SWAMI DAYANANDA COLLEGE OF ARTS & SCIENCE, MANJAKKUDI.

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DEPARTMENT OF MATHEMATICS

16SCCMM6:

CLASSICAL ALGEBRA AND THEORY OF NUMBERS

CLASS:

II – B.Sc., MATHEMATICS

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CLASSICAL ALGEBRA
AND
THEORY OF NUMBERS

UNIT-I :-

Relation between roots and coefficients of polynomial equation - Symmentric functions - sum of the rth powers of the roots.

UNIT-II :-

Newton's theorem on the sum of the power of the roots - Transformations of equation - Diminshing, increasing and multiplying the roots by a constant - Reciprocal equation - To increase or decrease the roots of the equation by a given quantity.

UNIT-III :-

Form of quotient and remainder - Removal of terms - To form of an equation whose roots are any power - Transformation in general - Descart's rule of sign.

UNIT-IV :-

Inequalities - elementary principal - Geometric and Arithmetic means - Weierstrass inequalities cauchy inequalities - Applications to maxima and minima.

UNIT-V :-

Theory of numbers - Prime and composite numbers - divisions of a given number N - Euler's function (N) and its value - The highest power of a prime P contained in N! - Congruences - Fermat's, Wilson's and Lagrange's theorems.

TEXT BOOK:-

1. Algebra volume I
2. Algebra volume II } → T.K. Manickavasagam Pillai

2016

UNIT - I

THEORY OF EQUATIONS

Let us considers $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$

This is a polynomial in x of degree n provided a_0 is $\neq 0$.

The equation is obtained by putting $f(x) = 0$ that is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \rightarrow ②$. is called an algebraic equation of degree.

Any value of x for which the polynomial $f(x)$ vanishes is called a root of the equation $f(x)=0$ that is if 'a' is a root of the equation $f(x)=0$, then $f(a)=0$.

RESULT :-

(i) If $f(a)=0$ the polynomial $f(x)$ has the factor $(x-a)$ that is if 'a' be the root of equation $f(x)=0$ then $(x-a)$ is a factor of the polynomial $f(x)$.

(ii) If $f(a)$ and $f(b)$ are of different signs then atleast one root of the equation $f(x)=0$ must lie between a and b.

(iii) If $f(x)=0$ is an equation of odd degree it has atleast one real root whose sign is opposite to that of the last term.

(iv) If $f(x)=0$ is of even degree and the constant term is negative the eqn has atleast one positive root and atleast one negative root.

(v) And equation of the n^{th} degree has n roots only that is $f(x)=0$ cannot have more than n roots.

(vi) In an equation with rational coefficients irrational roots occur in pairs that is $a+\sqrt{b}$ is a root then $a-\sqrt{b}$ is also a root.

(vii) In an equation with real coefficients imaginary roots occur in pairs that is if $\alpha+i\beta$ is a root then $\alpha-i\beta$ is also a root.

Relations between the roots and co-efficients of equations :-

Let us consider the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n=0 \rightarrow ①$$

Let $\alpha_1, \alpha_2 \dots \alpha_n$ be its roots we have

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n=a_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n).$$

$$=a_0\left[x^n-\Sigma\alpha_1x^{n-1}+\Sigma\alpha_1\alpha_2x^{n-2}-\dots+(-1)^n\alpha_1\alpha_2\alpha_3\dots\alpha_n\right] \rightarrow ②$$

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where $\Sigma\alpha_1$ = sum of the roots taken one at a time

$$= \alpha_1 + \alpha_2 + \dots + \alpha_n$$

$\Sigma \alpha_1\alpha_2$ = sum of the product of the roots taken two at a time.

$$= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \dots$$

$\Sigma \alpha_1\alpha_2\alpha_3$ = sum of the product of the roots taken three at a time.

$$= \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\alpha_4 + \dots$$

Equating like coefficients on both sides of ② we get

(ie) $a_1 = -a_0 \Sigma\alpha_1$

$$\Sigma\alpha_1 = \frac{-a_1}{a_0}$$

$$a_2 = a_0 \Sigma\alpha_1\alpha_2$$

$$\Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}$$

$$a_3 = -a_0 \Sigma\alpha_1\alpha_2\alpha_3$$

$$\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}.$$

and finally we get

$$a_n = (-1)^n (\alpha_1\alpha_2\dots\alpha_n) a_0$$

$$(\alpha_1\alpha_2\dots\alpha_n) = (-1)^n \frac{a_n}{a_0}$$

PROBLEMS:-

1) If α and β are the roots of $2x^2+3x+5=0$ find $\alpha+\beta$ and $\alpha\beta$.

Soln:-

Here $a_0=2$; $a_1=3$; $a_2=5$

By using the relation between roots and coeffcients,

$$\alpha+\beta = -\frac{a_1}{a_0}$$

$$= -3/2$$

$a_0=2$

$a_1=3$

$a_2=5$

$$\alpha\beta = \frac{a_2}{a_0}$$

$$= 5/2 .$$

2) If α, β, γ are the roots of $2x^3+3x^2+5x+6=0$ find $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\alpha\beta\gamma$

Soln:-

Here $a_0=2$; $a_1=3$; $a_2=5$; $a_3=6$

$$\Sigma\alpha = -\frac{a_1}{a_0}$$

$$= -3/2$$

$$\Sigma\alpha\beta = \alpha\beta+\alpha\gamma+\beta\gamma = \frac{a_2}{a_0}$$

$$= 5/2$$

$\alpha\beta\gamma = (-1)^3 \frac{a_3}{a_0}$

$= (-1)^3 6/2$

$= -6/2$

$= -3.$

3) Solve the equation $x^3+6x+20=0$, one root being $1+3i$

Soln:-

Given equation $x^3+6x+20=0$ is cubic
Hence we have three roots.

One root is $1+3i = \alpha$ (say)

Therefore another root is $1-3i = \beta$ ($\because$ since complex root)
To find third root (γ).

Now sum of the roots taken one at a time =

$= \alpha_1+\beta+\gamma = -\frac{a_1}{a_0}$ [Here $a_0=1$; $a_1=0$].

$\alpha+\beta+\gamma = \frac{0}{1}$

$1+3i+1-3i+\gamma = 0$

$\boxed{\gamma=-2}$

Hence the roots of given eqn are

$\{1+3i\}$, $1-3i$, -2.

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A) Solve the equation $3x^3-23x^2+72x-170=0$ having given that $3+\sqrt{-5}$ is a root.

<u>Soln:-</u>

Given equation is $3x^3-23x^2+72x-70=0$

Hence we have three roots.

One root is $3+i\sqrt{5}=\alpha$ (say)

$\therefore$ Another root is $3-i\sqrt{5}=\beta$ ($\because$ since complex root)

To find the third root (γ):-

Now sum of the roots taken one at a time

$$\alpha+\beta+\gamma=-\frac{a_1}{a_0} \qquad [\text{Here } a_0=3;\ a_1=-23]$$

$$3+\cancel{i\sqrt{5}}+3-\cancel{i\sqrt{5}}+\gamma=\frac{23}{3}$$

$$3+3+\gamma=\frac{23}{3}$$

$$6+\gamma=\frac{23}{3}$$

$$\gamma=\frac{23}{3}-6$$

$$\gamma=5/3$$

Hence the roots of the given eqn is

$3+i\sqrt{5},\ 3-i\sqrt{5},\ 5/3.$

5) Solve the eqn $x^4+4x^3+6x^2+4x+5=0$ given that $\sqrt{-1}$

5m

Soln:-

Given $x^4+4x^3+6x^2+4x+5=0 \rightarrow ①$

It is of degree 4.

Hence we have 4 roots.

One root is $\sqrt{-1}$ (or) i

$\therefore -i$ is also a root ($\because$ since complex roots)

Since $x=i$ and $x=-i$ are the roots we have

$(x-i)(x+i)$ is a factor of eqn ①

$\therefore (x^2+1)$ is a factor of eqn ①

Dividing equation ① by (x^2+1)

$$x^2+4x+5$$

$$x^2+1 \;\big)\; x^4+4x^3+6x^2+4x+5$$

$$\underset{(-)}{x^4} \underset{(-)}{+x^2}$$

$$4x^3+5x^2+4x+5$$

$$\underset{(-)}{4x^3} \underset{(-)}{+4x}$$

$$5x^2+5$$

$$\underset{(-)}{5x^2} \underset{(-)}{+5}$$

$$0$$

The quotient is $x^2+4x+5=0$

solve this equation we get

$$= \frac{-4 \pm \sqrt{16-4(5)(1)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$x = -2 \pm i$$

Hence the root of the given equation are

$i, -i, -2+i, -2-i$.

6) Solve the equation $x^4+2x^3-16x^2-22x+7=0$ which has a root $2+\sqrt{3}$

Soln:-

Given $x^4+2x^3-16x^2-22x+7=0$

It is of degree 4

Hence we have 4 roots.

One root is $2+\sqrt{3}$

$\therefore 2-\sqrt{3}$ is also a root ($\because$ since irrational root)

Since $x = 2+\sqrt{3}$ and $x = 2-\sqrt{3}$ are the roots we have $[x-(2+\sqrt{3})]\,[x-(2-\sqrt{3})] = [(x-2)-\sqrt{3}]\,[(x-2)+\sqrt{3}]$

$$= [x-2-\sqrt{3}][x-2+\sqrt{3}]$$

$$x^2 - 2x + \sqrt{3}x - 2x + 4 - 2\sqrt{3} - \sqrt{3}x + 2\sqrt{3} - 3$$

$$= x^2 - 2x - 2x + 4 - 2\sqrt{3} + \sqrt{3}x + 2\sqrt{3} - \sqrt{3}x - 3$$

$$= x^2 - 4x + 4 - 3$$

$$= x^2 - 4x + 1$$

$$\begin{array}{r|l}
 & x^2+6x+7 \\
\hline
x^2-4x+1 & x^4+2x^3-16x^2-22x+7 \\
 & x^4-4x^3+x^2 \\
 & \overset{(-)}{} \quad \overset{(+)}{} \quad \overset{(-)}{} \\
\cline{2-2}
 & \quad 6x^3-17x^2-22x+7 \\
 & \quad 6x^3-24x^2+6x \\
 & \quad \overset{(-)}{} \quad \overset{(+)}{} \quad \overset{(-)}{} \\
\cline{2-2}
 & \qquad 7x^2-28x+7 \\
 & \qquad 7x^2-28x+7 \\
 & \qquad \overset{(-)}{} \quad \overset{(+)}{} \quad \overset{(-)}{} \\
\cline{2-2}
 & \qquad\qquad 0 \\
\cline{2-2}
\end{array}$$

The Quotient is x^2+6x+7

solve this equation we get

$$= \frac{-6 \pm \sqrt{36-4(7)(1)}}{2}$$

$$= \frac{-6 \pm \sqrt{36-28}}{2}$$

$$= \frac{-6 \pm \sqrt{8}}{2}$$

$$= -6 \pm \sqrt{2}$$

$$x = -3 \pm \sqrt{2}$$

Hence the root of the given equation are

$2+\sqrt{3}, 2-\sqrt{3}, -3+\sqrt{2}, -3-\sqrt{2}$.

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FORM THE EQUATION

1. Form the equation with rational coefficients one roots of whose roots is $(\sqrt{2}+\sqrt{3})$

Soln:-

One root is $\sqrt{2}+\sqrt{3}$

ie, $x = \sqrt{2}+\sqrt{3}$

$x-\sqrt{2} = \sqrt{3}$

Squaring on both sides

$(x-\sqrt{2})^2 = (\sqrt{3})^2$

$x^2-2\sqrt{2}x+2 = 3$

$x^2-2\sqrt{2}x+2-3=0$

$x^2-2\sqrt{2}x-1=0$

$x^2-1 = 2\sqrt{2}x$

Squaring on both sides

$(x^2-1)^2 = 4\times2(x^2)$

$x^2-2x^2+1 = 8x^2$

$x^4-10x^2+1=0$

2. Form the equation with rational coefficients having $1+\sqrt{5}$ and $1+\sqrt{-5}$ as two of its roots.

Soln:-

Given $x = 1+\sqrt{5}$ and $x = 1+i\sqrt{5}$. are the roots of required equation.

i.e, $[x-(1+\sqrt{5})]$ and $[x-(1+i\sqrt{5})]$ are the factors of required equation.

Since complex and irrational roots occur in pairs we have

$x = 1-\sqrt{5}$ and $x = 1-i\sqrt{5}$ are also the roots of required equation.

i.e, $[x-(1-\sqrt{5})]$ and $[x-(1-i\sqrt{5})]$ are also the factors of the required equation.

$$[x-(1+\sqrt{5})]\,[x-(1-\sqrt{5})]\,[x-(1+i\sqrt{5})]\,[x-(1-i\sqrt{5})]=0$$

$$[(x-1)-\sqrt{5}]\,[(x-1)+\sqrt{5}]\,[(x-1)-i\sqrt{5}]\,[(x-1)+i\sqrt{5}]=0$$

$$[(x-1)^2-(\sqrt{5})^2]\,[(x-1)^2+(\sqrt{5})^2]=0$$

$$[x^2-2x+1-5]\,[x^2-2x+1+5]=0$$

$$[x^2-2x-4]\,[x^2-2x+6]=0$$

$$x^4-2x^3+6x^2-2x^3+4x^2-12x-4x^2+8x-24=0$$

$$x^4-4x^3+6x^2-4x-24=0$$

Which the required equation.

3. Solve the equation $x^3-12x^2+39x-28=0$ whose roots are in A.P

11 a)

5m. Soln:-

Given equation is cubic hence it has three roots.

Let the roots be $\alpha-d, \alpha, \alpha+d$

Now sum of the roots,

$$(\alpha-d)+\alpha+(\alpha+d) = \frac{-a_1}{a_0} \quad \left\{ \because \text{ Here } \begin{array}{l} a_1 = -12 \\ a_0 = 1 \end{array} \right\}$$

$$3\alpha = \frac{-(-12)}{1}$$

$$3\alpha = 12$$

$$\boxed{\alpha = 4}$$

∴ $\alpha=4$ is a one root of given equation. by using the division we have.

$$\begin{array}{c|cccc} 4 & 1 & -12 & 39 & -28 \\ & 0 & 4 & -32 & 28 \\ \hline & 1 & -8 & 7 & \boxed{0} \end{array}$$

The reduced eqn is

$$x^2-8x+7=0$$

Solving the equation we get.

$$(x-1)(x-7)=0$$

$$x=1,7$$

Hence the roots of the given equation of

1, 4, 7.

10M * 4. Solve the equation $x^4 \mp 2x^3 - 21x^2 \pm 22x + 40 = 0$ whose roots are in A.P.

Soln:-

Given $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0 \rightarrow ①$

It has 4 roots.

Let the roots be $(a-3d), (a-d), (a+d), (a+3d)$

Now the sum of the roots

$$(a-3d)+(a-d)+(a+d)+(a+3d) = \frac{-a_1}{a_0}.$$

$$4a = \frac{-2}{1}$$

$$4a = -2$$

$$\boxed{a = -1/2} \rightarrow ②.$$

$a_0 = 1$

$a_1 = 2$

$a_2 = -21$

$a_3 = -22$

$a_4 = 40$

Product of the roots

$$(a-3d)(a-d)(a+d)(a+3d) = \frac{a_4}{a_0}$$

$$(a^2 - 9d^2)(a^2 - d^2) = \frac{40}{1}$$

By using eqn ②

$$(1/4 - 9d^2)(1/4 - d^2) = 40$$

$$\frac{1}{16} - \frac{1}{4}d^2$$

$$(1 - 36d^2)(1 - 4d^2) = 640$$

$$1-36d^2-4d^2+144d^4=640$$

$$144d^4-40d^2+1-640=0$$

$$144d^4-40d^2-639=0$$

$$d^2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{40\pm\sqrt{1600+368064}}{288}$$

$$d^2=\frac{40\pm\sqrt{369664}}{288}$$

$$=\frac{40\pm 608}{288}$$

$$=\frac{\overset{\cancel{162}\ \cancel{81}\ 9}{\cancel{648}}}{\underset{\cancel{72}\ \cancel{36}\ 4}{\cancel{288}}},-\frac{568}{288}$$

$d^2=9/4$ (or) $d^2=-1.97$; $d^2=\pm 3/2$ (or) $\pm\sqrt{1.97}$
$\downarrow$
{ imaginary which is not possible

$d=3/2, -3/2$

Case(i):- when $d=3/2$ & $a=-1/2$ the roots are

$d=3/2$; $\left(-1/2-9/2\right),\left(-1/2-3/2\right),\left(-1/2+3/2\right),\left(-1/2+9/2\right)$

$\Rightarrow(-5,-2,1,4)$

case(ii) when

$d=-3/2$; $a=-1/2$ the roots are

$\Rightarrow \left(-\frac{1}{2}+\frac{9}{2}\right), \left(-\frac{1}{2}+\frac{3}{2}\right), \left(-\frac{1}{2}-\frac{3}{2}\right), \left(-\frac{1}{2}-\frac{9}{2}\right)$

$\Rightarrow 4, 1, -2, -5.$

Hence the roots of the given equation is

$-5, -2, 1, 4.$

H.W:-

1) Solve the equation $32x^3-48x^2+22x-3=0$ whose roots in A.

Soln:-

Given equation is cubic hence it has three roots.

Let the roots be $\alpha-d, \alpha, \alpha+d$

Now the sum of the roots,

$$(\alpha-d)+\alpha+(\alpha+d) = \frac{-a_1}{a_0} \quad \left\{ \text{Here } a_1=-48; \; a_0 = 32 \right\}$$

$$3\alpha = \frac{\cancel{48}^{\,6\;3}}{\cancel{32}_{\,4\;2}}$$

$$3\alpha = \frac{3}{2}$$

$$\alpha = \frac{3}{6}$$

$$\boxed{\alpha = \frac{1}{2}}$$

$\therefore \alpha = \frac{1}{2}$ is a one root of given equation by using division we have.

$$\frac{1}{2}\left|\begin{array}{cccc} 32 & -48 & 22 & -3 \\ 0 & 16 & -16 & 3 \\ \hline 32 & -32 & 6 & \boxed{0} \end{array}\right.$$

The reduced eqn is

$$32x^2-32x+6=0$$

Solving the equation we get,

$$= \frac{32 \pm \sqrt{1024-4(6)(32)}}{2(32)}$$

$$= \frac{32 \pm \sqrt{1024-768}}{64}$$

$$= \frac{32 \pm \sqrt{256}}{64}$$

$$= \frac{32 \pm 16}{64}$$

$$= \frac{\cancel{48}^{3}}{\cancel{64}_{4}}, \frac{\cancel{16}^{1}}{\cancel{64}_{4}}$$

$$= 3/4, 1/4$$

Hence the root of the given equation is

$1/4, 1/2, 3/4$.

2) Solve the equation $x^3-6x^2+13x-10=0$ whose roots are in A.P

2ion) i)

<u>Soln:-</u>

Given equation is cubic hence it has three roots.

Let the roots be $\alpha-d, \alpha, \alpha+d$

Now sum of the roots.

$$(\alpha-d)+\alpha+(\alpha+d) = -\frac{a_1}{a_0} \quad \left\{ \because \text{Here } \begin{matrix} a_1 = -6 \\ a_0 = 1 \end{matrix} \right\}$$

$$3\alpha = \frac{6}{1}$$

$$\boxed{\alpha = 2}$$

$\therefore$ $\alpha = 2$ is a one root of given equation by using the division we have

$$\begin{array}{c|cccc} 2 & 1 & -6 & 13 & -10 \\ & 0 & 2 & -8 & 10 \\ \hline & 1 & -4 & 5 & \boxed{0} \end{array}$$

The reduced eqn is

$$x^2-4x+5=0$$

Solving the equation we get.

$$= \frac{4 \pm \sqrt{16-4(5)(1)}}{2}$$

$$= \frac{4 \pm \sqrt{16-20}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$= 2 \pm i$$

Hence the root of the given equation is $2, 2+i, 2-i$.

3) Solve the equation $8x^3 - 84x^2 + 262x - 231 = 0$ whose roots are in A.P.

Soln:-

Given equation is cubic hence it has three roots.

Let the roots be $\alpha-d, \alpha, \alpha+d$

Now sum of the roots

$$(\alpha-d) + \alpha + (\alpha+d) = -\frac{a_1}{a_0}$$

$$3\alpha = \frac{\overset{42\,21}{84}}{\cancel{8}\,\overset{}{4}_2} \qquad \left\{ \because \text{Here } \begin{array}{l} a_1 = -84 \\ a_0 = 8 \end{array} \right\}$$

$$3\alpha = \frac{21}{2}$$

$$\alpha = 21/6$$

$$\boxed{\alpha = 7/2}$$

$\therefore \alpha = 7/2$ is a one root of given equation by using the division we have

$$\begin{array}{c|cccc} 7/2 & 8 & -84 & 262 & -231 \\ & 0 & 28 & -196 & 231 \\ \hline & 8 & -56 & 66 & \boxed{0} \end{array}$$

The reduced equation is

$$8x^2 - 56x + 66 = 0$$

$$= \frac{56 \pm \sqrt{3136 - 2112}}{16}$$

$$= \frac{56 \pm \sqrt{1024}}{16}$$

$$= \frac{56 \pm 32}{16}$$

$$= \frac{\overset{11}{\cancel{88}}}{\underset{2}{\cancel{16}}}, \frac{\overset{3}{\cancel{24}}}{\underset{2}{\cancel{16}}}$$

$$= 11/2, 3/2$$

Hence the root of the given equation is

$11/2, 7/2, 3/2$.

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Q*:

1. Find the condition that the root of the equation $x^3 + px^2 + qx + r = 0$ may be in A.P

Soln:-

Given equation $x^3+px^2+qx+r=0 \rightarrow$ ①

Let the roots be $\{a-\cancel{3}d\},(a),(a+d)$

Now sum of the root is $a-d+a+a+d=\frac{-}{a}$

$\{\because a_0=1;\ a_1=p\}$.

$$3a=\frac{-p}{1}$$

$$\boxed{a=-p/3}$$

$\therefore$ $a=-p/3$ is a root of a given equation

$\therefore$ Put $x=-p/3$ in eqn ① we get

$$\left(-p/3\right)^3+p\left(-p/3\right)^2+q\left(-p/3\right)+r=0$$

$$\frac{-p^3}{27}+\frac{p^3}{9}-\frac{pq}{3}+r=0$$

$$\frac{1}{27}\left[-p^3+3p^3-9pq+27r\right]=0$$

$$2p^3-9pq+27r=0$$

H.W:-

1) Solve the equation $x^4-8x^3+16x^2+8x-15=0$ whose roots are in A.P

Soln:-

Given $x^4-8x^3+16x^2+8x-15=0 \rightarrow ①$

It has 4 roots

Let the roots be $(a-3d),(a-d),(a+d),(a+3d)$

Now the sum of the roots

$$(a-3d)+(a-d)+(a+d)+(a+3d) = -\frac{a_1}{a_0}$$

$$4a = \frac{8}{1}$$

$\{\because$ Here $a_0=1; a_1=-8\}$

$$4a = 8$$

$$\boxed{a=2} \rightarrow ②$$

$$(a-3d)(a-d)(a+d)(a+3d) = \frac{a_4}{a_0}$$

$$(a^2-9d^2)(a^2-d^2) = \frac{-15}{1}$$

By using eqn ②

$$(4-9d^2)(4-d^2)=-15$$

$$(4-9d^2)(4-d^2)=-15$$

$$16-4d^2-36d^2+9d^4=-15$$

$$9d^4-40d^2+16+15=0$$

$$9d^4-40d^2+31=0$$

$$d^2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{40\pm\sqrt{1600-4(9)(31)}}{18}$$

$$= \frac{40 \pm \sqrt{1600-116}}{18}$$

$$= \frac{40 \pm \sqrt{484}}{18}$$

$$= \frac{40 \pm 22}{18}$$

$$= \frac{62}{18} , \frac{18}{18}$$

$$d^2 = \frac{31}{9} , 1$$

$$d = \pm\frac{\sqrt{31}}{3} , \pm 1$$

$$d = 1, -1$$

Case (i) :-

when $d = 1$; $a = 2$

$$(2-3(1)), (2\mp 1), (2+1), (2+3(1)).$$

$$\Rightarrow -1, 1, 3, 5.$$

Case (ii): when $d=-1$; $a=2$

$$(2-3(-1)), (2+1), (2-1), (2+3(-1))$$

$$\Rightarrow 5, 3, 1, -1$$

Hence the root of the given equation is

$-1, 1, 3, 5$.

2. Find the value of k for which the roots of eqn $2x^3+6x^2+5x+k=0$ are in A.P.

Soln:-

Given equation $2x^3+6x^2+5x+k=0 \rightarrow$ ①

Let the roots be $a-d, a, a+d$

Now sum of the root is $a-d+a+a+d = -\frac{a_1}{a_0}$.

$$3a = \frac{-6}{2} \qquad \{\because a_0=2; a_1=6\}$$

$$3a = -3$$

$$\boxed{a=-1}$$

$\therefore a=-1$ is a root of a given equation

$\therefore$ Put $x=-1$ in eqn ① we get

$$2(-1)^3+6(-1)^2+5(-1)+k=0$$

$$2(-1)+6-5+k=0$$

$$-2+6-5+k=0$$

$$6-7+k=0$$

$$-1+k=0$$

$$\boxed{k=1}$$

28/6 **G.P:-**

1) Solve the equation $3x^3-26x^2+52x-24=0$ whose roots are in G.P.

Soln:-

Given $3x^3-26x^2+52x-24=0 \rightarrow ①$

Let the root be $\frac{a}{r}, a, ar$.

Now product of the roots is

$$\frac{a}{r} \times a \times ar = \frac{-a_3}{a_0} \quad \left\{ \because a_3=-24; \; a_0=3 \right\}$$

$$a^3 = \frac{24}{3}$$

$$a^3=8$$

$$\boxed{a=2}$$

$\therefore a=2$ is a root of the given equation

$\therefore x=2$ is a root of ①

$(x-2)$ is a factor of ①

By using the division we have

$$3x^3-26x^2+52x-24=0$$

$$\begin{array}{r|rrrr} 2 & 3 & -26 & 52 & -24 \\ & 0 & 6 & -40 & 24 \\ \hline & 3 & -20 & 12 & \boxed{0} \end{array}$$

The reduced eqn is

$$3x^2-20x+12=0.$$

$$=\frac{20 \pm \sqrt{400-4(12)(3)}}{6}$$

$$=\frac{20 \pm \sqrt{400-144}}{6}$$

$$=\frac{20 \pm \sqrt{256}}{6}$$

$$=\frac{20 \pm \sqrt{256}}{6}$$

$$=\frac{20 \pm 16}{6}$$

$$=\frac{36}{6}, \frac{4}{6}$$

$$x = 6, 2/3$$

$$x=6; \; x=2/3$$

The roots are $2/3, 2, 6$.

2) Find the condition that the roots of the equation $x^3 \mp px^2+qx-r=0$ may be in G.P

x 5m
a

Soln:-

The given $x^3-px^2+qx-r=0 \rightarrow$ ①

Let the root be $\frac{a}{r}$, a, ar

Now the product of the root is

$$\frac{a}{r} \times a \times ar = \frac{-a_3}{a_0} \qquad \{\because a_3 = r \,;\; a_0 = 1\}$$

$$a^3 = r$$

$$\boxed{a = r^{1/3}} \rightarrow ②$$

Since 'a' is a root of eqn ①

Put $x = a$ in eqn ① we get

$$a^3 - pa^2 + qa - r = 0$$

$$\cancel{r} - pa^2 + qa - \cancel{r} = 0$$

$x^3 - px^2 + qx - r = 0$

$$a(q - pa) = 0$$

$$q - pa = 0 \qquad (\because a \neq 0)$$

$$q = ap$$

$$\frac{q}{p} = a$$

$$\frac{q^3}{p^3} = a^3$$

$$\frac{q^3}{p^3} = r$$

$$\boxed{q^3 = rp^3}.$$

Hence the recured condition is $p^3 r = q^3$

3) Solve $x^3+x^2-16x+20=0$ the difference between two of its roots being seven.

Soln:-

The given eqn $x^3+x^2-16x+20=0 \to$ ①

Let the roots be $\alpha, \alpha+7, \beta$

Now the sum of the roots taken one at a time

$a_1 \quad a_2 \quad \alpha_1$

$\alpha+(\alpha+7)+\beta = \frac{-a_1}{a_0}$

$2\alpha+\beta+7=-1$ $\{\because a_1=1;\ a_0=1\}$

$2\alpha+\beta=-8 \to$ ②

Sum of the roots taken two at a time

$\{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3=\frac{a_2}{a_0}\}$

$\alpha(\alpha+7)+\alpha\beta+(\alpha+7)\beta=\frac{-16}{1}$

$\alpha^2+7\alpha+\alpha\beta+\alpha\beta+7\beta=-16$

$\alpha^2+2\alpha\beta+7\alpha+7\beta=-16$

$\alpha^2+2\alpha\beta+7(\alpha+\beta)=-16 \to$ ③

By eqn ②

$\beta=-8-2\alpha$ in ③.

