $\log(1/x) = t$ $\qquad e^t = \frac{1}{x}$

$t = \log 1/x$

$1/x = e^t$

$x = e^{-t}$

$dx = -e^{-t}\,dt$

x	0	1
t	$\log(1/0)$ ∞	$\log(1/1)$ 0

$$\int_0^1 x^m (\log(1/x))^n dx = \int_\infty^0 (e^{-t})^m t^n (-e^{-t})\,dt$$

$$= \int_0^\infty e^{-mt} \cdot t^n e^{-t}\,dt$$

$$= \int_0^\infty e^{-(m+1)t}\, t^n\,dt$$

put $(m+1)t = y$ $\qquad dt = \frac{1}{m+1}\,dy$

$t = \frac{y}{m+1}$

$$\int_0^\infty e^{-y} \frac{y^n}{(m+1)^n} \cdot \frac{1}{(m+1)}\,dy = \frac{1}{(m+1)^{n+1}} \int_0^\infty y^{(n+1)-1} e^{-y}\,dy$$

$$= \frac{1}{(m+1)^{n+1}} \, \Gamma(n+1)$$

2. $\int_0^\infty e^{-x^2} dx$

Put $x^2 = t$

x	0	∞
t	0	∞

$2x\,dx = dt$

$dx = 1/2 \frac{1}{\sqrt{t}}\,dt$

$$\int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}}\,dt$$

$$= 1/2 \int_0^\infty t^{1/2} e^{-t}\,dt$$

$$= 1/2\, \Gamma(1/2) = \sqrt{\pi}/2$$

3. Evaluate $\int_0^1 x^{-9}(1-x)^8 dx$

$= \beta(8,9)$

$= \dfrac{\Gamma(8)\Gamma(9)}{\Gamma(17)}$

$= \dfrac{7!\,8!}{16!}$ (since $\Gamma(n+1) = n!$)

4 $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$

$= \frac{1}{2}\beta\left(\frac{7+1}{2}, \frac{5+1}{2}\right)$

$= \frac{1}{2}\beta(4,3)$

$= \frac{1}{2}\dfrac{\Gamma(4)\Gamma(3)}{\Gamma(7)}$

$= \frac{1}{2}\dfrac{3!\,2!}{6!} = \frac{1}{2}\dfrac{1\times2\times3\times2\times1}{1\times2\times3\times4\times5\times6}$

$= 1/120$

5. $\int_0^{\pi/2} \sin^{10}\theta \, d\theta$

$= \frac{1}{2}\dfrac{\Gamma(11/2)\Gamma(1/2)}{\Gamma(11/2+1/2)}$ $\left(\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n+1}\theta \, d\theta = \frac{1}{2}\beta(m,n)\right)$

$2m-1 = 10$ $2n-1 = 0$

$2m = 1$ $2n = 1$

$m = 11/2$ $n = 1/2$

$= \frac{1}{2}\beta(11/2, 1/2)$

$= \frac{1}{2}\dfrac{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left[\Gamma(1/2)\right]^2}{\Gamma(6)}$

$= \frac{1}{36}\dfrac{9\times7\times5\times3\,(\sqrt{\pi})^2}{5!}$

$= \dfrac{63\pi}{512}.$

6) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$= \int_0^{\pi/2} \sqrt{\dfrac{\sin\theta}{\cos\theta}}\, d\theta$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$

$2m-1 = 1/2$ $\quad 2n-1 = -1/2$

$2m = 3/2$ $\quad 2n = 1/2$

$m = 3/4$ $\quad n = 1/4$

$$= \tfrac{1}{2} B\left(\tfrac{3}{4}, \tfrac{1}{4}\right)$$

$$= \tfrac{1}{2} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)}$$

$$= \tfrac{1}{2}\Gamma(1-1/4)\Gamma(1/4)$$

Since $\left(\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}\right)$

$$= \tfrac{1}{2} \frac{\pi}{\sin \frac{1}{4}\pi}$$

$$= \frac{\pi}{2\sin \pi/4} = \frac{\pi}{2 \times \frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}}$$

7) Express $\int_0^1 x^m(1-x^n)^{10}\, dx$ interms of gamma functions & Evaluate.

