SWAMI DAYANANDA COLLEGE OF ARTS & SCIENCE, MANJAKKUDI.

(Affiliated to Bharathidasan University, Tiruchirappalli)
UGC Recognized under section 2(f) & 12(B)

DEPARTMENT OF MATHEMATICS

16SCCMM9:
NUMERICAL METHODS

CLASS:
III – B.Sc., MATHEMATICS

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NUMERICAL METHOD WITH MAT LAB PROGRAMMING

13/6/19

UNIT - IV :-

Curve fitting - linear and parabolic curve by the method of least squares principles - solving Algebric and transcendental equations - Bisection method, false position method and Newton - Raphson method - solving simultanous algebraic equations - Gauss - seidal methods - Gauss elimination methods.

UNIT - V :-

Interpolation - Newton's forward and backward difference formulae - Legranges interpulation formulae - Numerical integration using Trapezaidal and simpson's $1/3$ rd rule - solution of ODE's - Euler method and Runge - Kutta fourth order method.

AUTHOR :-

M.K. Venkatraman numerical methods in science and engineering 5^{th} edition.

14/6/19

UNIT - IV

Curve fitting :-

The principles of least squares :-

We have discribed,

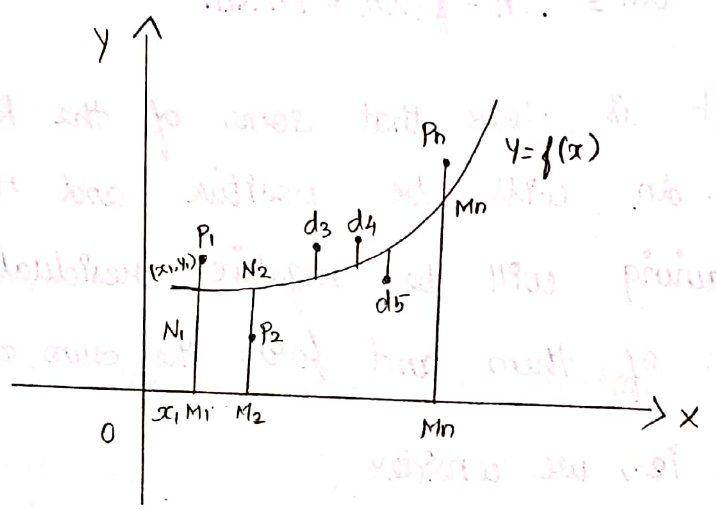
1) The Graphical method.

2) The method of group averages to determine the constants that occur in the equation choose an to represent a given data.

In the graphical method of fitting a straight line $y = a + bx$ to a given data the constant b is the slope which can be calculated with the help of any two points on the lines.

In the method of group averages different groupings of the observation can be made. Hence it is clearly that these two methods will give different values of the constant. Depending on the judgement of the individual.

The method of the least squares has the advantages of giving a unique set of values to the constant.



Let $(x_1, y_1), (x_2, y_2) \ldots\ldots (x_n, y_n)$ be n sets of observation of related data and $y = f(x) \rightarrow (1)$ be the suggested relationship between x and y.

When $x = x_1$ and the observed value $y = y_1 = P_1M_1$ from the relationship (1) $y = f(x_1) = N_1M_1$ and this is known as the excepted value of y. The expression $d_1 = y_1 - f(x_1)$ which is the difference between the observed and calculated values of y is called a resudial. Thus we have a resudial $d_2, d_3 \ldots d_n$ for all the remaining observations.

$$d_1 = y_1 - f(x_1) = P_1M_1 - N_1M_1 = P_1N_1$$

$$d_2 = y_2 - f(x_2) = P_2N_2 \ldots$$

$$d_n = y_n - f(x_n) = P_nN_n.$$

17/6/19

It is clear that some of the Residuals $d_1, d_2 \ldots d_n$ will be positive and the remaining will be negative residuals, we square each of them and form the sum of the squares

ie., we consider

$$E = d_1^2 + d_2^2 + \ldots + d_n^2$$

$$= [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \ldots + [y_n - f(x_n)]^2$$

The quantity E is clearly a measure of how well the curve $y = f(x)$ fits the set of points as a whole.

For E will be zero iff each of the points $P_1, P_2, \ldots$ lie on $y = f(x)$ and it will decrease in value depending on the closeness of the points P to the curve.

Hence, "The best representative curve to the set of point is that for which E, sum of the square of the residuals is a minimum" This is known as the least square criterion or the principle of least square.

Fitting a straight line:

Let $(x_1, y_1)(x_2, y_2) \ldots (x_n, y_n)$ be n sets of observation of related data and $y = ax+b$ the equation to the line of the best fit for them.

We have to find the constants a and b for any x_i the expected value of y (ie, the value calculated from the equation) is ax_i+b and the observed value of y is y_i

Hence residual d_i = observed value - expected value.

$$d_i = y_i - f(x_i)$$

$$d_i = y_i - (ax_i+b) \quad ; i = 1, 2, \ldots, n$$

Let E be the sum of square of the residual

$$E = \sum (y_i - f(x_i))^2 \quad i = 1, 2, \ldots n$$

$$E = \sum_{i=1}^{n} [y_i - (ax_i+b)]^2$$

$$E = [y_1 - (ax_1 + b)]^2 + [y_2 - (ax_2+b)]^2 + \dots + [y_n - (ax_n + b)]^2$$

E is the function of the parameters a and b for E to be minimum the condition are $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$ partially differentiating E with respect to a, $\frac{\partial E}{\partial a} = 2[y_1 - (ax_1+b)](-x_1) + 2[y_1 - (ax_2)+b](-x_2) + \dots + 2[y_n - (ax_n+b)](-x_n)$

Equating this to 0

$$2[y_1 - (ax_1+b)](-x_1) + 2[y_1 - (ax_2+b)](-x_2) + \dots + 2[y_n - (ax_n+b)](x_n)$$

$$x_1[y_1 - (ax_1+b)] + x_2[y_2 - (ax_2+b)] + \dots + x_n[y_n - (ax_n+b)] = 0$$

$$[x_1y_1 - ax_1^2 - bx_1] + [x_2y_2 - ax_2^2 - bx_2] + \dots + [x_ny_n - ax_n^2 - bx_n] = 0$$

$$[x_1y_1 + x_2y_2 + \dots + x_ny_n] - a[x_1^2 + x_2^2 + \dots + x_n^2] - b[x_1 + x_2 + \dots + x_n] = 0$$

$$\sum_{i=1}^{n} x_iy_i - a\sum_{i=1}^{n}[x_i]^2 - b\sum_{i=1}^{n}[x_i] = 0 \rightarrow ①$$

Partially differentating E with respect to b

$$\frac{\partial E}{\partial b} = 2[y_1-(ax_1+b)](-1) + 2[y_2-(ax_2+b)](-1) + \dots + 2[y_n-(ax_n+b)](-1) = 0$$

Equating this to zero, we get

$$-1[y_1-(ax_1+b)] + (-1)[y_2-(ax_2+b)] + \dots + (-1)[y_n-(ax_n+b)] = 0$$

$$-[-y_1+ax_1+b-y_2+ax_2+b+\dots-y_n+ax_n+b] = 0$$

$$-a\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i + nb = 0$$

$$a\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + nb$$

$$a\sum_{i=1}^{n} x_i + nb = \sum_{i=1}^{n} y_i$$

21/6/19

$$y - ax_1 - b + y_2 - ax_2 - b + \dots + y_n - ax_n - b = 0$$

$$(y_1+y_2+\dots y_n) - a(x_1+x_2+\dots+x_n) - nb = 0$$

$$\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i - nb = 0$$

$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} x_i + nb \rightarrow (2)$$

Eqn (1) and (2) are two simultanous linear equations from which a and b can be solved.

Thus we get the equation of the line best fitting the data as $y = ax+b$

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Equation ① and ② are called normal equations.

This is of the form ~~or~~ into

$a\sum x^2 + b\sum x = \sum xy$ and

$a\sum x + nb = \sum y$

Problem:-

1. Using the method of least square to fit a straight line to the following data.

x :	0	5	10	15	20
y :	7	11	16	20	26

Estimate the value of y when $x = 25$

Solution :-

Let the straight line fit $y = ax + b \rightarrow$ ①

The normal equations are

$$a\sum x^2 + b\sum x = \sum xy \rightarrow ① \text{ and}$$

$$a\sum x + nb = \sum y \rightarrow ②$$

x	y	x^2	xy
0	7	0	0
5	11	25	55
10	16	100	160
15	20	225	300
20	26	400	520.
$\Sigma x = 50$	$\Sigma y = 80$	$\Sigma x^2 = 750$	$\Sigma xy = 1035$

① $\Rightarrow a(750) + b(50) = 1035$

② $\Rightarrow a(50) + 5b = 80$

① $\div 5$

① $\Rightarrow a(150) + b(10) = 207$

② $\times 2 \Rightarrow a(100) + 10b = 160$

$a(150) = +47$

$a = \frac{47}{50}$

$\boxed{a = 0.94}$

② $\Rightarrow 0.94(50) + 5b = 80$

$\frac{47}{\cancel{50}}(\cancel{50}) + 5b = 80$

$47 + 5b = 80$

$5b = 80 - 47$

$5b = 33$

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$$b = \frac{33}{5}$$

$$\boxed{b = 6.6}$$

Putting these values in (I), the line of best fit is $y = 0.94x + 6.6 \rightarrow$ (II)

Putting $x = 25$ in eqn (II).

$$y = 0.94(25) + 6.6$$

$$= 30.1$$

$$\boxed{\therefore y = 30}$$

Hence when $x = 25$ expected value of $y = 30$

2. Find a suitable change of variables x and y in the relation $y = a + bxy$ so that the relation between new variables may be linear. Hence find the constant a and b if the following set of values satisfy approximately the above relation.

x :	-4	1	2	3
y :	4	6	10	8

Soln:-

The given relation is $y = a + bxy \rightarrow$ (1)

Dividing eqn ① by xy

$$\frac{1}{x} = \frac{a}{xy} + b \rightarrow ②$$

Putting $\frac{1}{x} = v$; $\frac{1}{xy} = u$

$\therefore$ ② $\Rightarrow$ $v = au + b \rightarrow ③$

The normal equations are

$$a \sum u^2 + b \sum u = \sum uv \rightarrow ④$$

$$a \sum u + nb = \sum v \rightarrow ⑤$$

x	y	$u = \frac{1}{xy}$	$v = \frac{1}{x}$	u^2	uv
-4	4	-0.0625	-0.25	0.0039	0.0156
1	6	0.1667	1	0.0277	0.1667
2	10	0.05	0.5	0.0025	0.025
3	8	0.0416	0.33	0.0017	0.0137
		$\sum u = 0.1958$	$\sum v = 1.5800$	$\sum u^2 = 0.0358$	$\sum uv = 0.2210$

④ $\Rightarrow$ $a(0.0358) + b(0.1958) = 0.2210$

⑤ $\Rightarrow$ $a(0.1958) + 4b = 1.5800$

0.7836

④ $\times 4$ $\Rightarrow$ $a(0.1432) + b(0.\cancel{78}32) = 0.8840$

0.7836

⑤ $\times 0.1958$ $\Rightarrow$ $\underset{(-)}{a}(0.0383) \underset{(-)}{+} b(0.7832) = \underset{(-)}{\cancel{6.3200}}\ 0.3094$

$$a(0.1049) = 5.4360 \quad 0.5746$$

$$\boxed{a = 5.4776}$$

(5) $\Rightarrow (5.4776)(0.1958) + 4b = 1.58$

$$1.0725 + 4b = 1.58$$

$$4b = 1.58 - 1.0725$$

$$b = \frac{0.5075}{4}$$

$$\boxed{b = 0.1269}$$

3. Find by the method of least squares is straight line that the best fit the data in the following cases.

i)

x :	1	2	3	4	5
y :	16	19	23	26	30

ii)

x :	0	1	2	3	4
y :	1	1.8	3.3	4.5	6.3

iii)

x :	1	2	3	4	5
y :	14	27	40	55	68

1) **Soln:-**

Let the straight line fit $y = ax+b \to ①$

The normal equations are

$$a\sum x^2 + b\sum x = \sum xy \to ②$$

$$a\sum x + nb = \sum y \to ③$$

x	y	x^2	xy
1	16	1	16
2	19	4	38
3	23	9	69
4	26	16	104
5	30	25	150
$\sum x = 15$	$\sum y = 114$	$\sum x^2 = 55$	$\sum xy = 377$

② $\Rightarrow a(55) + b(15) = 377 \to ④$

③ $\Rightarrow a(15) + 5b = 114 \to ⑤$

④ $\Rightarrow a(55) + b(15) = 377$

③ $\times 5 \Rightarrow a(45) + b(15) = 342$

$a(10) = 35$

$a = 35/10 \Rightarrow \boxed{a = 3.5}$

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⑤ $\Rightarrow 3.5(15) + 5b = 114$

$52.5 + 5b = 114$

$5b = 114 - 52.5$

$b = \dfrac{61.5}{5}$

$\boxed{b = 12.3}$

ii) Soln:-

Let the straight line fit $y = ax + b \rightarrow$ ⓪

The normal equations are

$a \sum x^2 + b \sum x = \sum xy \rightarrow$ ①

$a \sum x + nb = \sum y \rightarrow$ ②

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x = 10$	$\sum y = 16.9$	$\sum x^2 = 30$	$\sum xy = 47.1$

① $\Rightarrow a(30) + b(10) = 47.1 \rightarrow$ ③

② $\Rightarrow a(10) + 5b = 16.9 \rightarrow$ ④

③ $\Rightarrow a(30) + b(\cancel{10}) = 47.1$

④ $\times 2 \Rightarrow \underline{a(20) + \cancel{b}(10) = 33.8}$
$(-) \quad (-) \quad (-)$

$a(10) = 13.3$

$\boxed{a = 1.33}$

④ $\Rightarrow 1.33(10) + 5b = 16.9$

$13.3 + 5b = 16.9$

$5b = 16.9 - 13.3$

$5b = 3.6$

$b = \frac{3.6}{5}$

$\boxed{b = 0.72}$

iii) Soln:-

Let the straight line fit $y = ax + b \rightarrow$ (I)

The normal equations are

$a\Sigma x^2 + b\Sigma x = \Sigma xy \rightarrow$ ①

$a\Sigma x + nb = \Sigma y \rightarrow$ ②

x	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\Sigma x = 15$	$\Sigma y = 204$	$\Sigma x^2 = 55$	$\Sigma xy = 748$

① $\Rightarrow a(55) + b(15) = 748 \rightarrow$ ③

② $\Rightarrow a(15) + 5b = 204 \rightarrow$ ④

③ $\Rightarrow a(55) + b(15) = 748$

④ $\times 3 \Rightarrow a(45) + b(15) = 612$

$$a(10) = 136$$

$$a = \frac{136}{10}$$

$$\boxed{a = 13.6}$$

(4) $\Rightarrow$ $a(15) + 5b = 204$

$13.6\,(15) + 5b = 204$

$204 + 5b = 204$

$5b = 204 - 204$

$5b = 0$

$\boxed{b = 0}$

24/6/19

Fitting a parabola :-

Let $(x_1,y_1),(x_2,y_2),\ldots\ldots(x_n,y_n)$ be n sets of observations of related data and $y = ax^2+bx+c$ the equation of the parabola of best fit for them.