$\alpha^2+2\alpha(-8-2\alpha)+7(\alpha-8-2\alpha)=-16$

$\alpha^2-16\alpha-4\alpha^2+7\alpha-56-14\alpha=-16$

$-3\alpha^2-23\alpha-40=0$

$3x^2+23\alpha+40=0$

$$= \frac{-23 \pm \sqrt{(23)^2-4(3)(40)}}{6}$$

$$= \frac{-23 \pm \sqrt{529-480}}{6}$$

$$= \frac{-23 \pm \sqrt{49}}{6}$$

$$= \frac{-23 \pm 7}{6}$$

$$= \frac{-30}{6}, \frac{-16}{6}$$

$\alpha = -5, -8/3$

Since α is a root of eqn ① it should satisfy the given equation.

But $\alpha = -8/3$ does not satisfy eqn ①

$\therefore$ $\alpha = -5$ is a root of eqn ①

$2(-5)+\beta=-8$

$-10+\beta=-8$

$$\boxed{\beta = 2}$$

Hence the root of the eqn is $-5, 2, 2$.

H.W:-

1) Solve the equation $27x^3+42x^2-28x-8=0$ whose roots are in G.P.

Soln:-

The given $27x^3+42x^2-28x-8=0 \rightarrow ①$

Let the root be $\frac{a}{r}, a, ar$

Now product of the root is

$$\frac{a}{r} \times a \times ar = -\frac{a_3}{a_0} \qquad \left\{ \because \begin{array}{l} a_3 = -8 \\ a_0 = 27 \end{array} \right\}$$

$$a^3 = \frac{8}{27}$$

$$\boxed{a = 2/3}$$

$\therefore a = 2/3$ is a root of the given equation

$\therefore x = 2/3$ is a root of ①

$(x - 2/3)$ is a factor of ①

By using the division we have

$$27x^3 + 42x^2 - 28x - 8 = 0$$

$$\begin{array}{c|cccc} & 27 & 42 & -28 & -8 \\ 2/3 & 0 & 18 & 40 & 8 \\ \hline & 27 & 60 & 12 & \boxed{0} \end{array}$$

$27x^3 + 60x^2 + 12x$

$27x^2 + 60x + 12 = 0$

$$= \frac{-60 \pm \sqrt{3600-1296}}{54}$$

$$= \frac{-60 \pm \sqrt{2304}}{54}$$

$$= \frac{-60 \pm 48}{54}$$

$$= \frac{-12}{54}, \frac{-108}{54}$$

$x = -2/9, -2$

Hence the roots are $-2/9, 2/3, -2$

2. Solve the equation $x^3 - 7x^2 + 14x - 8 = 0$ whose roots are in G.P

Soln:-

Given $x^3 - 7x^2 + 14x - 8 = 0 \rightarrow$ ①

Let the roots be $\frac{a}{r}, a, ar$

Now product of the root is

$$\frac{a}{r} \times a \times ar = -\frac{a_3}{a_0} \quad \left\{ \because a_3 = 8;\ a_0 = 1 \right\}$$

$$a^3 = \frac{8}{1}$$

$$\boxed{a=2}$$

$\therefore$ $a = 2$ is a root of the given equation.

$\therefore$ $x = 2$ is a root of ①

$(x-2)$ is a factor of ①

By using the division we get,

$$x^3-7x^2+14x-8=0$$

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 14 & -8 \\ & 0 & 2 & -10 & 8 \\ \hline & 1 & -5 & 4 & \boxed{0} \end{array}$$

$$x^2-5x+4=0$$

$$(x-1)(x-4)=0$$

$$x = 1, 4$$

Hence the roots are 1, 2, 4.

29/6.

1. Solve $2x^3-x^2-22x-24=0$ two of the roots being in the ratio 3:4.

✓

Soln:

Given $2x^3-x^2-22x-24=0 \rightarrow$ ①

Let the roots be $3\alpha, 4\alpha, \beta$

Now sum of the root is

$$3\alpha+4\alpha+\beta = -\frac{a_1}{a_0} \qquad \left\{\because a_0 = 2 ; a_1 = -1 ; a_2 = -22\right\}$$

$$7\alpha+\beta = 1/2$$

$$\beta = 1/2 - 7\alpha \rightarrow ②.$$

Sum of the roots taken two at a time

$$\left[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3 = \frac{a_2}{a0}\right]$$

$$3\alpha 4\alpha + 3\alpha\beta + 4\alpha\beta = \frac{-22}{2}$$

$$12\alpha^2 + 7\alpha\beta = -11 \rightarrow ③$$

By using ② & ③ we get,

$$12\alpha^2 + 7\alpha(1/2 - 7\alpha) = -11$$

$$12\alpha^2 + \frac{7}{2}\alpha - 49\alpha^2 + 11 = 0$$

$$-37\alpha^2 + \frac{7}{2}\alpha + 11 = 0$$

$$-74\alpha^2 + 7\alpha + 22 = 0$$

$$\alpha = \frac{-7 \pm \sqrt{49 - 4(22)(-74)}}{2(-74)}$$

$$\alpha = \frac{-7 \pm \sqrt{49 + 6512}}{-148}$$

$$= \frac{-7 \pm \sqrt{6561}}{-148}$$

$$= \frac{-7 \pm 81}{-148}$$

$$= \frac{7 \pm 81}{148}$$

$$= \frac{\overset{\cancel{44}\,22}{\cancel{88}}}{\underset{\cancel{74}\,37}{\cancel{148}}}, \frac{-\overset{1}{\cancel{74}}}{\underset{2}{\cancel{148}}}$$

$$= \frac{22}{37}, -\frac{1}{2}$$

$$\alpha = \frac{22}{37}, -\frac{1}{2} \rightarrow ④$$

By using ④ in ② we get,

$$\beta = \frac{1}{2} - \frac{7}{2}$$

$$= \frac{-8}{2}$$

$$\boxed{\beta = -4}$$

$$\beta = \frac{1}{2} - 7\left(\frac{22}{37}\right)$$

$$= \frac{1}{2} - \frac{154}{37}$$

$$= \frac{74-154}{\cancel{74}}$$

$$= \frac{37-154}{74}$$

$$= \frac{37-308}{74}$$

$$\boxed{\beta = \frac{-271}{74}}$$

Now the product of the roots

$(3\alpha)(4\alpha)(\beta) = -\frac{a_3}{a_0}$ $\{a_3 = -24 ; a_0 = 2\}$

$12\alpha^2\beta = \frac{24}{2}$

$12\alpha^2\beta = 12$

$\alpha^2\beta = 1 \rightarrow ⑤$

The value of α and β which satisfy eqn ⑤ is $\alpha = -1/2$, $\beta = 4$.

Hence the root are $-3/2, -2, 4$.

2. Solve $x^4 - 8x^3 + 7x^2 + 36x - 36 = 0$ given that the product of two of the roots is negative of the product of the remaing 2.

Soln:-

Given $x^4 - 8x^3 + 7x^2 + 36x - 36 = 0 \rightarrow ①$

Let the roots be $\alpha, \beta, \gamma, \delta$

Since the product of two roots = $-$ { product of remaining 2 roots }

$$\alpha\beta = -\gamma\delta \rightarrow ②.$$

Now the product of the given roots

$$\alpha\beta\gamma\delta = \frac{a_4}{a_0} \quad \{\because a_4 = -36 \; ; a_0 = 1\}$$

$$\alpha\beta(-\alpha\beta) = \frac{-36}{1}$$

$$\alpha^2\beta^2 = 36$$

$$\alpha\beta = 6 \text{ (or) } -6$$

$$\gamma\delta = -6 \text{ (or) } 6.$$

The factors corresponding to these roots are of the form $(x^2 - px - 6 = 0)$, $(x^2 - qx + 6 = 0)$

$\therefore \; x^2 - px - 6 = 0;$

$x^2 - qx + 6 = 0$

$$\therefore x^4 - 8x^3 + 7x^2 + 36x - 36 = (x^2 - px - 6)(x^2 - qx + 6)$$

Equating like coefficients we get

$-8 = -q - p$ ($\because$ coefficient of x^3)

$p + q = 8$

$36 = -6p + 6q$ ($\because$ coefficient of x)

$-p + q = 6$

$x^2 - $(sum of the roots) x + product of the roots.

Solve this term.

$$p + q = 8$$
$$-p + q = 6$$
$$\overline{\quad 2q = 14 \quad}$$
$$\boxed{q = 7}$$

$\Rightarrow p + 7 = 8.$

$p = 8 - 7$

$\boxed{p = 1}$

$\therefore p = 1 ; q = 7$

Given equation can be written as

$(x^2 - x - 6)\ (x^2 - 7x + 6) = 0$

$(x-3)(x+2)\ (x-6)(x-1) = 0$

$\therefore x = 1, 3, 6, -2.$

Hence the roots are $-2, 1, 3, 6.$

Symmetric function of the roots :-

A symmetric function of the roots of an equation is a function involving all the roots of an equation such that expression remains unaltered when two of the roots are interchanged.

30/6

1. If $\alpha \beta \gamma$ are the roots of the equation $x^3-px^2+qx-r=0$ find the value of

(i) $\Sigma\alpha^2$ $(=\alpha^2+\beta^2+\gamma^2)$

(ii) $\Sigma\alpha^3$ $(=\alpha^3+\beta^3+\gamma^3)$

(iii) $\Sigma\alpha^2\beta$ $(=\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma+\gamma^2\beta+\gamma^2\alpha+\beta^2\alpha)$

(iv) $\Sigma\alpha^2\beta^2$ $(=\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)$.

Soln:-

Given $\overset{a_0}{x^3}\overset{a_1}{-px^2}\overset{a_2}{+qx}\overset{a_3}{-r}=0 \rightarrow ①$

Let α, β, γ are the root of eqn ①

Now the roots taken one at a time

$$\alpha+\beta+\gamma = \frac{-a_1}{a_0} = \frac{p}{1} \qquad \left\{\because \begin{array}{ll} a_0=1 & ; a_2=q \\ a_1=-p & ; a_3=-r \end{array}\right\}$$

$$\alpha+\beta+\gamma = p \rightarrow ②$$

Now the roots taken two at a time.

$$\alpha\beta+\beta\gamma+\alpha\gamma = \frac{a_2}{a_0} = \frac{q}{1}$$

$$\alpha\beta+\beta\gamma+\alpha\gamma = q \rightarrow ③$$

Now the product of the roots.

$$\alpha\beta\gamma = \frac{-a_3}{a_0} = \frac{r}{1}$$

$$\alpha\beta\gamma = r \rightarrow ④.$$

(i) $\Sigma\alpha^2 = \alpha^2+\beta^2+\gamma^2$

$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$= p^2 - 2q$ (∵ using ② and ③).

$\Sigma\alpha^2 = p^2 - 2q \rightarrow ⑤$

(ii) $\Sigma\alpha^3 = \alpha^3+\beta^3+\gamma^3$.

$(a+b+c)^3 = a^3+b^3+c^3+3ab^2+3ac^2+3a^2b+3$

$+3a^2c+3bc^2+6abc$

$$= (\alpha+\beta+\gamma)[\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\alpha\gamma)]+3\alpha\beta\gamma]$$

$$= p[p^2-2q-(q)]+3r.$$

$= p[p^2-2q-q]+3r$ (∵ using ②, ③, ④, ⑤)

$$= p[p^2-3q]+3r$$

$$= p^3-3pq+3r$$

(iii) $\Sigma\alpha^2\beta = (\alpha^2\beta+\overset{+\alpha\beta\gamma}{\alpha^2\gamma}+\beta^2\gamma+\overset{+\alpha\beta\gamma}{\gamma^2\beta}+\overset{+\alpha\beta\gamma}{\gamma^2\alpha}+\beta^2\alpha)\begin{matrix}+3\alpha\beta\gamma\\-3\alpha\beta\gamma\end{matrix}$

$(\alpha^2\beta+\alpha^2\gamma+\alpha\beta\gamma)+(\beta^2\gamma+\beta^2\alpha+\alpha\beta\gamma)+(\gamma^2\beta+\gamma^2\alpha+\alpha\beta\gamma)-3\alpha\beta\gamma$

$$= (\alpha+\beta+\gamma)[\alpha\beta+\beta\gamma+\alpha\gamma]-3\alpha\beta\gamma$$

(∵ using ②, ③, ④.

$= pq - 3r$

(iv) $\sum \alpha^2\beta^2 = (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)$

$= (\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

(∵ Using ②, ③, ④).

$= q^2 - 2pr$

2. If $\alpha\beta\gamma$ are the root of the equation $x^3 + px^2 + qx + r = 0$ find the value of

(i) $\sum \alpha^2$ $(= \alpha^2 + \beta^2 + \gamma^2)$

(ii) $\sum 1/\alpha$ $(= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})$

(iii) $\sum \frac{1}{\alpha\beta}$ $(= \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma})$

(iv) $\sum \alpha^2\beta^2$ $(= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)$

(v) $\sum \alpha^3$ $(= \alpha^3 + \beta^3 + \gamma^3)$

Soln:-

Given $x^3 + px^2 + qx + r = 0 \rightarrow$ ①

Let α, β, γ are the roots of eqn ①

Sum of the root taken one at a time

$$\alpha + \beta + \gamma = -\frac{a_1}{a_0} = \frac{-p}{1}$$

$\alpha + \beta + \gamma = -p \rightarrow$ ②.

Sum of the root taken two at a time

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{a_2}{a_0}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{q}{1}$$

$\alpha\beta+\beta\gamma+\alpha\gamma = q \rightarrow ③$

Product of the roots,

$$\alpha\beta\gamma = \frac{-a_3}{a_0} = \frac{-r}{1}$$

$$\alpha\beta\gamma = -r \rightarrow ④$$

(i) $\Sigma\alpha^2 = \alpha^2+\beta^2+\gamma^2.$

$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma).$$

$$= (-p)^2 - 2q \qquad (\because \text{Using } ② \text{ and } ③).$$

$$\Sigma\alpha^2 = p^2 - 2q \rightarrow ⑤.$$

(ii) $\Sigma 1/\alpha = 1/\alpha + 1/\beta + 1/\gamma$

$$= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma} \qquad (\because \text{Using } ③ \text{ and } ④)$$

$$\Sigma 1/\alpha = -\frac{q}{r} \rightarrow ⑥$$

(iii) $\Sigma 1/\alpha\beta = 1/\alpha\beta + 1/\alpha\gamma + 1/\beta\gamma$

$$= \frac{\gamma+\beta+\alpha}{\alpha\beta\gamma} \qquad (\because \text{Using } ④, ③)$$

$$= \frac{-p}{-r}$$

$\sum 1/\alpha\beta = \frac{p}{r}$ → ⑦

(iv) $\sum\alpha^2\beta^2 = \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2.$

$= (\alpha\beta+\beta\gamma+\alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= q^2 + 2r(-p)$

$\sum\alpha^2\beta^2 = q^2 - 2rp$ → ⑧.

(v) $\sum\alpha^3 = \alpha^3+\beta^3+\gamma^3.$

$= (\alpha+\beta+\gamma)[\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\alpha\gamma)]+3\alpha\beta\gamma$

$= -p[p^2-2q-(+q)]-3r.$

$= -p[p^2-2q-q]-3r$

$= -p[p^2-3q]-3r$

$\sum\alpha^3 = -p^3+3pq-3r.$

$\sum\alpha^3 = p^3-3pq+3r$ → ⑨

3. If $\alpha\beta\gamma$ are the roots of $x^3+px^2+qx+r=0$ find the value of $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)$

Soln:-

Given $x^3+px^2+qx+r=0$ → ①

Let α, β, γ are the roots of eqn ①

Sum of the root one at a time.

$$\alpha+\beta+\gamma = -\frac{a_1}{a_0} = \frac{-p}{1}$$

$$\alpha+\beta+\gamma = -p \rightarrow ②$$

Sum of the root two at a time

$$\alpha\beta+\beta\gamma+\alpha\gamma = \frac{a_2}{a_0} = \frac{q}{1}$$

$$\alpha\beta+\beta\gamma+\alpha\gamma = q \rightarrow ③$$

Product of the roots.

$$\alpha\beta\gamma = \frac{-a_3}{a_0} = \frac{-r}{1}$$

$$\alpha\beta\gamma = -r \rightarrow ④$$

$$\Rightarrow (\alpha^2\beta^2+\beta^2+\alpha^2+1)\ (\gamma^2+1)$$

$$= \alpha^2\beta^2\gamma^2+\beta^2\gamma^2+\alpha^2\gamma^2+\gamma^2+\alpha^2\beta^2+\beta^2+\alpha^2+1$$

$$= (\alpha\beta\gamma)^2+(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)+(\alpha^2+\beta^2+\gamma^2)+1$$

By using the results ⑧.

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = q^2-2pr$$

$$\alpha^2+\beta^2+\gamma^2 = p^2-2q$$

$$\therefore (\alpha^2+1)(\beta^2+1)(\gamma^2+1) = (-r)^2+(q^2-2pr)+(p^2-2q)+1$$

$$= r^2+q^2-2pr+p^2-2q+1$$

$$=(q^2-2q+1)+(r^2-2pr+p^2)$$

$$=(q-1)^2+(p-r)^2$$

4. If $\alpha\beta\gamma$ are the roots of $x^3-px^2+qx-r=0$ find the values of

(i) $\sum \frac{\beta^2+\gamma^2}{\beta\gamma}$ (or) $\sum\left(\frac{\beta}{\gamma}+\frac{\gamma}{\beta}\right)$

(ii) $\sum(\beta-\gamma)^2$

(iii) $\sum(\beta^2+\beta\gamma+\gamma^2)$

Soln:-

Given $x^3-px^2+qx-r=0 \rightarrow ①$

Let $\alpha\beta\gamma$ are the roots of ①

$$\alpha+\beta+\gamma=\frac{-a_1}{a_0}=\frac{-(-p)}{1}$$

$$\alpha+\beta+\gamma=p \rightarrow ②$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=\frac{a_2}{a_0}=\frac{q}{1}$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=q \rightarrow ③$$

$$\alpha\beta\gamma=\frac{-a_3}{a_0}=\frac{-(-r)}{1}$$

$$\alpha\beta\gamma=r \rightarrow ④$$

(i) $\sum \frac{\beta^2+\gamma^2}{\beta\gamma}$

$$\sum \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\alpha^2+\gamma^2}{\alpha\gamma} + \frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$= \frac{\alpha(\beta^2+\gamma^2)+\beta(\gamma^2+\alpha^2)+\gamma(\alpha^2+\beta^2)}{\alpha\beta\gamma}$$

$$= \frac{\alpha^2\beta+\alpha\beta^2+\alpha^2\gamma+\alpha\gamma^2+\beta^2\gamma+\beta\gamma^2}{\alpha\beta\gamma}$$

$$= \frac{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma}{\alpha\beta\gamma}$$

$$\sum \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{pq-3r}{r}.$$

(ii) $\sum(\beta-\gamma)^2$

$$\sum(\beta-\gamma)^2 = (\beta-\gamma)^2 + \cancel{(\alpha-\gamma)}(\gamma-\alpha)^2 + (\alpha-\beta)^2$$

$$= 2(\alpha^2+\beta^2+\gamma^2) - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= 2[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)] - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= 2(p^2-2q)-2q$$

$$= 2p^2-4q-2q$$

$$= 2p^2-6q$$

$$\sum(\beta-\gamma)^2 = 2p^2-6q$$

(pFi) $\sum(\beta^2+\beta\gamma+\gamma^2)$

$$\sum(\beta^2+\beta\gamma+\gamma^2) = (\beta^2+\beta\gamma+\gamma^2)+(\gamma^2+\gamma\alpha+\alpha^2)+(\alpha^2+\alpha\beta+\beta^2)$$

$$= 2(\alpha^2+\beta^2+\gamma^2)+(\alpha\beta+\beta\gamma+\gamma\alpha).$$

$$= 2[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)] + (\alpha\beta+\beta\gamma+\gamma\alpha).$$

$$= 2(p^2-2q)+q$$

$$= 2p^2-4q+q$$

$$\sum(\beta^2+\beta\gamma+\gamma^2) = 2p^2-3q$$

H.W:-

1) Solve $x^4-8x^3+14x^2+8x-15=0$ given that the sum of the two roots = the sum of the other two.

Soln:-

Given $x^4-8x^3+14x^2+8x-15=0 \rightarrow ①$

Let the root $\alpha, \beta, \gamma, \delta$

$\alpha+\beta = \gamma+\delta \rightarrow ②$.

Now sum of the root are,

$$\alpha+\beta+\gamma+\delta = \frac{-a_1}{a_0} \qquad \left\{\because a_0=1;\ a_1=-8;\ a_2=14;\ a_3=8\right\}$$

$$\alpha+\beta+\gamma+\delta = \frac{8}{1}$$

$$\alpha+\beta+\gamma+\delta = 8.$$

$\alpha+\beta+\alpha+\beta=8$

$2\alpha+2\beta=8$

$2(\alpha+\beta)=8$

$\alpha+\beta=4 \,;\, \gamma+\delta=4.$

The factors corresponding to these roots are of the form $\{x^2-4x+p=0 \,;\, x^2-4x+q=0$

$x^4-8x^3+14x^2+8x-15=(x^2-4x+p)(x^2-4x+q)$

Equating like coefficients we get x^2

$14=16+p+q$

$p+q=-2 \rightarrow ③$

Equating constant term.

$pq=-15$

$p=-15/q \rightarrow ④$

Sub in ③

$-15/q+q=-2$

$-15+q^2=-2q$

$q^{2}\overset{5\,-3}{\boxed{+2q}}-15=0$ $\qquad 5\times3=15$

$q^2+5q-3q-15=0$ $\qquad 5+-3=2$

$(q-3)(q+5)=0$

$q = 3, -5$

$p = -5, 3$

$\therefore (x^2 - 4x + 3)(x^2 - 4x - 5) = 0$

$(x-3)(x-1) \,;\, (x-5)(x+1)$

$x = 3, 1, 5, -1$

$\therefore$ Hence the roots are $-1, 1, 3, 5$.

2|7. Formation of equation by symmetric roots :-

1. If α, β, γ are the roots of the equation

✓ $x^3 + ax^2 + bx + c = 0$ form the equation whose roots are $\alpha\beta, \beta\gamma, \gamma\alpha$.

Soln:-

The relations between the roots and coefficients are

$\alpha + \beta + \gamma = -a$ $\quad a_0 = 1$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$ $\quad a_1 = a$

$\alpha\beta\gamma = -c$ $\quad a_2 = b$

$a_3 = c$.

The required equation is,

$$[(x - \alpha\beta)(x - \beta\gamma)](x - \alpha\gamma) = 0$$

$[x^2 - \alpha\beta x - \beta\gamma x + \alpha\beta^2\gamma](x - \alpha x) = 0.$

$x^3 - \alpha x^2 - \alpha\beta x^2 - \alpha^2\beta x$
$- \beta\gamma x^2 + \alpha\beta\gamma x + \alpha^2\beta^2\gamma x$
$- \alpha^2\beta^2\gamma^2$

ie, $x^3 - x^2(\alpha\beta + \beta\gamma + \alpha\gamma) + (\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)x$
$- \alpha^2\beta^2\gamma^2 = 0.$

ie, $x^3 - x^2(\alpha\beta + \beta\gamma + \alpha\gamma) + x\alpha\beta\gamma(\alpha + \beta + \gamma) - (\alpha\beta\gamma)^2 = 0$

$x^3 - bx^2 + (-c)(-a)x - (-c)^2 = 0.$

ie, $x^3 - bx^2 + acx - c^2 = 0$

2. If α, β, γ are the roots if $x^3+px^2+qx+r=0$ form the equation whose roots are.

$\beta+\gamma-2\alpha, \gamma+\alpha-2\beta, \alpha+\beta-2\gamma$

Soln:-

we have $\alpha+\beta+\gamma=-p$

$x^3 - S_1x^2 + S_2x - S_3 =$

$\alpha\beta+\beta\gamma+\alpha\gamma=q$

$\alpha\beta\gamma=-r$

$S_1 \rightarrow$ sum of the roots taken one at a time

In the required equation

$S_2 \rightarrow$ sum of the roots taken two at a time

$S_1 =$ Sum of the roots

$S_3 \rightarrow$ product of the roots

$= \beta+\gamma-2\alpha+\alpha-2\beta+\gamma+\alpha+\beta-2\gamma$

$=0.$

$S_2 =$ Sum of the products of the roots taken two at a time.

$= \underset{+\alpha-\alpha}{(\beta+\gamma-2\alpha)}\underset{+\beta-\beta}{(\gamma+\alpha-2\beta)} + \underset{+\alpha-\alpha}{(\beta+\gamma-2\alpha)}\underset{+\gamma-\gamma}{(\alpha+\beta-2\gamma)} +$

$\underset{+\gamma-\gamma}{(\alpha+\beta-2\gamma)}$ ~~$(\alpha+\beta-2\gamma)$~~

$\underset{+\beta-\beta}{(\gamma+\alpha-2\beta)}.$

$= (\alpha+\beta+\gamma-3\alpha)(\alpha+\beta+\gamma-3\beta) +$ ~~a similar terms~~

$+ (\alpha+\beta+\gamma-3\gamma)(\alpha+\beta+\gamma-3\alpha) + (\alpha+\beta+\gamma-3\beta)(\alpha+\beta+\gamma-3\gamma)$

$= (-p-3\alpha)(-p-3\beta) + (-p-3\alpha)(-p-3\gamma) + (-p-3\gamma)(-p$

$= (p+3\alpha)(p+3\beta) + (p+3\alpha)(p+3\gamma) + (p+3\gamma)(p+3\beta)$

$(p^2+3\alpha p+3\alpha\beta+9\alpha\beta) + (p^2+3\gamma p+3\alpha p+9\alpha\gamma) + (p^2+3\beta p+3 \quad +9\beta\gamma)$

$= 3p^2+6p(\alpha+\beta+\gamma)+9(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 3p^2+6p(-p)+9q.$

$= 9q-3p^2$

$S_3 =$ Product of the roots.

$= (\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma).$

$\quad +\alpha-\alpha \qquad +\beta-\beta \qquad +\gamma-\gamma$

$= (\alpha+\beta+\gamma-3\alpha)(\alpha+\beta+\gamma-3\beta)(\alpha+\beta+\gamma-3\gamma)$

$= (-p-3\alpha)(-p-3\beta)(-p-3\gamma)$

$= -[(p+3\alpha)(p+3\beta)(p+3\gamma)] \Rightarrow -[(p^2+3(\gamma+\alpha)p+9\alpha\gamma)(p+3\gamma)]$

$= -\{p^3+3p^2(\alpha+\beta+\gamma)+9p(\alpha\beta+\beta\gamma+\alpha\gamma)+27\alpha\beta\gamma\}.$

$= -\{p^3+3p^2(-p)+9pq-27\overset{r}{\cancel{\gamma}}\} \Rightarrow -[p^3+3(\gamma+\alpha)p^2+9\alpha\gamma p$

$+3\beta p^2+9\beta(\gamma+\alpha)p$

$+27\alpha\beta\gamma]$

$= +2p^3-9pq+27r$

Hence the required equation is

$$x^3-S_1x^2+S_2x-S_3=0.$$

ie, $x^3+(9q-3p^2)x-(2p^3-9pq+27r)=0.$

5m

Horner's method :-

Sum of the powers of the roots of an equation.

Let $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_n$ be the roots of an equation $f(x)=0$.

The sum of the r^{th} powers of the roots.

ie, $\alpha_1^r+\alpha_2^r+\ldots+\alpha_n^r.$

is usually denoted by S_r. We can easily see that

Sr constitutes a symmetric function of the roots and hence we can calculate the value of Sr by the described methods, ~~described~~ in the previous article.

When r is greater than 4] the calculation of S_r by the previous method becomes tedious and in those cases, the following two methods can be used profitably.

We have $f(x) = (x-\alpha_1)+(x-\alpha_2)+\dots+(x-\alpha_n)$.

Taking logarithms on both sides and differentiating we get.

$$\log f(x) = \log(x-\alpha_1)+\log(x-\alpha_2)+\dots+\log(x-\alpha_n)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\dots+\frac{1}{x-\alpha_n}.$$

$$\frac{x f'(x)}{f(x)} = \frac{x}{x-\alpha_1}+\frac{x}{x-\alpha_2}+\dots+\frac{x}{x-\alpha_n}.$$

$$= \frac{1}{1-\frac{\alpha_1}{x}}+\frac{1}{1-\frac{\alpha_2}{x}}+\dots+\frac{1}{1-\frac{\alpha_n}{x}}.$$

$$= \left(1-\frac{\alpha_1}{x}\right)^{-1}+\left(1-\frac{\alpha_2}{x}\right)^{-1}+\dots+\left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots+x^n+\dots$$

$$= \left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}+\dots+\frac{\alpha_1^n}{x^n}\right)+\dots+\left(1+\frac{\alpha_2}{x}+\frac{\alpha_2^2}{x}+\dots\right.$$

$$\cdots\cdots \frac{\alpha_2^{r}}{x^{r}}) + \cdots + \cdots + (1 + \frac{\alpha_n}{x} + \frac{\alpha_n^2}{x^2} + \cdots + \frac{\alpha_n^{r}}{x^{r}} + \cdots$$

$$\Rightarrow (1 + \frac{\alpha_2}{x} + \frac{\alpha_2^2}{x^2} + \cdots + \frac{\alpha_2^{r}}{x^{r}} + \cdots)$$

$1+1+\cdots+1$ n times

$$= n + (\Sigma\alpha_1)\,1/x + (\Sigma\alpha_1^2)\,1/x^2 + \cdots + (\Sigma\alpha_1^{r})\,1/x^{r} + \cdots$$

$$= n + S_1 \cdot 1/x + S_2 \cdot 1/x^2 + \cdots + S_r \cdot 1/x^{r} + \cdots$$

$\therefore S_r =$ coefficient of $1/x^{r}$ in the expansion of $\dfrac{x f'(x)}{f(x)}$.