$$\int_0^1 x^5(1-x^2)^{10}\, dx$$

Put $x^n = y$ $\quad x = y^{1/n}$

$n x^{n-1} dx = dy$

x	0	1
y	0	1

$$\int_0^1 x^m (1-x^n)^p \, dx = \int_0^1 y^{m/n}(1-y)^p \frac{dy}{n y^{\frac{n-1}{n}}}$$

$$= \tfrac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}} (1-y)^p \, dy$$

$$= \tfrac{1}{n} B\left(\frac{m-n+1}{n}+1, p+1\right)$$

$$= \tfrac{1}{n} B\left(\frac{m-n+1+n}{n}, p+1\right)$$

$$= \tfrac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$$

$$= \tfrac{1}{n} \Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)$$

$$= \frac{\tfrac{1}{n}\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}, p+1\right)}$$

$$= \int_0^1 x^5(1-x^3)^{10}$$

$m=5,\ n=3\ \ p=10$

The integral becomes.

$$= \frac{1/3\ \Gamma\left(\frac{5+1}{3}\right)\Gamma(10+1)}{\Gamma\left(\frac{5+1}{3}+10+1\right)}$$

$$= 1/3\ \frac{\Gamma(2)\ \Gamma(11)}{\Gamma(13)}$$

$$= 1/3\ \frac{1!\ 10!}{12!}$$

$$= 1/3 \times \frac{1}{11\times 12}$$

$$= \frac{1}{396}$$

8) Prove that $\displaystyle\int_0^{\pi/2} \frac{\cos^{2m-1}\theta\ \sin^{2n-1}\theta\, d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}$

$$= \frac{\beta(m,n)}{2a^m b^n}$$

$$L.H.S = \int_0^{\pi/2} \frac{\cos^{2m-1}\theta\ \sin^{2n-1}\theta\ d\theta / (\cos^2\theta)^{m+n}}{(a\cos^2\theta\ b\sin^2\theta)^{m+n}/(\cos^2\theta)^{m+n}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\cos^{2m-1}\theta\ \sin^{2n-1}\theta\ d\theta / \cos^{2m+2n+1-1}}{a\cos^{2m+2n}\theta\ \cos^{-2m-2n}\theta + b\tan^{2(m+n)}\theta}$$

$$= \int_0^{\pi/2} \frac{\cos^{2m-1}\theta\ \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta}\cdot\frac{1}{\cos^{2m+1}}\, d\theta}{a+b(\tan\theta)^{2(m+n)}}$$

$$= \int_0^{\pi/2} \frac{\cos^{-2}\theta\ (\tan\theta)^{2n-1}\, d\theta}{a+b(\tan\theta)^{2(m+n)}}$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta\ (\tan\theta)^{2n-1}\, d\theta}{a+b(\tan\theta)^{2(m+n)}}$$

Put $t = \tan\theta$ $\quad dt = \sec^2\theta \, d\theta$

θ	0	$\pi/2$
t	0	α

the integral becomes

$$= \int_0^\infty \frac{t^{2n-1}\, dt}{(a+bt^2)^{m+n}}$$

Put $\sqrt{b}\, t = \sqrt{ay}$

$$t = \sqrt{\frac{ay}{b}} \qquad bt^2 = ay$$

$$y = b/a \; t^2$$

$$dt = \tfrac{1}{2}\sqrt{a/by}\; dy$$

the integral becomes

t	0	∞
y	0	∞

$$= \int_0^\infty \frac{\left(\sqrt{\frac{ay}{b}}\right)^n \cdot (1/2)\sqrt{\frac{a}{by}}\; dy}{\left(\sqrt{ay/b}\right)(a+ay)^{m+n}}$$

$$= \frac{1}{2}\int_0^\infty \frac{(a/b)^n\, y^n \sqrt{1/y}\; dy}{a^{m+n}\sqrt{y}\,(1+y)^{m+n}}$$

$$= \frac{1}{2}\int_0^\infty \frac{(a/b)^n\, y^n\, y^{-1/2}\, y^{-1/2}\, dy}{a^{m+n}(1+y)^{m+n}}$$

$$= \frac{1}{2}\, \frac{a^n}{b^n a^{m+n}} \int_0^\infty y^{n-1}(1+y)^{-(m+n)}\, dy$$

$$= \frac{1}{2a^m b^n}\int_0^\infty y^{n-1}(1+y)^{-(m+n)}\, dy$$

$$= \cancel{\frac{1}{2a^m}} = \frac{1}{2a^m b^n}\, \beta(m,n)$$

Hence proved//.