We have to find the constants a,b,c

for any x_i the expected value (i.e., value calculated from the equation) is $ax_i^2+bx_i+c$ and the observed value of y is y_i.

Hence the residual $d_i = \overset{\text{Observed value - expected value}}{y_i - (ax_i^2+bx_i+c)}$

$i = 1,2\ldots\ldots n$

Let E be the some of the squares of the residual.

i.e., $E = \sum_{i=1}^{n} [y_i - (ax_i^2+bx_i+c)]^2$

$E = [y_1-(ax_1^2+bx_1+c)+[y_2-(ax_2^2+bx_2+c)]+[y_3-(ax_3^2+bx_3+c]+\ldots+[y_n-(ax_n^2$

$+ bx_n + c]^2$.

E is a function of the parameters a, b and c

For E to be minimum the conditions are

$$\frac{\partial E}{\partial a} = 0, \quad \frac{\partial E}{\partial b} = 0 \text{ and } \frac{\partial E}{\partial c} = 0.$$

Partially differentiating E with respect to 'a'.

$$\frac{\partial E}{\partial a} = 2[y_1 - (ax_1^2 + bx_1 + c)] - (x_1^2) + 2[y_2 - (ax_2^2 + bx_2 + c)] - (x_2^2) + \dots + 2[y_n - (ax_n^2 + bx_n + c)] - (x_n^2).$$

Equating this to zero we get.

$$-2\sum_{i=1}^{n} \{[y_i - (ax_i^2 + bx_i + c)].(x_i^2)\} = 0$$

$$\sum_{i=1}^{n} [y_i - (ax_i^2 + bx_i + c)](x_i)^2 = 0$$

$$\sum_{i=1}^{n} [(y_i (x_i)^2) - (ax_i^4 + bx_i^3 + cx_i^2)] = 0$$

$$\sum_{i=1}^{n} y_i x_i^2 - a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = 0$$

$$\sum y_i x_i^2 = a\sum x_i^4 + b\sum x_i^3 + c\sum x_i^2 \to (1)$$

Partially differentiating E with respect to 'b'

$$\frac{\partial E}{\partial b} = 2[y_1 - (ax_1^2 + bx_1 + c)] - (x_1) + 2[y_2 - (ax_2^2 + bx_2 + c)] - (x_2) + \dots + 2[y_n - (ax_n^2 + bx_n + c)] - (x_n)$$

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$$-2\sum_{i=1}^{n}\{[y_i-(ax_i^2+bx_i+c)](x_i)\}=0$$

Equating this to zero we get

$$\sum_{i=1}^{n}[y_i-(ax_i^2+bx_i+c)](x_i)=0.$$

$$\sum_{i=1}^{n}[y_i(x_i)-(ax_i^3+bx_i^2+cx_i)]=0.$$

$$\sum_{i=1}^{n}y_ix_i-a\sum_{i=1}^{n}x_i^3+b\sum_{i=1}^{n}x_i^2+c\sum_{i=1}^{n}x_i=0.$$

$$\sum y_ix_i=a\sum x_i^3+b\sum x_i^2+c\sum x_i^2 \rightarrow (2).$$

Partially differentiating E with respect to 'c'

$$\frac{\partial E}{\partial c}=2[y_1-(ax_1^2+bx+c)]\cdot(-1)+2[y_2-(ax_2^2+bx+c)]\cdot(-1)$$
$$+\dots+2[y_n-(ax_n^2+bx_n+c]\cdot(-1)$$

$$-2\sum_{i=1}^{n}\{[y_i-(ax_i^2+bx_i+c)](-1)\}=0$$

Equating this to zero we get

$$\sum_{i=1}^{n}[y_i-(ax_i^2+bx_i+c)](-1)=0$$

$$\sum_{i=1}^{n}[-y_i-(-ax_i^2-bx_i-c)]=0$$

$$\sum_{i=1}^{n}[y_i-(ax_i^2+bx_i+c)]=0$$

$\sum y_i - a\sum x_i^2 - b\sum x_i - nc = 0$

$\sum y_i = a\sum x_i^2 + b\sum x_i + nc \rightarrow (3)$

Hence equations (1), (2) and (3) are the normal equations it can be written as

$\sum x^2 y = a\sum x^4 + b\sum x^3 + c\sum x^2$

$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$.

$\sum y = a\sum x^2 + b\sum x + nc$

By solving these normal equations we get the values of a, b and c and hence the equation to the best fitting parabola.

Problem:-

1. The following table gives the levels of prizes in certain years fit a second degree parabola to the data.

assumed value 1875 1585 3760/2 1880 74 100 174/2 87.

Year	:	1875	1876	1877	78	79	80
Prize level	:	88	87	81	78	74	79
Year	:	81	82	83	84	85	
Prize level	:	85	84	90	92	100	

Soln:-

For the year x take the origin at 1880 and for the prize level y take the origin at 87

The normal equations are

$$\Sigma x^2y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 \rightarrow (1)$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \rightarrow (2)$$

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc \rightarrow (3)$$

x	y	$X = x-1880$	$Y = y-87$	X^2	X^3	X^4	XY	X^2Y
1875	88	-5	1	25	-125	625	-5	25
76	87	-4	0	16	-64	256	0	0
77	81	-3	-6	9	-27	81	18	-54
78	78	-2	-9	4	-8	16	18	-36
79	74	-1	-13	1	-1	1	13	-13
80	79	0	-8	0	0	0	0	0
81	85	1	-2	1	1	1	-2	-2
82	84	2	-3	4	8	16	-6	-12
83	90	3	3	9	27	81	9	27
84	92	4	5	16	64	256	20	80
85	100	5	13	25	125	625	65	325
Total		$\Sigma X = 0$	$\Sigma Y = -19$	$\Sigma X^2 = 110$	$\Sigma X^3 = 0$	$\Sigma X^4 = 1958$	$\Sigma XY = 130$	$\Sigma X^2Y = 3$[illegible]

① $\Rightarrow$ $340 = a(1958) + b(0) + c(110)$

② $\Rightarrow$ $130 = a(0) + b(110) + c(0)$

③ $\Rightarrow$ $-19 = a(110) + b(0) + (11)c.$

① $\Rightarrow$ $340 = a(1958) + c(\cancel{110})$

③ $\times 10 \Rightarrow -190 = a(1100) + \cancel{c}(110)$

(+) −

$550 = a(858)$

$\boxed{a = 0.6177}$

① $\Rightarrow$ $340 = (0.6177)(1958) + c(110)$

3169.6104

$340 = 1209.4566 + c(110)$

$340 = 1209.4566 + c(110)$

-2829.6104

$-869.4566 = c(110)$

$\boxed{c = -7.9042}$

③ $\Rightarrow$ $-19 = (0.6177)(110) + 11(-7.9042)$

$-19 = 67.9470 - 86.9462$

② $\Rightarrow 130 = b(110)$

$$\boxed{b = 1.1818}$$

Hence the best fitting parabola $Y = aX^2 + bX + c$

$\Rightarrow Y = 0.6177X^2 + 1.1818X - 7.9042$

where, $Y = y - 87$; $X = x - 1880$

$\therefore y - 87 = 0.6177X^2 + 1.1818X - 7.9043$

$\therefore y = 0.6177X^2 + 1.1818X + 79.0957$

$(X \to x - 1880)$

2. Find by the method of least squares the parabola that best fits the data in following cases taking x as the independent variable.

i)

X	:	0	1	2	3	4
Y	:	1	5	10	22	38

ii)

X	:	1	2	3	4	5
Y	:	5	12	26	60	97

iii)

X	:	0	2	4	6	8	10
Y	:	1	3	13	31	57	91

P) **Soln:-**

The normal equations are

$$\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2 \rightarrow (1)$$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x \rightarrow (2)$$

$$\sum y = a\sum x^2 + b\sum x + nc \rightarrow (3)$$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	5	1	1	1	5	5
2	10	4	8	16	20	40
3	22	9	27	81	66	198
4	38	16	64	256	152	608
$\sum x = 10$	$\sum y = 76$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 243$	$\sum x^2y = \cancel{100}\ 851$

$(1) \Rightarrow 851 = a(354) + b(100) + c(30)$

$(2) \Rightarrow 243 = a(100) + b(30) + c(10)$

$(3) \Rightarrow 76 = a(30) + b(10) + 5c$

$(1) \Rightarrow 851 = a(354) + b(100) + c(30)$

$(2) \times 3 \Rightarrow 729 = a(300) + b(90) + c(30)$

$\quad\quad - \quad\quad - \quad\quad -$

$122 = 54a + 10b$

$② \Rightarrow 243 = a(100) + b(30) + c(10)$

$③ \times 3 \Rightarrow 228 = a(90) + b(30) + 15c$

$15 = a(10) + 5c$

$② \Rightarrow 243 = a(100) + b(30) + c(10)$

$③ \times 2 \Rightarrow -152 = a(60) + b(20) + c(10)$

$91 = a(40) + b(10) \rightarrow ④$

$① \Rightarrow 851 = a(354) + b(100) + c(30)$

$② \times 3 \Rightarrow 729 = a(300) + b(90) + c(30)$

$122 = a(54) + b(10) \rightarrow ⑤$

solve ④ & ⑤

$a(40) + b(10) = 91$

$a(54) + b(10) = 122$

$a(-14) = -31$

$\boxed{a = 2.2143}$

$④ \Rightarrow 2.2143(40) + b(10) = 91$

$88.5720 + b(10) = 91$

$b(10) = 2.4280$

$\boxed{b = 0.2428}$

$③ \Rightarrow 76 = (30)(2.2143) + (0.2428)(10) + 5C$

$76 = 66.4290 + 2.4280 + 5C$

$$76 = 68.8570 + 5C$$

$$76 - 68.8570 = 5C$$

$$7.1430 = 5C$$

$$\boxed{C = 1.4286}$$

Hence the best fitting parabola $y = ax^2 + bx +$

$$y = 2.2143x^2 + 0.2428x + 1.4286.$$

(ii) Soln:-

The normal equations are

$$\sum x^2 y = a\sum x^4 + b\sum x^3 + c\sum x^2 \rightarrow (1)$$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x \rightarrow (2)$$

$$\sum y = a\sum x^2 + b\sum x + nc \rightarrow (3)$$

x	y	x^2	x^3	x^4	xy	x^2y
1	5	1	1	1	5	5
2	12	4	8	16	24	48
3	26	9	27	81	78	234
4	60	16	64	256	240	960
5	97	25	125	625	485	2425
$\Sigma x = 15$	$\Sigma y = 200$	$\Sigma x^2 = 55$	$\Sigma x^3 = 225$	$\Sigma x^4 = 979$	$\Sigma xy = 832$	$\Sigma x^2y = 3672$

① $\Rightarrow$ $3672 = a(979) + b(225) + c(55)$

② $\Rightarrow$ $832 = a(225) + b(55) + c(15)$

③ $\Rightarrow$ $200 = a(55) + b(15) + 5c$

② $\Rightarrow$ $832 = a(225) + b(55) + c(15)$

③ $\times 3 \Rightarrow$ $600 = a(165) + b(45) + c(15)$

$232 = a(60) + b(10) \rightarrow$ ④

① $\Rightarrow$ $3672 = a(979) + b(225) + c(55)$

③ $\times 11 \Rightarrow$ $2200 = a(605) + b(165) + c(55)$

$1472 = a(374) + b(60) \rightarrow$ ⑤

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(4) × (6) ⇒ $1392 = a(360) + b(60)$

(5) ⇒ $1472 = a(374) \pm b(60)$

$$-80 = a(-14)$$

$$a = 80/14$$

$$\boxed{a = 5.7143}$$

(3) ⇒ $200 = 5.7143(55)$

(4) ⇒ $232 = 5.7143(60) + b(10)$

$$232 = 342.8580 + b(10)$$

$$-110.8580 = b(10)$$

$$\boxed{b = -11.0858}$$

(3) ⇒ $200 = 5.7143(55) + (-11.0858)(15) + 5c$

$$200 = 314.2865 - 166.2870 + 5C$$

$$200 - 147.9995 = 5C$$

$$52.0005 = 5C$$

$$\boxed{C = 10.4001}$$

Hence the best fitting of parabola

$y = ax^2 + bx + c$.

$$y = 5.7143x^2 - 11.0858x + 10.4001$$

(iii) Soln:-

The normal equations are

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 \rightarrow ①$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \rightarrow ②$$

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc. \rightarrow ③$$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
2	3	4	8	16	6	12.
4	13	16	64	256	52	208
6	31	36	216	1296	186	1116
8	57	64	512	4096	456	3648
10	91	100	1000	10,000	910	9100
$\Sigma x = 30$	$\Sigma y = 196$	$\Sigma x^2 = 220$	$\Sigma x^3 = 1800$	$\Sigma x^4 = 15,664$	$\Sigma xy = 1610$	$\Sigma x^2y = 14,084$

$$① \Rightarrow 14,084 = a(15,664) + b(1800) + c(220)$$

$$② \Rightarrow 1610 = a(1800) + b(220) + c(30)$$

$$③ \Rightarrow 196 = a(220) + b(30) + 6c$$

$① \times 3 \Rightarrow 42252 = a(46992) + b(5400) + \cancel{c(660)}$

$② \times 22 \Rightarrow 35420 = a(39600) + b(4840) + \cancel{c(660)}$

$a(7392) + b(560) = 6832 \rightarrow ④$

$② \Rightarrow 1610 = a(1800) + b(220) + \cancel{c(30)}$

$③ \times 5 \Rightarrow 980 = a(1100) + b(150) + \cancel{c(30)}$

$630 = a(700) + b(70) \rightarrow ⑤$

Solve ④ & ⑤

$④ \Rightarrow a(7392) + b(560) = 6832$

$⑤ \times 8 \Rightarrow a(5600) + b(560) = 5040$

$a(1792) = 1792$

$\boxed{a = 1}$

$⑤ \Rightarrow 700 + b(70) = 630$

$b(70) = -70$

$\boxed{b = -1}$

$③ \Rightarrow 196 = 220 - 30 + 6c$

$196 = 190 + 6c$

$196 - 190 = 6c$

$$6 = 6c$$

$$\boxed{c = 1}$$

$$Y = x^2 - x + 1.$$

Hence the best fitting parabola $Y = ax^2 + bx + c$

$$Y = x^2 - x + 1.$$

1/7/19

Solution of Algebraic and Transcendental Equations :-

If $f(x)$ is a quadratic, cubic are biquadratic expression then algebraic formulae are available for expressing the roots interms of the coefficients.

On the other hand when $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions, algebraic methods are not available and recourse must be taken to find the roots by approxiate methods.

Now $f(x)$ is the algebraic function of the form

$$f_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

are called polynomials and we have some special methods for determining their roots.

A non-algebraic function is called a Transcendental function.

Example :-

$$f(x) = \log x^3 - 0.7$$

$$\phi(x) = e^{-0.5x} - 5x$$

$$\phi(x) = \sin^2 x - x^2 - 2, \text{ etc.,}$$

BISECTION METHOD :-

If a function $f(x)$ is continous between a and b and $f(a)$ and $f(b)$ are of opposite signs then there exists atleast one root between a & b.

For defintenes let $f(a)$ be negative and $f(b)$ be positive then the roots lies between a & b and let it approximate value is given by

$$x_0 = \frac{a+b}{2}$$

If $f(x_0) = 0$ we conclude that x_0 is a root of the equation $f(x) = 0$ otherwise the root lies between either x_0 and b or x_0 and a depending on whether $f(x_0)$ is negative or positive.