Example :-

Find the sum of the cubes of the roots of the equation $x^5 = x^2 + x + 1$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

Soln:-

The equation can be written in the form

$$f(x) \Rightarrow x^5 - x^2 - x - 1 = 0$$

$r = 3.$

$S_r =$ coefficient of $1/x^{r}$ in the expansion of $\dfrac{x f'(x)}{f(x)}$

$S_3 =$ coefficient of $1/x^3$ in the expansion of

$$\frac{x(5x^4 - 2x - 1)}{x^5 - x^2 - x - 1} = \frac{5x^5 - 2x^2 - x}{x^5 - x^2 - x - 1} = \frac{x^5(5 - 2/x^3 - 1/x^4)}{x^5(1 - 1/x^3 - 1/x^4 - 1/x^5)}$$

$$\frac{x\frac{d}{dx}(x^5 - x^2 - x - 1)}{x^5 - x^2 - x - 1}$$

$$= \text{coefficient of } 1/x^3 \text{ in } \frac{5 - 2/x^3 - 1/x^4}{1 - 1/x^3 - 1/x^4 - 1/x^5}.$$

$$= \text{coefficient of } 1/x^3 \text{ in } (5 - 2/x^3 - 1/x^4)(1 - 1/x^3 - 1/x^4 - 1/x^5)^{-1}$$

$$[1 - [1/x^3 + 1/x^4 + 1/x^5]]^{-1}$$

$$= \text{coefficient of } 1/x^3 \text{ in } (5 - 2/x^3 - 1/x^4)$$

$(1-x)^{-1} = 1 + x + x^2 + \cdots$

($\because$ omitting higher power)

$$\{(1 + 1/x^3 + 1/x^4 + 1/x^5) + (1/x^3 + 1/x^4 + 1/x^5) + \cdots\}$$

$= \text{coefficient of } 1/x^3 \text{ in } (5-2/x^3-1/x^4)(1+1/x$

$= 3.$

7/7.

UNIT-II

Newton's theorem on the sum of the powers of the roots :-

2m. Let $\alpha_1, \alpha_2 \dots \alpha_n$ be the roots of the equation

$$f(x) = x^n + P_1 x^{n-1} + P_2 x^{n-2} + \dots + P_n = 0$$

and let be $s_r = \alpha_1^r + \alpha_2^r + \dots + \alpha_n^r$,

So that $s_0 = n$.

Thus $s_r + P_1 s_{r-1} + P_2 s_{r-2} + \dots + r P_r = 0$, if $r<n$

and $s_r + P_1 s_{r-1} + P_2 s_{r-2} + \dots + P_n s_{r-n} = 0$, if $r \geq n$

1. Show that the sum of the Eleventh powers of the roots of $x^7+5x^4+1=0$ is zero.

Proof:-

Given $x^7+5x^4+1=0 \rightarrow ①$

Since 11 is greather than 7, the degree of

$= \text{coefficient of } 1/x^3 \text{ in } (5-2/x^3-1/x^4)(1+1/x$

$= 3.$

7/7.

UNIT-II

Newton's thearem on the sum of the powers of the roots :-

X 2m. Let $\alpha_1, \alpha_2 \dots \alpha_n$ be the roots of the equation

$$f(x) = x^n + P_1 x^{n-1} + P_2 x^{n-2} + \dots + P_n = 0$$

and let be $s_r = \alpha_1^r + \alpha_2^r + \dots + \alpha_n^r$,

So that $s_0 = n$.

Thus $S_r + P_1 S_{r-1} + P_2 S_{r-2} + \dots + r P_r = 0$, if $r < n$

and $s_r + P_1 S_{r-1} + P_2 S_{r-2} + \dots + P_n S_{r-n} = 0$, if $r \geq n$

1. Show that the sum of the Eleventh powers of the roots of $x^7 + 5x^4 + 1 = 0$ is zero.

Proof:-

Given $x^7 + 5x^4 + 1 = 0 \rightarrow$ ①

Since 11 is greather than 7, the degree of the equation we have to use the equation in Newton's theorem,

$$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \cdots + P_n S_{r-n} = 0$$

If we assume the equation

$$x^7 + P_1 x^6 + P_2 x^5 + P_3 x^4 + P_4 x^3 + P_5 x^2 + P_6 x + P_7 = 0 \rightarrow ②.$$

In eqn ① & ② we get.

$$P_3 = 5 ; P_7 = 1$$

$$P_1 = P_2 = P_4 = P_5 = P_6 = 0$$

$$\therefore S_{11} + P_1 S_{10} + P_2 S_9 + P_3 S_8 + P_4 S_7 + P_5 S_6 + P_6 S_5 + P_7 S_4 = 0$$

$$S_{11} + 5S_8 + S_4 = 0 \rightarrow ③.$$

Again,

$$S_8 + P_1 S_7 + P_2 S_6 + P_3 S_5 + P_4 S_4 + P_5 S_3 + P_6 S_2 + P_7 S_1 = 0$$

$$S_8 + 5S_5 + S_1 = 0 \rightarrow ④.$$

Using the 1st theorem

$$S_5 + P_1 S_4 + P_2 S_3 + P_3 S_2 + P_4 S_1 + P_5 5 = 0$$

(ie) $S_5 + 5S_2 = 0 \rightarrow ⑤$

$n = 4$

Again $S_4 + P_1 S_3 + P_2 S_2 + P_3 S_1 + 4P_4 = 0$ $S_0 = n$

(ie) $S_4 + 5S_1 = 0 \rightarrow ⑥$

Again $S_2+P_1S_1+2P_2=0$

(i) $S_2=0 \to ⑦$

Also $S_1=0 \to ⑧$

From eqn ⑥, ⑦, ⑧ we get $S_4=0$

From ⑤ & ⑦ we get $S_5=0$

From ④ we get $S_5=0$

From ④ we get $S_8=0$

Substituting the values of S_4, S_8 in ③ we get $S_{11}=$

Hence the sum of the 11th powers of the root of $x^7+5x^4+1=0$ is zero.

2. If $a+b+c+d=0$ S.T $\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \cdot \frac{a^3+b^3+c^3+d^3}{3}$.

Proof:-

Since $a+b+c+d=0$, we can consider that a, b, c, d are the roots of the equation.

$x^4+P_1x^3+P_2x^2+P_3x+P_4=0$ where $P_1=0$

From Newton's theorem on the sum of powers of the roots, we get

$$S_5 + P_1 S_4 + P_2 S_3 + P_3 S_2 + P_4 S_1 = 0 \rightarrow ①$$

$$S_4 + P_1 S_3 + P_2 S_2 + P_3 S_1 + \overset{P_4 S_0}{4P_4} = 0 \rightarrow ②$$

$$S_3 + P_1 S_2 + P_2 S_1 + \overset{S_0}{3P_3} = 0 \rightarrow ③$$

$$S_2 + P_1 S_1 + 2P_2 = 0 \rightarrow ④$$

$$S_1 + P_1 = 0 \rightarrow ⑤$$

From ⑤ we get $S_1 = 0$

From ④ we get

$$S_2 = -2P_2$$

From ③ $S_3 + 0 + 0 + 3P_3 = 0$

$$S_3 = -3P_3.$$

From ①

$$S_5 + (0) + P_2(-3P_3) + P_3(-2P_2) + P_4(0) = 0$$

$$S_5 - 3P_2 P_3 - 2P_2 P_3$$

$$\therefore \quad S_5 = 5P_2 P_3$$

$$\frac{S_5}{5} = P_2 P_3 \qquad S_2 = -2P_2$$

$$\frac{S_5}{5} = \frac{S_2}{-2} \cdot \frac{S_3}{-3} \qquad \frac{S_2}{-2} = P_2 \; ; \; S_3 = -3P_3$$

$$\frac{S_5}{5} = \frac{S_2}{2} \cdot \frac{S_3}{3} \qquad P_3 = \frac{-S_3}{3}$$

$$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \cdot \frac{a^3+b^3+c^3+d^3}{3}.$$

3. Find $\frac{1}{\alpha^5}+\frac{1}{\beta^5}+\frac{1}{\gamma^5}$ where α, β, γ are

10/7. the roots of the equation $x^3+2x^2-3x-1=0$

Soln:- Given $\overset{\alpha,\beta,\gamma}{x^3+2x^2-3x-1=0} \rightarrow ①$

Put $x = 1/y$ in the equation than eqn ① becomes,

$$(1/y)^3+2(1/y)^2-3(1/y)-1=0$$

$$\frac{1}{y^3}+\frac{2}{y^2}-\frac{3}{y}-1=0$$

$$\frac{1+2y-3y^2-y^3}{y^3}=0$$

$$1+2y-3y^2-y^3=0$$

$$y^3+3y^2-2y-1=0 \quad (1/\alpha, 1/\beta, 1/\gamma)$$

By Newton's theorem,

$$S_r+P_1S_{r-1}+P_2S_{r-2}+P_3S_{r-3}+\cdots+P_nS_{r-n}=0$$

if $r \geq n$.

$$S_5+P_1S_4+P_2S_3+P_3S_2+P_4S_1=0 \qquad r=5,\ n=3$$

$\therefore P_1=3$

$$S_5+3S_4-2S_3-S_2=0 \rightarrow ②$$

$P_2=-2$

$P_3=-1$

$$\left.\begin{array}{l} S_4+3S_3-2S_2-S_1=0 \\ S_3+3S_2-2S_1-3=0 \end{array}\right\} \text{II}$$

$\therefore S_0 = n$

$$\left.\begin{array}{l} S_2+3S_1-4=0 \\ S_1+3=0 \\ \therefore S_1=-3 \end{array}\right\} \text{I}$$

$$S_2+3(-3)-4=0$$

$$S_2-9-4=0$$

$$\boxed{S_2=13}$$

$$S_3+3(13)-2(-3)-3=0$$

$$S_3+39+6-3=0$$

$$\boxed{S_3=-42}$$

$$S_4+3(-42)-2(13)+3=0$$

$$S_4+(-126)-26+3=0$$

$$S_4-126-26+3=0$$

$$S_4-149=0$$

$$\boxed{S_4=149}$$

$$S_5+3(149)-2(-42)-13=0$$

$$S_5+447+84-13=0$$

$$S_5 + 531 - 13 = 0$$

$$\boxed{S_5 = -518}$$

$$\therefore \frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = S_5$$

$$\Rightarrow -518.$$

TRANSFORMATIONS OF EQUATION

I H.W:-

1) calculate the sum of the cubes of the roots of the equation.

(i) $x^4 + 2x + 3 = 0$

Soln:-

The equation can be written as the form

$$f(x) = x^4 + 2x + 3 = 0$$

S_3 = coefficient of $1/x^3$ in the expansion of

$$\frac{x(4x^3+2)}{x^4+2x+3} = \frac{4x^4+2x}{x^4+2x+3} = \frac{x^4(4+2/x^3)}{x^4(1+2/x^3+3/x^4)}$$

= coefficient of $1/x^3$ in $\left(4+2/x^3\right)\left(1+2/x^3+3/x^4\right)^{-1}$

$\Rightarrow$ coefficient of $1/x^3$ in $\left(4+2/x^3\right)\left\{\left(1-2/x^3+3/x^4\right)+\right.$

$\left(\frac{2}{x^3}+\frac{3}{x^4}\right)^2 + \dots \}$.

$\Rightarrow$ coefficient of $\frac{1}{x^3}$ in $\left(4+\frac{2}{x^3}\right)\left(1-\frac{2}{x^3}+\dots\right\}$

$\Rightarrow -6$.

(ii) $x^3-6x^2+11x-6=0$

Soln:-

The equation can be written as the form

$$f(x) = x^3-6x^2+11x-6=0$$

S_3 = coefficient of $\frac{1}{x^3}$ in the expansion.

$$\frac{x(3x^2-12x+11)}{x^3-6x^2+11x+6} = \frac{3x^3-12x^2+11x}{x^3-6x^2+11x+6} = \frac{\cancel{x^3}\left(3-\frac{12}{x}+\frac{11}{x^2}\right)}{\cancel{x^3}\left(1-\frac{6}{x}+\frac{11}{x^2}-\frac{6}{x^3}\right)}$$

$=$ coefficient of $\frac{1}{x^3}$ in $\left(3-\frac{12}{x}+\frac{11}{x^2}\right)\left[1-\left(\frac{6}{x}-\frac{11}{x^2}+\frac{6}{x^3}\right)\right]^{-1}$

$\Rightarrow$ coefficient of $\frac{1}{x^3}$ in $\left(3-\frac{12}{x}+\frac{11}{x^2}\right)\left[1+\left(\frac{6}{x}-\frac{11}{x^2}+\frac{6}{x^3}\right)\right.$

$$\left. + \left(\frac{6}{x}-\frac{11}{x^2}+\frac{6}{x^2}\right)^2 + \left(\frac{6}{x}-\frac{11}{x^2}+\frac{6}{x^3}\right)^3 + \dots\right]$$

$\Rightarrow$ coefficient of $\frac{1}{x^3}$ in $\left(3-\frac{12}{x}+\frac{11}{x^2}\right)\left[1+\overset{\checkmark}{\left(\frac{6}{x}-\frac{11}{x^2}+\frac{6}{x^3}\right)}\right.$

$$+\overset{\checkmark}{\left(\frac{6}{x}-\frac{11}{x^2}\right)^2}+\overset{\times}{\left(\frac{6}{x^3}\right)^2}+\overset{\times}{2\left(\frac{6}{x}-\frac{11}{x^2}\right)\left(\frac{6}{x^3}\right)}+\overset{\checkmark}{\left(\frac{6}{x}-\frac{11}{x^2}\right)^3}$$

$$\left.+\overset{\times}{3\left(\frac{6}{x}-\frac{11}{x^2}\right)^2\left(\frac{6}{x^3}\right)}+\overset{\times}{3\left(\frac{6}{x}-\frac{11}{x^2}\right)\left(\frac{6}{x^3}\right)^2}+\overset{\times}{\left(\frac{6}{x^3}\right)^3}\right].$$

$\Rightarrow$ coefficient of $1/x^3$ in $(3-12/x+11/x^2)\left[1+(6/x-11/x\right.$

$+(6/x)^2+(11/x^2)^2-2(6/x)(11/x^2)+(6/x)^3-3(6/x)$

$$\left.+3(6/x)(11/x^2)^2-(11/x^2)^3+\cdots\right]$$

$\Rightarrow$ coefficient of $1/x^3$ in $3-12/x+11/x^2+18/x-33/x^2$

$$+18/x^3-72/x^2+\frac{132}{x^3}-\frac{72}{x^4}+\frac{66}{x^3}-\frac{121}{x^4}+\frac{66}{x^5}$$

$$+\frac{108}{x^2}-\frac{396}{x^3}+\frac{648}{x^3}-\frac{432}{x^3}+\frac{1584}{x^4}-\frac{2592}{x^4}+\frac{396}{x^4}$$

$$-\frac{1452}{x^5}+\frac{2376}{x^5}$$

{∵ omitting the higher powers}

$\Rightarrow$ coefficient of $1/x^3$ in $18+132+66-396+648-432$

$\Rightarrow 36.$

2. In the equation $x^4-x^3-7x^2+x+6=0$ find the values of S_4 and S_6.

Soln:-

Given $x^4-x^3-7x^2+x+6=0 \rightarrow ①$

By using Newton's theorem

$$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \dots + P_n S_{r-n} = 0$$

if $r \geq n$

S_1 S_2

$S_6 - S_5 - 7S_4 + S_3 + 6S_2 = 0$

$P_1 = -1$

$S_5 - S_4 - 7S_3 + S_2 + 6S_1 = 0$

$P_2 = -7$

$S_4 - S_3 - 7S_2 + S_1 + \overset{S_0}{24} = 0$

$P_3 = 1$

$P_4 = 6$

$S_3 - S_2 - 7S_1 + \overset{S_0}{3} = 0$ $S_0 = 3$

$S_0 = n$

$S_2 - S_1 - \overset{S_0}{14} = 0$ $S_0 = 2$; $P_2 = -7$

$S_1 - \overset{S_0}{1} = 0$

$\boxed{S_1 = 1}$

$S_2 - 1 - 14 = 0$

$S_2 - 15 = 0$

$\boxed{S_2 = 15}$

$S_3 - 15 - 7 + 3 = 0$

$S_3 - 19 = 0$

$\boxed{S_3 = 19}$

$S_4 - 19 - 105 + 1 + 24 = 0$

$S_4 - 99 = 0$

$\boxed{S_4 = 99}$

$S_5 - 99 - 133 + 15 + 6 = 0$

$S_5 - 211 = 0$

$\boxed{S_5 = 211}$

$S_6 - 211 - 693 + 19 + 90 = 0$

$S_6 - 795 = 0$

$\boxed{S_6 = 795}$

$\therefore$ $S_4 = 99$ and $S_6 = 795$.

3. If α, β, γ be the roots of the equation $x^3 - \overset{\times}{7x} + 7 = 0$ find $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$.

Soln:-

Given $x^3 - 7x + 7 = 0 \rightarrow ①$

Put $x = 1/y$ in the equation than eqn ① becomes

$$(1/y)^3 - 7(1/y)^2 + 7 = 0$$

$$\frac{1}{y^3} - \frac{7}{y^2} + 7 = 0$$

$$\frac{1 - 7y^2 + 7y^3}{y^3} = 0$$

$$1 - 7y^2 + 7y^3 = 0$$

$7y^3-7y^2+1=0$

$(\div 7)$

$y^3-y^2+1/7=0$

By Using Newton's theorem

$S_r+P_1S_{r-1}+P_2S_{r-2}+\dots+P_nS_{r-n}=0$ if $r\geq n$.

$P_1=-1$

$P_2=0$

$P_3=1/7$

$S_0=n$

$S_4-S_3+1/7S_1=0$

$S_3-S_2+1/7\,S_0=0$

$S_3-S_2+3/7=0$

$S_2-S_1=0$

$S_1=S_2$

$S_1-S_0=0$

$S_1-1=0$

$\boxed{S_1=1}$

$\boxed{S_2=1}$

$S_3-1+3/7=0$

$S_3-4/7=0$

$\boxed{S_3=4/7}$

$S_4-4/7+1/7=0$

$S_4-3/7=0$

$\boxed{S_4=3/7}$

$\therefore S_4 = 3/7$.

4\. Show that the sum of the nineth power of the roots of $x^3+3x+9=0$ is zero.

Proof:-

Given $x^3+3x+9=0 \to ①$

By Using Newton's theorem

$$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \cdots + P_n S_{r-n} = 0$$

if $r \geq n$

$$x^3 + P_1x^2 + P_2x + P_3 = 0 \to ②.$$

$S_9 + 3S_7 + 9S_6 = 0$

$S_7 + 3S_5 + 9S_4 = 0$; $S_6 + 3S_4 + 9S_3 = 0$

$S_5 + 3S_3 + 9S_2 = 0$; $S_4 + 3S_2 + 9S_1 = 0$

$S_3 + 3S_5 + 9S_6 = 0.$

$P_1 = 0$

$P_2 = 3$

$P_3 = 9$

$\boxed{S_1 = 0}$; $S_2 + 3S_0 = 0$

$\boxed{S_2 = -6}$

$S_3 + 27 = 0$

$\boxed{S_3 = -27}$

$S_5 - 81 - 54 = 0$; $\boxed{S_5 = 135}$

$S_4 + 3S_2 + 9S_1 = 0$

$S_4 - 18 = 0$

$\boxed{S_4 = 18}$

$S_6 + 54 - 243 = 0$

$\boxed{S_6 = 189}$

$S_7 + 405 + 162 = 0$

$S_7 + 567 = 0$

$\boxed{S_7 = -567}$

$S_9 + 3(-567) + 9(189) = 0$

$S_9 - 1701 + 1701 = 0$

$\boxed{S_9 = 0}$

$\therefore S_9 = 0.$

12/7. TRANSFORMATIONS OF EQUATION:-

Let a_0 is a $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0 \rightarrow ①$

Be a given equation.

Let its roots be $\alpha_1, \alpha_2 \ldots \alpha_n$.

It is possible to transform this equation into another equation whose roots are the roots of eqn ① with a given relation.

I To transform an equation into another equation whose roots are the roots of the given equation with their signs changed.

To transform on equation,

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \rightarrow ①$

into another equation whose roots are the roots of eqn ① with their signs changed than just change the sign of the odd powers of x.

1. If the roots of $x^3 - 12x^2 + 23x + 36 = 0$ are $-1, 4, 9$ find the equation whose roots are $1, -4, -9$.

Soln:-

Given $x^3 - 12x^2 + 23x + 36 = 0 \rightarrow ①$.

The roots are $-1, 4, 9$.

Now we find an equation whose roots are $1, -4, -9$.

That is, to find an equation whose roots are the roots of eqn ① but the signs are changed.

Hence in equation ① we have to

change the sign of odd powers of x.

Hence the required eqn is $-x^3-12x^2-23x+36=0$

$\therefore x^3+12x^2+23x-36=0$

II. To transform an equation into another equation whose roots are 'm' times those of the given equation.

x

$y=mx$

$x=y/m$

To transform an equation,

$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n=0 \rightarrow ①$

into another equation whose roots are m times those of the given equation then just multiply the successive coefficients begining with the second by ~~m~~ $m, m^2, m^3 \dots$ etc.

1. Multiply the roots of the equation $x^4+2x^3+4x^2+6x+8=0$ by $1/2$.

Soln:-

Given $x^4+2x^3+4x^2+6x+8=0 \rightarrow ①$

To multiply the roots of eqn ① by $1/2$.

we have to multiply the successive coefficients begining with the second by $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4$

$x^4+(1/2)2x^3+(1/2)^2 4x^2+(1/2)^3 6x+(1/2)^4 8=0$

$x^4+x^3+x^2+3/4\,x+1/2=0$

$\Rightarrow 4x^4+4x^3+4x^2+3x+2=0$

I **H.w:-** ∴ which is required equation

2) Find the eqn whose roots are $-1,-6,2,-3$ if the roots of the eqn $x^4-8x^3+7x^2+36x-36=0$ are $1,-2,3,6$.

Soln:-

Given $x^4-8x^3+7x^2+36x-36=0 \rightarrow ①$

The roots are $-1,-6,2,-3$

Now we find an equation whose roots are $1,-2,3,6$.

That is, to find an equation whose roots are the roots of eqn ① but the signs are changed.

Hence in equation ① we have to change the sign of odd powers of x.

Hence the required eqn is

$x^4+8x^3+7x^2-36x-36=0$

$\therefore x^4+8x^3+7x^2-36x-36=0$

3) Find the eqn whose roots are equal in magnitude but opposite in sign to the roots of the equation $x^{10}-12x^8+40x^4-15x+20=0$

Soln:-

Given $x^{10}-12x^{8}+40x^{4}-15x+20=0 \rightarrow ①$

The roots are equal in magnitude

Now we find an equation whose roots are in opposite sign.

That is, to find an equation whose roots are the roots of eqn ① but the signs are changed.

Hence in equation ① we have to change the sign of odd powers of x.

Hence the required eqn is

$x^{10}-12x^{8}+40x^{4}+15x+20=0$

$\therefore x^{10}-12x^{8}+40x^{4}+15x+20=0$

16/7.

I 2) Transform the eqn $3x^3+4x^2+5x-6=0$ into one in which the coeffient of x^3 is unity.

Soln:-

Given $3x^3+4x^2+5x-6=0 \rightarrow ①$

Multiply the roots of eqn ① by 3 we get

That is we have to multiply the successive coefficient begining with the second term by $3, 3^2, 3^3$ . .

$$3x^3 + 4(3x^2) + 5(3x^3) \cancel{+} - 6 = 0$$

$\div$ by 3

$$x^3 + 4x^2 + 15x - 54 = 0 .$$

3) Remove the fractional coefficient from the equation

$$x^3 - \tfrac{1}{4}x^2 + \tfrac{1}{3}x - 1 = 0$$

Soln:-

Given $x^3 - \tfrac{1}{4}x^2 + \tfrac{1}{3}x - 1 = 0 \rightarrow ①$

Multiply the roots of eqn ① by m we get

$$x^3 - (m)\,x^2/4 + (m^2)\,x/3 - (m^3) = 0$$

Let $m = 12$ ($\because$ LCM of 3 and 4)

then the required equation is

$$x^3 - (12)\frac{x^2}{4} + (12)^2\frac{x}{3} - (12^3) = 0$$

$$x^4 - 3x^2 + 48x - 1728 = 0$$

II. To transform an equation into another equation whose roots are the reciprocals of the roots of the given equation.

condition :-

To transform an equation of the n^{th} degree into another whose roots are the reciprocals of the roots of the given equation, then we have to change x to $1/x$ in the given equation and multiply the resulting equation by x^n.

1) If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+px^3+qx^2+rx+s=0$ find the equation whose roots are $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$

Soln:-

Given $x^4+px^3+qx^2+rx+s=0 \rightarrow ①$

It roots are $\alpha, \beta, \gamma, \delta$ to find the equation whose roots are $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$ (reciprocal of the roots of ①)

Now, we have to change x to $1/x$, we get

$$\frac{1}{x^4}+\frac{p}{x^3}+\frac{q}{x^2}+\frac{r}{x}+s=0$$

Multiply by $x^4 \Rightarrow 1+px+qx^2+rx^3+sx^4=0$

$$\therefore sx^4+rx^3+qx^2+px+1=0.$$

2) If 1, 2, 3, 6 are the roots of the equation $x^4-12x^3+47x^2-72x+36=0$ find an equation whose roots are $1, 1/2, 1/3, 1/6$.

Soln:-

Given $x^4-12x^3+47x^2-72x+36=0 \rightarrow ①$

If roots are 1, 2, 3, 6 to find the eqn whose roots are $1, 1/2, 1/3, 1/6$ (reciprocal of the roots of eqn ①)

Now, we have to change x to $1/x$, we get

$$\frac{1}{x^4}-\frac{12\cancel{x^3}}{x^3}+\frac{47}{x^2}-\frac{72}{x}+36=0$$

Multiply of $x^4 \Rightarrow$

$$1-12x+47x^2-72x^3+36x^4=0$$

$$36x^4-72x^3+47x^2-12x+1=0$$

3) Solve the eqn $6x^3\cancel{+}-11x^2-3x+2=0$ given that its roots are in H.P

Soln:-

Given $6x^3-11x^2-3x+2=0 \rightarrow ①$

Its roots are in H.P.

changing x into $1/x$ in eqn ①, we get

$$\frac{6}{x^3}-\frac{11}{x^2}-\frac{3}{x}+2=0$$

$\times x^3$ $$6x-11x-3x^2+2x^3=0$$

$$\therefore \overset{a_0}{2x^3}-\overset{a_1}{3x^2}-\overset{a_2}{11x}+\overset{a_3}{6}=0 \rightarrow ②$$

Now, the roots of ② are in A.P

(∵ since H.P is reciprocal of A.P)

Let the roots be

$$(\alpha-d)+\alpha+(\alpha+d)=\frac{-a_1}{a_0}$$

$$3\alpha=\frac{3}{2}$$

$$\alpha=\frac{3}{6}$$

$$\boxed{\alpha \Rightarrow 1/2}$$

$\therefore \alpha=1/2$ is a one root of given equation by using the division we have

$$\begin{array}{c|cccc} 1/2 & 2 & -3 & -11 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline & 2 & -2 & -12 & \boxed{0} \end{array}$$

The reduced equation is.

$$2x^2-2x-12=0$$

$$=\frac{2\pm\sqrt{4-4(-12)(2)}}{4}$$

$$=\frac{2\pm\sqrt{4+96}}{4}$$

$$=\frac{2\pm\sqrt{100}}{4}$$

$$=\frac{2\pm 10}{4}$$

$$=\frac{12}{4},\ \frac{-8}{4}$$

$$=3,-2.$$

$\therefore$ The roots of ② is $\frac{1}{2}, -2, 3$

17|7

Hence the roots of ① is, $2, -\frac{1}{2}, \frac{1}{3}$

H.W.