We designed is new interval as (a_1, b_1) whose length is $\frac{|b-a|}{2}$ as before this is bisected at x_1 and the new interval will be exactly half the length of the previous one.

We repeat this process until the latest interval is as small as desired. (say ε)

It is clear that the interval with this is reduced by a factor of one-half at each step and at the end of the n^{th} step. The new interval will be $[a_n, b_n]$ of length $\frac{|b-a|}{2^n}$

Then we have $\frac{|b-a|}{2^n} \leq \varepsilon \rightarrow (1)$

The eqn (1) gives the number of iteration required to archieve an accuracy ε.

PROBLEM :-

1. Find a real root of the eqn

$f(x) = x^3 - x - 1 = 0$

Soln:-

Given $f(x) = x^3 - x - 1 = 0$

$8 - 2 - 1$

$f(0) = -1 \Rightarrow -ve$

$f(1) = -1 \Rightarrow -ve$

$f(2) = 5 \Rightarrow +ve$

$\therefore$ The root lies between 1 and 2

Let $a = 1$; $b = 2$

$\therefore x_0 = \frac{a+b}{2}$

n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ $f(x) = x^3 - x - 1$
0	1	2	$x_0 = \frac{1+2}{2} = 1.5$	0.8750 (+ve)
1	1	1.5	$x_1 = \frac{1+1.5}{2} = 1.25$	−0.2969 (-ve)
2	1.25	1.5	$x_2 = \frac{1.25+1.5}{2} = 1.3750$	0.2246 (+ve)
3.	1.25	1.375	$x_3 = \frac{1.25+1.375}{2} = 1.3125$	−0.0515 (-ve)
4.	1.3125	1.375	$x_4 = \frac{1.3125+1.375}{2} = 1.3438$	0.0828 (+ve)
5.	1.3125	1.3438	$x_5 = \frac{1.3125+1.3438}{2} = 1.3282$	0.0149 (+ve)
6.	1.3125	1.3282	$x_6 = \frac{1.3125+1.3282}{2} = 1.3204$	−0.0183 (-ve)
7.	1.3204	1.3282	$x_7 = \frac{1.3204+1.3282}{2} = 1.3243$	−0.0018 (-ve)
8.	+1.3243	1.3282	$x_8 = 1.3263$	0.0068 (+ve)
9.	1.3243	1.3263	$x_9 = 1.3253$	0.0025 (+ve)
10.	1.3243	1.3253	$x_{10} = 1.3248$	0.0003 (+ve)

Hence the root of the given equations is 1.3253

2. Find the root of the equation $x^3 - 2x - 5$

Soln:-

Given $f(x) = x^3 - 2x - 5 = 0$

$f(0) = -5 = -ve$

$f(1) = 1 - 2 - 5 = -6 = -ve$

$f(2) = 8 - 4 - 5 = -1 = -ve$

$f(3) = 27 - 6 - 5 = 16 = +ve$

$\therefore$ The root lies between 2 and 3

Let $a = 2$; $b = 3$

$\therefore x_0 = \frac{a+b}{2}.$

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n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ $f(x) = x^3 - 2x - 5$
0	2	3	$x_0 = \frac{2+3}{2} = 2.5$	5.6250 (+ve)
1	2	2.5	$x_1 = 2.25$	1.8906 (+ve)
2.	2	2.25 ~~1.8906~~	$x_2 = \cancel{1.94}\ 2.125$	0.3457 (+ve)
3.	2	2.125	$x_3 = 2.0625$	-0.3513 (-ve)
4	2.0625	2.125	$x_4 = 2.0938$	-0.0084 (-ve)
5	2.0938	2.125	$x_5 = 2.1094$	0.1671 (+ve)
6	2.0938	2.1094	$x_6 = 2.1016$	0.0790 (+ve)
7	2.0938	2.1016	$x_7 = 2.0977$	0.0352 (+ve)
8	2.0938	2.0977	$x_8 = 2.0958$	0.0139 (+ve)
9	2.0938	2.0958	$x_9 = 2.0948$	0.0028 (+ve)

Hence the root of the given equation is 2.0948

H.W:-

3. Find the root of the equation

$x^3 + x^2 + x + 7 = 0$

Soln:

Given $f(x) = x^3 + x^2 + x + 7 = 0$

$f(0) = 7 = +ve.$

$f(-1) = -1 + 1 + 1 + 7 = 8\ +ve$

$$f(-2) = -8+4-2+7 = 1 \quad +ve$$

$$f(-3) = -27+9-3+7 = -14 \quad -ve.$$

n	(+ve) a	b −(ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ $f(x) = x^3+x^2+x+7$
0	-2	-3	$x_0 = \frac{-2-3}{2} = -2.5$	-4.8750
1	-2	-2.5	$x_1 = -2.2500$	-1.5781
2	-2	-2.25	$x_2 = -2.1250$	-0.2051
3	-2	-2.1250	$x_3 = -2.0625$	0.4177
4	-2.0625	-2.1250	$x_4 = -2.0938$	0.1110
5	-2.0938	-2.1250	$x_5 = -2.1094$	-0.0458
6	-2.0938	-2.1094	$x_6 = -2.1016$	0.0329
7	-2.1016	-2.1094	$x_7 = -2.1055$	-0.0063
8	-2.1016	-2.1055	$x_8 = -2.1036$	0.0128
9	-2.1036	-2.1055	$x_9 = -2.1046$	0.0027
10	-2.1046	-2.1055	$x_{10} = -2.1051$	-0.0023
11	-2.1046	-2.1051	$x_{11} = -2.1049$	-0.0003
12.	-2.1046	-2.1049	$x_{12} = -2.1048$	0.0007

∴ The roots lies between -2 and -3

Let $a = -2$ and $b = -3$

$$x_0 = \frac{a+b}{2}$$

4. Find the root of the equation $x^3-4x-9=0$

Soln:-

Given $f(x) = x^3-4x-9 = 0$

$f(0) = -9 = -ve$

$f(1) = 1-4-9 = -12 = -ve$

$f(2) = 8-8-9 = -9 = -ve$

$f(3) = 27-12-9 = 6 = +ve$

$\therefore$ The root lies between 2 and 3

Let $a = 2$; $b = 3$

$\therefore x_0 = \frac{a+b}{2}$.

n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ $f(x) = x^3-4x-9$.
0	2	3	$x_0 = 2.5$	-3.3750
1	2.5	3	$x_1 = 2.75$	0.7969
2.	2.75	2.75	$x_2 = 2.6250$	-1.4121
3.	2.625	2.75	$x_3 = 2.6875$	-0.3391
4.	2.6875	2.75	$x_4 = 2.7188$	0.2218
5.	2.6875	2.7188	$x_5 = 2.7032$	-0.0547
6.	2.7032	2.7188	$x_6 = 2.7110$	0.0806
7.	2.7032	2.7110	$x_7 = 2.7071$	0.0103

8.	2.7032	2.7071	$x_8 = 2.7052$	-0.0239
9	2.7052	2.7071	$x_9 = 2.7062$	-0.0059
10	2.7062	2.7071	$x_{10} = 2.7067$	0.0031
11	2.7062	2.7067	$x_{11} = 2.7065$	-0.0005
12	2.7065	2.7067	$x_{12} = 2.7066$	0.0013
13	2.7065	2.7066	$x_{13} = 2.7066$	0.0013

$\therefore$ Hence the root of the given equation is 2.7066

5. $x^3 - x - 4 = 0$

Soln:-

Given $f(x) = x^3 - x - 4 = 0$

$f(0) = -4 = -ve$

$f(1) = 1 - 1 - 4 = -4 = -ve$

$f(2) = 8 - 2 - 4 = 2 = +ve$

$\therefore$ The root lies between 1 and 2.

Let $a = 1$; $b = 2$.

$$x_0 = \frac{a+b}{2}.$$

n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ $f(x) = x^3 - x - 4$
0	1	2	$x_0 = 1.5$	-2.1250
1	1.5	2	$x_1 = 1.75$	-0.3906
2	1.75	2	$x_2 = 1.875$	0.7168
3	1.75	1.875	$x_3 = 1.8125$	0.1418
4	1.75	1.8125	$x_4 = 1.7813$	-0.1292
5	1.7813	1.8125	$x_5 = 1.7969$	0.0050
6	1.7813	1.7969	$x_6 = 1.7891$	-0.0624
7	1.7891	1.7969	$x_7 = 1.7930$	-0.0288
8	1.7930	1.7969	$x_8 = 1.7950$	-0.0115
9	1.7950	1.7969	$x_9 = 1.7960$	-0.0028
10	1.7960	1.7969	$x_{10} = 1.7965$	0.0015
11	1.7960	1.7965	$x_{11} = 1.7963$	-0.0002
12	1.7963	1.7965	$x_{12} = 1.7964$	0.0007
13	1.7963	1.7964	$x_{13} = 1.7964$	0.0007

Hence the roots of the given equation is 1.7964

6. $x^3 - 18 = 0$

Soln:-

Given $f(x) = x^3 - 18 = 0$

$f(0) = -18 = -ve$

$f(1) = 1 - 18 = -17 = -ve$

$f(2) = 8 - 18 = -10 = -ve$.

$f(3) = 27 - 18 = 9 = +ve$.

$\therefore$ The root lies between 2 & 3.

$a = 2$; $b = 3$

$$x_0 = \frac{a+b}{2}.$$

n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ x^3-18.
0	2	3	$x_0 = 2.5$	-2.3750
1	2.5	3	$x_1 = 2.75$	2.7969
2.	2.5	2.75	$x_2 = 2.625$	0.0879
3.	2.5	2.625	$x_3 = 2.5625$	-1.1736
4.	2.5625	2.625	$x_4 = 2.5938$	-0.5494
5.	2.5938	2.625	$x_5 = 2.6094$	-0.2327
6.	2.6094	2.625	$x_6 = 2.6172$	-0.0729

7.	2.6172	2.625	$x_7 = 2.6211$	0.0074
8.	2.6172	2.6211	$x_8 = 2.6192$	-0.0317
9	2.6192	2.6211	$x_9 = 2.6202$	-0.0112
10.	2.6202	2.6211	$x_{10} = 2.6207$	-0.0009
11.	2.6207	2.6211	$x_{11} = 2.6209$	0.0033
12.	2.6207	2.6209	$x_{12} = 2.6208$	0.0012
13.	2.6207	2.6208	$x_{13} = 2.6208$	-0.0012.

Hence the given equation is

2.6208

7. $x^3 - x^2 - 1 = 0$

Soln:-

Given $f(x) = x^3 - x^2 - 1 = 0$

$f(0) = -1 = -ve$

$f(1) = 1-1-1 = -ve$

$f(2) = 8-4-1 = 3 = +ve$

$\therefore$ The root lies between 1 & 2.

$a = 1$; $b = 2$

$$x_0 = \frac{a+b}{2}$$

n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ x^3-x^2-1
0	1	2	$x_0 = 1.5$	0.1250
1	1	1.5	$x_1 = 1.25$	-0.6094
2	1.25	1.5	$x_2 = 1.3750$	-0.2910
3.	1.3750	1.5	$x_3 = 1.4375$	-0.0959
4.	1.4375	1.5	$x_4 = 1.4688$	0.0114
5	1.4375	1.4688	$x_5 = 1.4532$	-0.0429
6	1.4532	1.4688	$x_6 = 1.4610$	-0.0160
7.	1.4610	1.4688	$x_7 = 1.4649$	-0.0024
8	1.4649	1.4688	$x_8 = 1.4669$	0.0047
9	1.4649	1.4669	$x_9 = 1.4659$	0.0012
10	1.4649	1.4659	$x_{10} = 1.4654$	-0.0006

Hence the root of the given eqn is 1.4654

8. $x^3+x^2-1=0$.

Soln:-

Given $f(x) = x^3+x^2-1=0$

$f(0) = 0+0-1 = -1 = -ve$

$f(1) = 1+1-1 = 1 = +ve.$

The root lies b/w 0 & 1

$a=0 \quad b=1$

$\therefore x_0 = \frac{a+b}{2}$

n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ x^3+x^2-1
0	0	1	$x_0 = 0.5$	-0.6250
1	0.5	1	$x_1 = 0.75$	-0.0156
2	0.75	1	$x_2 = 0.8750$	0.4355
3	0.75	0.8750	$x_3 = 0.8125$	0.1965
4	0.75	0.8125	$x_4 = 0.7813$	0.0874
5	0.75	0.7813	$x_5 = 0.7657$	0.0352
6	0.7657	0.7657 ~~0.7813~~	$x_6 =$ 0.7579 ~~0.7735~~	0.0098 ~~0.0611~~
7	0.75	0.7579	$x_7 = 0.7540$	-0.0028
8	0.7540	0.7579	$x_8 = 0.7560$	0.0036
9	0.7540	0.7560	$x_9 = 0.7550$	0.0004
10	0.7540	0.7550	$x_{10} = 0.7545$	-0.0012

Hence the root of the given eqn is 0.7545

9. $x^3 - 3x - 5 = 0$

Soln:- Given $f(x) = x^3 - 3x - 5 = 0$

$f(0) = -5 = -ve.$

$f(1) = 1 - 3 - 5 = -7 = -ve$

$f(2) = 8 - 6 - 5 = 3 = -ve$

$f(3) = 27 - 9 - 5 = 13 = +ve$

$\therefore$ The roots lies b/w 2 & 3

n	$a_{(-ve)}$	$b_{(+ve)}$	$x_0 = \frac{a+b}{2}$	$f(x_0)$ x^3-3x-5
0	2	3	$x_0 = 2.5$	3.1250
1	2	2.5	$x_1 = 2.25$	-0.3594
2	2.25	2.5	$x_2 = 2.3750$	1.2715
3	2.25	2.3750	$x_3 = 2.3125$	0.4290
4	2.25	2.3125	$x_4 = 2.2813$	0.0287
5	2.25	2.2813	$x_5 = 2.2657$	-0.1664
6	2.2657	2.2813	$x_6 = 2.2735$	-0.0692
7	2.2735	2.2813	$x_7 = 2.2774$	-0.0203

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8	2.2774	2.2813	$x_8 = 2.2794$	0.0048
9	2.2774	2.2794	$x_9 = 2.2784$	-0.0078
10	2.2784	2.2794	$x_{10} = 2.2789$	-0.0015
11	2.2789	2.2794	$x_{11} = 2.2865$	0.0945
12	2.2789	2.2865	$x_{12} = 2.2827$	0.0464

Hence the root of the given equation is 2.2827

10. $x^3 - 5x + 3$

Soln:-

$$f(x) = x^3 - 5x + 3 = 0$$

$$f(0) = 3 = +ve$$

$$f(1) = 1-5+3 = -1 = -ve.$$

The roots lies b/w 0 & 1

$$a=0 \; ; \; b=1$$

$$\therefore x_0 = \frac{a+b}{2}$$

n	a (+ve)	b (-ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ x^3-5x+3
0	0	1	$x_0 = 0.5$	0.6250
1	0.5	1	$x_1 = 0.75$	-0.3281
2	0.5	0.75	$x_2 = 0.6250$	0.1191
3	0.6250	0.75	$x_3 = 0.6875$	-0.1125
4	0.6250	0.6875	$x_4 = 0.6563$	0.0012
5	0.6563	0.6875	$x_5 = 0.6719$	-0.0562
6	0.6563	0.6719	$x_6 = 0.6641$	-0.0276
7	0.6563	0.6641	$x_7 = 0.6602$	-0.0132
8	0.6563	0.6602	$x_8 = 0.6583$	-0.0062
9	0.6563	0.6583	$x_9 = 0.6573$	-0.0025
10	0.6563	0.6573	$x_{10} = 0.6568$	-0.0007

Hence the root of the given equation is 0.6568

11. $x^3+x-1=0$

Soln:-

Given $f(x) = x^3+x-1 = 0$

$f(0) = -1 = -ve$

$f(1) = 1+1-1 = +ve.$

The roots lies b/w 0 & 1

$$x_0 = \frac{a+b}{2}$$

$a = 0$; $b = 1$

n	a (-ve)	b (+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$ x^3+x-1
0	0	1	$x_0 = 0.5$	-0.3750
1	0.5	1	$x_1 = 0.75$	0.1719
2	0.5	0.75	$x_2 = 0.6250$	-0.1309
3	0.625	0.75	$x_3 = 0.6875$	0.0125
4	0.625	0.6875	$x_4 = 0.6563$	-0.3610
5	0.6563	0.6875	$x_5 = 0.6719$	-0.0248
6	0.6719	0.6875	$x_6 = 0.6797$	-0.0063
7	0.6797	0.6875	$x_7 = 0.6836$	0.0031
8	0.6797	0.6836	$x_8 = 0.6817$	-0.0015
9	0.6817	0.6836	$x_9 = 0.6827$	0.0009
10	0.6817	0.6827	$x_{10} = 0.6822$	-0.0003

11.	0.6822	0.6827	$x_{11} = 0.6825$	0.0.004
12.	0.6822	0.6825	$x_{12} = 0.6824$	0.0002
13.	0.6822	0.6824	$x_{13} = 0.6823$	-0.0001

Hence the root in the given eqn is 0.6823

6/7/19

METHOD OF FALSE POSITION(OR) REGULA FALSI METHOD:-

This is the oldest method for finding the real root of non-linear equation $f(x)=0$ and closely resembles the bisection method in this method also known as Regula falsi (or) the method of chords.