H.P 1. solve the eqn $81x^3-18x^2-36x+8=0$ whose roots are in H.P

[illegible]

Soln:- Given $81x^3-18x^2-36x+8=0 \rightarrow ①$

$\therefore$ Its roots are in A·P

change x into $1/x$ in eqn ① we get

$$81/x^3 - 18/x^2 - 36/x + 8 = 0 \Rightarrow 81 - 18x - 36x^2 + 8x^3 = 0$$

$$8x^3 - 36x^2 - 18x + 81 = 0 \rightarrow ②$$

Now the roots of ② are in A·P

Let the roots be $(\alpha - d) + \alpha + (\alpha + d) = \frac{-a_1}{a_0}$

$$3\alpha = \frac{\cancel{36}\,\cancel{9}\,3}{\cancel{8}\,4\,2}$$

$$\boxed{\alpha = 3/2}$$

$\alpha = 3/2$ is one root of given eqn by using division we have

$$\begin{array}{c|cccc} 3/2 & 8 & -36 & -18 & 81 \\ & 0 & 12 & -36 & -81 \\ \hline & 8 & -24 & -54 & \boxed{0} \end{array}$$

The reduced eqn is $8x^2 - 24x - 54 = 0$

$$\Rightarrow \frac{24 \pm \sqrt{576 + 1728}}{16}$$

$$\Rightarrow \frac{24 \pm \sqrt{2304}}{16} \Rightarrow \frac{24 \pm 48}{16}$$

$$\Rightarrow \frac{\cancel{72}\,9}{\cancel{16}\,2}, \frac{-\cancel{24}\,\cancel{6}\,3}{\cancel{16}\,\cancel{4}\,2} \Rightarrow 9/2, -3/2$$

Hence the roots is $2/3, 2/9, -2/3$

H.W:-

II

4. Multiply the roots of $x^3 - 3x + 1 = 0$ by 10.

Soln:-

Given $x^3 - 3x + 1 = 0 \rightarrow ①$

To multiply the roots of eqn ① by 10 we have to multiply the successive coefficients begining with the second by $10, (10)^2, (10)^3$.

$$x^3 - 3x + 1 = 0$$

$$x^3 + 10(0x^2) - 3x(10)^2 + 1(10)^3 = 0$$

$$x^3 - 3x(10)^2 + 1(10)^3 = 0$$

$$x^3 - 300x^2 + 1000 = 0$$

∴ Which is the required equation.

17/7 To increase or decrease the roots of a given equation by a given quantity.

Increasing

(To increase the roots of an equation by h diminish the roots of that equation by $-h$)

1. Diminish the roots of $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$ by 2.

Soln:-

$$x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$$

2	1	-5	7	-4	5	
	0	2	-6	2	-4	
2	1	-3	1	-2	1	(constant term)
	0	2	-2	-2		
2	1	-1	-1	-4		(x)
	0	2	2			
2	1	1	1			(x^2)
	0	2				
2	1	3				(x^3)
	0					(x^4)
	1					

Hence the required equation is

$$\therefore x^4+3x^3+x^2-4x+1=0$$

2. Increase the roots of the equation $3x^4+7x^3-15x^2+x-2=0$ by 7, and find the transformed equation.

Soln:-

Given $3x^4+7x^3-15x^2+x-2=0$

Increasing by 7 the roots of the given equation is the same as decreasing the roots by -7.

-7	3	7	-15	1	-2	
	0	-21	98	-581	4060	
-7	3	-14	83	-580	4058	
	0	-21	245	-2296		
-7	3	-35	328	-2876		

-7	0	-21	392
-7	3	-56	720
	0	-21	
-7	3	-77	
	0		
	3		

Hence the required equation is

$$3x^4 - 77x^3 + 720x^2 - 2876x + 4058 = 0$$

3. If α, β, γ are the roots of the equation $x^3 - 6x^2 + 12x - 8 = 0$ find the equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$.

Soln:-

Given $x^3 - 6x^2 + 12x - 8 = 0 \rightarrow ①$

2	1	-6	12	-8
	0	2	-8	8
2	1	-4	4	0
	0	2	-4	
2	1	-2	0	
	0	-2		
	1	0		

The transformed equation is $x^3 = 0$

(ie) the roots are $= 0, 0, 0$

(ie) $\alpha-2=0$; $\beta-2=0$; $\gamma-2=0$

(ie) $\alpha=2$; $\beta=2$; $\gamma=2$.

4. If α,β,γ are the roots of $8x^3-4x^2+6x-1=0$ find the eqn whose roots are $\alpha+1/2, \beta+1/2, \gamma+1/2$.

Soln:-

Here we have to increase the roots of the given equation by $1/2$.

(ie) diminish the roots of the given equation by $-1/2$

$$
\begin{array}{r|rrrr}
-1/2 & 8 & -4 & 6 & -1 \\
 & 0 & -4 & 4 & -5 \\
\hline
 & 8 & -8 & 10 & \boxed{-6} \\
-1/2 & 0 & -4 & 6 & \\
\hline
 & 8 & -12 & \boxed{16} & \\
-1/2 & 0 & -4 & & \\
\hline
 & 8 & \boxed{-16} & & \\
-1/2 & 0 & & & \\
\hline
 & 8 & & &
\end{array}
$$

The equation whose roots are $\alpha+1/2, \beta+1/2, \gamma+1/2$ is

$$8x^3-16x^2+16x-6=0$$

H.W :-

2. Solve the eqn $6x^3-11x^2+6x-1=0$ whose roots in

10m 16) **Soln:-**

Given $6x^3-11x^2+6x-1=0 \rightarrow$ ①

It roots are H.P

change x into $1/x$ in eqn ① we get

$$\frac{6}{x^3}-\frac{11}{x^2}+\frac{6}{x}-1=0$$

$$6-11x+6x^2-x^3=0$$

$$-x^3+6x^2-11x+6=0$$

$$x^3-6x^2+11x-6=0 \rightarrow ②$$

Now the roots of ② are in A.P (∵ since H.P is reciprocal of A.P)

Let the roots be

$$(\alpha-d)+\alpha+(\alpha+d)=\frac{-a_1}{a_0}$$

$$3\alpha=\frac{6}{1}$$

$$\alpha=6/3$$

$$\boxed{\alpha=2}$$

∴ $\alpha=2$ is one root of given equation then using the division we have.

$$\begin{array}{c|cccc} 2 & 1 & -6 & 11 & -6 \\ & 0 & 2 & -8 & 6 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

$x^2-4x+3=0$

$$\Rightarrow \frac{4 \pm \sqrt{16-12}}{2}$$

$$\Rightarrow \frac{4 \pm \sqrt{4}}{2}$$

$$= \frac{4 \pm 2}{2}$$

$$\Rightarrow \frac{6}{2}, \frac{2}{2}$$

$\Rightarrow 3, 1$

Hence is roots is $1, \frac{1}{2}, \frac{1}{3}$.

3. Solve the eqn $\overset{a_0}{15x^4} - \overset{a_1}{8x^3} - \overset{a_2}{14x^2} + \overset{a_3}{8x} - \overset{a_4}{1} = 0$ given that the roots are in H.P

Soln:-

Given $15x^4-8x^3-14x^2+8x-1=0 \rightarrow ①$

It root are H.P

change x into $1/x$ in eqn ① we get

$$\frac{15}{x^4} - \frac{8}{x^3} - \frac{14}{x^2} + \frac{8}{x} - 1 = 0$$

$$15 - 8x - 14x^2 + 8x^3 - x^4 = 0$$

$$-\overset{a_0}{x^4} + \overset{a_1}{8x^3} - \overset{a_2}{14x^2} - \overset{a_3}{8x} + \overset{a_4}{15} = 0$$

$x^4 - 8x^3 + 14x^2 + 8x - 15 = 0 \rightarrow ②$

Now the roots ② are in A.P

Let the root be

$(\alpha-3d) + (\alpha-d) + (\alpha+d) + (\alpha+3d) = \frac{-a_1}{a_0}$

$4\alpha = \frac{8}{1}$

$\boxed{\alpha = 2} \rightarrow ③$

$\therefore \alpha = 2$ is one root of the given eqn by using division we

$(\alpha-3d)(\alpha-d)(\alpha+d)(\alpha+3d) = \frac{a_4}{a_0}$

$(\alpha^2 - 9d^2)(\alpha^2 - d^2) = \frac{-15}{1}$

By using eqn ③

$(4-9d^2)(4-d^2) = -15$

$(4-9d^2)(4-d^2) = -15$

$16 - 4d^2 - 36d^2 + 9d^4 = -15$

$9d^4 - 40d^2 + 16 + 15 = 0$

$9d^4 - 40d^2 + 31 = 0$

$$d^2 = \frac{40 \pm \sqrt{1600-4(9)(31)}}{18}$$

$$= \frac{40 \pm \sqrt{1600+116}}{18}$$

$$= \frac{40 \pm \sqrt{484}}{18} \Rightarrow \frac{40 \pm 22}{18}$$

$$= \frac{62}{18}, \cancel{\frac{18}{18}}^{1}$$

$$d^2 = \frac{31}{9}, 1$$

$$d = \pm\frac{\sqrt{31}}{3}, \pm 1$$

$$d = 1, -1$$

Case (i)

when $d=1$; $a=2$

$[2-3(1)], [2-1], [2+1], [2+3(1)]$

$\Rightarrow -1, 1, 3, 5$

Case (ii)

when $d=-1$; $a=2$

$[2-3(-1)], [2+1], [2-1], [2+3(-1)]$

$\Rightarrow 5, 3, 1, -1$

Hence the roots is $-1, 1, 1/3, 1/5$.

4. Find the condition that the roots of $x^3-3px^2+3qx+r=0$ may be in H.P

Soln:-

Given $x^3-3px^2+3qx+r=0 \rightarrow ①$

It is the roots in H.P

change x into $1/x$ in eqn ① we get

$$1/x^3 - 3p/x^2 + 3q/x + r = 0$$

$$1-3px+3qx^2+rx^3=0$$

$$rx^3+3qx^2-3px-1=0 \rightarrow ②$$

Now the roots of ② in A.P

Let the roots be

$$(\alpha-d)+\alpha+(\alpha+d) = \frac{-a_1}{a_0}$$

$$3\alpha = \frac{-3q}{r}$$

$$\boxed{\alpha = -q/r}$$

$\alpha = -q/r$ is a roots of given equation.

Put $x=-q/r$ in eqn ① we get

$$r\left(-q/r\right)^3 + 3q\left(-q/r\right)^2 + 3p\left(-q/r\right) + 1 = 0$$

$$r\left(\frac{-q^3}{r^3}\right) + 3q\left(\frac{q^2}{r^2}\right) - \frac{3pq}{r} + 1 = 0$$

$$-\frac{\cancel{r}q^3}{r^{\cancel{3}}} + \frac{3q^3}{r^2} - \frac{3pq}{r} + 1 = 0$$

$$\frac{-q^3}{r^2} + \frac{3q^3}{r^2} - \frac{3pq}{r} + 1 = 0$$

$$\frac{1}{r^2}\left[-q^3 + 3q^3 - 3pqr + r^2\right] = 0$$

$$2q^3 - 3pqr + r^2 = 0.$$

5. Removal the fractional coefficient from

$$x^5 - 1/3\,x^4 + \frac{25}{27}x^2 + \frac{14}{81}x - 8/81 = 0$$

Soln:-

Given $x^5 - 1/3\,x^4 + 25/27\,x^2 + \frac{14}{81}x - \frac{8}{81} = 0 \rightarrow ①$

$$x^5 - \frac{1}{3}x^4 + 0x^3 + \frac{25}{27}x^2 + \frac{14}{81}x - \frac{8}{81} = 0$$

Multiply the roots of eqn ① by m we get

Let $m = 3$, then the required eqn is

$$x^5 - (m)\,x^4/3 + (m^3)\,\frac{25x^2}{27} + (m^4)\,\frac{14x^4}{81} - (m^5)\frac{8}{81} = 0$$

$$x^5 - (3)\,\frac{x^4}{3} + (3^3)\,\frac{25x^2}{27} + (3^4)\,\frac{14x}{81} - (3^5)\,\frac{8}{81} = 0$$

$$x^5 - x^4 + 25x^2 + 14x - 24 = 0.$$

6. Remove the fractional coefficients from

$$x^5 + \frac{4}{3}x^4 + \frac{2}{9}x^3 + \frac{1}{12}x^2 + \frac{1}{36} = 0$$

$$\begin{array}{r|l} 3 & 3,9,12,36 \\ \hline 3 & 1,3,4,12 \\ \hline 2 & 1,1,4,4 \\ \hline 2 & 1,1,2,2 \\ \hline & 1,1,1,1 \end{array} \qquad 3 \times 2$$

Soln:-

Given $x^5 + \frac{4}{3}x^4 + \frac{2}{9}x^3 + \frac{1}{12}x^2 + 0x + \frac{1}{36} = 0 \rightarrow ①$

Multiply the roots of eqn ① we get

$$x^5 + (m)\frac{4x^4}{3} + (m^2)\frac{2x^3}{9} + (m^3)\frac{x^2}{12} + \frac{(m^5)}{36} = 0$$

Let $m = 6$.

then the required eqn is

$$x^5 + (6)\frac{4x^4}{3} + (36)\frac{2x^3}{9} + (216)\frac{x^2}{12} + \frac{7776}{36} = 0$$

$$x^5 + 8x^4 + 8x^3 + 18x^2 + 216 = 0.$$

Diminish Method :-

7. $2x^5 - x^3 + 10x - 8 = 0$ by 5.

Soln:-

$$2x^5 - x^3 + 10x - 8 = 0$$

$$\begin{array}{c|cccccc}
5 & 2 & 0 & -1 & 0 & 10 & -8 \\
 & 0 & 10 & 50 & 245 & 1225 & 6175 \\ \hline
5 & 2 & 10 & 49 & 245 & 1235 & \boxed{6167} \\
 & 0 & 10 & 100 & 745 & 4950 & \\ \hline
5 & 2 & 20 & 149 & 990 & \boxed{6185} & \\
 & 0 & 10 & 150 & 1495 & & \\ \hline
5 & 2 & 30 & 299 & \boxed{2485} & & \\
 & 0 & 10 & 200 & & & \\ \hline
5 & 2 & 40 & \boxed{499} & & & \\
 & 0 & 10 & & & & \\ \hline
5 & 2 & \boxed{50} & & & & \\
 & 0 & & & & & \\ \hline
 & 2 & & & & &
\end{array}$$

Hence the required equation is

$$2x^5+50x^4+499x^3+2485x^2+6185x+6167=0$$

8. $x^5-4x^4+3x^3-4x+6=0$ by 3

Soln:-

$$x^5-4x^4+3x^3-4x+6=0$$

$$\begin{array}{c|cccccc}
3 & 1 & -4 & 3 & 0 & -4 & 6 \\
 & 0 & 3 & -3 & 0 & 0 & -12 \\ \hline
3 & 1 & -1 & 0 & 0 & -4 & \boxed{-6} \\
 & 0 & 3 & 6 & 18 & 54 & \\ \hline
3 & 1 & 2 & 6 & 18 & \boxed{50} & \\
 & 0 & 3 & 15 & 63 & & \\ \hline
3 & 1 & 5 & 21 & \boxed{81} & & \\
 & 0 & 3 & 24 & & & \\ \hline
3 & 1 & 8 & \boxed{45} & & & \\
 & 0 & 3 & & & & \\ \hline
3 & 1 & \boxed{11} & & & &
\end{array}$$

$$\begin{array}{|c}\hline 0 \\ 1\end{array}.$$

Hence the required eqn is

$$x^5+11x^4+45x^3+81x^2+50x-6=0.$$

9. $x^4-5x^3+7x^2-17x+11=0$ by 2.

Soln:-

Given $x^4-5x^3+7x^2-17x+11=0$

$$\begin{array}{c|ccccc}
2 & 1 & -5 & 7 & -17 & 11 \\
 & 0 & 2 & -6 & 26 & 18 \\ \hline
2 & 1 & -3 & 13 & 9 & \boxed{29} \\
 & 0 & 2 & -2 & 22 & \\ \hline
2 & 1 & -1 & 11 & \boxed{31} & \\
 & 0 & 2 & 2 & & \\ \hline
2 & 1 & 1 & \boxed{13} & & \\
 & 0 & 2 & & & \\ \hline
2 & 1 & \boxed{3} & & & \\
 & 0 & & & & \\ \hline
 & 1 & & & &
\end{array}$$

$$\begin{array}{c|ccccc}
2 & 1 & -5 & 7 & -17 & 11 \\
 & 0 & 2 & -6 & 2 & -30 \\ \hline
2 & 1 & -3 & 1 & -15 & \boxed{-19} \\
 & 0 & 2 & -2 & -2 & \\ \hline
2 & 1 & -1 & -1 & \boxed{-17} & \\
 & 0 & 2 & 2 & & \\ \hline
2 & 1 & 1 & \boxed{1} & & \\
 & 0 & 2 & & & \\ \hline
2 & 1 & \boxed{3} & & &
\end{array}$$

$$\begin{array}{|l} 0 \\ \hline 1 \end{array}$$

Hence the required eqn is

$$x^4+3x^3+x^2-17x-19=0$$

Increase method :-

10. $x^4-x^3-10x^2+4x+24=0$ by 2

Soln:-

Given $x^4-x^3-10x^2+4x+24=0$

Increasing by 2 the roots of the given eqn is the same as decreasing the roots by -2

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -10 & 4 & 24 \\ & 0 & -2 & +6 & +8 & -24 \\ \hline -2 & 1 & -3 & -4 & +12 & \boxed{0} \\ & 0 & -2 & 10 & -12 & \\ \hline -2 & 1 & -5 & 6 & \boxed{0} & \\ & 0 & -2 & 14 & & \\ \hline -2 & 1 & -7 & \boxed{20} & & \\ & 0 & -2 & & & \\ \hline -2 & 1 & \boxed{-9} & & & \\ & 0 & & & & \\ \hline & 1 & & & & \end{array}$$

Hence the required eqn is

$$x^4-9x^3+20x^2=0$$

11. $4x^4+32x^3+83x^2+76x+21=0$ by 2

Given:-

$$4x^4+32x^3+83x^2+76x+21=0$$

Increasing by 2 the roots of the given eqn is the same as decreasing root by -2.

$$\begin{array}{r|rrrrr}
-2 & 4 & 32 & 83 & 76 & 21 \\
 & 0 & -8 & -48 & -70 & -12 \\
\hline
 & 4 & 24 & 35 & 6 & \boxed{9} \\
-2 & 0 & -8 & -32 & -6 & \\
\hline
 & 4 & 16 & 3 & \boxed{0} & \\
-2 & 0 & -8 & -16 & & \\
\hline
 & 4 & 8 & \boxed{-13} & & \\
-2 & 0 & -8 & & & \\
\hline
 & 4 & \boxed{0} & & & \\
-2 & 0 & & & & \\
\hline
 & 4 & & & &
\end{array}$$

$$x^4-13x^3+9=0$$

12. $3x^5-5x^3+7=0$ by $+4$

Soln:-

Given $3x^5-5x^3+7=0$

Increasing by 4 the roots of the given eqn is same as decreasing root -4:

$$\begin{array}{r|rrrrrr}
-4 & 3 & 0 & -5 & 0 & 0 & 7 \\
 & 0 & -12 & 48 & -172 & 688 & -2752 \\
\hline
-4 & 3 & -12 & 43 & -172 & 688 & \boxed{-2745} \\
 & 0 & -12 & 96 & -556 & 2912 & \\
\hline
-4 & 3 & -24 & 139 & -728 & \boxed{3600} & \\
 & 0 & -12 & +144 & -1132 & & \\
\hline
-4 & 3 & -36 & +283 & \boxed{-1860} & & \\
 & 0 & -12 & 192 & & & \\
\hline
-4 & 3 & -48 & \boxed{475} & & & \\
 & 0 & -12 & & & & \\
\hline
-4 & 3 & \boxed{-60} & & & & \\
 & 0 & & & & & \\
\hline
 & 3 & & & & &
\end{array}$$

Hence the required eqn is

$$3x^5 - 60x^4 + 475x^3 - 1860x^2 + 3600x - 2745 = 0.$$

RECIPROCAL EQUATIONS:-

18/7

(A) 2m If an equation remains unaltered when x is changed into $1/x$ (reciprocal of x) then it is called a reciprocal equation.

$x^n + P_1x^{n-1} + P_2x^{n-2} + \dots\dots + P_n = 0 \to ①$ be a reciprocal eq.

STANDARD FORMS OF RECIPROCAL EQUATIONS:-

TYPE I:-

Reciprocal equation of degree 4

with like and unlike signs for its coefficients

$x^4+x^3+x^2+x+1=0 \to$ like

$x=-1$ must be a root

$x=1$ must be a root

$\Downarrow$

$x^4-x^3+x^2-x\overset{-}{+}1=0 \to$ unli

is not reciprocal

TYPE-II :-

Reciprocal equation of odd degree with like signs for its coefficients. In this case $x=-1$ is a root of the given equation.

TYPE-III :-

Reciprocal equation of odd degree with unlike signs for its coefficients. In this case $x=1$ is a root of the given equation.

TYPE-IV :-

Reciprocal equation of even degree with unlike signs for its coefficients and the middle term is absent. In this case $x=1,-1$ are the root of the given equation.

T-I.

1. Solve $x^4-10x^3+26x^2-10x+1=0$

Soln:-

Given $x^4-10x^3+26x^2-10x+1=0 \to ①$

This is a reciprocal equation of degree 4 with like signs (T-I)

Dividing ① by x^2, we get.

$$x^2-10x+26-\frac{10}{x}+\frac{1}{x^2}=0$$

$$\left(x^2+1/x^2\right)-10\left(x+1/x\right)+26=0$$

Let $x+1/x=u$

$$x^2+1/x^2=u^2-2$$

$$\Rightarrow (u^2-2)-10u+26=0$$

$$u^2-10u+24=0$$

$$(u-6)(u-4)=0$$

$\therefore u=4,6.$

$$x+1/x=4$$

$$x^2+1=4x$$

$$x^2-4x+1=0$$

$$x=\frac{4\pm\sqrt{16-4(1)(1)}}{2}$$

$$=\frac{4\pm\sqrt{16-4}}{2}$$

$$=\frac{4\pm\sqrt{12}}{2}$$

$$=\frac{4\pm 2\sqrt{3}}{2}$$

$$x+1/x=6$$

$$x^2+1=6x$$

$$x^2-6x+1=0$$

$$x=\frac{6\pm\sqrt{36-4(1)(1)}}{2}$$

$$=\frac{6\pm\sqrt{32}}{2}$$

$$=\frac{6\pm 4\sqrt{2}}{2}$$

$$\Rightarrow 3\pm 2\sqrt{2}.$$

$x = 2 \pm \sqrt{3}$; $x = 3 \pm 2\sqrt{2}$.

∴ The roots are $2 \pm \sqrt{3}$, $3 \pm 2\sqrt{2}$.

T-II Reciprocal eqn of odd degree with like signs for its coefficients.

For this type $x = -1$ is a root of the given reciprocal equation.

Now dividing the given reciprocal eqn by $x+1$ we get a reciprocal eqn of degree 4 which is clearly of type I.

1. Solve $x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0$

Soln:

Given $x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0 \rightarrow ①$

This is a reciprocal eqn of degree 5 (odd degree) with like signs (T-II)

∴ $x = -1$ is a root of eqn ①

$$\begin{array}{r|rrrrrr} -1 & 1 & 4 & 1 & 1 & 4 & 1 \\ & 0 & -1 & -3 & 2 & -3 & -1 \\ \hline & 1 & 3 & -2 & 3 & 1 & \boxed{0} \end{array}$$

∴ The reduced eqn is

$x^4 + 3x^3 - 2x^2 + 3x + 1 = 0$

This is a reciprocal eqn of degree 4 with like signs (T-I).

Dividing ① by x^2 we get.

$$x^2 + 3x - 2 + \frac{3}{x} + \frac{1}{x^2} = 0$$

$$\left(x^2 + 1/x^2\right) + 3\left(x + 3/x\right) - 2 = 0$$

Let $x + 1/x = u$

$$x^2 + 1/x^2 = u^2 - 2$$

$$(u^2 - 2) + 3u - 2 = 0$$

$$u^2 - 2 + 3u - 2 = 0$$

$$u^2 \cancel{+3u} - 4 + 3u = 0$$

$$u^2 + 3u - 4 = 0$$

$$(u-1)(u+4) = 0$$

$$u = 1, -4$$

$x + 1/x = 1$

$x^2 + 1 = x$

$x^2 - x + 1 = 0$

$$x = \frac{1 \pm \sqrt{1 \mp 4(1)(1)}}{2}$$

$x + 1/x = -4$

$x^2 + 1 = -4x$

$x^2 + 4x + 1 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(1)}}{2}$$

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$$= \frac{1 \pm \sqrt{1 \mp 4}}{2} \qquad\qquad = \frac{-4 \pm \sqrt{12}}{2}$$

$$= \frac{1 \pm i\sqrt{3}}{2} \qquad\qquad = \frac{-4 \pm 2\sqrt{3}}{2}$$

$$x = \frac{1 \pm i\sqrt{3}}{2} \qquad\qquad x = -2 \pm \sqrt{3}$$

Hence the roots are $\frac{1 \pm i\sqrt{3}}{2}$; $-2 \pm \sqrt{3}$., -1

2. Solve $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$

Soln:-

Given $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0 \rightarrow ①$

This is reciprocal eqn of degree 5 (odd degree) with like signs (T-II)

$\therefore$ $x-1$ is a root of eqn ①

$$\begin{array}{r|rrrrrr} -1 & 6 & 11 & -33 & -33 & 11 & 6 \\ & 0 & -6 & -5 & 38 & -5 & -6 \\ \hline & 6 & 5 & -38 & 5 & 6 & \lfloor 0 \end{array}$$

$$6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$$

This is reciprocal eqn of degree 4 with like sign (T-I)

Dividing by x^2 we get

$$6x^2+5x-38+\frac{5}{x}+\frac{6}{x^2}=0$$

$$6\left(x^2+1/x^2\right)+5\left(x+1/x\right)-38=0$$

Let $x+1/x=u$

$$x^2+1/x^2=u^2-2$$

$$6(u^2-2)+5u-38=0$$

$$6u^2-12+5u-38=0$$

$$6u^2+5u-50=0$$

$$\Rightarrow \frac{-5\pm\sqrt{25-4(6)(-50)}}{2(6)}$$

$$\Rightarrow \frac{-5\pm\sqrt{25+1200}}{12}$$

$$\Rightarrow \frac{-5\pm\sqrt{1225}}{12}$$

$$\Rightarrow \frac{-5\pm 35}{12}$$

$$\Rightarrow \frac{\cancel{30}^{5}}{\cancel{12}_{2}},\ \frac{-\cancel{40}^{\cancel{20}\,10}}{\cancel{12}\,\cancel{6}\,3}$$

$$u \Rightarrow \frac{5}{2},\ \frac{-10}{3}$$

$x+1/x=5/2$; $x+1/x=-10/3$

$x^2+1 = \frac{5}{2}x$; $x^2+1 = -\frac{10}{3}x$

$x^2 - \frac{5}{2}x + 1 = 0$; $x^2 + \frac{10}{3}x + 1 = 0$

$2x^2 - 5x + 2 = 0$; $3x^2 + 10x + 3 = 0$

$\Rightarrow \frac{5 \pm \sqrt{25 - 4(2)(2)}}{4}$; $\frac{-10 \pm \sqrt{100 - 4(3)(3)}}{6}$

$\Rightarrow \frac{5 \pm \sqrt{25-16}}{4}$; $\frac{-10 \pm \sqrt{100-36}}{6}$

$\Rightarrow \frac{5 \pm \sqrt{9}}{4}$; $\frac{-10 \pm \sqrt{64}}{6}$

$\Rightarrow \frac{5 \pm 3}{4}$; $\frac{-10 \pm 8}{6}$

$\Rightarrow 8/4, 2/4$; $-2/6, -18/6$

$x = 2, 1/2$

; $x \Rightarrow -1/3, -3.$

T-I. Hence the roots are $2, 1/2, -1/3, -3, -1$

2. Solve $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$

Soln:-

Given $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0 \rightarrow ①$

This is a reciprocal equation of degree 4 with like sign (T-I)

Dividing x^2 by eqn ①, we get

$$4x^2 - 20x + 33 - \frac{20}{x} + \frac{4}{x^2} = 0$$

$$4\left(x^2+\frac{1}{x^2}\right)-20\left(x+\frac{1}{x}\right)+33=0$$

let $x+\frac{1}{x}=u$

$$x^2+\frac{1}{x^2}=u^2-2$$

$$4(u^2-2)-20u+33=0$$

$$4u^2-8-20u+33=0$$

$$4u^2-20u+25=0$$

$$\Rightarrow \frac{20 \pm \sqrt{400-4(4)(25)}}{2(4)}$$

$$\Rightarrow \frac{20 \pm \sqrt{400-400}}{8}$$

$$\Rightarrow \frac{20 \pm 0}{8}$$

$$\Rightarrow \frac{20\;\cancel{10}\;5}{8\;\cancel{4}\;2}$$

$$u \Rightarrow \frac{5}{2}$$

$x+\frac{1}{x} \Rightarrow \frac{5}{2}$; $x+\frac{1}{x} \Rightarrow \frac{5}{2}$

$x^2+1=\frac{5}{2}x$; $x^2+1 \Rightarrow \frac{5}{2}x$

$x^2-\frac{5}{2}x+1=0$; $x^2-\frac{5}{2}x+1=0$

$2x^2-5x+2=0$; $2x^2-5x+2=0$

$$\Rightarrow \frac{5 \pm \sqrt{25-4(2)(2)}}{4} \quad ; \quad \frac{5 \pm \sqrt{25-4(2)(2)}}{4}$$

$$\Rightarrow \frac{5 \pm \sqrt{25-16}}{4} \quad ; \quad \frac{5 \pm \sqrt{25-16}}{4}$$

$$\Rightarrow \frac{5 \pm \sqrt{9}}{4} \quad ; \quad \frac{5 \pm \sqrt{9}}{4}$$

$$\Rightarrow \frac{5 \pm 3}{4} \quad ; \quad \frac{5 \pm 3}{4}$$

$x = 8/4, 2/4 \quad ; \quad x = 8/4, 2/4$

$x = 2, 1/2 \quad ; \quad x = 2, 1/2$

Hence the eqn roots are $2, 1/2, 2, 1/2$.