We choose two points a and b such that $f(a)$ and $f(b)$ are of opposites signs.

Hence the roots must lie b/w in this two points

$$x_r = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

1. Find the real root of the equation $f(x) = x^3 - 2x - 5 = 0$

Soln:-

Given $f(x) = x^3 - 2x - 5 = 0$

$f(0) = -5 = -ve$

$f(1) = 1-2-5 = -6 = -ve$

$f(2) = 8-4-5 = -1 = -ve$

$f(3) = 27-6-5 = 16 = +ve$.

$\therefore$ The root lies b/w 2 & 3

n	(-ve) a	b (+ve)	$x_n = \frac{af(b) - bf(a)}{f(b)-f(a)}$	$f(x_n)$ x^3-2x-5
1	2	3	$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2(16) - 3(-1)}{16+1} = \frac{35}{17} = 2.0588$	-0.3911
2	2.0588	3	$x_2 = \frac{2.0588 f(3) - 3f(2.0588)}{f(3) - f(2.0588)} = \frac{2.0588(16) - 3(-0.3911)}{16 - (-0.3911)} = \frac{32.9408 + 1.1733}{16.3911} = \frac{34.1141}{16.3911} = 2.0813$	-0.1468

3	2.0813	3	$x_3 = \dfrac{2.0813 f(3) - 3f(2.0813)}{f(3) - f(2.0813)}$ $= \dfrac{2.0813(16) - 3(-0.1468)}{16 - (-0.1468)}$ $= \dfrac{33.7412}{16.1468}$ $= 2.0897$	-0.0540
4	2.0897	3	$x_4 = \dfrac{2.0897 f(3) - 3f(2.0897)}{f(3) - f(2.0897)}$ $= \dfrac{2.0897(16) - 3(-0.0540)}{16 - (-0.0540)}$ $= \dfrac{33.5972}{16.0540}$ $= 2.0928$	-0.0195
5	2.0928	3	$x_5 = \dfrac{2.0928 f(3) - 3f(2.0928)}{f(3) - f(2.0928)}$ $= \dfrac{2.0928(16) - 3(-0.0195)}{16 - (-0.0195)}$ $= \dfrac{33.5433}{16.0195}$ $= 2.0939$	-0.0073

6	2.0939	3	$x_6 = \frac{2.0939 f(3) - 3 f(2.0939)}{f(3) - f(2.0939)}$ $= \frac{2.0939(-0.0807) - 3 f(-0.0073)}{16 - f(-0.0073)}$ $= \frac{33.5243}{16.0073}$ $= 2.0943$	-0.0028

Hence the root of the given equation is 2.0943

2. Given that the equation $x^{2.2} = 69$ has the root b/w 5 & 8 use the method of regula falsi to determine it

Soln:-

Given $f(x) = x^{2.2} - 69 = 0$

$f(0) = 0 - 69 = -ve$

$f(1) = 1 - 69 = 68 = +ve$

$f(2) = 2^{2.2} - 69 = 4.5948 - 69 = -64.4052$

$f(5) = 5^{2.2} - 69 = -34.5068 = -ve$

$f(6) = 6^{2.2} - 69 = -17.4851 = -ve$

$f(7) = 7^{2.2} - 69 = 3.3129 = +ve$

$f(8) = 8^{2.2} - 69 = 28.0059 = +ve$

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n	a (-ve)	b (+ve)	$x_n = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$	$x^{2.2} - 69$ $f(x_n)$
1	5	8	$x_1 = \dfrac{5 f(8) - 8 f(5)}{f(8) - f(5)}$ $= \dfrac{5(28.0059) - 8(-34.5068)}{28.0059 + 34.5068}$ $= \dfrac{416.0839}{62.5127}$ $= 6.6560$	-4.2754
2.	6.6560	8	$x_2 = \dfrac{6.6560 f(8) - 8 f(6.6560)}{f(8) - f(6.6560)}$ $= \dfrac{6.6560(28.0059) - 8(-4.2754)}{28.0059 + 4.2754}$ $= \dfrac{220.6105}{32.2813}$ $= 6.8340$	-0.4062
3.	6.8340	8	$x_3 = \dfrac{6.8340 f(8) - 8 f(6.8340)}{f(8) - f(6.8340)}$ $= \dfrac{6.8340(28.0059) - 8(-0.4062)}{(28.0059) + 0.4062}$ $= \dfrac{194.6419}{28.4121}$ $= 6.8507$	-0.0369

4	6.8507	8	$x_4 = \dfrac{6.8507 f(8) - 8f(6.8507)}{f(8) - f(6.8507)}$ $= \dfrac{6.8507(28.0059)+8(0.0369)}{28.0059+0.0369}$ $= \dfrac{192.1552}{28.0428}$ $= +6.8522$	-0.0037

Hence the root of the given equation is 6.8522.

1) $x \log_{10} x = 1.2$ $(1, 2, 3) \to 2.742$.

2) $xe^{x} = 3$ $(0.1, 1.5) \to 1.048$.

8/7/19

7. Solve the equation $x \tan x = -1$ by regula falsi method starting with $x_0 = 2.5$ and $x_1 = 3.0$ correct to three decimal places.

Soln:-

Given $f(x) = x \tan x + 1 = 0$

$f(2.5) = 2.5 \tan 2.5 + 1 = -0.8676$

$f(3) = 3 \tan 3 + 1 = 0.5724$

The roots lies b/3 2.5 & 3

$a = 2.5 \quad b = 3$

n	(-ve) a	b (+ve)	$x_r = \frac{a f(b) - b f(a)}{f(b) - f(a)}$	$f(x_0)$
1	2.5	3	$x_1 = \frac{2.5 f(3) - 3 f(2.5)}{f(3) - f(2.5)}$	
			$= \frac{2.5(0.5724) - 3(0.8676)}{0.5724 + 0.8676}$	0.0082
			$= \frac{4.0338}{1.44}$	
			$= 2.8013$	
2	2.5	2.8013	$x_2 = \frac{2.5 f(2.8013) - 2.8013 f(2.5)}{\cancel{2.8013}\ 0.0082 - f(2.5)}$	
			$= \frac{2.5(0.0082) - 2.8013(-0.8676)}{\cancel{2.8013}\ 0.0082 + 0.8676}$	0.0003
			$= \frac{2.4509}{\cancel{3.6689}\ 0.8758}$	
			$= 2.7985$	
3	2.5	2.7985	$x_3 = \frac{2.5 f(2.7985) - 2.7985 f(2.5)}{f(2.7985) - f(2.5)}$	
			$= \frac{2.5(0.0003) - 2.7985(-0.8676)}{0.0003 + 0.8676}$	0.0000
			$= \frac{2.4287}{0.8679} = 2.7984$	

Hence the root of the given equation is 2.7984

4. $x \log_{10} x = 1.2$

Soln:-

Given $x \log_{10} x - 1.2 = 0$

$f(1) = 1 \log_{10} 1 - 1.2 = -1.2 = -ve$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979 = -ve$

$f(3) = 3 \log_{10} 3 - 1.2 = 0.2314 = +ve.$

The roots lies b/w 2 & 3

$a = 2 \; ; \; b = 3$

$$x_r = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

n	$a_{(-ve)}$	$b_{(+ve)}$	$x_r = \frac{a f(b) - b f(a)}{f(b) - f(a)}$	$f(x_0)$ $x \log_{10} x = 1.2$
1	2	3	$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)}$ $= \frac{2(0.2314) + 3(0.5979)}{0.2314 + 0.5979}$ $= \frac{2.2565}{0.8293} = 2.7210$	-0.0171
2	2.7210	3	$x_2 = \frac{2.7210(0.2314) + 3(0.0171)}{0.2314 + 0.0171}$ $= \frac{0.6809}{0.2485} = 2.7400$	-0.0006

3	2.7400	3	$x_3 = \dfrac{2.7400(0.2314)+3(0.0006)}{0.2314+0.0006} = \dfrac{0.6358}{0.2320} = 2.7405$	-0.0001
4	2.7405	3	$x_4 = \dfrac{2.7405(0.2314)+3(0.0001)}{0.2314+0.0001} = \dfrac{0.6345}{0.2315} = 2.7408$	0.0001

Hence the root of given eqn is

2.7408

5 $xe^x = 3$

ie'

Soln:-

Given $f(x) = xe^x - 3 = 0$

$f(0) = -3 = -ve$

$f(1) = -0.2817 = -ve$

$f(1.5) = 3.7255 = +ve.$

The root lies b/w 1 & 1.5

$a = 1$; $b = 1.5$

n	$a_{(-ve)}$	$b_{(+ve)}$	$x_r = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x_0)$ $xe^x - 3 = 0$
1.	1	1.5	$x_1 = \frac{1(3.7225) + 1.5(0.2817)}{3.7225 + 0.2817}$ $= \frac{4.1451}{4.0042}$ $= 1.0352$	-0.0852
2	1.0352	1.5	$x_2 = \frac{1.0352(3.7225) + 1.5(0.0852)}{3.7225 + 0.0852}$ $= \frac{3.9813}{3.8077}$ $= 1.0456$	-0.0252
3	1.0456	1.5	$x_3 = \frac{1.0456(3.7225) + 1.5(0.0252)}{3.7225 + 0.0252}$ $= \frac{3.9300}{3.7477}$ $= 1.0486$	-0.0077

Hence the root of the given eqn is

1.0486

9/7/19 Newton - Raphson Method:-

When the derivative of $f(x)$ is a simple expression and easily found the roots of $f(x)=0$ can be computed rapidly by a process called the Newton - Raphson Method.

This method is a particular form the iteration method and can be derived as follows.

Let $x = x_0$ be an approximate value of one root of equation $f(x)=0$.

If $x = x_1$ is the exact root then $f(x_1)=0$. also $\rightarrow$ ①

Also, $x_1 - x_0$ will be small.

Let $x_1 - x_0 = h$ then $x_1 = x_0 + h \rightarrow$ ②

Putting ② in ①

$$f(x_0+h) = 0$$

ie,

$$f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0 \rightarrow (3)$$

(Taylor's theorem)

Since h is small we can omit h^2 and higher powers of h and from (3) we have

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow h = \frac{-f(x_0)}{f'(x_0)} \rightarrow (4)$$

Putting this value of h by using (4) in (2) we get.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \rightarrow (5)$$

The value x_1 given by equation (5) will be a closer approximation to the root of $f(x) = 0$ than x_0.

Similarly starting with x_1 we can get a better approximation x_2 to the root given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$ and so on.

Thus we get the general formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad n = 0, 1, 2 \dots$$

is known as Newton Raphson formula

Convergence of Newton's method and rate of convergence :-

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = \phi(x_n) \rightarrow ①$$

Now above equation ~~source~~ how that this is really an iteration method.

The general form eqn ① is

$$x = \phi(x) \rightarrow ②$$

By using the convergence condition of an iteration method in eqn ② we get

If $\phi(x)$ is converges then $|\phi'(x)| < 1$

here

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\phi(x) = 1 - \left[\frac{f'(x) \cdot f'(x) - f(x) f''(x)}{(f'(x))^2}\right]$$

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$$= \frac{(f'(x))^2 - (f'(x))^2 + f(x)\, f''(x)}{f'(x)^2}$$

$$= \frac{f(x)\, f''(x)}{f'(x)^2}$$

① $x = e^{-x}$

② $x^{\sin 2} - 4 = 0$

③ $x^3 - 5x + 3 = 0$

④ $x^4 + x^2 - 80 = 0$

⑤ $x^3 + 3x^2 - 3 = 0$

⑥ $x + \log x = 2$

$$|\phi'(x)| = \left| \frac{f(x) \cdot f''(x)}{(f'(x))^2} \right|$$

Since, $|\phi'(x)| < 1$

$$\therefore \left| \frac{f(x) \cdot f''(x)}{(f'(x))^2} \right| < 1$$

$$|f(x) \cdot f''(x)| < |(f'(x))^2|$$

Hence Newton formula converges if

$$|f(x) \cdot f''(x)| < \{f'(x)\}^2$$

1. Using Newton Raphson Method find correct to 4 decimal places, the root of the eqn

$x^3 - 2x - 5 = 0$

Soln:-

Given $f(x) = x^3 - 2x - 5 = 0$

$$f'(x) = 3x^2 - 2$$

By newton raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$f(0) = -5 = -ve$

$f(1) = 1-2-5 = -6 = -ve$

$f(2) = 8-4-5 = -1 = -ve$

$f(3) = 27-6-5 = 16 = +ve$

Here, $|f(2)| = |-1| = 1$

$|f(3)| = |16| = 16$

so, $|f(2)| < |f(3)|$

Let $x_0 = 2$

The root lies b/w 2 & 3

Let us assume that $x_0 = 2.6$

n	x_n	$f(x_n)$ x^3-2x-5	$f'(x_n)$ $3x^2-2$	$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)}$
~~0~~	~~$x_0 = 2.6$~~	~~7.3760~~	~~18.2800~~	~~$x_1 = 2.6 - \frac{7.3760}{18.2800}$~~ ~~$x_1 = 2.1965$~~
0	$x_0 = 2$	$-1.$	10	$x_1 = 2 + \frac{1}{10}$ $x_1 = 2.1000$
1	$x_1 = 2.1$	0.0610	11.23	$x_2 = 2.1 - \frac{0.0610}{11.23}$ $= 2.0946$
2	$x_2 = 2.0946$	0.0005	11.1620	$x_3 = 2.0946 - \frac{0.0005}{11.1620}$ $= 2.0946$

$\therefore$ The root of the given eqn with correct to 4 decimal places is 2.0946

H.W

1. $x = e^{-x}$

Soln:-

Given $f(x) = x - e^{-x} = 0$

$f'(x) = 1 + e^{-x}$

By newton raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(0) = -1 = -ve$

$f(1) = 1 - e^{-1} = 1 - 0.3679 = 0.6321 = +ve$

$f(2) = 2 - e^{-2} = 1.8647 = +ve$

$f(3) = 3 - e^{-3} =$

$|f(0)| = |-1| = 1$

$|f(1)| = |0.6321| = 0.6321$

$\therefore |f(1)| < |f(0)|$

$0.6321 < 1$

~~Let $x_0 = 1$~~

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	$x_0 = 1$	0.6321	1.3679	$x_1 = 1 - \frac{0.6321}{1.3679}$ $x_1 = 0.5379$
1	0.5379	-0.0461	1.5840	$x_2 = 0.5379 + \frac{0.0461}{1.5840}$ $x_2 = 0.5670$
2	0.5670	-0.0002	1.5672	$x_3 = 0.5670 + \frac{0.0002}{1.5672}$ $x_3 = 0.5671$
3	0.5671	-0.0001	1.5672	$x_4 = 0.5671 + \frac{0.0001}{1.5672}$ $x_4 = 0.5672$
4	0.5672	0.0001	1.5671	$x_5 = 0.5672 + \frac{0.0001}{1.5671}$ $x_5 = 0.5671$

$\therefore$ Hence the root of the given equation is 0.5671 with 4 decimal places.

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2. $x^3-5x+3=0$

Soln:-

Given $f(x)= x^3-5x+3=0$

$f'(x)= 3x^2-5$

By newton raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(0) = +3 = +ve$

$f(1) = 1-5+3 = -1 = -ve$

$f(2) = 8-10+3 = 1 = +ve$

The root lies b/w 0 & 1

Here $|f(0)| = |3| = 3$

$|f(1)| = |-1| = 1$

(ie) $|f(0)| > |f(1)|$

Let $x_0 = 1$

n	x_n	$f(x_n)$ x^3-5x+3	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f'(x_n)$ $3x^2-5$
0	$x_0=1$	-1	$x_1 = 1-\frac{-1}{-2}$ $x_1 = 0.5$	-2
1	$x_1=0.5$	0.6250	$x_2 = 0.5+\frac{0.6250}{4.2500}$ $x_2 = 0.6471$	-4.2500
2	$x_2=0.6471$	0.0355	$x_3 = 0.6471+\frac{0.0355}{3.7438}$ $x_3 = 0.6566$	-3.7438
3	$x_3=0.6566$	0.0001	$x_4 = 0.6566+\frac{0.0001}{3.7066}$ $x_4 = 0.6566$	-3.7066

$\therefore$ The root of the given equation with correct to 4 decimal places is 0.6566

3. $x^4+x^2-80=0$

Soln:

Given $f(x) = x^4+x^2-80=0$

$$f'(x) = 4x^3+2x$$

By newton raphson formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(0) = -80 = -ve$

$f(1) = 1+1-80 = -78 = -ve$

$f(2) = 16+4-80 = -60 = -ve$

$f(3) = 81+9-80 = 10 = +ve$.

The root lies b/w ② & ③.

Here $|f(2)| = |-60| = 60$

$|f(3)| = |10| = 10$.

(ie) $|f(2)| > |f(3)|$

Let $x_0 = 3$

x^4+x^2-80 $\quad 4x^3+2x$ $\quad x^4+x^2-80$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	3	10	114	$x_1 = 3 - \frac{10}{114}$ $x_1 = 2.9123$
1	2.9123	0.4172	104.6272	$x_2 = 2.9123 - \frac{0.4172}{104.6272}$ $x_2 = 2.9083$
2.	2.9083	-0.0005	104.2126	$x_3 = 2.9083 + \frac{0.0005}{104.2126}$ $x_3 = 2.9083$

∴ The root ~~is~~ of the given equation with correct to 4 decimal places is 2.9083

4: $x^3 + 3x^2 - 3 = 0$

Soln:-

Given $f(x) = x^3 + 3x^2 - 3 = 0$.

$$f'(x) = 3x^2 + 6x$$

By newton raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(0) = -3 = -ve$$

$$f(1) = 1 + 3 - 3 = 1 = +ve.$$

The root lies b/w 0 & 1

Here $|f(0)| = |-3| = 3$

$|f(1)| = 1 = 1$.

(ie) $|f(0)| > |f(1)|$

Let $x_0 = 1$

n	x_n	$f(x_n)$ x^3+3x^2-3	$f'(x_n)$ $3x^2+6x$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	$x_0=1$	1	9	$x_1 = 1 - \frac{1}{9}$ $x_1 = 0.8889$
1	$x_1=0.8889$	0.0728	7.7038	$x_2 = 0.8889 - \frac{0.0728}{7.7038}$ $x_2 = 0.8795$
2	$x_2=0.8795$	0.0009	7.5976	$x_3 = 0.8795 - \frac{0.0009}{7.5976}$ $x_3 = 0.8794$
3	$x_3=0.8794$	0.0001	7.5964	$x_4 = 0.8794 - \frac{0.0001}{7.5964}$ $x_4 = 0.8794$

$\therefore$ The root of the given equation with correct to 4 decimal places is 0.8794

5. $x + \log x = 2$

Soln:-

Given $f(x) = x + \log x - 2 = 0$

$f'(x) = 1 + \frac{1}{x}$

By newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(0) = -2 = -ve$

$f(1) = 1 + 0 - 2 = -1 = -ve$

$f(2) = 2 + 0.3010 - 2 = 0.3010$

Here $|f(1)| = |-1| = 1$

$|f(2)| = |0.3010| = 0.3010$

(ie) $|f(1)| > |f(2)|$

Let $x_0 = 0.3010 = 2$

n	x_n	$f(x_n)$ $x+\log x-2$	$f'(x_n)$ $1+1/x$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	2	0.3010	1.5	$x_1 = 2 - \frac{0.8010}{1.5} = 1.7993$
1	1.7993	0.0544	1.5558	$x_2 = 1.7993 - \frac{0.0544}{1.5558} = 1.7643$
2	1.7643	0.0109	1.5668	$x_3 = 1.7643 - \frac{0.0109}{1.5668} = 1.7573$
3	1.7573	0.0021	1.5691	$x_4 = 1.7573 - \frac{0.0021}{1.5691} = 1.7560$
4	1.7560	0.0005	1.5695	$x_5 = 1.7560 - \frac{0.0005}{1.5695} = 1.7557$
5	1.7557	0.0002	1.5696	$x_6 = 1.7557 - \frac{0.0002}{1.5696} = 1.7556$
6.	1.7556	0.0000	1.5696	$x_7 = 1.7556 - \frac{0.0000}{1.5696} = 1.7556.$

∴ The root of the given equation with correct to 4 decimal places is 1.7556.

5. $x^{\sin 2} - 4 = 0$

Soln:-

Given $f(x) = x^{\sin 2} - 4 = 0$

$$f'(x) = \sin 2 \, x^{\sin 2 - 1}$$

By newton raphson formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(0) = -4 = -ve$

$f(1) = -3 = -ve$

$f(2) = -2.1219 = -ve$

$f(3) = -1.2845 = -ve$

$f(4) = -0.4726 = -ve$

$f(5) = 0.3209 = +ve.$

Here

$|f(4)| = |-0.4726| = 0.4726.$

$|f(5)| = |0.3209| = 0.3209.$

$|f(4)| > |f(5)|$

$x^{\sin 2} - 4$ $\quad \sin 2 \cdot x^{(\sin 2 - 1)}$ $\quad x^{(\sin 2 - 1)} = 0.8642$ ...

n	(x_n)	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	5	0.3209	0.7858	$x_1 = 5 - \frac{0.3209}{0.7858}$ $x_1 = 4.5916$
1	4.5916	-0.0013	0.7919	$x_2 = 4.5916 + \frac{0.00..}{0.79..}$ $x_2 = 4.5932$
2	4.5932	0.0000	0.7919	$x_3 = 4.5932 - \frac{0.0000}{0.7919}$ $x_3 = 4.5932$

$\therefore$ The root of the given equation with correct to 4 decimal places is 4.5932

16/7/19

1. Use the newton raphson method to find the root of the equation $x \sin x + \cos x = 0$

Soln:-

Given $f(x) = x \sin x + \cos x = 0$

$f'(x) = x \cos x + \sin x - \sin x$

$f'(x) = x \cos x$.

n	(x_n)	$f(x_n)$ $x\sin x + \cos x$	$f'(x_n)$ $x\cos x$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	π	-1	-3.1416	$x_1 = \pi - \frac{1}{3.1416}$ $x_1 = 2.8233$
1	2.8233	-0.0662	-2.6815	$x_2 = 2.8233 - \frac{0.0662}{2.6815}$ $x_2 = 2.7986$
2	2.7986	-0.0006	-2.6356	$x_3 = 2.7986 - \frac{0.0006}{2.6356}$ $x_3 = 2.7984$
3	2.7984	0.0000	-2.6352	$x_4 = 2.7984 + \frac{0.0000}{2.6352}$ $x_4 = 2.7984$

$\therefore$ The root of the given equation with correct to 4 decimal places is 2.7984.

2. Solve $\sin x = 1 - x$.

Soln:-

Given $f(x) = \sin x - 1 + x$

$f'(x) = \cos x + 1$.

n	(x_n)	$f(x_n)$ ($\sin x - 1 + x$)	$f'(x_n)$ ($\cos x + 1$)	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	$\pi/3$	0.9132	1.5	$x_1 = \pi/3 - \frac{0.9132}{1.5}$ $x_1 = 0.4384$
1	0.4384	−0.1371	1.9054	$x_2 = 0.4384 + \frac{0.1371}{1.9054}$ $x_2 = 0.5104$
2.	0.5104	−0.0011	1.8725	$x_3 = 0.5104 + \frac{0.0011}{1.8725}$ $x_3 = 0.5110$
3.	0.5110	0.0000	1.8723	$x_4 = 0.5110$

∴ The roots of the given equation with correct to 4 decimal places is 0.5110

7. Solve $4(x-\sin x)=1$

Soln:-

Given $4(x-\sin x)=1$

$$f(x) = 4x - 4\sin x - 1 = 0$$

$$f'(x) = 4 - 4\cos x$$

n	(x_n)	$f(x_n)$ $4x-4\sin x-1$	$f'(x_n)$ $4-4\cos x$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	$\pi/2$	1.2832	4	$x_1 = \frac{\pi}{2} - \frac{1.2832}{4}$ $x_1 = 1.2500$
1	1.25	0.2041	2.7387	$x_2 = 1.25 - \frac{0.2041}{2.7387}$ $x_2 = 1.1755$
2	1.1755	0.0105	2.4597	$x_3 = 1.1755 - \frac{0.0105}{2.4597}$ $x_3 = 1.1712$
3.	1.1712	-0.0001	2.4438	$x_4 = 1.1712 + \frac{0.0001}{2.4438}$ $x_4 = 1.1712.$

$\therefore$ The roots of the given equation with correct to 4 decimal places is 1.1712

4). $x - \cos x = 0$.

Soln:-

Given $f(x) = x - \cos x = 0$

$f'(x) = 1 + \sin x$

n	(x_n)	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	$\pi/2$	1.5708	2	$x_1 = \pi/2 - \frac{1.5708}{2}$ $x_1 = 0.7854$
1	0.7854	0.0783	1.7071	$x_2 = 0.7854 - \frac{0.0783}{1.7071}$ $x_2 = 0.7395$
2	0.7395	0.0007	1.6739	$x_3 = 0.7395 - \frac{0.0007}{1.6739}$ $x_3 = 0.7931$
3.	0.7931	0.0915	1.7125	$x_4 = 0.7931 - \frac{0.0915}{1.7125}$ $x_4 = 0.7397$
4	0.7397	0.0010	1.6741	$x_5 = 0.7397 - \frac{0.0010}{1.6741}$ $x_5 = 0.7391$
5	0.7391	0.0000	1.6736	$x_6 = 0.7391 - \frac{0.0000}{1.6736}$ $x_6 = 0.7391$

$\therefore$ The root of the gn eqn with correct to 4 decimal places is 0.7391

5) $\sin x = x/2$

Soln:-

Given $f(x) = \sin x - x/2 = 0$

$f'(x) = \cos x - 1/2$

n	(x_n)	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	$\pi/2$	0.2146	-0.5	$x_1 = \pi/2 + \frac{0.2146}{0.5}$ $x_1 = 2$
1	2	-0.0907	-0.9161	$x_2 = 2 - \frac{0.0907}{0.9161}$ $x_2 = 1.9010$
2	1.9010	-0.0045	-0.8242	$x_3 = 1.9010 - \frac{0.0045}{0.8242}$ $x_3 = 1.8955$
3	1.8955	0.0000	-0.8190	$x_4 = 1.8955 + \frac{0.0000}{0.8190}$ $x_4 = 1.8955$

∴ The root of the given eqn with the correct to 4 decimal places 1.8955

17/7/2019

ITERATION METHOD:-

To describe this method for finding the roots of the equation $f(x)=0 \rightarrow (1)$ we rewrite this equation in the form $x=\phi(x) \rightarrow (2)$

There are many ways of doing this

For example:-

$x^3+x^2-1=0$ can be expressed as either of the form:

$$x=(1-x^2)^{1/3} \text{ or } x=\sqrt{1-x^3} \text{ or } x=\frac{1}{x^2+x} \ldots$$

Let x_0 be an approximate value of desired root and substituting it for 'x' on the right side of equation (2) we obtain first approximate $x_1=\phi(x_0)$.

The successive approximations are then given by $x_2=\phi(x_1)$, $x_3=\phi(x_2)$, ... $x_n=\phi(x_{n-1})$

The sequence of the approximations $x_0, x_1, x_2, \ldots$ does not always converge.

Let $\in$ be a root of $f(x)=0$ and let I be an interval containing the point ϵ.

Let $\phi(x)$ & $\phi'(x)$ be continuous in I, where $\phi(x)$ is defined by the equation $x = \phi(x_0)$ which equivalent to $f(x) = 0$. Then if $|\phi'(x)| < 1$, for all x in I.

The sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ defined by $x_{n+1} = \phi(x_n)$ converges to the root ϵ provided that the initial approximation x_0 is choosen in I.

PROBLEM:-

1) Find the real root of the equation $x^3+x^2-1=0$ on the interval $[0,1]$ with an accuracy of 10^{-4}.