T-III:-

2017.

Reciprocal equation of odd degree with unlike signs for its coefficient.

For this type $x=1$ is a root or $x-1$ is a factor of the given reciprocal equation. Now dividing the given reciprocal equation by $x-1$, we get a reciprocal equation of degree 4. which can be solved by using (typeI).

1. Solve the equation. $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$

Soln:-

Given $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0 \rightarrow ①$

This is a reciprocal equation of odd degree with unlike signs (T-III).

$\therefore$ $x=1$ is a root of eqn ①.

1	6	-1	-43	43	1	-6	
	0	6	5	-38	5	6	
	6	5	-38	5	6	$\boxed{0}$	

The reduced eqn is

$$6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0 \rightarrow ②.$$

This is a reciprocal equation of degree 4 with like sign (T-I)

Dividing x^2 by ①, we get.

$$6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$$

$$6\left(x^2 + 1/x^2\right) + 5\left(x + 1/x\right) - 38 = 0$$

Let $x + 1/x = u$

$$x^2 + 1/x^2 = u^2 - 2$$

$$6(u^2 - 2) + 5u - 38 = 0$$

$$6u^2 - 12 + 5u - 38 = 0$$

$$6u^2 + 5u - 50 = 0$$

$$\Rightarrow \frac{-5 \pm \sqrt{25 - 4(6)(-50)}}{2(6)}$$

$$\Rightarrow \frac{-5 \pm \sqrt{25 + 1200}}{12}$$

$$\Rightarrow \frac{-5 \pm \sqrt{1225}}{12}$$

$$\Rightarrow \frac{-5 \pm 35}{12}$$

$$\Rightarrow \frac{\cancel{30}^{5}}{\cancel{12}_{2}}, \frac{-\cancel{40}^{20\,10}}{\cancel{12}_{6\,3}}$$

$$u = 5/2, -10/3$$

$$x + 1/x = 5/2 \quad ; \quad x + 1/x = -10/3$$

$$x^2 + 1 = 5/2\,x \quad ; \quad x^2 + 1 = -10/3\,x$$

$x^2 - \frac{5}{2}x + 1 = 0$; $x^2 + \frac{10}{3}x + 1 = 0$

$2x^2 - 5x + 2 = 0$; $3x^2 + 10x + 3 = 0$

$\Rightarrow \frac{5 \pm \sqrt{25-16}}{4}$; $\frac{-10 \pm \sqrt{100-36}}{6}$

$\Rightarrow \frac{5 \pm \sqrt{9}}{4}$; $\frac{-10 \pm \sqrt{64}}{6}$

$\Rightarrow \frac{5 \pm 3}{4}$; $\frac{-10 \pm 8}{6}$

$\Rightarrow 8/4, 2/4$; $-2/6, -18/6$

$\Rightarrow 2, 1/2$; $-1/3, -3$

$x \Rightarrow 2, 1/2$; $x = -1/3, -3$

Hence the roots are $2, 1/2, -3, -1/3, 1$

10m T-IV:-

Reciprocal equations with even degree and the terms equidistant from the first and last have opposite signs and the middle term is absent.

For this type $x = 1$ and $x = -1$ are the roots of the given reciprocal equation.

Now dividing the given equation by

$(x-1)$ and $(x+1)$ we get a reciprocal eqn o

degree 4 which can be solved by using

type-I.

.*. 1. Solve the equation $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x$

$-6 = 0$

Soln:-

Given $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0 \rightarrow$

This is a reciprocal eqn of even degree with unlike sign and its middle term is absent (T-IV).

$\therefore$ $x = 1$ and $x = -1$ are the roots of eqn ①

$$\begin{array}{r|rrrrrrrl}
1 & 6 & -35 & 56 & 0 & -56 & 35 & -6 & \\
 & 0 & 6 & -29 & 27 & 27 & -29 & 6 & \\
\hline
 & 6 & -29 & 27 & 27 & -29 & 6 & \boxed{0} & \\
-1 & 0 & -6 & 35 & -62 & 35 & -6 & & \\
\hline
 & 6 & -35 & 62 & -35 & 6 & \boxed{0} & &
\end{array}$$

$\therefore$ The reduced eqn is

$$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0 \rightarrow ②$$

This is a reciprocal eqn of degree 4 with like sign (T-I).

Dividing x^2 by ①, we get

$$6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$$

$$6\left(x^2 + 1/x^2\right) - 35\left(x + 1/x\right) + 62 = 0$$

Let $x + 1/x = u$

$$x^2 + 1/x^2 = u^2 - 2$$

$$6\left(u^2 - 2\right) - 35u + 62 = 0$$

$$6u^2 - 12 - 35u + 62 = 0$$

$$6u^2 - 35u + 50 = 0$$

$$\Rightarrow \frac{35 \pm \sqrt{1225 - 4(6)(50)}}{2(6)}$$

$$\Rightarrow \frac{35 \pm \sqrt{1225 - 1200}}{12}$$

$$\Rightarrow \frac{35 \pm \sqrt{25}}{12}$$

$$\Rightarrow \frac{35 \pm 5}{12}$$

$$\Rightarrow \frac{\cancel{40}\,\cancel{20}\,10}{\cancel{12}\,\cancel{6}\,3}, + \frac{\cancel{30}\,5}{\cancel{12}\,2}$$

$u = 10/3, +5/2$

$x + 1/x = 10/3$; $x + 1/x = +5/2$

$x^2 + 1 = 10/3\, x$; $x^2 + 1 = +5/2\, x$

$x^2 - 10/3\, x + 1 = 0$; $x^2 - 5/2\, x + 1 = 0$

$3x^2 - 10x + 3 = 0$; $2x^2 - 5x + 2 = 0$

$\Rightarrow (2x-1)(3x-9)$; $(2x-1)(2x-4) = 0$

$\rightarrow x \Rightarrow 3, 1/3$; $x \Rightarrow 2, 1/2$

Hence the root are $2, 3, 1/2, 1/3, 1, -1$.

T-III.

2. Solve $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$

Soln:-

Given $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0 \rightarrow ①$

This is a reciprocal equation of odd degree with (T-III)

$\therefore x = 1$ is a root of eqn ①.

$$\begin{array}{c|cccccc} 1 & 1 & -5 & 9 & -9 & 5 & -1 \\ & 0 & 1 & -4 & 5 & -4 & 1 \\ \hline & 1 & -4 & 5 & -4 & 1 & \boxed{0} \end{array}$$

The reduced eqn is

$$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0 \rightarrow ②.$$

This is a reciprocal eqn of degree 4 with like sign (T-I).

Dividing x^2 by ①, we get

$$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$$

$$x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2} = 0$$

$$(x^2 + 1/x^2) - 4(x + 1/x) + 5 = 0$$

Let $x + 1/x = u$

$$x^2 + 1/x^2 = u^2 - 2$$

$-1 \times -3 = 3$

$-1 + 3 = -4$

$$(u^2 - 2) - 4u + 5 = 0$$

$$u^2 - 4u + 3 = 0$$

$$(u-1)(u-3) = 0$$

$$u = 1, 3$$

$x + 1/x = 1$; $x + 1/x = 3$

$x^2 + 1 = x$; $x^2 + 1 = 3x$

$x^2 - x + 1 = 0$; $x^2 - 3x + 1 = 0$

$= \frac{1 \pm \sqrt{1-4}}{2}$; $\frac{3 \pm \sqrt{9-4(1)(1)}}{2}$

; $\frac{3 \pm \sqrt{5}}{2}$

$x = \frac{1 \pm i\sqrt{3}}{2}$

Hence the roots are $\frac{1 \pm i\sqrt{3}}{2}$, $\frac{1 - i\sqrt{3}}{2}$,

$\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$, -1

T-IV.

2. Solve $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x \overset{-}{\cancel{+}} 6 = 0$

Soln:- Given $6x^6 - 25x^4 - 31x^4 - 31x^2 + 25x - 6 = 0 \rightarrow$

This is a reciprocal eqn of even degree with unlike sign and its middle term is absent (T-IV)

$\therefore x = 1$ and $x = -1$ are the roots of eqn ①

$$\begin{array}{r|rrrrrrr}
1 & 6 & -25 & 31 & 0 & -31 & 25 & -6 \\
 & 0 & 6 & -19 & 12 & 12 & -19 & 6 \\
\hline
 & 6 & -19 & 12 & 12 & -19 & 6 & \boxed{0} \\
-1 & 0 & -6 & 25 & -37 & 25 & -6 & \\
\hline
 & 6 & -25 & 37 & -25 & 6 & \boxed{0} &
\end{array}$$

∴ The reduced eqn is

$$6x^4-25x^3+37x^2-25x+6=0 \rightarrow ②$$

Dividing x^2 by ① we get.

$$6x^2-25x+37-\frac{25}{x}+\frac{6}{x^2}=0$$

$$6\left(x^2+1/x^2\right)-25\left(x+1/x\right)+37=0$$

Let $x^2+1/x^2=u$

$$x^2+1/x^2=u^2-2$$

$$6\left(u^2-2\right)-25u+37=0$$

$$6u^2-12-25u+37=0$$

$$6u^2-25u+25=0$$

$$\Rightarrow \frac{25\pm\sqrt{625-4(6)(25)}}{2(6)}$$

$$\Rightarrow \frac{25\pm\sqrt{625-600}}{}$$

$$\Rightarrow \frac{25 \pm \sqrt{25}}{12}$$

$$\Rightarrow \frac{25 \pm \sqrt{5}}{12}$$

$$\Rightarrow \frac{\cancel{30}^{5}}{\cancel{12}_{2}}, \frac{\cancel{20}^{\cancel{10}\,5}}{\cancel{12}\,\cancel{6}_{3}}$$

$$a \Rightarrow 5/2, 5/3.$$

$x + 1/x = 5/2 \quad ; \quad x + 1/x = 5/3$

$x^2 + 1 = 5/2\, x \quad ; \quad x^2 + 1 = 5/3\, x$

$x^2 - 5/2\, x + 1 = 0 \quad ; \quad x^2 - 5/3\, x + 1 = 0$

$2x^2 - 5x + 2 = 0 \quad ; \quad 3x^2 - 5x + 3 = 0$

$$\Rightarrow \frac{5 \pm \sqrt{25 - 4(2)(2)}}{4} \quad ; \quad \frac{5 \pm \sqrt{25 - 4(3)(3)}}{6}$$

$$\Rightarrow \frac{5 \pm \sqrt{25 - 16}}{4} \quad ; \quad \frac{5 \pm \sqrt{25 - 36}}{6}$$

$$\Rightarrow \frac{5 \pm \sqrt{9}}{4} \quad ; \quad \frac{5 \pm \sqrt{-11}}{6}$$

$\Rightarrow \frac{5\pm 3}{4}$; $\frac{5\pm i\sqrt{11}}{6}$

$\Rightarrow 8/4, 2/4$; $\frac{5\pm i\sqrt{11}}{6}$

$\Rightarrow 2, 1/2$; $\frac{5\pm i\sqrt{11}}{6}$

Hence the roots are $2, 1/2; \frac{5+i\sqrt{11}}{3}, \frac{5-i\sqrt{11}}{3}, 1, -1.$

UNIT-III

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FORM OF THE QUOTIENT AND REMAINDER WHEN A POLYNOMIAL IS DIVIDED BY A POLYNOMIAL:-

1) Find the Quotient and Remainder when $3x^3+8x^2+8x+12=0$ is divided by $x-4$

Soln:-

Given $3x^3+8x^2+8x+12=0 \rightarrow$ ①

$$\begin{array}{c|cccc} 4 & 3 & 8 & 8 & 12 \\ & 0 & 12 & 80 & 352 \\ \hline & 3 & 20 & 88 & \boxed{364} \end{array}$$

Quotient is $3x^2+20x+88$ and the remainder is 364

2) Find the Quotient and Remainder when $2x^6+3x^5-15x^2+2x-4$ is divided by $x+5$

Soln:-

Given $2x^6+3x^5-15x^2+2x-4 \Rightarrow$ ①

$$\begin{array}{c|ccccccc} -5 & 2 & 3 & 0 & 0 & -15 & 2 & -4 \\ & 0 & -10 & 35 & -175 & 875 & -4300 & 21490 \\ \hline & 2 & -7 & 35 & -175 & 860 & -4298 & \boxed{+21486} \end{array}$$

Quotient $2x^5-7x^4+35x^3-175x^2+860x-4298$

$\Rightarrow \frac{5\pm 3}{4} \quad ; \quad \frac{5\pm i\sqrt{11}}{6}$

$\Rightarrow 8/4, 2/4 \quad ; \quad \frac{5\pm i\sqrt{11}}{6}$

$\Rightarrow 2, 1/2 \quad ; \quad \frac{5\pm i\sqrt{11}}{6}$

Hence the roots are $2, 1/2 ; \frac{5+i\sqrt{11}}{3}, \frac{5-i\sqrt{11}}{3}, 1, -1.$

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Quotient is $3x^2+20x+88$ and the remainder is 364

2) Find the Quotient and Remainder when $2x^6+3x^5-15x^2+2x-4$ is divided by $x+5$

Soln:-

Given $2x^6+3x^5-15x^2+2x-4 \Rightarrow ①$

$$\begin{array}{c|ccccccc} -5 & 2 & 3 & 0 & 0 & -15 & 2 & -4 \\ & 0 & -10 & 35 & -135 & 875 & -4300 & 21490 \\ \hline & 2 & -7 & 35 & -135 & 860 & -4298 & \boxed{+21486} \end{array}$$

Quotient $2x^5-7x^4+35x^3-175x^2+860x-4298$

Remainder 21486

3) Find the quotient and Remainder when $x^4-5x^3+7x^2-4x+5$ is divided by $x-2$

Soln:-

Given $x^4-5x^3+7x^2-4x+5 \rightarrow ①$

$$\begin{array}{c|ccccc} 2 & 1 & -5 & 7 & -4 & 5 \\ & 0 & 2 & -6 & 2 & -4 \\ \hline & 1 & -3 & 1 & -2 & \boxed{1} \end{array}$$

Quotient x^3-3x^2+x-2

Remainder 1

4) S.T the eqn $x^4-3x^3+4x^2-2x+1=0$ can be transformed into a reciprocal equation by diminishing the roots by unity and hence solve the equation.

Proof:-

Given $x^4-3x^3+4x^2-2x+1=0 \rightarrow ①$

This is a reciprocal eqn of degree 4 with signs

$$\begin{array}{c|ccccc} 1 & 1 & -3 & 4 & -2 & 1 \\ & 0 & 1 & -2 & 2 & 0 \\ \hline 1 & 1 & -2 & 2 & 0 & \boxed{1} \\ & 0 & 1 & -1 & 1 & \\ \hline 1 & 1 & -1 & 1 & \boxed{1} & \\ & 0 & +1 & 0 & & \\ \hline 1 & 1 & 0 & \boxed{1} & & \\ & 0 & 1 & & & \\ \hline & 1 & \boxed{1} & & & \end{array}$$

$x^4+x^3+x^2+x+1=0 \rightarrow ②$

$\therefore$ It is reciprocal equation.

This is a reciprocal eqn of degree 4 with like signs (T-I).

Divide ② by x^2, we get

$$x^2+x+1+\frac{1}{x}+1/x^2=0$$

Let $x+1/x=u$

$$x^2+1/x^2=u^2-2.$$

$$\left(x^2+1/x^2\right)+\left(x+1/x\right)+1=0.$$

$$u^2-2+u+1=0$$

$$u^2+u-1=0.$$

$$u \Rightarrow \frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2}$$

$$\Rightarrow \frac{-1\pm\sqrt{1+4}}{2}$$

$$\Rightarrow \frac{-1\pm\sqrt{5}}{2}$$

$$u=\frac{-1+\sqrt{5}}{2},\ \frac{-1-\sqrt{5}}{2}$$

$$x+1/x=\frac{-1+\sqrt{5}}{2} \quad ; \quad x+1/x=\frac{-1-\sqrt{5}}{2}$$

$$x^2+1 = \frac{-1+\sqrt{5}}{2}x \quad ; \quad x^2+1 = \frac{-1-\sqrt{5}}{2}x.$$

$$2x^2+2 = (-1+\sqrt{5})x \quad ; \quad 2x^2+2 = (-1-\sqrt{5})x$$

$$2x^2-(-1+\sqrt{5})x+2=0 \quad ; \quad 2x^2-(-1-\sqrt{5})x+2=0$$

$$b=-(-1+\sqrt{5});\ a=2;\ c=2 \quad ; \quad b=-(-1-\sqrt{5});\ a=2;\ c=2$$

$$x = \frac{(-1+\sqrt{5})\pm\sqrt{(-1+\sqrt{5})^2-16}}{4} \quad ; \quad \Rightarrow \frac{(-1-\sqrt{5})\pm\sqrt{(-1-\sqrt{5})^2-16}}{4}$$

$$x = \frac{(-1+\sqrt{5})\pm\sqrt{1+5-2\sqrt{5}-16}}{4} \quad ; \quad \frac{(-1-\sqrt{5})\pm\sqrt{1+5+2\sqrt{5}-16}}{4}$$

$$x = \frac{(-1+\sqrt{5})\pm\sqrt{-2\sqrt{5}-10}}{4} \quad ; \quad x \Rightarrow \frac{(-1-\sqrt{5})\pm\sqrt{2\sqrt{5}-10}}{4}$$

∴ The roots of original equation are these roots increased by 1.

ie, $1+\frac{(-1+\sqrt{5})\pm\sqrt{-2\sqrt{5}-10}}{4}$; $1+\frac{(-1-\sqrt{5})\pm\sqrt{2\sqrt{5}-10}}{4}$

$$\frac{4+(-1+\sqrt{5})\pm\sqrt{-2\sqrt{5}-10}}{4} \quad ; \quad \frac{4+(-1-\sqrt{5})\pm\sqrt{2\sqrt{5}-10}}{4}$$

$$\frac{4-1+\sqrt{5}\pm\sqrt{-2\sqrt{5}-10}}{4} \quad ; \quad \frac{4-1-\sqrt{5}\pm\sqrt{2\sqrt{5}-10}}{4}$$

$$\frac{(3+\sqrt{5})\pm\sqrt{-2\sqrt{5}-10}}{4} \quad ; \quad \frac{(3-\sqrt{5})\pm\sqrt{2\sqrt{5}-10}}{4}$$

Hence the roots are $\frac{(3+\sqrt{5})\pm\sqrt{-2\sqrt{5}-10}}{4}$ and

$\frac{(3-\sqrt{5})\pm\sqrt{2\sqrt{5}-10}}{4}$.

5. Find the eqn whose roots are the roots $x^4-x^3-10x^2+4x+24=0$ increased by 2 and hence solve the equation.

<u>Soln:</u>

$$\text{Given } x^4-x^3-10x^2+4x+24=0 \rightarrow ①$$

Increased by 2 the roots of the given eqn is the same as decreasing the roots by -2

-2	1	-1	-10	4	24
	0	-2	6	8	-24
	1	-3	-4	12	$\boxed{0}$
-2	0	-2	10	-12	
	1	-5	6	$\boxed{0}$	
-2	0	-2	14		
	1	-7	$\boxed{20}$		
-2	0	-2			
	1	$\boxed{-9}$			
-2	0				
	1				

$$x^4-9x^3+20x^2=0 \rightarrow ②.$$

The transformed eqn is

$$x^4-9x^3+20x^2=0$$

∴ It is reciprocal equation

This is a reciprocal eqn. of degree 4 with like signs (T-I).

Divided x^2 by ② we get:

$x^2 - 9x + 20 = 0.$

~~Let $x + 1/x = u$~~

~~$x^2 + 1/x^2 = u^2 - 2.$~~

$(x-4)\ (x-5) = 0.$

$x = 4, 5$

Hence the roots are 0, 0, 4, 5.

∴ The roots of original equation are there roots decreased by 2.

ie, $0-2,\ 0-2,\ 4-2,\ 5-2.$

$-2,\ -2,\ 2,\ 3$

Hence the roots are $-2, -2, 2, 3$

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6. S.T the equation $x^4 + 5x^3 + 9x^2 + 5x - 1 = 0$ can be transformed into a reciprocal eqn by increasing the roots by 2 and hence solve the eqn

Soln:-

Given $x^4+5x^3+9x^2+5x-1=0 \rightarrow ①$

Increased by 2 the roots of the given eqn is the same as decreasing the roots by -2

$$\begin{array}{r|rrrrr}
-2 & 1 & 5 & 9 & 5 & -1 \\
 & 0 & -2 & -6 & -6 & 2 \\
\hline
 & 1 & 3 & 3 & -1 & \boxed{1} \\
-2 & 0 & -2 & -2 & -2 & \\
\hline
 & 1 & 1 & 2 & \boxed{-3} & \\
-2 & 0 & -2 & 2 & & \\
\hline
 & 1 & -1 & \boxed{3} & & \\
-2 & 0 & -2 & & & \\
\hline
 & 1 & \boxed{-3} & & & \\
-2 & 0 & & & & \\
\hline
 & 1 & & & &
\end{array}$$

The transformed eqn is

$x^4-3x^3+3x^2-3x+1=0 \rightarrow ②$

∴ It is reciprocal equation.

This is a reciprocal eqn of degree 4 with like signs (T-I)

Divided x^2 by ② we get

$$x^2-3x+3-\frac{3}{x}+\frac{1}{x^2}=0$$

$$\left(x^2+1/x^2\right)-3(x+1/x)+3=0.$$

Let $x+1/x=u$

$x^2+1/x^2=u^2-2.$

$$u^2 - 2 - 3u + 3 = 0$$

$$u^2 - 3u + 1 = 0.$$

$$\Rightarrow \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2}$$

$$\Rightarrow \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Rightarrow \frac{3 \pm \sqrt{5}}{2}$$

$$u = \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}.$$

$$x + 1/x = \frac{3+\sqrt{5}}{2} \quad ; \quad x + 1/x = 3-\sqrt{5}/2$$

$$x^2 + 1 = \frac{3+\sqrt{5}}{2}x \quad ; \quad x^2 + 1 = \frac{3-\sqrt{5}}{2}x$$

$$2x^2 + 2 = (3+\sqrt{5})x \quad ; \quad 2x^2 + 2 = (3-\sqrt{5})x$$

$$2x^2 - (3+\sqrt{5})x + 2 = 0 \quad ; \quad 2x^2 - (3-\sqrt{5})x + 2 = 0.$$

$$x = \frac{(3+\sqrt{5}) \pm \sqrt{(3+\sqrt{5})^2 - 4(2)(2)}}{4} \quad ; \quad x = \frac{(3-\sqrt{5}) \pm \sqrt{(3-\sqrt{5})^2 - 16}}{4}$$

$$x = \frac{(3+\sqrt{5}) \pm \sqrt{9+5+6\sqrt{5}-16}}{4} \quad ; \quad x = \frac{(3-\sqrt{5}) \pm \sqrt{9+5-6\sqrt{5}-16}}{4}$$

$$x = \frac{(3+\sqrt{5}) \pm \sqrt{6\sqrt{5}-2}}{4} \quad ; \quad x = \frac{(3-\sqrt{5}) \pm \sqrt{-6\sqrt{5}-2}}{4}$$

∴ The roots of original equation are these roots increased by 2.

ie, $\frac{(3+\sqrt{5}) \pm \sqrt{6\sqrt{5}-2}}{4} - 2$; $\frac{(3-\sqrt{5}) \pm \sqrt{-6\sqrt{5}-2}}{4} - 2$.

$$\Rightarrow \frac{(3+\sqrt{5}) \pm \sqrt{6\sqrt{5}-2} - 8}{4} \quad ; \quad \frac{(3-\sqrt{5}) \pm \sqrt{-6\sqrt{5}-2} - 8}{4}$$

$\Rightarrow$ ~~$(3+\sqrt{5})$~~ +

$$\Rightarrow \frac{(-5+\sqrt{5}) \pm \sqrt{6\sqrt{5}-2}}{4} \quad ; \quad \frac{(-5-\sqrt{5}) \pm \sqrt{-6\sqrt{5}-2}}{4}$$

Hence the roots are $\frac{(-5+\sqrt{5}) \pm \sqrt{6\sqrt{5}-2}}{4}$ and $\frac{(-5-\sqrt{5}) \pm \sqrt{-6\sqrt{5}-2}}{4}$.

7. Find the equation whose root exceed by 2 the roots of the eqn $4x^4+32x^3+83x^2+76x+21=0$ and hence solve the eqn.

Soln:-

Given $4x^4+32x^3+83x^2+76x+21=0 \rightarrow ①$

Increased by 2 the roots of the given eqn is the same as decreasing the roots by -2.

$$\begin{array}{r|rrrrr}
 & 4 & 32 & 83 & 76 & 21 \\
-2 & 0 & -8 & -48 & -70 & -12 \\
\hline
 & 4 & 24 & 35 & 6 & \boxed{9} \\
-2 & 0 & -8 & -32 & -6 & \\
\hline
 & 4 & 16 & 3 & \boxed{0} & \\
-2 & 0 & -8 & -16 & & \\
\hline
 & 4 & 8 & \boxed{-13} & & \\
-2 & 0 & -8 & & & \\
\hline
 & 4 & \boxed{0} & & & \\
-2 & 0 & & & & \\
\hline
 & 4 & & & &
\end{array}$$

$4x^4-13x^2+9=0$

The transformed eqn is

$$4x^4-13x^2+9=0 \quad \rightarrow ②$$

This is reciprocal eqn of degree 4 with like signs (T-II)

Divi

$$4x^4-13x^2+9=0$$

$\therefore x^2=u$

$$4u^2-13u+9=0$$

$$\begin{array}{c} 36 \\ \wedge \\ -4\ -9 \end{array}$$

$$4u^2-4u-9u+9=0$$

$$4u(u-1)-9(u-1)=0$$

$$(4u-1)(u-1)=0$$

$$(4u-9)(u-1)=0$$

$u = 9/4 \; ; \; u = 1$

$\therefore x^2 = 9/4 \; ; \; x^2 = 1$

$x = \pm 3/2 \; ; \; x = \pm 1$

The roots of the original equation are those roots increased by 2.

ie, $x = (3/2 - 2) \; ; \; (-3/2 - 2) \; ; \; (1-2) \; ; \; (-1-2)$

$x = -1/2, -7/2, -1, -3$

Hence the roots are $-1/2, -7/2, -1, -3$

Removal of terms :-

One of the cheif uses of this transformation is to remove a certain speified term from an equation. Such a step always help to find the solutions of an equation.

Let the given equation be

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

Then if $y = x - h$, we obtain the new equation. $x = (y-h)$

$$a_0 (y+h)^n + a_1 (y+h)^{n-1} + a_2 (y+h)^{n-2} + \dots + a_n = 0$$

$a_0 (y^n + nC_1 y^{n-1} h + n(n-1) y^{n-2} h^2$

which, when arranged is descending powers of y, becomes

$$a_0 (y+h)^n + a_1 (y+h)^{n-1} + a_2 (y+h)^{n-2} +$$

$$a_0 y^n + (n a_0 h + a_1) y^{n-1} + \left\{ \frac{n(n-1)}{2} a_0 h^2 + (n-1) a_1 h + a_2 \right\} y^{n-2} + \dots = 0)$$

If the term to be removed is the second, we put

$$\boxed{n a_0 h + a_1 = 0 \text{ so that } h = \frac{-a_1}{n a_0}}$$

If the term to be removed is the third, we put

$$\boxed{\frac{n(n-1)}{2!} a_0 h^2 + (n-1) a_1 h + a_2 = 0}$$

and so obtain a quadratic to find h; and similarly we may any other assigned term.

Example 1:-

27/7 Find the relation between the coefficient in the eqn $x^4+px^3+qx^2+rx+s=0$ in order that the coefficient of x^3 and x may be removable by the same transformation.

Soln:-

Let us reduce the roots of the equation by h.

Instead of x substitute x+h. H

The transformed equation is $(x+h)$ $n=4$

$$(x+h)^4 + p(x+h)^3 + q(x+h)^2 + r(x+h) + s = 0 \quad a_0=1$$

$a_1=p$

$a_2=q$

$a_3=r$

$a_4=s$

ie, $n a_0 h + a_1$ $\quad \frac{n(n-1)}{2!} a_0 h^2 + (n-1) a_1 h + a_2$

$$x^4 + (4h+p)x^3 + \left(\frac{4(4-1)}{2}(1)h^2 + (4-1)ph + q\right)x^2$$

$$+ \left(\frac{4(4-1)(4-2)}{6}(1)h^3 + \frac{4(4-1)(4-2)}{2}ph^2 + (4-2)qh + r\right)x$$

$$+ (h^4 + ph^3 + qh^2 + rh + s) = 0$$

$\frac{n(n-1)(n-2)}{3!} a_0 h^3$

$+ \frac{(n-1)(n-2)}{2!} a_1 h^2$

$+ (n-2) a_2 h$

$+ a_3$

ie, $x^4+(4h+p)x^3+(6h^2+3hp+q)x^2+(4h^3+3h^2p+2hq+r)$

$x+(h^4+ph^3+qh^2+rh+s)=0$

The coefficients of x^3 and x in the transformed equation are zero.

$4h+p=0 \rightarrow ①; \quad 4h^3+3h^2p+2hq+r=0 \rightarrow ②$

$h=-p/4$

$② \Rightarrow 4(-p/4)^3+3(-p/4)^2p+2(-p/4)q+r=0$

$$\frac{-4p^3}{4^3}+\frac{3p^3}{4^2}-\frac{2pq}{4}+r=0$$

$$-\frac{p^3}{4^2}+\frac{3p^3}{16}-\frac{2pq}{4}+r=0$$

$$\frac{-p^3}{16}+\frac{3p^3}{16}-\frac{2pq}{4}+r=0$$

$$-p^3+3p^3-8pq+16r=0$$

$$\therefore p^3-4pq+8r=0$$

2. Solve the eqn $x^4+20x^3+143x^2+430x+462=0$ by removing its second term.

Soln:-

Let us reduce the roots of the equation by h.

Instead of x substitute $x+h$

The transformed equation is

$$(x+h)^4 + 20(x+h)^3 + 143(x+h)^2 + 430(x+h) + 462 = 0$$

ie, (By using second term (naoh+an)) $\left(\frac{n(n-1)}{2!} a_0 h^2 + (n-1) a_1 h + a_2\right) x^2$

$$x^4 + (4h+20)x^3 + \left(\frac{4(4-1)}{2}(1)h^2 + (4-1)20h + 143\right)x^2$$

$$+\left(\frac{\overset{2}{\cancel{4}}\overset{3}{(4-1)}\overset{2}{(4-2)}}{\cancel{6}\,3}(1)h^3 + \frac{\overset{3}{(4-1)}\overset{2}{(4-2)}}{\cancel{2}}20h^2 + \overset{2}{(4-2)}143h + 430\right)x$$

$$+(h^4 + 20x^3 + 143x^2 + 430x + 462) = 0.$$

ie,

$$x^4 + (4h+20)x^3 + (6h^2+60h+143)x^2 + (4h^3+60h^2+286h$$

$$+430)x + (h^4+20x^3+143x^2+430+462) = 0$$

The coefficient of x^3 and x in the transformed equation are zero By removing 2nd term, we get

$4h+20=0 \rightarrow$ ① ; $4h^3+60h^2+286h+430=0 \rightarrow$ ②

$h = -20/4$; $\boxed{h=-5}$

Hence the remove the 2nd second increase the roots of the eqn by 5

-5	1	20	143	430	462
	0	-5	-75	-340	-450
-5	1	15	68	90	12
	0	-5	-50	-90	
-5	1	10	18	0	
	0	-5	-25		
-5	1	5	-7		
	0	-5			
	1	0			

$$-5 \,\big|\, \underline{0}$$
$$1$$

The reduced eqn
$$x^4 - 7x^2 + 12 = 0$$

$x^2 = u$ y

$y^2 u^2 - 7uy + 12 = 0$ $\quad y^2 - 7y + 12 = 0$

$(y-4)(y-3) = 0$

$y = 4, 3$

$x^2 = 4$; $x^2 = 3$

$x = \pm 2$; $x = \pm\sqrt{3}$

Hence the roots are, greater than the roots of original eqn by 5 $2, -2, +\sqrt{3}, -\sqrt{3}$.

∴ These roots of original equation are there roots decreased by 5.

ie,

$2-5, -2-5, \sqrt{3}-5, -\sqrt{3}-5$

$-3, -7, \sqrt{3}-5, -\sqrt{3}-5$.

Hence the roots are $-3, -7, \sqrt{3}-5, -\sqrt{3}-5$.

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TO FORM AN EQUATION WHOSE ROOTS ARE ANY POWER OF THE ROOTS OF A GIVEN EQUATION.

The method of forming such equations is illustrated in the following examples.

Example 1:

Find the equation whose roots are the squares of the roots of the equation.

·($x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = 0$)

Let $\alpha_1, \alpha_2, \dots \alpha_n$ be the roots of equation. Then we have $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n) \to ①$

If we transform the equation into another whose roots are the negatives of the roots of this eqn, we get.

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots = (+x+\alpha_1)(+x+\alpha_2)\dots(+x+\alpha_n) \to$$

①×② ∴ $(x^2-\alpha_1^2)(x^2-\alpha_2^2)\dots(x^2-\alpha_n^2)$

(a^2-b^2)

$$= (x^n + p_2 x^{n-2} + p_4 x^{n-4} + \dots)^2 - (p_1 x^{n-1} + p_3 x^{n-3} + \dots)^2$$

It is evident that the left-hand side when expanded contains only even powers of x.

Replacing x^2 by y, we get

$$(y-\alpha_1)^2 (y-\alpha_2)^2 \dots (y-\alpha_n)^2 = y^n + (2p_2 - p_1^2) y^{n-1} + \dots$$

∴ $y^n + (2p_2 - p_1^2) y^{n-1} + \dots = 0$ will have roots

$\alpha_1^2, \alpha_2^2, \dots \alpha_n^2$.

Example 2:

Find the equation whose roots are the squares of the roots of $x^4 + x^3 + 2x^2 + x + 1 = 0$

Let the roots be $\alpha, \beta, \gamma, \delta$

Then $x^4 + x^3 + 2x^2 + x + 1 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ -①

Changing x into $-x$, we get -

$x^4-x^3+2x^2-x+1=(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)\rightarrow$ ①. ①×②

$\therefore (x^4+2x^2+1)^2-(x^3+x)^2=(x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)$

$(x(x+1))^2$

Put $x^2=y$

Then $(y^2+2y+1)^2-y(y+1)^2=(y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$.

$\therefore$ The equation whose roots are the squares of the roots are the given equation is $(y^2+2y+1)^2-y(y+1)^2=0$

$(a^2+b^2+c)^2=a^4+b^2+c^2+2ab+2bc+2ca$

$y^4+4y^2+1+4y^3+4y+2y^2-$

ie, $y^4+3y^3+4y^2+3y+1=0$.

$y(y^2+2y+1)=0$

$y^4+4y^3+6y^2+4y+1-y^3-2y^2-y=0$

$y^4+4y^2+1+4y^3+4y+2y^2$

$y^4+3y^3+4y^2+3y+1=0$

$-y^3-2y^2+y=0$

Example 3:

(I.Y.) Find the equation whose roots are the cubes of the roots of $x^4-x^3+2x^2+3x+1=0$. If the cube roots of the unity are $1, \omega, \omega^2$ then $\boxed{(P+Q+R)(P+\omega Q+\omega^2 R)(P+\omega^2 Q+\omega R)=P^3+Q^3+R^3-3PQR.}$

Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation

Then,

$$x^4-x^3+2x^2+3x+1=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

ie, $(1-x^3)+x(3+x^3)+2x^2=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\rightarrow$ ①

changing x to ωx in the eqn, we get

$\Rightarrow (1-\omega^3x^3)+\omega x(\omega^3x^3+3)+2(\omega^2x^2)=0$

$$(1-x^3)+\omega x(3+x^3)+2\omega^2x^2$$

$$=(\omega x-\alpha)(\omega x-\beta)(\omega x-\gamma)(\omega x-\delta)\rightarrow ②$$

Since $\omega^3=1$

Changing x into $\omega^2 x$, we get

$\Rightarrow (1-x^3)+\omega^2x(x^3+3)+2\omega x^2=0$

$$(1-x^3)+\omega^2x(3+x^3)+2\omega x^2$$

$$=(\omega^2x-\alpha)(\omega^2x-\beta)(\omega^2x-\gamma)(\omega^2x-\delta)\rightarrow ③$$

Since $\omega^4=\omega$

Multiplying ①, ②, ③ we get

$$\{(1-x^3)+x(3+x^3)+2x^2\}.\{(1-x^3)+\omega x(3+x^3)+2\omega^2x^2\}$$

$$.\{(1-x)^3+\omega^2x(3+x^3)+2\omega x^2\}$$

$$\Rightarrow \overset{\to\text{Product symbol}}{\Pi}(x-\alpha)(\omega x-\alpha)(\omega^2x-\alpha)$$

ie, $(P+Qx+Rx^2)(P+Q\omega x+R\omega^2x^2).(P+Q\omega^2x+R\omega x^2)$

$$\Rightarrow \Pi\{-\alpha^3+\alpha^2x\overset{0}{(1+\omega+\omega^2)}-\alpha x^2\overset{0}{(\omega+\omega^3+\omega^2)}+\overset{(1)}{\omega^3}x^3\}$$

$$\Rightarrow \Pi(x^3-\alpha^3).$$

where $P=1-x^3$, $Q=\cancel{x}(3+x^3)$, $R=2\cancel{x^2}$.

ie,

$$P^3+Q^3x^3+R^3x^6-3x^3P.Q.R$$

$$=(x^3-\alpha^3)(x^3-\beta^3)(x^3-\gamma^3)(x^3-\delta^3)$$

ie,

$$(1-x^3)^3+(3+x^3)^3x^3+8x^6-6x^3(1-x^3)(3+x^3)$$

$$=(x^3-\alpha^3)(x^3-\beta^3)(x^3-\gamma^3)(x^3-\delta^3)$$

Put $x^3=y$

Then $(1-y)^3+(3+y)^3y+8y^2-6y(1-y)(3+y)$

$$\Rightarrow (y-\alpha^3)^3(y-\beta^3)(y-\gamma^3)(y-\delta^3)$$

∴ The eqn whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ is

$(1-y)^3+(3+y)^3y+8y^2-6y(1-y)(3+y)=0$

$(1-3y+3y^2-y^2)+(27+3(9)(y)+3(3)(y^2)+y^3)y+8y^2-6y(3-3y+y-y^2)$

i.e, $y^4+14y^3+50y^2+6y+1=0$.

$\Rightarrow 1-3y+3y^2-y^3+27y+27y^2+9y^3+y^4+8y^2-18y+18y^2-6y^2+6y^3$

H.W: $\Rightarrow y^4+14y^3+50y^2+6y+1=0$

9\. Solve the following eqn by removing the second term each. $x^4-12x^3+48x^2-72x+35=0$

soln:-

Let us reduce the roots of the equation by h.

Instead of x substitute x+h.

The transformed equation is

$(x+h)^4-12(x+h)^3+48(x+h)^2-72(x+h)+35=0$

i.e,

$x^4+(4h-12)x^3+\dots=0$

i.e,

By removing second term, we get

$4h-12=0$

$\boxed{h=3}$

Hence remove the second term by decrease the roots of the eqn by +3

3	1	-12	48	-72	35	
	0	3	-27	63	-27	
3	1	-9	21	-9	8	
	0	3	-18	9		
3	1	-6	3	0		
	0	3	-9			
3	1	-3	-6			
	0	3				
	1	0				

The reduced eqn is

$x^4-6x^2+8=0$

$x^2=y$

$y^2-6y+8=0$

$(y-4)(y-2)=0$

$y=4,2$

$x^2=4$; $x^2=2$

$x=\pm 2$; $x=\pm\sqrt{2}$

Hence the roots are

$2,-2,\sqrt{2},-\sqrt{2}$.

∴ The roots of the original eqn are these increased by 3

$2+3,-2+3,\sqrt{2}+3,-\sqrt{2}+3$

$5,1,\sqrt{2}+3,-\sqrt{2}+3$.

8. $x^4+4x^3+5x^2+2x-6=0$

Soln:-

Let us reduce the roots of the eqn by h.

Instead of x substitute $x+h$

The transformed eqn is,

$(x+h)^4+4(x+h)^3+5(x+h)^2+2(x+h)-6=0$

ie, By removing second term, we get

$x^4+(4h+4)x^3+\dots\dots=0$

$4h+4=0$

$\boxed{h=-1}$

Hence remove the second term increased the roots by 1

$$\begin{array}{r|rrrrr}
-1 & 1 & 4 & 5 & 2 & -6 \\
 & 0 & -1 & -3 & -2 & 0 \\
\hline
 & 1 & 3 & 2 & 0 & \boxed{-6} \\
-1 & 0 & -1 & -2 & 0 & \\
\hline
 & 1 & 2 & 0 & \boxed{0} & \\
-1 & 0 & -1 & -1 & & \\
\hline
 & 1 & 1 & \boxed{-1} & & \\
-1 & 0 & -1 & & & \\
\hline
 & 1 & \boxed{0} & & &
\end{array}$$

The reduced eqn is

$x^4-x^2-6=0$

$x^2=y$

$y^4-y^2-6=0$

$(y+2)(y-3)$

$y=-2,3$

$x^2=\pm i\sqrt{2}$; $x=\pm\sqrt{3}$

Hence the roots are

$\pm i\sqrt{2}, \pm\sqrt{3}$

∴ The roots of the original eqn are there decreased by 1

$\Rightarrow -1 \pm \sqrt{3}, -1 \pm i\sqrt{2}$.

9. $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$

Soln:-

Let us reduce the roots of the eqn by h.

Instead of x substitute x+h

The transformed eqn is

$(x+h)^4 + 16(x+h)^3 + 83(x+h)^2 + 152(x+h) + 84 = 0$

ie,

$x^4 + (4h+16)x^3 + \cdots = 0$

$4h + 16 = 0$

$\boxed{h = -4}$

Hence remove the second term increased the root by 4.

$$\begin{array}{r|rrrrr}
-4 & 1 & 16 & 83 & 152 & 84 \\
 & 0 & -4 & -48 & -140 & -48 \\ \hline
 & 1 & 12 & 35 & 12 & \boxed{36} \\
-4 & 0 & -4 & -32 & -12 & \\ \hline
 & 1 & 8 & 3 & \boxed{0} & \\
-4 & 0 & -4 & -16 & & \\ \hline
 & 1 & 4 & \boxed{-13} & & \\
-4 & 0 & -4 & & & \\ \hline
 & 1 & \boxed{0} & & &
\end{array}$$

The reduced eqn is

$x^4 - 13x^2 + 36 = 0$

$x^2 = y$

$y^2 - 13y^2 + 36 = 0$

$(y-9)(y-4) = 0$

$y = 9, 4$

$x^2 = 9 \ ; x^2 = 4$

$x = \pm 3 \ ; \ x = \pm 2$

Hence the roots are

$3, -3, 2, -2$

∴ The roots of the original eqn are there decreased by 4

$3-4, -3-4, 2-4, -2-4.$

$\Rightarrow -7, -6, -2, -1.$

10. $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0$

Soln:-

Let us reduce the roots of the eqn by h

Instead of x substitute x+h

The transformed eqn is

$(x+h)^4 - 8(x+h)^3 + 19(x+h)^2 - 12(x+h) + 2 = 0$

ie,

$x^4 + (4h-8)x^3 + \cdots = 0$

$4h - 8 = 0$

$\boxed{h=2}$

Hence remove the second term decreased by to 2

$$\begin{array}{c|cccccc} 2 & 1 & -8 & 19 & -12 & 2 \\ & 0 & 2 & -12 & 14 & 4 \\ \hline & 1 & -6 & 7 & 2 & \boxed{6} \\ 2 & 0 & 2 & -8 & -2 & \\ \hline & 1 & -4 & -1 & \boxed{0} & \\ 2 & 0 & 2 & -4 & & \\ \hline 2 & 1 & -2 & \boxed{-5} & & \\ & 0 & 2 & & & \\ \hline & 1 & \boxed{0} & & & \end{array}$$

$\therefore \; x^4 - 5x^2 + 6 = 0$

$x^2 = y$

$y^4 - 5y^2 + 6 = 0$

$(y-2)(y-3) = 0$

$y = \pm 2, 3$

$x^2 = 2 \;;\; x^2 = 3$

$x = \pm\sqrt{2} \;;\; x = \pm\sqrt{3}$

Hence the roots are

$\pm\sqrt{2}, \pm\sqrt{3}$

$\therefore$ The roots of the original eqn are the roots increased by 2

$\Rightarrow 2 \pm \sqrt{2} \,,\; 2 \pm \sqrt{3}$

11. $x^3 - 12x^2 + 48x - 72 = 0$

soln:-

Let us reduce the roots of the eqn by h.

Instead of x substituting x+h

The transformed eqn is

$(x+h)^3 - 12(x+h)^2 + 48(x+h) - 72 = 0$.

ie, $x^3 + (3h-12)x^2 + \dots = 0$

$3h - 12 = 0$

$\boxed{h = 4}$

Hence remove the second term decreased by 4

$$\begin{array}{c|cccc}
4 & 1 & -12 & 48 & -72 \\
 & 0 & 4 & -32 & 64 \\
\hline
 & 1 & -8 & 16 & \boxed{-8} \\
4 & 0 & 4 & -16 & \\
\hline
 & 1 & -4 & \boxed{0} & \\
4 & 0 & 4 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

$x^3 - 8 = 0$

$x^3 = 8$

$x = 2$

$(x-2)(x^2+2x+4) = 0$

$x = 2 ; x = \frac{-2 \pm \sqrt{4-16}}{2}$

$x = 2; \quad x = -2 \pm \sqrt{-12}$

$$x = \frac{-2 \pm i2\sqrt{3}}{2}$$

$x = 2; \quad x = -1 \pm i\sqrt{3}$

Hence the roots are

$2, -1 \pm i\sqrt{3}.$

∴ The roots of the original eqn of the roots are increased by 4

$2+4, 4-1+i\sqrt{3}, 4-1-\sqrt{3}$

$\to 6, 3 \pm i\sqrt{3}.$

2|8 Transformation in General :-

Let $\alpha_1, \alpha_2, \dots \alpha_n$ be the roots of the equation $f(x)=0$, It is required to find an equation whose roots are.

$\phi(\alpha_1), \phi(\alpha_2) \dots \phi(\alpha_n)$

The relation between a root x of $f(x)=0$ and a root y of the required equation is $y=\phi(x)$

Now if x be eliminated between $f(x)=0$ and $y=\phi(x)$, an equation in y is obtained which is the required equation.

By means of the relations between the roots and coefficients of an equation we can establish a relation between the corresponding roots given and the required equations.

Example -1 :-

If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$, from the equation whose roots are $\alpha-\frac{1}{\beta\gamma}, \beta-\frac{1}{\gamma\alpha}, \gamma-\frac{1}{\alpha\beta}$

Soln:- we have $\alpha-\frac{1}{\beta\gamma}$

$$\Rightarrow \alpha-\frac{\alpha}{\alpha\beta\gamma} \quad \left(-\frac{a_n}{a_0}\right) \to \text{Product of the roots} \quad -\frac{r}{1}$$

$$\Rightarrow \alpha-\frac{\alpha}{-r} \text{ since } \alpha\beta\gamma=-r$$

$$\Rightarrow \alpha+\frac{\alpha}{r}$$

$$\therefore y = x+\frac{x}{r}$$

$\therefore$ The required equation is obtained by eliminating x between the equations.

$$y = x+\frac{x}{r} \to ① \qquad y = x+\frac{x}{r}$$

$$x^3+Px^2+qx+r=0 \to ② \qquad y = x(1+1/r)$$

From ①, we get $x = \frac{yr}{1+r}$ $\qquad \frac{y}{1+1/r} = x \to \frac{yr}{1+r} = x$

Substituting this value of x in the equation ②

we get, $\frac{(yr)^{1/3}}{(1+r)^{1/3}} + P\frac{(yr)^2}{(1+r)^2} + q\frac{(yr)}{(1+r)} + r = 0$ $\quad y^3r^3 + Py^2r^2(1+r) + q(1+r)^2yr + (1+r)^3 r = 0$

$\div r \quad y^3r^2 + Pr(1+r)y^2 + q(1+r)^2y + (1+r)^3 = 0$

$$r^2y^3 + Pr(1+r)y^2 + q(1+r)^2y + (1+r)^3 = 0$$

Example - 2 :-

If a, b, c be the roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are $bc-a^2, ca-b^2, ab-c^2$.

Soln:-

we have $bc-a^2 = \frac{abc}{a} - a^2$

$= \frac{-r}{a} - a^2$ since $abc = -r$

Hence the required equation is obtained by eliminating x between the equations.

$y = -\frac{r}{x} - x^2 \rightarrow ①$

$y = \frac{-r-x^3}{x}$

$xy = -r-x^3$

and $x^3+px^2+qx+r=0 \rightarrow ②$

$x^3+xy+r=0$

From ①, we get $x^3+xy+r=0 \rightarrow ③$

subtracting ③ from ②, we get ②-③

$\cancel{x^3}+px^2+qx+\cancel{r}-\cancel{x^3}-xy-\cancel{r}=0$

$px^2+qx-xy=0$

(ie) $x(px+q-y)=0$

(ie) $x=0$ or $px+q-y=0$

x cannot be equal to zero

$\therefore px+q-y=0$

$\therefore x = \frac{y-q}{p}$

substituting this value of x in equation ②, we get

$$\left(\frac{y-q}{p}\right)^3 + p\left(\frac{y-q}{p}\right)^2 + q\left(\frac{y-q}{p}\right) + r = 0$$

$\frac{(y-q)^3}{p^3} + \frac{p(y-q)^2}{p^2} + q\frac{(y-q)}{p} + r = 0$ | $(y-q)^3 + p^2(y-q)^2 + qp^2(y-q) + rp^3$

(ie) $y^3 + (p^2-3q)y^2 + (3q^2-p^2q)y + p^3r - q^3 = 0$

$y^3 - q^3 - 3y^2q + 3yq^2 + p^2(y^2+q^2-2yq) + qp^2y - q^2p^2 + rp^3 = 0.$

Example-3:- $y^3 - q^3 - 3y^2q + 3yq^2 + y^2p^2 + p^2q^2 - 2yp^2q + qp^2y - q^2p^2 + rp^3 = 0$

13. b) If α, β, γ be the roots of the equation

$\therefore y^3 + (p^2-3q)y^2 + (3q^2-p^2q)y + rp^3 - q^3 = 0$

$x^3 - 6x + 7 = 0$ from an equation whose roots are

$\therefore y^3 + (p^2-3q)y^2 + (3q-p^2)qy + p^3r - q^3 = 0$

$\alpha^2 + 2\alpha + 3,\ \beta^2 + 2\beta + 3,\ \gamma^2 + 2\gamma + 3.$

Soln:-

Here we have to eliminate x between the equations.

$$x^3 - 6x + 7 = 0 \rightarrow ①$$

and $y = x^2 + 2x + 3$

(ie) $x^2 + 2x + (3-y) = 0 \rightarrow ②$ $\quad x^3 + 2x^2 + x(3-y) = 0$

Multiplying ② by x and subtracting ① from it, we get

② - ① $\quad -x^3 + 6x - 7 + x^3 + 2x^2 + x(3-y) = 0$

$$2x^2 + (9-y)x - 7 = 0 \rightarrow ③$$

$2x^2 + (9-y)x - 7 = 0$

From ② & ③, we get

$$\begin{array}{cccc} & x^2 & x & \text{constant} \\ 2 & 3-y & 1 & 2 \\ 9-y & -7 & 2 & 9-y \end{array}$$

$$\frac{x^2}{-14-(9-y)(3-y)} = \frac{x}{7+2(3-y)} = \frac{1}{(9-y)-4}$$

$-14-(27-9y-3y+y^2) = 9-4-y$ $\quad \Rightarrow -14y - 27 + 9y + 3y - y^2$

$\frac{x^2}{}$ $\Rightarrow 14 - 27 + 12y - y^2$

so that $(13-2y)^2 = (5-y)(-y^2+12y-41)$

$-14 - 27 + 9y + 3y - y^2 = 5 - y$

(ie) $y^3 - 21y^2 + 153y - 374 = 0$

$(-y^2+12y-41) = 5-y$

Example - 4 :-

If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$, find the value of $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)$.

Soln:- Given $x^3+px^2+qx+r=0 \to ①$

Let $y = x^2+1 \to ②$ $\quad x^2=y-1$

For that, eliminate x between ① & ②

② $\Rightarrow x(x^2+q)+(px^2+r)=0$

$$x(x^2+q) = -(px^2+r) \qquad (x^2=y-1)$$

ie, $x(y-1+q) = -(p(y-1)+r)$ $\quad$ ($\because$ by ①)

squaring on both sides, we get

$$x^2(y-1+q)^2 = [p(y-1)+r]^2.$$

$$(y-1)(y-1+q)^2 = (p(y-1)+r)^2$$

ie, $y^3+y^2\,\text{term}+y\,\text{term}-(q-1)^2-(p-r)^2=0$

The roots of the equation are $\alpha^2+1, \beta^2+1, \gamma^2+1$

$\therefore$ Product the roots

$$(\alpha^2+1)(\beta^2+1)(\gamma^2+1) = (q-1)^2+(p-r)^2$$

Example-5:-

If α is a root of $x^2(x+1)^2 - K(x-1)(2x^2+x+1)=$

P.T $\frac{\alpha+1}{\alpha-1}$ is also a root.

24/8 Example - 5 :-

If α is a root of $x^2(x+1)^2 - K(x-1)(2x^2+x+1)=0$

18) 10m Prove that $\frac{\alpha+1}{\alpha-1}$ is also a root.

Soln:-

From the equation whose roots are

$\frac{\alpha+1}{\alpha-1}, \frac{\beta+1}{\beta-1}, \frac{\gamma+1}{\gamma-1}, \frac{\delta+1}{\delta-1}$

For that, eliminate x between the equations

$$y = \frac{x+1}{x-1} \rightarrow ①$$

$(x-1)y = x+1$

$xy - y = x+1$

$xy - y - 1 = x$

$xy - y - 1 - x = 0$

$x(y-1) - (y+1) = 0$

and $x^2(x+1)^2 - K(x-1)(2x^2+x+1) = 0 \rightarrow ②$.

From ①, we get $x = \frac{y+1}{y-1}$

Substituting this value of x in ②, we get

$$\left(\frac{y+1}{y-1}\right)^2 \left\{\frac{y+1}{y-1}+1\right\}^2 - K\left\{\frac{y+1}{y-1}-1\right)\left\{2\left(\frac{y+1}{y-1}\right)^2 + \frac{y+1}{y-1}+1\right\} = 0$$

$y-1$ $-y+1$ $y-1$

$$\frac{(y+1)^2(2y)^2}{(y-1)^4} - k\left(\frac{2(y-1)}{(y-1)^2}\right)\left(\frac{2(y+1)^2+(y+1)(y-1)+(y-1)^2}{(y-1)^2}\right) = 0.$$

(ie) $(y+1)^2(2y)^2 - K \cdot 2 \cdot (y-1)\left\{2(y+1)^2 + y^2 - 1 + (y-1)^2\right\} = 0$

(ie) $4y^2(y+1)^2 - K \cdot 2(y-1)(4y^2+2y+2) = 0$

(ie) $y^2(y+1)^2 - K(y-1)(2y^2+y+1) = 0$

We get the same equation as the original equatio

$\therefore \frac{\alpha+1}{\alpha-1}$ is a root of

$x^2(x+1)^2 - K(x-1)(2x^2+x+1) = 0$

Example-6:-

Find the equation whose roots are the squares of the difference of the roots of the equation $x^3+px+q=0$ (p and q being real). Hence deduce the condition that all the roots of the cubic shall be real.

Soln:-

Let α, β, γ be the roots of the equation $x^3+px+q=0$ we have to form the equation whos roots are

$$(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$$

$$(\beta-\gamma)^2 = \beta^2+\gamma^2-2\beta\gamma$$

$$= \alpha^2+\beta^2+\gamma^2-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}$$

$$= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}$$

Hence we have $\alpha+\beta+\gamma=0$, $\alpha\beta+\beta\gamma+\alpha\gamma=p$, $\alpha\beta\gamma=-q$

$$\therefore (\beta-\gamma)^2 = -2p-\alpha^2+\frac{2q}{\alpha}$$

Hence to get the transformed equation eliminate x between the equations.

$$y=-2p-x^2+\frac{2q}{x} \rightarrow ①$$

$xy = -2px-x^3+2q$

$$\text{and } x^3+px+q=0 \rightarrow ②$$

$x^3+x(y+2p)-2q =$

① can be written as

③ → ②

$$x^3+(y+2p)x-2q=0 \rightarrow ③$$

Subtracting ② from ③, we get $(y+p)x-3q=0$

$$\therefore x = \frac{3q}{y+p}$$

substituting this value of x in ②, we get

$$\left(\frac{3q}{y+p}\right)^3+p\left(\frac{3q}{y+p}\right)+q=0$$

simplifying $y^3+6py^2+9p^2y+4p^3+27q^2=0$

$$\therefore (\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-(4p^3+27q^2)$$

If α,β,γ are real, then $\alpha-\beta, \beta-\gamma, \gamma-\alpha$ are real and may be positive or negative.