Soln:

Given $f(x) = x^3+x^2-1 = 0$ $\qquad x^3+x^2-100=0$

$$x^3+x^2 = 1 \qquad x^3+x^2=100$$

$$x^2(1+x) = 1 \qquad x^2(1+x)=100$$

$$x^2 = \frac{1}{1+x} \qquad x^2 = \frac{100}{1+x}$$

$$\therefore x = \frac{1}{\sqrt{1+x}} \qquad x = \frac{10}{\sqrt{1+x}}$$

$$\phi x = \frac{10}{\sqrt{1+x}}$$

$$\therefore \phi(x) = \frac{1}{\sqrt{1+x}} \qquad \frac{u}{v} \qquad \frac{vu'-uv'}{v^2}$$

$$\phi'(x) = \frac{\sqrt{1+x}\,(0) - 1 \cdot \frac{1}{2}\left(\sqrt{1+x}\right)}{\left(\sqrt{1+x}\right)^2}$$

$$= \frac{-1}{2} \cdot \frac{1}{(1+x)^{3/2}}$$

$$|\phi'(x)| = \frac{1}{2} \cdot \frac{1}{(1+x)^{3/2}}$$

$$\therefore |\phi'(x)| < 1 \text{ on } [0,1]$$

$\frac{1}{\sqrt{1+x}}$

Let $x_0 = 0.6$

n	x_n	$x_{n+1} = \phi(x_n)$ $\quad x = \frac{1}{\sqrt{1+x}}$
0	$x_0 = 0.6$	$x_1 = \frac{1}{\sqrt{1+0.6}}$ $x_1 = 0.7906$
1	$x_1 = 0.7906$	$x_2 = 0.7473$ $\quad x_2 = \frac{1}{\sqrt{1+0.7906}}$
2	$x_2 = 0.7473$	$x_3 = 0.7565$ $\quad x_3 = \frac{1}{\sqrt{1+0.7473}}$
3	$x_3 = 0.7565$	$x_4 = 0.7545$ $\quad x_4 = \frac{1}{\sqrt{1+0.7565}}$
4	$x_4 = 0.7545$	$x_5 = 0.7550$ $\quad x_5 = \frac{1}{\sqrt{1+0.7545}}$
5	$x_5 = 0.7550$	$x_6 = 0.7549$ $\quad x_6 = \frac{1}{\sqrt{1+0.7550}}$
6	$x_6 = 0.7549$	$x_7 = 0.7549$ $\quad x_7 = \frac{1}{\sqrt{1+0.7549}}$

2. $xe^x = 1$, the root lies b/w 0 & 1

Soln:-

$$f(x) = xe^x - 1 = 0$$

$$x = \frac{1}{e^x} \Rightarrow \phi(x) = e^{-x}$$

$$\phi'(x) = -e^{-x}$$

$$|\phi'(x)| = e^{-x}$$

$|\phi'(x)| < 1$ on $[0, 1]$

Let $x_0 = 0.6$

n	x_n	$x_{n+1} = \phi(x_n)$ $\frac{1}{e^x}$
0	$x_0 = 0.6$	$x_1 = e^{-0.6}$ $x_1 = 0.5488$
1	$x_1 = 0.5488$	$x_2 = e^{-0.5488}$ $x_2 = 0.5776$
2	$x_2 = 0.5776$	$x_3 = e^{-0.5776}$ $x_3 = 0.5612$
3.	$x_3 = 0.5612$	$x_4 = 0.5705$
4.	$x_4 = 0.5705$	$x_5 = 0.5652$
5	$x_5 = 0.5652$	$x_6 = 0.5682$

6.	$x_6 = 0.5682$	$x_7 = 0.5665$
7.	$x_7 = 0.5665$	$x_8 = 0.5675$
8.	$x_8 = 0.5675$	$x_9 = 0.5669$
9.	$x_9 = 0.5669$	$x_{10} = 0.5673$
10.	$x_{10} = 0.5673$	$x_{11} = 0.5671$
11.	$x_{11} = 0.5671$	$x_{12} = 0.5672$
12.	$x_{12} = 0.5672$	$x_{13} = 0.5671$

The roots of given eqn is 0.5671

3. $2x = \cos x + 3$

Soln:-

Given $f(x) = 2x - \cos x + 3$

$$2x = \cos x + 3$$

$$x = \frac{\cos x + 3}{2}$$

$$\therefore \phi(x) = \frac{\cos x + 3}{2}$$

$$\phi'(x) = \frac{-\sin x}{2}$$

$$|\phi'(x)| = \frac{\sin x}{2} = 0.5$$

$$|\phi'(x)| < 1$$

$$x_0 = \pi/2$$

$\frac{(\cos(x)+3)}{2}$

n	x_n	$x_{n+1} = \phi(x_n)$
0	$x_0 = \pi/2$	$x_1 = \frac{\cos(\pi/2)+3}{2} = 1.5$
1	$x_1 = 1.5$	$x_2 = 1.5354$
2	$x_2 = 1.5354$	$x_3 = 1.5177$
3.	$x_3 = 1.5177$	$x_4 = 1.5265$
4.	$x_4 = 1.5265$	$x_5 = 1.5221$
5.	$x_5 = 1.5221$	$x_6 = 1.5243$
6.	$x_6 = 1.5243$	$x_7 = 1.5232$
7.	$x_7 = 1.5232$	$x_8 = 1.5238$
8.	$x_8 = 1.5238$	$x_9 = 1.5235$
9.	$x_9 = 1.5235$	$x_{10} = 1.5236$
10.	$x_{10} = 1.5236$	$x_{11} = 1.5236$
11.	$x_{11} = 1.5236$	$x_{12} = 1.5236$

The roots of given eqn is 1.5236.

4. $\cos x = 3x - 1$

Soln:-

Given $f(x) = \cos x - 3x + 1 = 0$

$$3x = \cos x + 1$$

$$x = \frac{\cos x + 1}{3}$$

$$\phi(x) = \frac{\cos x + 1}{3}$$

$$\phi'(x) = \frac{-\sin x}{3}$$

$$|\phi'(x)| = \frac{\sin x}{3}$$

$$|\phi'(x)| < 1$$

$$x_0 = \pi/2$$

n	x_n	$x_{n+1} = \phi(x_n)$
0	$x_0 = \pi/2$	$x_1 = \frac{\cos(\pi/2)+1}{3}$ $x_1 = 0.3333$
1	$x_1 = 0.3333$	$x_2 = 0.6483$
2	$x_2 = 0.6483$	$x_3 = 0.5990$
3.	$x_3 = 0.5990$	$x_4 = 0.6086$
4	$x_4 = 0.6086$	$x_5 = 0.6068$
5	$x_5 = 0.6068$	$x_6 = 0.6072$
6.	$x_6 = 0.6072$	$x_7 = 0.6071$
7.	$x_7 = 0.6071$	$x_8 = 0.6071$

The roots of the given equation 0.6071

5. $x = \frac{1}{(x+1)^2}$. the root lies b/w 0 & 1.

Soln:-

Given $f(x) = x - \frac{1}{(x+1)^2}$

$$x = \frac{1}{(x+1)^2}$$

$$\phi(x) = \frac{1}{(x+1)^2}$$

$$\phi'(x) = \frac{-2}{(x+1)^3}$$

$$|\phi'(x)| < 1$$

$$x_0 = 0.6$$

n	x_n	$x_{n+1} = \phi(x_n)$
1	$x_0 = 0.6$	$x_1 = \frac{1}{(0.6+1)^2}$ $= 0.3906$
2	$x_1 = 0.3906$	$x_2 = 0.5171$
3	$x_2 = 0.5171$	$x_3 = 0.4345$
4	$x_3 = 0.4345$	$x_4 = 0.4860$
5	$x_4 = 0.4860$	$x_5 = 0.4529$
6	$x_5 = 0.4529$	$x_6 = 0.4737$
7	$x_6 = 0.4737$	$x_7 = 0.4604$
8	$x_7 = 0.4604$	$x_8 = 0.4689$

9.	$x_8 = 0.4689$	$x_9 = 0.4635$
10	$x_9 = 0.4635$	$x_{10} = 0.4669$
11	$x_{10} = 0.4669$	$x_{11} = 0.4647$
12.	$x_{11} = 0.4647$	$x_{12} = 0.4661$
13.	$x_{12} = 0.4661$	$x_{13} = 0.4652$
14.	$x_{13} = 0.4652$	$x_{14} = 0.4658$
15.	$x_{14} = 0.4658$	$x_{15} = 0.4654$
16.	$x_{15} = 0.4654$	$x_{16} = 0.4657$
17.	$x_{16} = 0.4657$	$x_{17} = 0.4655$
18.	$x_{17} = 0.4655$	$x_{18} = 0.4656$
19.	$x_{18} = 0.4656$	$x_{19} = 0.4656$

The roots of the given equation is 0.4656

24/7/19 Solutions of linear algebraic equations :-

Gauss elimination method : (Direct method)

Basically the most effective direct solution techniques currently being used are applications of gauss elimination methods.

In this method the given system is transformed into an equivalent system with upper triangular coefficients matrix. That is a matrix in which elements below the diagonal elements are zero which can be solved by Back substitution.

25/7/19

1. Solve the system of Equation by Gauss elimination method $x+2y+z=3$

$2x+3y+3z=10$

$3x-y+2z=13$

Soln:-

Matrix form the given system is

$$AX=B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

Now the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_2 \to R_2 - 2R_1$

$$\begin{array}{cccc} 2 & 3 & 3 & +10 \\ -2 & -4 & -2 & -6 \\ \hline 0 & -1 & -1 & 4 \end{array}$$

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & +1 & +4 \\ 3 & -1 & 2 & 13 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & +1 & 4 \\ 0 & -7 & -1 & -4 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$R_3 \to R_3 - 3R_1$

$$\begin{array}{cccc} 3 & -1 & 2 & 13 \\ -3 & -6 & -3 & -9 \\ \hline 0 & -7 & -1 & -4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -8 & -28 \end{bmatrix} \quad \begin{array}{l} R_2 \to -R_2 \\ R_3 \to R_3 + 7R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -7 & -1 & -4 \\ 0 & 7 & -7 & -24 \\ \hline 0 & 0 & 8 & 28 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad R_3 \to \frac{R_3}{-8}$$

$R_3 \to \frac{R_3}{-8}$

By back substitution method

$$x + 2y + z = 3$$

$$y - z = -4$$

$$\boxed{z = 3}$$

$$y - 3 = -4$$

$$y = -4 + 3$$

$$\boxed{y = -1}$$

$$x + 2y + 3 = 3$$

$$x - 2 + 3 = 3$$

$$x + 1 = 3$$

$$x = 3 - 1$$

$$\boxed{x = 2}$$

2) $10x+y+z=12$

$x+10y+z=12$

$x+y+10z=12$

Soln:-

Matrix form the given system is

$$AX=B$$

$$\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$$

Now the argumented matrix is

$$\begin{bmatrix} 10 & 1 & 1 & 12 \\ 1 & 10 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{bmatrix}$$

$$[A,B] \sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 1 & 10 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{bmatrix} \quad R_2 \to R_2 - 10R_1$$

1 10 1 12

$$\sim \begin{bmatrix} 1 & 10 & 1 & 12 \\ 10 & 1 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 10 & 1 & 12 \\ 0 & -99 & -9 & -108 \\ 0 & -9 & 9 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 10R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 10 & 1 & 12 \\ 0 & -99 & -9 & -108 \\ 0 & 0 & 108 & 108 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

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$$\sim \begin{bmatrix} 1 & 10 & 1 & 12 \\ 0 & 1 & 9/99 & 108/99 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad R_2 \to -\frac{R_2}{99}$$

$$\sim \begin{bmatrix} 1 & 10 & 1 & 12 \\ 0 & 1 & 1/11 & 12/11 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad R_3 \to \frac{R_3}{108}$$

By back substitution method,

$$x + 10y + z = 12$$

$$y + \frac{1}{11} z = \frac{12}{11}$$

$$\boxed{z = 1}$$

$$y + \frac{1}{11} = \frac{12}{11}$$

$$y = \frac{12}{11} - \frac{1}{11}$$

$$\boxed{y = 1}$$

$$x + 10 + 1 = 12$$

$$x = 12 - 11$$

$$\boxed{x = 1}$$

Assignment NO: 02

1. $x - y + z = 1$

$-3x + 2y - 3z = -6$

$2x - 5y + 4z = 5$

Soln:-

Matrix form of the given system is

$$AX = B$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix}$$

Now the augmented matrix,

$$[A, B] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{bmatrix} \qquad R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -3 & 2 & 3 \end{bmatrix} \qquad R_3 \to R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 12 \end{bmatrix} \qquad R_3 \to \frac{R_3}{2}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

By back substitution method,

$$x - y + z = 1$$

$$\boxed{y = 3}$$

$$\boxed{z=6}$$

$$x-3+6=1$$

$$x=1-3$$

$$\boxed{x=-2}$$

2. $28x+4y-z=32$

$x+3y+10z=24$

$2x+17y+4z=35$

Soln:

Matrix form of the given system is

$$AX=B$$

$$\begin{bmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 24 \\ 35 \end{bmatrix}$$

Now the augmented matrix,

$$[A,B] = \begin{bmatrix} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 10 & 24 \\ 28 & 4 & -1 & 32 \\ 2 & 17 & 4 & 35 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 10 & 24 \\ 0 & -80 & -281 & -640 \\ 0 & 11 & -16 & -13 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 28R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 10 & 24 \\ 0 & 1 & 3.5125 & 8 \\ 0 & 11 & -16 & -13 \end{bmatrix} \quad R_2 \to \frac{-R_2}{80}$$

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$$\therefore \sim \begin{bmatrix} 1 & 3 & 10 & 24 \\ 0 & 1 & 3.5125 & 8 \\ 0 & 0 & 54.6375 & 101 \end{bmatrix} \quad R_3 \to R_3 - 11R_2$$

By back substitution method,

$$X + 3Y + 10Z = 24$$

$$Y + 3.5125Z = 8$$

$$(54.6375)Z = 101$$

$$\boxed{Z = 1.8485}$$

$$Y + 3.5125(1.8485) = 8$$

$$Y + 6.4929 = 8$$

$$\boxed{Y = 1.5071}$$

$$X + 3(1.5071) + 10(1.8485) = 24$$

$$X + 4.5213 + 18.4850 = 24$$

$$X + 23.0063 = 24$$

$$\boxed{X = 0.9937}$$

2. $10x - 2y + 3z = 23$

$2x + 10y - 5z = -33$

$3x - 4y + 10z = 41$

Soln:-

Matrix form of the given system $Ax = B$

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

Now, the augumented matrix is,

$$[A,B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix} \quad R_1 \to \frac{R_1}{10}$$

$$\sim \begin{bmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 52/5 & -28/5 & -188/5 \\ 0 & -17/5 & 91/10 & -341/10 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 1 & -7/13 & -47/13 \\ 0 & 1 & -91/34 & -341/34 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2/\tfrac{52}{5} \\ R_3 \to \dfrac{R_3}{-17/5} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 1 & -7/13 & -47/13 \\ 0 & 0 & \frac{-945}{442} & \frac{-2835}{442} \end{bmatrix} \quad R_3 \to R_3 - R_2$$