$\therefore (\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$ are all positive.

$\therefore (\beta-\gamma)^2(\alpha-\gamma)^2(\alpha-\beta)^2$ is positive

Here ① $(\beta-\gamma)^2(\alpha-\gamma)^2(\alpha-\beta)^2$ is positive

(ie) $4p^3+27q^2$ is negative.

② $\Rightarrow (\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2$ is positive

(ie) $-6p$ is +ve

(ie) p is -ve

$4p^3+27q^2$ is negative implies that p is -ve.

$\therefore$ The condition for the roots of the equation to be real is

$4p^3+27q^2$ is negative.

28/8 **Descarete's rule of signs**

5m (This rule states that, "an equation $f(x)=0$ cannot have more positive real roots than the number of changes in the signs of the coefficient of $f(x)$".)

If any polynomial $\phi(x)$ is multiplied by a factor $(x-a)$, where 'a' is positive, then there will be atleast one more change in the signs of the coefficients than in the original polynomial

Let the signs of the polynomial $\phi(x)$ be

++--+-++-+--+ ✓

Let us multiply this polynomial by $(x-a)$. We write down only the signs of the terms which occur in the multiplication of $\phi(x)$ by $(x-a)$

$\phi(x)$: ++--+-++-+--+ (x^0) constant

++--+-++-+

$x-a$: +-

--++-+--+-

$(x-a)\phi(x)$: + ± - ± + - + ± - + - ± + -

Result:

Let us take the most unfavourable case and all ambiguities are replaced by continuations then the signs of the term are ++--+-++-+-++- ✓

Thus even in the most unfavourable case there is one more change of sign than the number of changes of sign in the original polynomial.

Thus we conclude that the effect of multiplying by $(x-a)$ is to introduce atleast one change of sign.

The product of all the factors

corresponding to negative and imaginary roots of $f(x)=0$ be a polynomial $f(x)$]. The effect of multiplying $f(x)$ by each of the factors $x-\alpha, x-\beta, x-\gamma, \dots$ corresponding to the positive roots α, β, γ is to introduce atleast one change of sign for each, so that when the complete product is formed containing all the roots. we've the resulting polynomial which has atleast as many changes of signs as it has positive roots. This is Descarte's rule of signs.

Descarete's rule of signs for negative roots of an equation.

The number of negative roots of an equation $f(x)=0$ is not greater than the number of changes of signs in the terms of $f(-x)$.

Let $f(x) = (x-\alpha_1), (x-\alpha_2) \dots (x-\alpha_n)$

Then $f(-x) = (-x-\alpha_1), (-x-\alpha_2) \dots (-x-\alpha_n)$

$\therefore$ The roots of $f(-x)=0$ are $\alpha_1, \alpha_2, \dots \alpha_n$

$\therefore$ The (-ve) roots of $f(x)=0$ are the roots of $f(-x)=0$

$\therefore$ The maximum no. of $\bar{+}$ve roots of $f(x)=0$ are given by the maximum no. of +ve roots of $f(-x)=0$

1) Discuss the nature of the roots of the equation $x^7 + 8x^5 - x + 9 = 0$

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Soln:-

Given $x^7 + 8x^5 - x + 9 = 0 \rightarrow$ ①

This series of signs of the terms of ① are $+ + - +$.

Here, there are two changes of sign

Therefore, the given equation ① has atmost two positive roots.

changing x into $-x$ in eqn ① we get

$f(-x) = -x^7 - 8x^5 + x + 9 = 0 \rightarrow$ ②

The series of signs of eqn ② are $- - + +$.

There is only one change

∴ Equation ① has atmost one negative roots and another four roots are imaginary also eqn ① has atmost four imaginary roots. Hence the given

H.W:-

d) P.T $f(x) = x^4+15x^2+7x-11=0$ has atmost two imaginary roots.

Soln:-

Given $x^4+15x^2+7x-11=0 \rightarrow ①$

This series of signs of the terms of ① are $+++-$

Here, there are one changes of sign

∴ The given eqn ① has atmost one positive roots.

Change x into $-x$ in eqn ① we get

$f(-x) = x^4+15x^2-7x+11=0 \rightarrow ②$

The series of signs of eqn ② are $++-+$

There is two change of sign.

∴ Equation ① has atmost one positive roots and one negative roots and also eqn ① has atmost two imaginary roots.

3) $x^5-6x^2-4x+5=0$

✓

Soln:-

Given $x^5-6x^2-4x+5=0 \rightarrow ①$

This series of signs of the terms of ①

are + - - +.

Here, there are two changes of sign.

∴ The ~~there are~~ given eqn ① has atmost two positive roots.

change x into $-x$ in eqn ① we get

$f(-x) = -x^5 - 6x^2 + 4x + 5 = 0 \rightarrow$ ②.

The series of signs of eqn ② are

- - + +

There is only one change.

∴ Equation ② has atmost one negative roots and also eqn ① has two imaginary roots.

4) $x^7 - 3x^4 + 2x^3 - 1 = 0$

Soln:-

Given $x^7 - 3x^4 + 2x^3 - 1 = 0$ — ①

This series of signs of the terms of ① are + - + -

Hence there are three changes of sign.

∴ The given eqn ① has atmost three positive roots.

change x into $-x$ in eqn ① we get

$f(-x) = -x^7 - 3x^4 - 2x^3 - 1 = 0 \rightarrow$ ②

The series of signs of eqn ② are

$- - - -$

There is no change of sign.

∴ Equation ② has no change of sign and also eqn ① has four imaginary roots.

5) $x^6 + 2x^2 - 5x + 1 = 0$

Soln:

Given $x^6 + 2x^2 - 5x + 1 = 0$ — ①

This series of signs of the terms of ① are $+ + - +$

Hence there are two changes of sign.

∴ The given eqn ① has atmost two positive roots.

change x into $-x$ in eqn ① we get

$f(x) = +x^6 + 2x^2 + 5x + 1 = 0$ — ②

The series of signs of eqn ② are

$+ + + +$

There is no change of sign.

∴ Equation ② has no change of sign.

and also eqn ① has four imaginary roots.

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UNIT - IV

INEQUALITIES

The following elementary principles of inequalities can easily be proved.

(1) If $a > b$, then $a + x > b + x$ and $a - x > b - x$.

(2) If $a > b$, then $-a < -b$.

(3) If $a > b$, then $ma > mb$ and $-ma < -mb$ (m being +

(4) If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, …, $a_n > b_n$, then

$$a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n.$$

and $a_1 a_2 \cdots a_n > b_1 b_2 \cdots b_n$.

(5) If $a > b$, then $a^m > b^m$ and $a^{-m} < b^{-m}$ (m being +ve

(6) If $b_1, b_2, \ldots\ldots b_n$ be all +ve, the fraction

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$

is not less than the least and not greater than the greatest of the n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots \frac{a_n}{b_n}$.

$\therefore$ Equation ③ has no change of sign.
and also eqn ① has four imaginary roots.

1|9

UNIT - IV

INEQUALITIES

The following elementary principles of inequalities can easily be proved.

(1) If $a > b$, then $a+x > b+x$ and $a-x > b-x$.

(2) If $a > b$, then $-a < -b$.

(3) If $a > b$, then $ma > mb$ and $-ma < -mb$ (m being +

(4) If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, ..., $a_n > b_n$, then

$$a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n.$$

and $a_1 a_2 \cdots a_n > b_1 b_2 \cdots b_n$.

(5) If $a > b$, then $a^m > b^m$ and $a^{-m} < b^{-m}$ (m being +ve

(6) If $b_1, b_2, \ldots b_n$ be all +ve, the fraction

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$

is not less than the least and not greater than the greatest of the n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots \frac{a_n}{b_n}$.

(7) If $a_1, a_2, \dots a_n ; b_1, b_2, \dots b_n$ be all positive $\left\{\frac{a_1+a_2+\dots+a_n}{b_1+b_2+\dots+b_n}\right\}^{1/n}$ is not less than the least and not greater than the greatest among the fractions.

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots \frac{a_n}{b_n}.$$

If a, b, x are positive numbers, prove that

$$1 < \frac{a+x}{b+x} < \frac{a}{b} \text{ if } a > b \text{ and } \frac{a}{b} < \frac{a+x}{b+x} < 1 \text{ if } a < b$$

Proof:-

$$\text{Now, } \frac{a+x}{b+x} - \frac{a}{b} = \frac{b(a+x) - a(b+x)}{b(b+x)}$$

$$= \frac{ba + bx - ab - ax}{b(b+x)}$$

$$\Rightarrow \frac{x(b-a)}{b(b+x)}$$

If $a > b \Rightarrow b < a$ then $b - a < 0$

$$\frac{a+x}{b+x} - \frac{a}{b} < 0$$

$$\frac{a+x}{b+x} < \frac{a}{b}, \text{ if } a > b \rightarrow ①$$

If $a < b \Rightarrow b > a$ then $b - a > 0$

$$\frac{a+x}{b+x} - \frac{a}{b} > 0$$

$$\frac{a+x}{b+x} > \frac{a}{b}$$

$$\Rightarrow \frac{a}{b} < \frac{a+x}{b+x}, \text{ if } a < b \rightarrow ②.$$

Let us consider, $1 - \frac{a+x}{b+x} = \frac{b+x-a-x}{b+x}$

$$= \frac{b-a}{b+x}$$

If $a > b \Rightarrow b < a$ then $b - a < 0$

$$1 - \frac{a+x}{b+x} < 0$$

$$\Rightarrow 1 < \frac{a+x}{b+x}, \text{ if } a > b \rightarrow ③$$

If $a < b \Rightarrow b > a$ then $b - a > 0$

$$1 - \frac{a+x}{b+x} > 0 \Rightarrow 1 > \frac{a+x}{b+x}, \text{ if } a < b \rightarrow ④.$$

From ① & ③, we get.

$$\boxed{1 < \frac{a+x}{b+x} < \frac{a}{b}, \text{ if } a > b}$$

From ② & ④, we get.

$$\boxed{\frac{a}{b} < \frac{a+x}{b+x} < 1, \text{ if } a < b}$$

∴ Hence proved.

Divya

2. P.T $\frac{1}{2\sqrt{n+1}} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$

<u>Proof:-</u>

Let $u_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \rightarrow ①$

$\therefore u_n < \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-1}\frac{2n}{2n+1} \to ②$

Since, $\frac{1}{2} < \frac{2}{3}, \frac{3}{4} < \frac{4}{5}, \ldots, \frac{2n-1}{2n} < \frac{2n}{2n+1}$

Multiplying ① and ②, we get

$$u_n^2 < \left(\frac{1}{\cancel{2}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{5}}{\cancel{6}}\cdots\frac{2n-1}{2n}\right)\left(\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{4}}{\cancel{5}}\cdot\frac{\cancel{6}}{7}\cdots\frac{2n}{2n+1}\right)$$

$$u_n^2 < \frac{1}{2n+1} \Rightarrow u_n < \frac{1}{\sqrt{2n+1}} \to ③$$

Also $(2n+1)u_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-3}{2n-2}\cdot\frac{2n-1}{2n}(2n+1)$

$$= \frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2n-1}{2n-2}\cdot\frac{2n+1}{2n} \to ④$$

$$(2n+1)u_n > \frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n+2}{2n+1} \to ⑤$$

$u^2 n$

$\tilde{u_n}$

Since, $\frac{3}{2} > \frac{4}{3}, \frac{5}{4} > \frac{6}{5}, \ldots, \frac{2n-1}{2n-2} > \frac{2n}{2n-1}$

Multiplying ④ & ⑤, we get

$$(2n+1)^2 u_n^2 > \left(\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2n-1}{2n-2}\cdot\frac{2n+1}{2n}\right)\left(\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n+2}{2n+1}\right).$$

$$> \frac{2n+2}{2} = \cancel{\frac{(2)2(n+1)}{2}}$$

$$\therefore u_n^2 > \frac{n+1}{(2n+1)^2}$$

$$\therefore u_n > \frac{\sqrt{n+1}}{2n+1} > \frac{\sqrt{n+1}}{\underset{2(n+1)}{2n+2}} = \frac{1}{2\sqrt{n+1}}$$

$$\Rightarrow u_n > \frac{1}{2\sqrt{n+1}} \to ⑥$$

From ⑤ & ⑥, we get

$$\frac{1}{2\sqrt{n+1}} < u_n < \frac{1}{\sqrt{2n+1}}$$

$$\therefore \frac{1}{2\sqrt{n+1}} < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

Madhu 3. P.T, $\frac{1}{2} < \left(\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\right)^{1/n} < 1$

Proof:-

The fractions $\frac{1}{2} < \frac{3}{4} < \cdots < \frac{2n-1}{2n}$ are in the ascending order.

$$\therefore \frac{1}{2} < \left(\frac{1\cdot 3\cdots\cdots 2n-1}{2\cdot 4\cdots\cdots 2n}\right)^{1/n} < \frac{2n-1}{2n}$$

But $\frac{2n-1}{2n} = 1 - \frac{1}{2n} < 1$

$$\therefore \frac{1}{2} < \left(\frac{1\cdot 3\cdots\cdots 2n-1}{2\cdot 4\cdots\cdots 2n}\right)^{1/n} < 1$$

5/9

Akash 1) If a, b, c are positive and not all equal then $(a+b+c)(bc+ca+ab) > 9abc$

Proof:-

$$(a+b+c)(bc+ca+ab) - 9abc$$

$$= abc + a^2c + a^2b + b^2c + abc + ab^2 + bc^2$$
$$+ c^2a + abc - 9abc$$

$$= ab^2 + ac^2 + a^2b + bc^2 + a^2c + b^2c - 6abc$$

$= (ab^2 + ac^2 - 2abc) + (a^2b + bc^2 - 2abc) + (a^2c + b^2c -$

$2abc)$

$= a(b^2+c^2-2bc) + b(a^2+c^2-2ac) + c(a^2+b^2-2ab)$

$= a(b-c)^2 + b(a-c)^2 + c(a-b)^2$

> 0

$(a+b+c)(bc+ca+ab) - 9abc > 0$

$\therefore (a+b+c)(bc+ca+ab) > 9abc.$

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1. S.T $(b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 \geq bc+ca+ab$

Ranevel.

Proof:-

LHS

$(b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2$

$= b^2+c^2+a^2+2bc-2ab-2ab+c^2+a^2+b^2+2ac-2ab$
$\quad -2bc+a^2+b^2+c^2+2ab-2bc-2ac.$

$= a^2+b^2+c^2+a^2+b^2-2ab+b^2+c^2-2bc+a^2+c^2-2ac$

$= a^2+b^2+c^2+(a-b)^2+(b-c)^2+(c-a)^2$

$\geq a^2+b^2+c^2$

$\geq \frac{a^2}{2} + \frac{a^2}{2} + \frac{b^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} + \frac{c^2}{2}$

$= \frac{a^2}{2} + \frac{b^2}{2} - ab + \frac{b^2}{2} + \frac{c^2}{2} - bc + \frac{c^2}{2} + \frac{a^2}{2} - ca$
$\quad + ab + bc + ca$

$= \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 + ab+bc+ca$

$\geq ab+bc+ca$

$\therefore (b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 \geq ab+bc+ca$

2. P.T if $n > 2$, $(n!)^2 > n^n$

Proof :-

$n! = 1.2 \cdots (n-1)n$

$n! = n(n-1)(n-2) \cdots r \; 3.2.1$

$\times$ly the above two,

$(n!)^2 = (n!)(n!) \quad (1.2 \cdots (n-1)n)(n(n-1)(n-2) \cdots 2.1)$

$= (1.n)(2(n-1))(3.(n-2)) \cdots (r(n-r+1)) \cdots$

$(n-1)2 \cdots n.1 \rightarrow$ ①.

$r(n-r+1) > n$ if $n - r(n-r+1) < 0$

$\quad 3-r > 2$

ie, if $n - rn + r^2 - r < 0$

ie, if $r^2 - rn - r + n < 0$

ie, if $r(r-1) - n(r-1) < 0$

ie, if $(r-n)(r-1) < 0$

ie, if $1 < r < n$

Put $r = 1, 2, 3, \cdots n$ we get.

$1.n = n$

$2.(n-1) > n$

$3.(n-2) > n$

$\vdots \quad \vdots$

$(n-1).2 > n$

$n.1 = n$

Multiplying all these inequalities, we get,

$1.n.2(n-1).3(n-2) \cdots n.1 > n^n$

ie, $(n!)^2 > n^n$

3\. If $a_1, a_2, \dots a_n$ be an arithmatical progression S.T $a_1^2 + a_2^2, \dots a_n^2 > a_1^n . a_2^n$. Deduce that if $n>2$, $(n!)^2 > n^n$

Yuvasri

<u>Proof:-</u>

Let d be the common difference of A.P

consider $a_1, a_2, \dots a_n$ be an A.P

$$\text{ie, } a_n = a_1 + (n-1)d \qquad (\because t_r = a+(n-1)d)$$

$$a_r . a_{n-r+1} = \{a_1+(r-1)d\}\{a_1+(n-r)d\}$$

$(n-r)(r-1)d^2$

$$= a_1^2 + (r-1)d\,a_1 + (n-r)da_1 + (n-r).(r-1)d^2$$

~~$(r-1)d^2$~~

$$= a_1^2 + (r-1+n-r)da_1 + (n-r)(r-1)d^2$$

$$= a_1^2 + (n-1)da_1 + (n-r)(r-1)d^2$$

$$> a_1^2 + (n-1)da_1$$

$$> a_1\{a+(n-1)d\}$$

$$\text{ie, } a_r . a_{n-r+1} > a_1 a_n$$

Put $r = 1, 2, 3 \dots n$ we get

$$a_1 . a_n = a_1 . a_n$$

$$a_2 . a_{n-1} > a_1 . a_n$$

$$a_3\, a_{n-2} > a_1 . a_n$$

$$\vdots \qquad \vdots$$

$$a_n . a_1 = a_1 . a_n$$

Multiplying, we get,

$a_1^2 \cdot a_2^2 \cdot \dots \cdot a_n^2 > (a_1 \cdot a_n)^n$

Let us take the A.P 1, 2, 3, ... n

$1^2, 2^2, 3^2, \dots\dots n^2 > 1^n \cdot n^n$.

$(1 \cdot 2 \cdot 3 \cdot \dots \cdot n 2) > n^n$

$\therefore (n!)^2 > n^n$

hayathri 4. S.T $1! \cdot 3! \cdot 5! \dots\dots (2n-1)! > (n!)^n$

✓ 5m **Proof :-**

14a) $(n-r)!\, r! > [n-(r+1)]!\, (r+1)!$ if $(n-r) > (r+1)$

i.e, if $n > 2r+1$

$\therefore (n-1)! \cdot 1! > (n-2)! \cdot 2! > (n-3)! \cdot 3! \dots\dots$

so long as $n > 2r+1$

From this, we get the inequalities

$2n!\; 0! > (2n-1)! \cdot 1! > (2n-2)!\, 2! >$

$(2n-3)! \cdot 3! > \dots > n! \cdot n!$

$\therefore (2n-1)!\, 1! > n!\, n!$

$(2n-3)!\, 3! > n! \cdot n!$

$(2n-5)!\, 5! > n!\, n!$

. . . .

. . . .

$1!\, (2n-1)! > n! \cdot n!$

Multiplying this, we get,

$$[1!.3!.5!\ldots\ldots(2n-1)!]^2 > (n!)^{2n}$$

$$\therefore 1!.3!.5!\ldots\ldots(2n-1)! > (n!)^n$$

24/9.

1. S.T $(x^m+y^m)^n < (x^n+y^n)^m$ if $m>n$

Proof:-

Let $x>y$

Then $(x^m+y^m)^n - (x^n+y^n)^m$

$$= [x^m(1+y^m/x^m)]^m - [x^n(1+y^n/x^n)]^m$$

$$= x^{mn}\{1+(y/x)^m\}^n - x^{mn}\{1+(y/x)^n\}^m$$

$$= x^{mn}\{[1+(y/x)^m]^n - [1+(y/x)^n]^m\}$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots+x^n$$

$$= x^{mn}\{1+n(y/x)^m+\frac{n(n-1)}{2!}(y/x)^{2m}\ldots\ldots-1-m(y/x)^n$$

$$-\frac{m(m-1)}{2!}(y/x)^{2n}\ldots\}$$

$$= x^{mn}\{[n(y/x)^m-m(y/x)^n]+[\frac{n(n-1)}{2!}(y/x)^{2m}-$$

$$\frac{m(m-1)}{2!}(y/x)^{2n}]+\ldots\}$$

It is given that $n<m$

$$(y/x)^m < (y/x)^n \text{ since } y/x<1$$

$$n(y/x)^m < m(y/x)^n$$

$$n(y/x)^m - m(y/x)^n < 0$$

|||ly

$$\frac{n(n-1)}{2!}\left(y/x\right)^{2m} - \frac{m(m-1)}{2!}\left(y/x\right)^{2m} < 0$$

and so on

$$\therefore (x^m+y^m)^n - (x^n+y^n)^m < 0$$

$$\text{ie, } (x^m+y^m)^n < (x^n+y^n)^m .$$

25/9

Geometric and Arithmetic means :-

The Arithmetic mean of n quantities 2m✓ 8) is the sum of all the quantities divided by n, and the geometric mean of n +ve quantities is the n^{th} root of their product. Thus, if $a_1, a_2, a_3, \ldots\ldots a_n$ be the n quantiti then their arithmetic mean is $\frac{a_1+a_2+\cdots+a_n}{n}$ then and their geometric mean is $(a_1a_2..a_n)$

AM ≥ GM

1. The arithmetic mean of n positive quantities which are not all equal to one another, is greater than their geometric mean.

<u>Proof :</u>

Let A be the arithmetic mean and

G be the geometric mean of the n positive quantities $a_1, a_2, a_3 \ldots a_n$. Then by definition

$$A = \frac{a_1+a_2+\cdots+a_n}{n} \text{ and } G = (a_1 a_2 \cdots a_n)^{1/n}$$

We have to prove

AM > GM

Suppose that any two of these quantities unchanged take $\frac{1}{2}(a_1+a_2)$ and $\frac{1}{2}(a_1+a_2)$ instead of a_1 and a_2

Say a_1 and a_2 are unequal.

Then keeping all the other quantities unchanged take $\frac{1}{2}(a_1+a_2)$ and $\frac{1}{2}(a_1+a_2)$ instead of a_1 and a_2.

The sum of these quantities.

$$= \frac{1}{2}(a_1+a_2) + \frac{1}{2}(a_1+a_2) + a_3 + a_4 + \cdots + a_n$$

$= a_1 + a_2 + a_3 + a_4 + \cdots + a_n$ which is unchanged.

But the product of the quantities

$\frac{a_1+a_2}{2} \cdot \frac{a_1+a_2}{2} > a_1 a_2$ except when $a_1 \neq a_2$.

Hence so long as any two of the factors

are unequal, the continued product can be increased without altering the sum and therefore all the factors must be equal to one another when their continued product has its greatest possible value. Since the sum of all the n factors is unchanged and is equal to $a_1+a_2+\dots+a_n$ each factor is equal to $\frac{a_1+a_2+\dots+a_n}{n}$ when all the factors are equal

$$\therefore \frac{a_1+a_2+\dots+a_n}{n}, \frac{a_1+a_2+\dots+a_n}{n} \dots n \text{ factors} > a_1a_2\dots a_n$$

$$\left(\frac{a_1+a_2+\dots+a_n}{n}\right)^n > a_1a_2\dots a_n$$

$$\text{(ie)} \ \frac{a_1+a_2+\dots+a_n}{n} > (a_1a_2\dots a_n)^{1/n}$$

ie, AM > GM

Example-1:-

S.T $n^n > 1.3.5 \dots (2n-1)$

Proof:-

WKT

AM > GM

we have $\frac{1+3+5+\dots(2n-1)}{n} > (1.3.5\dots 2n-1)^{1/}$

Now $1+3+5+\cdots(2n-1) = n\{1+\overset{n}{(2n-1)}\} = n^2$, $a+(n-1)d$

$\frac{n^2}{n} > (1.3.5.\cdots 2n-1)^{1/n}$ $\frac{n(n+1)}{2}$ $\frac{n(2n+1-1)}{2}$

(ie) $n > \{1.3.5\ldots(2n-1)\}^{1/n}$ $= \frac{2n^2}{2}$

(ie) $n^n > 1.3.5\ldots\ldots(2n-1)$ $= n^2$

Example - 2 :-

If x and y are positive quantities whose sum is 4, show that $(x+1/x)^2+(y+1/y)^2 \not< 12.\frac{1}{2}\ldots$

Proof:

$$(x+1/x)^2+(y+1/y)^2 = x^2+y^2+1/x^2+1/y^2+4$$

$$\frac{1/x^2+1/y^2}{2} > \left(1/x^2y^2\right)^{1/2} \quad (\because AM > GM)$$

(ie) $1/x^2 + 1/y^2 > 2 \cdot 1/xy$

Also, $\frac{x+y}{2} > \sqrt{xy}$. but $x+y=4$

$\therefore \frac{4}{2} > \sqrt{xy}$ (ie) $2 > \sqrt{xy}$ (ie) $4 > xy$

$2 > \frac{xy}{2}$

$\therefore \frac{2}{xy} > \frac{1}{2}$

Hence, $\frac{1}{x^2}+\frac{1}{y^2} > 1/2$

$x+y=4$

$y=4-x$

$y^2=(4-x)^2$

$$\begin{aligned}x^2+y^2 &= x^2+(4-x)^2\\ &= 2x^2-8x+16\\ &= 2(x^2-4x+4)+8\\ &= 2(x-2)^2+8\\ &\geq 8\end{aligned}$$

$$(x+1/x)^2+(y+1/y)^2 \ge 8+1/2+4 \text{ (ie) } \ge 12.1/2 \ \frac{25}{2}$$

Example - 3 :-

S.T if $s = a_1 + a_2 + \cdots + a_n$

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n} > \frac{n^2}{n-1} \text{ unless } a_1=a_2=\cdots$$

a_n (1/2)

Proof :-

AM > GM

$$\frac{1}{n}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n}\right) > \left\{\left(\frac{s}{s-a_1}\right)\left(\frac{s}{s-a_2}\right)\cdots\left(\frac{s}{s-a_n}\right)\right\}^{1/n} \to ①$$

unless $\frac{s}{s-a_1} = \frac{s}{s-a_2} = \cdots = \frac{s}{s-a_n}$ (ie) unless

$a_1 = a_2 = \cdots = a_n$

$$\text{Also } 1/n\left(\frac{s-a_1}{s}+\frac{s-a_2}{s}+\cdots+\frac{s-a_n}{s}\right) >$$

$$\left\{\frac{s-a_1}{s}\cdot\frac{s-a_2}{s}\cdots\frac{s-a_n}{s}\right\}^{1/n} \to ②.$$

Multiplying ① x ② we get

$$\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n}\right)$$

$$\times\left(\frac{s-a_1}{s}+\frac{s-a_2}{s}+\cdots+\frac{s-a_n}{s}\right) > 1$$

(ie) $\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}\right\} + \dots + \frac{s}{s-a_n}\right)\times$

$$\left\{\frac{ns-(\overbrace{a_1+a_2+\dots+a_n}^{s})}{s}\right\}>1$$

(ie) $\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s_1-a_2}+\dots+\frac{s}{s-a_n}\right)\times\left\{\frac{ns-s}{s}\right\}>1$

(ie) $\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}\right)(n-1)>1$

(ie) $\frac{s}{sa_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s_1-a_n}>\frac{n^2}{n-1}$

1/10 Example 4 :-

If $x_1.x_2\dots x_n=y^n$, show that $(1+x_1)(1+x_2)\dots$ $\dots(1+x_n)\not<(1+y)^n$.

Proof :-

$(1+x_1)(1+x_2)\dots(1+x_n)$

$=1+(x_1+x_2+\dots+x_n)+\Sigma x_1x_2+\Sigma x_1x_2x_3+\dots$ $\dots+(x_1x_2\dots x_n)$

$$\frac{x_1+x_2+\dots x_n}{n}\geq(x_1x_2\dots x_n)^{1/n}$$

ie, $\geq(y^n)^{1/n}$ ie, $\geq y$

$\therefore x_1+x_2+\dots+x_n\geq ny$

Σx_1x_2 consists of nC_2 terms out of which $(n-1)$ terms will contains x_1, $n-1$ terms contais x_2 factors.