By back substitution method

$$x - y/5 + 3z/10 = 23/10$$

$$y - \frac{7z}{13} = -\frac{47}{13}$$

$$\frac{-945}{442} z = -\frac{2835}{442}$$

$$\boxed{z = 3}$$

$$y = \frac{-47}{13} - \frac{7(3)}{13}$$

$$\boxed{y=-2}$$

$$x = \frac{23}{10} + \frac{(-2)}{5} - \frac{3(3)}{10}$$

$$\boxed{x=1}$$

4. $3x+y-z=3$

$2x-8y+z=-5$

$x-2y+9z=8$

Soln:-

Matrix form of the given system

$AX=B$

Now, the augmented matrix is,

$$[A,B] = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 9 & 8 \\ 2 & -8 & 1 & -5 \\ 3 & 1 & -1 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 9 & 8 \\ 0 & -4 & -17 & -21 \\ 0 & 7 & -28 & -21 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2-2R_1 \\ R_3 \to R_3-3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 9 & 8 \\ 0 & 1 & 17/4 & 21/4 \\ 0 & 1 & -4 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to \frac{R_2}{-4} \\ R_3 \to \frac{R_3}{7} \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 9 & 8 \\ 0 & 1 & 17/4 & 21/4 \\ 0 & 0 & -33/4 & -33/4 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

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By back substitution method

$$x - 2y + 9z = 8$$

$$y + \frac{17z}{4} = \frac{21}{4}$$

$$-\frac{33}{4} z = -\frac{33}{4}$$

$$\boxed{z = 1}$$

$$y = \frac{21}{4} - \frac{17}{4} = 1$$

$$\boxed{y = 1}$$

$$x = 8 + 2(1) - 9(1)$$

$$\boxed{x = 1}$$

6. $x_1 + x_2 + x_3 + x_4 = 2$

$x_1 + x_2 + 3x_3 - 2x_4 = -6$

$2x_1 + 3x_2 - x_3 + 2x_4 = 7$

$x_1 + 2x_2 + x_3 - x_4 = -2$

Soln:-

Matrix form of the given system is

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \\ -2 \end{bmatrix}$$

Now, the augmented matrix

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & 0 & -2 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & -3 & 2 & 7 \\ 0 & 1 & 0 & -2 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 & -4 \\ 0 & 0 & -3 & 2 & 7 \\ 0 & 0 & 2 & -3 & -8 \end{bmatrix} \quad R_3 \to R_3 - R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 & -4 \\ 0 & 0 & -1 & 2/3 & 7/3 \\ 0 & 0 & 1 & -3/2 & -8/2 \end{bmatrix} \quad \begin{matrix} R_3 \to \frac{R_3}{3} \\ R_4 \to \frac{R_4}{2} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 & -4 \\ 0 & 0 & -1 & 2/3 & 7/3 \\ 0 & 0 & 0 & -5/6 & -5/3 \end{bmatrix} \quad R_4 \to R_4 + R_3$$

By back substitution method,

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_2 - 2x_4 = -4$$

$$-x_3 + 2/3\, x_4 = 7/3$$

$$-5/6\, x_4 = -5/3$$

$$x_4 = -5/3 \times -6/5$$

$$\boxed{x_4 = 2}$$

$$-x_3 + 4/3 = 7/3$$

$$-x_3 = 7/3 - 4/3 = 3/3$$

$$\boxed{x_3 = -1}$$

$$x_2 - 4 = -4$$

$$\boxed{x_2 = 0}$$

$$x_1 + 0 - 1 + 2 = 2$$

$$x_1 + 1 = 2$$

$$x_1 = 2 - 1$$

$$\boxed{x_1 = 1}$$

6. $3x_1 + x_2 + x_3 = 4$

$x_1 + 4x_2 - x_3 = -5$

$x_1 + x_2 - 6x_3 = -12$

Soln:

Matrix form of the given equation is

$$AX = B$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & -1 \\ 1 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -12 \end{bmatrix}$$

Now the augmented matrix

$$[A, B] = \begin{bmatrix} 3 & 1 & 1 & 4 \\ 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & -5 \\ 3 & 1 & 1 & 4 \\ 1 & 1 & -6 & -12 \end{bmatrix} \quad R_1 \rightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -11 & 4 & 19 \\ 0 & -3 & -5 & -7 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -1 & 4/11 & 19/11 \\ 0 & 1 & 5/3 & 7/3 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow \frac{R_2}{11} \\ R_3 \rightarrow \frac{-R_3}{3} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -1 & 4/11 & 19/11 \\ 0 & 0 & 67/33 & 134/33 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2$$

By back substitution method,

$$x + 4y - z = -5$$

$$-y + 4/11\, z = 19/11$$

$$\frac{67}{33} z = \frac{134}{33}$$

$$Z = \frac{134}{33} \times \frac{33}{67}$$

$$\boxed{Z=2}$$

$$-y + 8/_{11} = 19/_{11}$$

$$-y = 11/_{11}$$

$$\boxed{y=-1}$$

$$x - 4 + 2 = -5$$

$$x - 6 = -5$$

$$x = -5+6$$

$$\boxed{x=1}$$

7) $2x_1 + 4x_2 + 2x_3 = 15$

$2x_1 + x_2 + 2x_3 = -5$

$4x_1 + x_2 - 2x_3 = 0$

Soln:-

Matrix form of the given equation is

$$AX = B.$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \\ 0 \end{bmatrix}$$

Now the augmented matrix

$$[A,B] = \begin{bmatrix} 2 & 4 & 2 & 15 \\ 2 & 1 & 2 & -5 \\ 4 & 1 & -2 & 0 \end{bmatrix} \qquad R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 15/2 \\ 2 & 1 & 2 & -5 \\ 4 & 1 & -2 & 0 \end{bmatrix} \qquad \begin{array}{l} R_3 \to R_3 - 4R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 4 & 1 & -2 & 0 \\ -4 & -8 & -4 & -30 \\ \hline 0 & -7 & -6 & -30 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & 2 & 5 \\ -2 & -4 & -2 & -15/2 \times 2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 15/2 \\ 0 & -3 & 0 & -20 \\ 0 & -7 & -6 & -30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 15/2 \\ 0 & 1 & 0 & 20/3 \\ 0 & 0 & -6 & 50/3 \end{bmatrix} \qquad R_2 \to \frac{R_2}{-3}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 15/2 \\ 0 & 1 & 0 & 20/3 \\ 0 & 0 & 1 & -25/9 \end{bmatrix} \qquad R_3 \to R_3 + 7R_2$$

By back substitution method,

$$x_1 + 2x_2 + x_3 = 15/2$$

$$x_2 = 20/3 \; ; \; \boxed{x_2 = 6.6667}$$

$$x_3 = -25/9 \; ; \; \boxed{x_3 = -2.7778}$$

$$x_1 + 2(6.6667) - 2.7778 = 7.5$$

$$x_1 + 10.5556 = 7.5$$

$$\boxed{x_1 = -3.0556}$$

RC 30/7

8/8/19

Gauss-Seidal Method :-

Consider

Iterative method :

This iterative method is not always successfull to all system of equations in this method is to succeed each equation of the system must posses one large coefficient and the large coefficient must be attached to a different unknowns in that equations.

These condition will be satisfied if the large coefficient are along the leading diagonal elements of the coefficient matrix.

When this condition is satisfied the system will be solvable by iterative method

i.e. The system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

will be solvable by iterative method is

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

Gauss-seidal method:-

(In this method we first verify the given system is diagonally dominant or not).

If it is not diagonally dominant we interchange the equation itself we obtain diagonaly dominant system then

Let us consider the system of equations are,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3.$$

Rewrite the above equations are.

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \rightarrow (1)$$ $\Rightarrow x_1 (y, z, $

(ii) $\Rightarrow x_2 (y_1, z_1)$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \rightarrow (2)$$ $\Rightarrow y_1 (z=0, x_1)$

ii $\Rightarrow y_2 (x_2, z_1)$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \rightarrow (3)$$ $\Rightarrow z_1 (x_1, y_1)$

ii $\Rightarrow z_2 (x_2, y_2)$

Iteration : 1

Put, $y=0$ & $z=0$ in (1)

$$x^* = \frac{1}{a_1}(d_1)$$

Put $x = x^*$, $z=0$ in (2)

$$y^* = \frac{1}{b_2}(d_2 - a_2 x^*)$$

Put $x = x^*$, $y = y^*$ in (3),

$$z^* = \frac{1}{c_3}(d_3 - a_3 x^* - b_3 y^*)$$

Iteration : 2

Put $y = y^*$, $z = z^*$ in (1)

$$x^{**} = \frac{1}{a_1}(d_1 - b_1 y^* - c_1 z^*)$$

Put $x = x^{**}$, $z = z^*$

$$y^{**} = \frac{1}{b_2}(d_2 - a_2 x^{**} - c_2 z^*)$$

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Put $x = x^{**}$, $y = y^{**}$

$$z^{**} = \frac{1}{c_3}(d_3 - a_3 x^{**} - b_3 y^{**})$$

Continuing this process until the convergency assured the convergency in the gauss seidal method is very fast.

21/8/19

1) Solve the following system of equations by Gauss-seidel method.

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = -3$$

Soln:-

The matrix form of given system of equation is

$$AX = B$$

$$\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

Here, the coefficient matrix is diagonally dominant.

Now

$$x = \frac{1}{10}\left[3 + 5y + 2z\right] \rightarrow ①$$

$$y = \frac{+1}{10}\left[3 + 4x + 3z\right] \rightarrow ②$$

$$z = -\frac{1}{10}\left[3 + x + 6y\right] \rightarrow ③$$

Iteration : 1

Put , $y = 0$ and $z = 0$ in eqn ①

$$x = \frac{1}{10}[3] = \frac{3}{10} = 0.3$$

$\Rightarrow$ ② $$y = \frac{1}{10}\left[3 + 4(0.3)\right] = \frac{1}{10}(3 + 1.2) = \frac{4.2}{10}$$

$$= 0.42$$

$\Rightarrow$ ③.

$$z = -\frac{1}{10}\left[3 + 0.3 + 6(0.42)\right]$$

$$= \frac{-1}{10}\left[3 + 0.3 + 2.52\right]$$

$$= -\frac{1}{10}\left[5.82\right]$$

$$= -0.582$$

$\therefore$ $x = 0.3$; $y = 0.42$; $z = -0.582$

Iteration : 2

Put $y = 0.42$, $z = -0.582$ in eqn ①

$$x = \frac{1}{10}\left[3 + 5(0.42) + 2(-0.582)\right] = \frac{3.9360}{10}$$

$$= 0.3936$$

② ⇒ $y = \frac{1}{10}\left[3 + 4(0.3936) + 3(-0.582)\right] \doteq 0.2828$

③ ⇒ $z = -\frac{1}{10}\left[3 + (0.3936) + 6(0.2828)\right] = -0.5090$

$\therefore x = 0.3936 \; ; \; y = 0.2828 \; ; \; z = -0.5090$

Iteration : 3

Put $y = 0.2828$, $z = -0.5090$ in eqn ①

$x = \frac{1}{10}\left[3 + 5(0.2828) + 2(-0.5090)\right] = 0.3396$

② ⇒ $y = \frac{1}{10}\left[3 + 4(0.3396) + 3(-0.5090)\right] = 0.2831$

③ ⇒ $z = -\frac{1}{10}\left[3 + 0.3396 + 6(0.2831)\right] = -0.5038$

$\therefore x = 0.3396 \; ; \; y = 0.2831 \; ; \; z = -0.5038$

Iteration : 4

Put $y = 0.2831$, $z = -0.5038$ in eqn ①

$x = \frac{1}{10}\left[3 + 5(0.2831) + 2(-0.5038)\right] = 0.3408$

② ⇒ $y = \frac{1}{10}\left[3 + 4(0.3408) + 3(-0.5038)\right] = 0.2852$

③ ⇒ $z = -\frac{1}{10}\left[3 + 0.3408 + 6(0.2852)\right] = -0.5052$

$\therefore x = 0.3408 \; ; \; y = 0.2852 \; ; \; z = -0.5052$

Iteration : 5

Put $y = 0.2852$, $z = -0.5052$ in eqn ①

$x = \frac{1}{10}\left[3 + 5(0.2852) + 2(-0.5052)\right] = 0.3416$

② ⇒ $y = \frac{1}{10}[3 + 4(0.3416) + 3(-0.5052)] = 0.2851$

③ ⇒

$$z = \frac{-1}{10}[3 + 0.3416 + 6(0.2851)] = -0.5052$$

$\therefore x = 0.3416;\ y = 0.2851;\ z = -0.5052$

Iteration : 6

Put $y = 0.2851$; $z = -0.5052$ in eqn ①

$$x = \frac{1}{10}[3 + 5(0.2851) + 2(-0.5052)] = 0.3415$$

② ⇒ $y = \frac{1}{10}[3 + 4(0.3415) + 3(-0.5052)] = 0.2850$

③ ⇒ $z = \frac{-1}{10}[3 + 0.3415 + 6(0.2850)] = -0.5052$

$\therefore x = 0.3415$; $y = 0.2850$; $z = -0.5052$

Iteration : 7

Put $y = 0.2850$; $z = -0.5052$ in eqn ①

$$x = \frac{1}{10}[3 + 5(0.2850) + 2(-0.5052)] = 0.3415$$

② ⇒ $y = \frac{1}{10}[3 + 4(0.3415) + 3(-0.5052)] = 0.2850$

③ ⇒ $z = \frac{-1}{10}[3 + 0.3415 + 6(0.2850)] = -0.5052$

$\therefore x = 0.3415$; $y = 0.2850$; $z = -0.5052$

1) $4x+2y+z=14$

$x+5y-z=10$

$x+y+8z=20$

Soln:-

The matrix form of given system of equations is

$$AX=B$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 1 & 5 & -1 \\ 1 & 1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ 20 \end{bmatrix}$$

Here, the coefficient matrix is diagonally dominent. Now,

$$x = \frac{1}{4}[14-2y-z] \rightarrow ① \quad 3.5-0.5y-0.25z$$

$$y = \frac{1}{5}[10-x+z] \rightarrow ② \quad 2-0.2x+0.2z$$

$$z = \frac{1}{8}[20-x-y] \rightarrow ③ \quad 2.5-0.125x-0.125y$$

Iteration :1

Put, $y=0$ and $z=0$ in eqn ①

$$x = \frac{14}{4} = 3.5$$

$$② \Rightarrow y = \frac{1}{5}(10-3.5) = 1.3$$

$$③ \Rightarrow z = \frac{1}{8}(20-3.5-1.3) = \frac{15.2}{8} = 1.9$$

$\therefore x=3.5$; $y=1.3$; $z=1.9$

Iteration : 2

Put, $y = 1.3$, $z = 1.9$ in eqn ①.

$$x = \frac{1}{4}[14 - 2(1.3) - 1.9] = 2.375$$

② $\Rightarrow$ $$y = \frac{1}{5}[10 - 2.375 + 1.9] = 1.9050$$

③ $\Rightarrow$ $$z = \frac{1}{8}[20 - 2.375 - 1.9050] = 1.965$$

$\therefore x = 2.375$; $y = 1.9050$; $z = 1.965$

Iteration : 3

Put, $y = 1.9050$, $z = 1.965$ in eqn ①

$$x = \frac{1}{4}[14 - 2(1.9050) - 1.965] = 2.0563$$

② $\Rightarrow$ $$y = \frac{1}{5}[10 - 2.0563 + 1.965] = 1.9817$$

③ $\Rightarrow$ $$z = \frac{1}{8}[20 - 2.0563 - 1.9817] = 1.9953$$

$\therefore x = 2.0563$; $y = 1.9817$; $z = 1.9953$

Iteration : 4

Put, $y = 1.9817$; $z = 1.9953$ in eqn ①

$$x = \frac{1}{4}[14 - 2(1.9817) - 1.9953] = 2.0103$$

② $\Rightarrow$ $$y = \frac{1}{5}[10 - 2.0103 + 1.9953] = 1.9970$$

③ $\Rightarrow$ $$z = \frac{1}{8}[20 - 2.0103 - 1.9970] = 1.9991$$

$\therefore x = 2.0103$; $y = 1.9970$; $z = 1.9991$

Iteration : 5

Put, $y = 1.9970$, $z = 1.9991$ in eqn ①

$$x = \frac{1}{4}[14 - 2(1.9970) - 1.9991] = 2.0017$$

② ⇒ $y = \frac{1}{5}[10 - 2.0017 + 1.9991] = 1.~~2497~~9995$

③ ⇒ $z = \frac{1}{8}[20 - 2.0017 - 1.9995] = 1.9999$

$\therefore x = 2.0017$; $y = 1.9995$; $z = 1.9999$

Iteration : 6

Put, $y = 1.9995$; $z = 1.9999$ in eqn ①.