$\Sigma\frac{x_1x_2}{nC_2}\geq(x_1^{n-1}.x_2^{n-1}\dots x_n^{n-1})^{1/nC_2}$ ie,

$$\geq (x_1 x_2 \ldots x_n)^{(n-1)^{2/n(n-1)}} \quad \text{ie, } (y^n)^{2/n} \quad \text{ie, } \geq y^2$$

$$\therefore \Sigma x_1 x_2 \geq nC_2 y^2$$

$|||^{ly}$

$$\Sigma x_1 x_2 x_3 > nC_3 y^3$$

$$\therefore (1+x_1)(1+x_2) \ldots (1+x_n) \geq 1 + ny + nC_2 y^2 + nC_3 y^3 + \ldots y^n \text{ ie } \geq (1+y)^n$$

Example-5:

If $a_1, a_2, a_3, \ldots a_n$ are positive and $(n-1)s = a_1 + a_2 + a_3 + \ldots + a_n$ then prove that

$$a_1 a_2 a_3 \ldots a_n \geq (n-1)^n (s-a_1)(s-a_2) \ldots (s-a_n)$$

Proof:-

$$(s-a_2) + (s-a_3) + \ldots + (s-a_n) = (n-1)s - a_2 - a_3 \ldots - a_n = a_1$$

$$\frac{(s-a_2) + (s-a_3) + \ldots + (s-a_n)}{n-1} \geq \{(s-a_2)(s-a_3) \ldots (s-a_n)\}^{1/(n-1)}$$

$$\text{ie, } \frac{a_1}{n-1} \geq \{(s-a_2)(s-a_3) \ldots (s-a_n)\}^{1/(n-1)}$$

$|||^{ly}$

$$\frac{a_2}{n-1} \geq \{(s-a_1)(s-a_3) \ldots (s-a_n)\}^{1/(n-1)}$$

$$\frac{a_3}{n-1} \geq \{(s-a_1)(s-a_2) \ldots (s-a_n)\}^{1/(n-1)}$$

……

$$\frac{a_n}{n-1} > \left\{(s-a_1)(s-a_3)\cdots\cdots(s-a_{n-1})\right\}^{1/(n-1)}$$

Multiplying these n inequalities ; we get

$$\left(\left((s-a_1)(s-a_2)\cdots(s-a_n)\right)^{n-1}\right)^{1/n-1}$$

$$\frac{a_1a_2\ldots\ldots a_n}{(n-1)^n} \geq (s-a_1)(s-a_2)\ldots\ldots(s-a_n)$$

ie, $a_1 a_2 \ldots\ldots a_n \geq (n-1)^n (s-a_1)(s-a_2)\ldots\ldots(s-a_n)$

1. If $a, b, c \ldots$ and $\alpha, \beta, \ldots\ldots$ be all positive then

$$\left(\frac{a\alpha+b\beta+c\gamma+\cdots}{a+b+c+\cdots}\right)^{a+b+c\cdots} > \alpha^a\beta^b\gamma^c\cdots\cdots$$

Proof :-

case(i),

First let us assumes that $a, b, c, \ldots$ are integers.

Take a quantitees each equal to α, b quantities each equal to β, c quantities each equal to $\gamma \ldots$ and so on.

The arithmetic mean of all inequalities is greater than their geometric mean.

The total number of quantities $= a+b+c+\cdots$

$\therefore$ Their arithmetic mean

$$= \frac{(\alpha+\alpha+\cdots\cdots a\text{ terms})+(\beta+\beta+\cdots+\beta\text{ terms})+\cdots\cdots}{a+b+c+\cdots\cdots}$$

$$= \frac{a\alpha+b\beta+c\gamma+\cdots}{a+b+c+\cdots\cdots}$$

Their geometric mean,

$$= \{\alpha.\alpha.\alpha \ldots a \text{ factors})(\beta.\beta \ldots b \text{ factors}) \ldots\}^{1/a+b}$$

$$= \{\alpha^a . \beta^b \ldots\}^{1/a+b+c \ldots}$$

$$\therefore \frac{a\alpha + b\beta + c\gamma + \ldots}{a+b+c+ \ldots} > (\alpha^a \beta^b \gamma^c \ldots)^{1/a+b+c}$$

case(ii)

If a, b, c are not integral, let m be the l.c.m of the denominators of a, b, c .....

Then ma, mb, mc, ... are all integers

$$\therefore \frac{ma\alpha + mb\beta + mc\gamma + \ldots}{ma + mb + mc + \ldots} > (\alpha^{ma} \beta^{mb} \gamma^{mc} \ldots)^{\frac{1}{(ma+mb+\ldots)}}$$

$$\text{ie, } \frac{a\alpha + b\beta + c\gamma + \ldots}{a+b+c+\ldots} > (\alpha^a \beta^b \gamma^c \ldots)^{\frac{1}{(a+b+c\ldots)}}$$

Hence for both integral and non-integral values for a, b, c, ... the result is true

2. S.T if a, b, c ..... k be n positives quantities

$$\left(\frac{a^2+b^2+\cdots+k^2}{a+b+\cdots+k}\right)^{a+b+\cdots k} > a^a b^b \ldots k^k > \left(\frac{a+b+\cdots k}{n}\right)^{a+b+\cdots k}$$

Proof :-

We have to prove that

$$\left(\frac{a\alpha + b\beta + c\gamma + \ldots}{a+b+c \ldots}\right) > (\alpha^a \beta^b \ldots)^{1/(a+b+\ldots)}$$

ie, $\left(\frac{a\alpha+b\beta+\cdots}{a+b+\cdots}\right)^{a+b+\cdots} > \alpha^a\beta^b\cdots \to ①$

In ①, put $\alpha = a$, $\beta = b$

$$\therefore \left(\frac{a^2+b^2+\cdots k^2}{a+b+\cdots k}\right)^{a+b+\cdots k} > a^a b^b \cdots k^k \to ②$$

In ①, put $\alpha = 1/a$, $\beta = 1/b, \cdots$

$$\therefore \left(\frac{1+1+\cdots n\text{ terms}}{a+b+\cdots k}\right)^{a+b+\cdots k} > (1/a)^a (1/b)^b \cdots (1/k)^k$$

$$\text{ie, } \left(\frac{a+b+\cdots k}{n}\right)^{a+b+\cdots k} < a^a b^b \cdots k^k \to ③$$

combining ② & ③, we get the required result.

6/10

Example - 2 :-

1. Show that if a, b, c are positive unequal quantities then $ax^{b-c} + bx^{c-a} + cx^{a-b} \nless a+b+c$.

Proof :-

$$\left(\frac{a\alpha+b\beta+c\gamma}{a+b+c}\right)^{a+b+c} > \alpha^a\beta^b\gamma^c$$

Put $\alpha = x^{b-c}$, $\beta = x^{c-a}$, $\gamma = x^{a-b}$

$$\therefore \left(\frac{ax^{b-c}+bx^{c-a}+cx^{a-b}}{a+b+c}\right) > (x^{b-c})^a . (x^{c-a})^b . (x^{a-b})^c$$

$$\text{ie, } > \{x^{(b-c)a} \; x^{(c-a)b} \; x^{(a-b)c}$$

$$\text{ie, } > x^{a(b-c)+b(c-a)+c(a-b)}$$

$$> x^0$$

> 1 taking $a+b+c$ root on both side.

$$\therefore \frac{ax^{b-c}+bx^{c-a}+cx^{a-b}}{a+b+c} > 1$$

$$\therefore ax^{b-c}+bx^{c-a}+cx^{a-b} > a+b+c$$

8) If $a, b, c \dots k$ are n positive quantities which are not all equal to one another and m is any rational number except 0 or 1, then

$$\frac{a^m+b^m+c^m+\dots+k^m}{n} \gtrless \left(\frac{a+b+c+\dots k}{n}\right)^m$$

according as m does not or does lie between 0 and 1.

Proof:-

By known result

$$\left(\frac{a\alpha+b\beta+c\gamma+\dots}{a+b+c+\dots}\right)^{a+b+c+\dots} > \alpha^a \beta^b \gamma^c \dots \to ①$$

Put in this inequality ① a^r for a, b^r for b, c^r for $c, \dots$ and a^{m-r} for α, b^{m-r} for $\beta, \dots$ where $m > r$.

$$\therefore \left(\frac{a^m+b^m+c^m+\dots+k^m}{a^r+b^r+c^r+\dots+k^r}\right)^{a^r+b^r+\dots+k^r}$$

$$> a^{(m-r)a^r} . b^{(m-r)b^r} \dots$$

(i.e) $$\left(\frac{a^m+b^m+c^m+\dots+k^m}{a^r+b^r+c^r+\dots+k^r}\right)^{a^r+b^r+\dots+k^r}$$

$$> (a^{a^r} b^{b^r} c^{c^r}\dots)^{m-r} \to ②$$

Again substitute in the inequality ①

$a^r, b^r, \ldots$ for $a, b \ldots$ respectively and

$a^{p-r}, b^{p-r} \ldots\ldots$ for $\alpha, \beta, \ldots\ldots$ respectively and $p<r$

Then, $\left(\frac{a^p+b^p+\cdots+k^p}{a^r+b^r+\cdots+k^r}\right)^{a^r+b^r+\cdots+k^r} > \left(a^{a^r}\cdot b^{b^r}\ldots\right)^{p-r}$

ie, $\left(\frac{a^r+b^r+\cdots\cdots+k^r}{a^p+b^p+\cdots+k^p}\right)^{a^r+b^r+\cdots+k^r} < \left(a^{a^r}\cdot b^{b^r}\ldots\right)^{r-p}$ → ③

∴ From inequalities ② & ③ we get

$$\left(\frac{a^m+b^m+\cdots+k^m}{a^r+b^r+\cdots+k^r}\right)^{\frac{1}{m-r}} > \left(\frac{a^r+b^r+\cdots+k^r}{a^p+b^p+\cdots+k^p}\right)^{1/r-p}$$

n

$b^{r-p}. b^{m-r}$

$b^{m-p}) \; b^{p-m}$

where $m>r>p$

(ie) $\left(\frac{a^m+b^m+\cdots+k^m}{a^r+b^r+\cdots+k^r}\right)^{r-p} > \left(\frac{a^r+b^r+\cdots+k^r}{a^p+b^p+\cdots+k^p}\right)^{m-r}$ → ④

(ie) $(a^m+b^m+\cdots+k^m)^{r-p} \cdot (a^r+b^r+\cdots+k^r)^{p-m} \times$

$(a^p+b^p+\cdots+k^p)^{m-r} > 1$

provided $m>r>p$ → ⑤

In this inequality ⑤ put $p=0$

Then when $m>r>0$ we get

$(a^m+b^m+\cdots+k^m)^r \, (a^r+b^r+\cdots+k^r)^{-m} \times$

$(a^0+b^0+\cdots+k^0)^{m-r} > 1$

(ie) $(a^m+b^m+\cdots+k^m)^r \quad (a^r+b^r+\cdots+k^r)^{-m}$

$(n)^{m-r} > 1$

(ie) $\left(\frac{a^m+b^m+\dots+k^m}{n}\right)^r > \left(\frac{a^r+b^r+\dots+k^r}{n}\right)^m$ $\to$ ⑥

In the inequality ⑥ put $m=1$

Then $\left(\frac{a+b+\dots+k}{n}\right)^r > \left(\frac{a^r+b^r+\dots+k^r}{n}\right)$

(ie) $\frac{a^r+b^r+\dots+k^r}{n} < \left(\frac{a+b+\dots+k}{n}\right)^r \to$ ⑦

where $1>r>0$ (ie) r lies between 0 and 1

In the inequality ⑥, Put $r=1$

Them $\frac{a^m+b^m+\dots+k^m}{n} > \left(\frac{a+b+c+\dots+k}{n}\right)^m \to$ ⑧

where $m>1$

In the inequality ⑥, Put $m=1$, $r=0$

$(a+b+\dots+k)^{-p}(n)^{p-1}(a^p+b^p+\dots+k^p) > 1$

(ie) $\frac{a^p+b^p+\dots+k^p}{n} > \left(\frac{a+b+\dots+k}{n}\right)^p \to$ ⑨

where $1>0>p$ (ie) p is negative

$\therefore$ From the inequalities ⑦, ⑧ & ⑨, we get

$\frac{a^x+b^x+\dots+k^x}{n} > \left(\frac{a+b+c+\dots+k}{n}\right)^x$ if $x<0$ or $x>1$

and $\frac{a^x+b^x+\dots+k^x}{n} < \left(\frac{a+b+c+\dots+k}{n}\right)^x$ if $0<x<1$

Hence we get the result.

2) If $a_1, a_2, \dots a_n$ are n positive numbers not all equal to one another, then

$$\frac{a_1^{p+q}+a_2^{p+q}+\dots+a_n^{p+q}}{n} \gtrless \frac{a_1^{p}+a_2^{p}+\dots+a_n^{p}}{n} \times$$

$$\frac{a_1^{q}+a_2^{q}+\dots+a_n^{q}}{n}$$ according as p and q have the same or opposite signs.

Proof:-

Case (i)

Let us suppose the p and q have the same sign. Then $a_1^p - a_2^p$ and $a_1^q - a_2^q$ are both positive, both negative or zero.

$$\therefore (a_1^p - a_2^p)(a_1^q - a_2^q) \geq 0$$

(ie) $a_1^{p+q} + a_2^{p+q} \geq a_1^p a_2^q + a_1^q a_2^p$

Writing down the nC_2 such inequalities obtained by taking all the combinations of the n numbers taken 2 at a time (every letter being taken with each of the $n-1$ other letters) and adding them, we have.

$$(n-1)(a_1^{p+q}+a_2^{p+q}+\dots+a_n^{p+q}) \geq \sum a_1^p a_2^q$$

(ie) $n(a_1^{p+q}+a_2^{p+q}+\dots+a_n^{p+q}) \geq \sum a_1^p a_2^q + \sum a_1^{p+q}$

(ie) $\geq (a_1^p + a_2^p + \dots + a_n^p)(a_1^q + a_2^q + \dots + a_n^q)$

$$\therefore \frac{a_1^{p+q} + a_2^{p+q} + \dots + a_n^{p+q}}{n} \geq \frac{a_1^p + a_2^p + \dots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \dots + a_n^q}{n}$$

Case(ii) :-

If p and q are of opposite signs.

$a_1^p - a_2^p$, $a_1^q - a_2^q$ have opposite signs.

$$\therefore (a_1^p - a_2^p)(a_1^q - a_2^q) \leq 0$$

The rest of the process being the same as above, with the only difference that the sign of the inequality is reversed we get finally.

$$\frac{a_1^{p+q} + a_2^{p+q} + \dots + a_n^{p+q}}{n} \leq \frac{a_1^p + a_2^p + \dots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \dots + a_n^q}{n}$$

Corollary :-

If m and r be positive integers and $m > r$.

Prove that $\dfrac{a_1^m + a_2^m + \dots + a_n^m}{n} > \dfrac{a_1^{m-r} + a_2^{m-r} + \dots + a_n^{m-r}}{n} \cdot \dfrac{a_1^r + a_2^r + \dots + a_n^r}{n}$

unless $a_1 = \dots = a_n$

Example - 1 :

S.T if a, b, c are three positive unequal quantities, then $\dfrac{a^8+b^8+c^8}{a^3b^3c^3} > \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

Proof :-

$$\frac{a^8+b^8+c^8}{3} > \frac{a^6+b^6+c^6}{3} \cdot \frac{a^2+b^2+c^2}{3}$$

but $\frac{a^6+b^6+c^6}{3} \geq (a^6b^6c^6)^{1/3}$ (ie) $\geq a^2b^2c^2$

AM $\geq$ GM.

and $a^2+b^2+c^2 \geq ab+bc+ca$

$\therefore \frac{a^8+b^8+c^8}{3} \geq \frac{a^2b^2c^2(ab+bc+ca)}{3}$

(ie) $a^8+b^8+c^8 \geq a^3b^3c^3(1/a+1/b+1/c)$

(ie) $\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

Example-2:-

P.T $8xyz \leq (y+z)(z+x)(x+y) \leq \frac{8}{3}(x^3+y^3+z^3)$

S **Proof:-**

$\frac{y+z}{2} \geq \sqrt{yz}$; $\frac{z+x}{2} \geq \sqrt{zx}$ and $\frac{x+y}{2} \geq \sqrt{xy}$

$\therefore \frac{(y+z)(z+x)(x+y)}{8} \geq \sqrt{yz}\cdot\sqrt{zx}\cdot\sqrt{xy}$ (ie) $\geq xyz$

$\therefore (y+z)(z+x)(x+y) \geq 8xyz$

$\frac{(y+z)+(z+x)+(x+y)}{3} \geq \{(y+z)(z+x)(x+y)\}^{1/3}$

$\therefore (y+z)(z+x)(x+y) \leq \left\{\frac{2(x+y+z)}{3}\right\}^3$

(ie) $\leq \frac{8}{27}(x+y+z)^3$

$\frac{x^3+y^3+z^3}{3} \geq \frac{x+y+z}{3}\cdot\frac{x+y+z}{3}\cdot\frac{x+y+z}{3}$

$\therefore (x+y+z)^3 \leq 9(x^3+y^3+z^3)$

$\therefore (y+z)(z+x)(x+y) \leq \frac{8}{27}\cdot 9(x^3+y^3+z^3)$

(ie) $< \frac{8}{3}(x^3+y^3+z^3)$

Hence we get $8xyz < (y+z)(z+x)(x+y) < \frac{8}{3}(x^3+y^3+z^3)$

Weirstrass inequalities :-

10m

Statement:- 1) If $a_1, a_2, a_3, \dots a_n$ are positive numbers whose sum is s, then (1) $(1+a_1)(1+a_2)\dots\dots(1+a_n) > 1+s$

10m

19) 2) $(1-a_1)(1-a_2)\dots\dots(1-a_n) > 1-s$

Proof :-

$(1+a_1)(1+a_2) = 1+a_1+a_2+a_1a_2$

$\therefore (1+a_1)(1+a_2) > 1+a_1+a_2$ since a_1 and $a_2 > 0$

$(1+a_1)(1+a_2)(1+a_3) > (1+a_1+a_2)(1+a_3)$

(ie) $> 1+a_1+a_2+a_3+a_1a_3+a_2a_3$

$> 1+a_1+a_2+a_3$ and so on.

$$\therefore (1+a_1)(1+a_2)\dots(1+a_n) > \underbrace{1+a_1+a_2+a_3+\dots+a_n}_{1+s}$$

(ie) $> 1+s$

2) Again $(1-a_1)(1-a_2) = 1-a_1-a_2+a_1a_2$

$\geq 1-a_1-a_2$ $\quad a_1a_2$ is >0

a_1 and

$(1-a_1)(1-a_2)(1-a_3) > (1-a_1-a_2)(1-a_3)$.

(ie) $> 1-a_1-a_2-a_3+a_1a_3+a_2a_3$

$> 1-a_1-a_2-a_3$ and so on

$$\therefore (1-a_1)(1-a_2)\dots\dots(1-a_n) > \underbrace{1-a_1-a_2-a_2-a_3\dots-a_n}_{1-S}$$

(ie) $> 1-s$.

10/10.

continue

1. If $a_1, a_2, \ldots a_n$ are positive and if $a_1, a_2, \ldots a_n$ are all less than 1, then

1) $(1+a_1)(1+a_2) \ldots\ldots (1+a_n) < \frac{1}{1-s}$ if $s<1$

2) $(1-a_1)(1-a_2) \ldots\ldots (1-a_n) < \frac{1}{1+s}$

$(1+a_1)(1-a_1) = 1-a_1^2$

$\therefore (1+a_1)(1-a_1) < 1 \quad \therefore 1+a_1 < \frac{1}{1-a_1}$

Similarily $(1+a_2)(1-a_2) < 1 \quad \therefore 1+a_2 < \frac{1}{1-a_2}$

$\therefore (1+a_3)(1-a_3) < 1 \therefore 1+a_3 < \frac{1}{1-a_3}$

$\ldots\ldots$

$(1+a_n)(1-a_n) < 1 \therefore 1+a_n < \frac{1}{1-a_n}$

$\therefore (1+a_1)(1+a_2) \ldots (1+a_n) < \frac{1}{(1-a_1)(1-a_2) \ldots (1-a_n)}$

By the previous article we know that

$(1-a_1)(1-a_2) \ldots\ldots (1-a_n) > 1-s$

$\therefore \frac{1}{(1-a_1)(1-a_2)\ldots(1-a_n)} < \frac{1}{1-s}$

$\therefore (1+a_1)(1+a_2) \ldots\ldots (1+a_n) < \frac{1}{1-s}$

Again

$(1+a_1)(1-a_1) = 1-a_1^2 < 1 \quad \therefore 1-a_1 < \frac{1}{1+a_1}$

Similarly

$1-a_2 < \frac{1}{1+a_2}, \; 1-a_3 < \frac{1}{1+a_3} \ldots\ldots 1-a_n < \frac{1}{1+a_n}$

$$\therefore (1-a_1)(1-a_2)\dots(1-a_n) < \frac{1}{(1+a_1)(1+a_2)\dots(1+a_n)}$$

By the previous articles, we know that

$$(1+a_1)(1+a_2)\dots(1+a_n) > 1+S$$

$$\therefore \frac{1}{(1+a_1)(1+a_2)\dots(1+a_n)} < \frac{1}{1+S}$$

$$\therefore (1-a_1)(1-a_2)\dots(1-a_n) < \frac{1}{1+S}$$

Cauchy's inequality :-

ⓘ Im 7 If $a_1, a_2 \dots a_n$, $b_1, b_2 \dots b_n$ are two sets of real number $\Sigma a_1^2 \Sigma b_1^2 (\Sigma a_1 b_1)^2$

The quadratic expression $ax^2+2bx+c$ is always positive.

If $(2b)^2 - 4ac < 0$ and $a > 0$

ie, if $b^2 - ac < 0$ and $a > 0$

consider the expression

$$(a_1x+b_1)^2 + (a_2x+b_2)^2 + \dots + (a_nx+b_n)^2$$

This expression is always positive for all values of x, since it is the same of the squares.

The expression is

$$x^2(a_1^2+a_2^2+\dots+a_n^2) + 2(a_1b_1+a_2b_2+\dots+a_nb_n)x + (b_1^2+b_2^2+\dots+b_n^2)$$

The coefficient of x^2 is $a_1^2+a_2^2+\dots+a_n^2$

which is +ve

Since the expression is positive for all values of x,

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 - (a_1^2 + a_2^2 + \dots + a_n^2) \times (b_1^2 + b_2^2 + \dots + b_n^2) < 0$$

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) \geq (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2$$

Applications to maxima and minima :-

1) If $a_1, a_2 \dots a_n$ are n positive variables, such that $a_1 + a_2 + \dots + a_n = K$ (constant) then $(a_1 a_2 \dots a_n)^{1/n}$ has the maximum value when $a_1 = a_2 = \dots = a_n$

Then maximum value of $(a_1 a_2 \dots a_n)^{1/n}$ is thus K/n

∴ The maximum value of $(a_1 a_2 \dots a_n)$ is $(K/n)^n$

2) If $a_1 a_2 \dots a_n = k_1$ (constant), then $a_1 + a_2 + \dots + a_n$ is least $a_1 = a_2 = \dots = a_n$ and the least value of $a_1 + a_2 + \dots + a_n$ is $n(k_1)^{1/n}$

Example-1:-

Find the greatest value of $a^m b^n c^p \dots$ when $a+b+c+\dots$ is constant $m, n, p \dots$ being positive integers.

Let $k = a^m b^n c^p \dots$

Then $\frac{k}{m^m n^n p^p} = \left(\frac{a}{m}\right)^m \cdot \left(\frac{b}{n}\right)^n \left(\frac{c}{p}\right)^p \dots$

$= \frac{a}{m}, \frac{a}{m}, \frac{a}{m} \ldots m$ factors $\frac{b}{n}, \frac{b}{n}, \frac{b}{n}$

$\ldots\ldots n$ factors $\times \frac{c}{p} \cdot \frac{c}{p} \cdot \frac{c}{p} \ldots p$ factors $\ldots$

The sum of all these factor $= a+b+c \ldots\ldots$

Since there are m factors each $\frac{a}{m}$, n factors each $\frac{b}{n}$ and so on.

$a+b+c \ldots\ldots =$ given constant, says, λ

The sum of all the factors is constant

$\therefore$ The product of the factors $\frac{k}{m^m n^n p^p} \ldots$ is greatest when all the factors are equal.

ie, when $\frac{a}{m} = \frac{b}{n} = \frac{c}{p} = \ldots$

$\therefore$ Each is equal to $\frac{a+b+\ldots}{m+n+p+\ldots} = \frac{\lambda}{m+n+p+\ldots}$

$a = \frac{m\lambda}{m+n+p+\ldots}, \; b = \frac{n\lambda}{m+n+p\ldots}, \; c = \frac{p\lambda}{m+n+p\ldots}$

The greatest value of k is

$$\left(\frac{m\lambda}{m+n+p\ldots}\right)^m \left(\frac{n\lambda}{m+n+p\ldots}\right)^n \left(\frac{p\lambda}{m+n+p\ldots}\right)^p \ldots$$

$$= \left(\frac{\lambda}{m+n+p+\ldots}\right)^{m+n+p+\ldots} \cdot m^m \cdot n^n \cdot p^p \ldots$$

Example -2 :-

If the perimeter of a triangle is given, show that the area is greatest when the triangle is equilateral.

Let a, b, c be the sides of the triangle and let

$a+b+c = 25$

If Δ is the area, then $\Delta^2 = s(s-a)(s-b)(s-c)$

$s-a+s-b+s-c = s = $ a constant

Hence the value of $(s-a)(s-b)(s-c)$ is greatest

when $s-a = s-b = s-c$

ie, when $a = b = c$.

$\therefore$ The value of Δ is greatest when $a=b=c$

Example - 3 :-

Find the maximum value of $(3-x)^5(2+x)^4$ when x lies between 3 and -2.

Let P be $(3-x)^5(2+x)^4$

Then $\frac{P}{5^5 4^4} = \left(\frac{3-x}{5}\right)^5 \left(\frac{2+x}{4}\right)^4$

$= \frac{3-x}{5} \cdot \frac{3-x}{5} \dots$ 5 factors $\times \frac{2+x}{4}$.

$\frac{2+x}{4} \dots$ 4 factors

Sum of the factors $= 5\left(\frac{3-x}{5}\right) + 4\left(\frac{2+x}{4}\right)$

$= 3-x+2+x$

$= 5$

Hence $\frac{P}{5^5 4^4}$ is greatest when all the factors are equal.

ie, when $\frac{3-x}{5} = \frac{2+x}{4}$ ie, when $x = \frac{2}{9}$

$\therefore$ P is greatest when $x = \frac{2}{9}$

$\therefore$ The greatest value of $P = \left(3-\frac{2}{9}\right)^5 \left(2+\frac{2}{9}\right)^4$

$\left(\frac{3-2/9}{5}\right)\left(\frac{2+2/9}{4}\right)$ $\left(\frac{25/9}{5}\right)^5 \cdot \left(\frac{20/9}{4}\right)^4$

$= \frac{(25)^5 (20)^4}{9^9} = \frac{5^{14} 4^4}{9^9}$

$= (5^2)^5 (4\times5)^4 = 5^{10}\times5^4\times4^4 = \frac{5^{14}\times4^4}{9^9}$

$\frac{5^5}{9^5} \cdot \frac{5^4}{9^4} \cdot 9^9$ $\frac{5^5 4^4}{9^9}$

Example 4:-

S.T the greatest value of $xyz(d-ax-by-cz)$ is $\frac{d^4}{4^4 abc}$ provided that all the factors are positive.

Let P be $xyz(d-ax-by-cz)$

Then $P\,abc = ax \cdot by \cdot cz\,(d-ax-by-cz)$.

~~Sum~~ Sum of the factors $= ax+by+cz+d-ax-by-cz$

$= d$

$=$ constant

Hence the product P abc is greatest when all the factors are equal.

ie, when $ax = by = cz = d-ax-by-cz$

Let $ax = by = cz = d-ax-by-cz = k$

Then $k = \frac{1}{4}d$

$$\therefore x = \frac{d}{4a}, \; y = \frac{d}{4b}, \; z = \frac{d}{4c}$$

Hence the greatest value of

$$P = \frac{d}{4a} \cdot \frac{d}{4b} \cdot \frac{d}{4c} \left(d - \frac{d}{4} - \frac{d}{4} - \frac{d}{4} \right)$$

$$= \frac{d^4}{4^4 abc}$$

Example - 5 :-

Find the least of value of $4x+3y$ for the positive values of x and y subject to the condition

✓ 5m. 14) b) $x^3y^2=6$

Since $x^3y^2=6$, if λ, μ are any constants we have $\lambda x . \lambda x . \lambda x . \mu y . \mu y = 6\lambda^3\mu^2$

Therefore,

$\lambda x + \lambda x + \lambda x + \mu y + \mu y$ is least

when $\lambda x = \mu y = (6\lambda^3\mu^2)^{1/5}$

Hence the least value of $3\lambda x + 2\mu y$ is $5(6\lambda^3\mu^2)^{1/5}$

putting $3\lambda = 4$ and $2\mu = 3$, it follow that the least value of $4x+3y$ is $5\left(6 \frac{4^3}{3^3} + \frac{3^2}{2^2}\right)^{1/5}$

ie, 10.