$x = \frac{1}{4}[14 - 2(1.9995) - 1.9999] = 2.0003$

② ⇒ $y = \frac{1}{5}[10 - 2.0003 + 1.9999] = 1.9999$

③ ⇒ $z = \frac{1}{8}[20 - 2.0003 - 1.9999] = 2$

$\therefore x = 2.0003$; $y = 1.9999$; $z = 2$

Iteration : 7

Put, $y = 1.9999$; $z = 2$ in eqn ①.

$x = \frac{1}{4}[14 - 2(1.9999) - 2] = 2.0001$

② ⇒ $y = \frac{1}{5}[10 - 2.0001 + 2] = 2$

③ ⇒ $z = \frac{1}{8}[20 - 2.0001 - 1.9999] = 2$

$\therefore x = 2.0001$; $y = 2$; $z = 2$

Iteration : 8

Put $y = 2$; $z = 2$ in eqn ①.

$x = \frac{1}{4}[14 - 2(2) - 2] = 2$

$y = \frac{1}{5}[10 - 2 + 2] = 2$

$z = \frac{1}{8}[20 - 2 - 2] = 2$

$\therefore x = 2$; $y = 2$; $z = 2$

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Iteration : 9

Put $y = 2$; $z = 2$ in eqn ①

$$x = \frac{1}{4}[14 - 2(2) - 2] = 2$$

② $\Rightarrow$ $y = \frac{1}{5}[10 - 2 + 2] = 2$

③ $\Rightarrow$ $z = \frac{1}{8}[20 - 2 - 2] = 2$

$x = 2$; $y = 2$; $z = 2$.

2) $3x - y + z = 1$

$3x + 6y + 2z = 0$

$3x + 3y + 7z = 4$

Soln:-

The matrix form of given system of equation is $AX = B$

$$\begin{bmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

Here, the coefficient matrix is diagonally dominent.

$x = \frac{1}{3}[1 + y - z]$ → ① $0.3 + 0.3y - 0.3$

$y = -\frac{1}{6}[+3x + 2z]$ → ② $0.5x + 0.33$

$z = \frac{1}{7}[4 - 3x - 3y]$ → ③ $0.5714 - 0.142$

Iteration 1 :-

Put $y = 0$ and $z = 0$ in eqn ①

$$x = \frac{1}{3}[1+0+0] = \frac{1}{3} = 0.3333$$

$$② \Rightarrow y = \frac{-1}{6}[3(0.3333)] = -0.1667$$

$$③ \Rightarrow z = \frac{1}{7}[4-3(0.3333)-3(-0.1667)] = 0.5$$

$\therefore x = 0.3333$; $y = -0.1667$; $z = 0.5$

Iteration: 2

Put $y = -0.1667$ and $z = 0.5$ in eqn ①.

$$x = \frac{1}{3}[1-0.1667-0.5] = 0.1111$$

$$② \Rightarrow y = \frac{-1}{6}[3(0.1111)+2(0.5)] = -0.2222$$

$$③ \Rightarrow z = \frac{1}{7}[4-3(0.1111)+3(0.2222)] = 0.6190$$

$\therefore x = 0.1111$; $y = -0.2222$; $z = 0.6190$

Iteration: 3

Put $y = -0.2222$ & $z = 0.6190$ in eqn ①

$$x = \frac{1}{3}[1-0.2222-0.6190] = 0.0529$$

$$② \Rightarrow y = -\frac{1}{6}[3(0.0529)+2(0.6190)] = -0.2328$$

$$③ \Rightarrow z = \frac{1}{7}[4-3(0.0529)+3(0.2328)] = 0.6485$$

$\therefore x = 0.0529$; $y = -0.2328$; $z = 0.6485$

Iteration: 4

Put $y = -0.2328$, $z = 0.6485$ in eqn ①

$$x = \frac{1}{3}[1-0.2328-0.6485] = 0.0396$$

$$② \Rightarrow y = \frac{-1}{6}[3(0.0396)+2(0.6485)] = -0.2360$$

$$③ \Rightarrow z = \frac{1}{7}[4-3(0.0396)+3(-0.2360)] = 0.6556$$

$\therefore x = 0.0396$; $y = -0.2360$; $z = 0.6556$

Iteration: 5

Put $y = -0.2360$; $z = 0.6556$ in eqn ①

$x = \frac{1}{3}(1 - 0.2360 - 0.6556) = 0.0361$

② $\Rightarrow y = \frac{-1}{6}(3(0.0361) + 2(0.6556)) = -0.2366$

③ $\Rightarrow z = \frac{1}{7}[4 - 3(0.0361) + 3(0.2366)] = 0.6574$.

$\therefore x = 0.0361$; $y = -0.2366$; $z = 0.6574$

Iteration-6:

Put $y = -0.2366$; $z = 0.6574$ in eqn ①

$x = \frac{1}{3}[1 - 0.2366 - 0.6574] = 0.0353$

② $\Rightarrow y = \frac{-1}{6}[3(0.0353) + 2(0.6574)] = -0.2368$.

③ $\Rightarrow z = \frac{1}{7}[4 - 3(0.0353) + 3(0.2368)] = 0.6577$

$\therefore x = 0.0353$; $y = -0.2368$; $z = 0.6577$

Iteration-7:

Put $y = -0.2368$; $z = 0.6577$ in eqn ①.

$x = \frac{1}{3}[1 - 0.2368 - 0.6577] = 0.0352$

② $\Rightarrow y = \frac{-1}{6}[3(0.0352) + 2(0.6577)] = -0.2368$

③ $\Rightarrow z = \frac{1}{7}[4 - 3(0.0352) + 3(0.2368)] = 0.6578$

$\therefore x = 0.0352$; $y = -0.2368$; $z = 0.6578$

Iteration - 8:-

Put $y = -0.2368$; $z = 0.6578$ in eqn ①

$$x = \frac{1}{3}[1 - 0.2368 - 0.6578] = 0.0351$$

$$② \Rightarrow y = \frac{-1}{6}[3(0.0351) + 2(0.6578)] = -0.2368$$

$$③ \Rightarrow z = \frac{1}{7}[4 - 3(0.0351) + 3(0.2368)] = 0.6579$$

$\therefore x = 0.0351$; $y = -0.2368$; $z = 0.6579$

Iteration - 9:-

Put $y = -0.2368$; $z = 0.6579$

$$x = \frac{1}{3}[1 - 0.2368 - 0.6579] = 0.0351$$

$$② \Rightarrow y = \frac{-1}{6}[3(0.0351) + 2(0.6579)] = -0.2369$$

$$③ \Rightarrow z = \frac{1}{7}[4 - 3(0.0351) + 3(0.2369)] = 0.6579$$

$\therefore x = 0.0351$; $y = -0.2369$; $z = 0.6579$.

Iteration - 10:-

Put $y = -0.2369$; $z = 0.6579$.

$$x = \frac{1}{3}[1 - 0.2369 - 0.6579] = 0.0351$$

$$② \Rightarrow y = -\frac{1}{6}[3(0.0351) + 2(0.6579)] = -0.2369$$

$$③ \Rightarrow z = \frac{1}{7}[4 - 3(0.0351) + 3(0.2369)] = 0.6579$$

$\therefore x = 0.0351$; $y = -0.2369$; $z = 0.6579$.

3) $x + y + 54z = 110$

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

Soln:-

The matrix form of given system of

equation is

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 54 \\ 27 & 6 & -1 \\ 6 & 15 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 110 \\ 85 \\ 72 \end{bmatrix}$$

Here, the coefficient matrix is diagonally dominant

$$x = \frac{1}{27}[85 - 6y + z] \rightarrow (1)$$

$$y = \frac{1}{15}[72 - 6x - 2z] \rightarrow (2)$$

$$z = \frac{1}{54}[110 - x - y] \rightarrow (3)$$

Iteration 1:

Put $y = 0$; $z = 0$ in eqn (1).

$$x = \frac{1}{27}[85 - 6(0) + (0)] = 3.1481$$

$$(2) \Rightarrow y = \frac{1}{15}[72 - 6(3.1481)] = 3.5408$$

$$(3) \Rightarrow z = \frac{1}{54}[110 - 3.1481 - 3.5408] = 1.9132.$$

$\therefore x = 3.1481$; $y = 3.5408$; $z = 1.9132$

Iteration 2:

Put $y = 3.5408$; $z = 1.9132$ in eqn (1)

$$x = \frac{1}{27}[85 - 6(3.5408) + 1.9132] = 2.4322$$

$$(2) \Rightarrow y = \frac{1}{15}[72 - 6(2.4322) - 2(1.9132)] = 3.5720$$

$$(3) \Rightarrow z = \frac{1}{54}[110 - 2.4322 - 3.5720] = 1.9258$$

$\therefore x = 2.4322$; $y = 3.5720$; $z = 1.9258$

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Iteration 3:

Put $y = 3.5720$; $z = 1.9258$ in eqn ①

$$x = \frac{1}{27}[85 - 6(3.5720) + 1.9258] = 2.4257$$

② ⟹ $$y = \frac{1}{15}[72 - 6(2.4257) - 2(1.9258)] = 3.5729$$

③ ⟹ $$z = \frac{1}{54}[110 - 2.4257 - 3.5729] = 1.9260.$$

$\therefore x = 2.4257$; $y = 3.5729$; $z = 1.9260$

Iteration 4:

Put $y = 3.5729$; $z = 1.9260$ in eqn ①

$$x = \frac{1}{27}[85 - 6(3.5729) + 1.9260] = 2.4255$$

② ⟹ $$y = \frac{1}{15}[72 - 6(2.4255) - 2(1.9260)] = 3.5730$$

③ ⟹ $$z = \frac{1}{54}[110 - 2.4255 - 3.5730] = 1.9260$$

$\therefore x = 2.4255$; $y = 3.5730$; $z = 1.9260$

Iteration 5:

Put $y = 3.5730$; $z = 1.9260$ in eqn ①.

$$x = \frac{1}{25}[85 - 6(3.5730) + 1.9260] = 2.4255$$

② ⟹ $$y = \frac{1}{15}[72 - 6(2.4255) - 2(1.9260)] = 3.5730$$

③ ⟹ $$z = \frac{1}{54}[110 - 2.4255 - 3.5730] = 1.9260.$$

$\therefore x = 2.4255$; $y = 3.5730$; $z = 1.9260.$

4) $8x - y + z = 18$

$2x + 5y - 2z = 3$

$x + y - 3z = -6$

Soln:-

The matrix form of given system of

equation is

$$Ax = B$$

$$\begin{pmatrix} 8 & -1 & 1 \\ 2 & 5 & -2 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 3 \\ -6 \end{pmatrix}$$

$$x = \frac{1}{8}[18 + y - z] \rightarrow (1)$$

$$y = \frac{1}{5}[3 - 2x + 2z] \rightarrow (2)$$

$$z = \frac{1}{3}[6 + x + y] \rightarrow (3)$$

Iteration 1:

Put $y = 0; z = 0$ in eqn (1)

$$x = \frac{1}{8}[18] = 2.25$$

$$(2) \Rightarrow y = \frac{1}{8}[3 - 2(2.25)] = -0.3$$

$$(3) \Rightarrow z = \frac{1}{3}[6 + 2.25 - 0.3] = 2.65$$

Iteration 2:

Put $y = -0.3$; $z = 2.65$ in eqn (1)

$$x = \frac{1}{8}[18 - 0.3 - 2.65] = 1.8813$$

$$(2) \Rightarrow y = \frac{1}{5}[3 - 2(1.8813) - 2(2.65)] = 0.9075$$

$$(3) \Rightarrow z = \frac{1}{3}[6 + 1.8813 + 0.9075] = 2.9296$$

$\therefore x = 1.8813$; $y = 0.9075$; $z = 2.9296$

Iteration 3:

Put $y = 0.9075$; $z = 2.9296$ in eqn (1)

$$x = \frac{1}{8}[18 + 0.9075 - 2.9296] = 1.9972$$

$$(2) \Rightarrow y = \frac{1}{5}[3 - 2(1.9972) + 2(2.9296)] = 0.973$$

$③ \Rightarrow z = \frac{1}{3}[6 + 1.9972 + 2.9296] = 2.9901$

$\therefore x = 1.9972 ; y = 0.9730 ; z = 2.9901.$

Iteration 4:

Put $y = 0.9730 ; z = 2.9901$ in eqn ①

$x = \frac{1}{8}[18 + 0.9730 - 2.9901] = 1.9979$

$② \Rightarrow y = \frac{1}{5}[3 - 2(1.9979) + 2(2.9901)] = 0.9969$

$③ \Rightarrow z = \frac{1}{3}[6 + 1.9979 + 0.9969] = 2.9983.$

$\therefore x = 1.9979 ; y = 0.9969 ; z = 2.9983.$

Iteration 5:

Put $y = 0.9969 ; z = 2.9983.$ in eqn ①

$x = \frac{1}{8}[18 + 0.9969 - 2.9983] = 1.9998$

$② \Rightarrow y = \frac{1}{5}[3 - 2(1.9998) + 2(2.9983)] = 0.9994$

$③ \Rightarrow z = \frac{1}{3}[6 + 1.9998 + 0.9994] = 2.9997$

$\therefore x = 1.9998 ; y = 0.9994 ; z = 2.9997$

Iteration 6:

Put $y = 0.9994 ; z = 2.9997$ in eqn ①

$x = \frac{1}{8}[18 + 0.9994 - 2.9997] = 2.$

$y = \frac{1}{5}[3 - 2(2) + 2(2.9997)] = 0.9999$

$z = \frac{1}{3}[6 + 2 + 0.9999] = 3.$

$\therefore x = 2 ; y = 0.9999 ; z = 3.$

Iteration 7:

Put $y = 0.9999 ; z = 3$ in eqn ①.

$x = \frac{1}{8}[18 + 0.9999 - 3] = 2$

② ⇒ $y = \frac{1}{5}[3-2(2)+2(3)] = 1$

③ ⇒ $z = \frac{1}{3}[6+2+1] = 3$

$\therefore x = 2;\ y = 1;\ z = 3$

Iteration: 8

Put $y = 1;\ z = 3$ in eqn ①

$x = \frac{1}{8}[18+1-3] = 2$

② ⇒ $y = \frac{1}{5}[3-2(2)+2(3)] = 1$

③ ⇒ $z = \frac{1}{3}[6+2+1] = 3$

$\therefore x = 2;\ y = 1;\ z = 3$

5) $10x_1 - 2x_2 - x_3 - x_4 = 3$

$-2x_1 + 10x_2 - x_3 - x_4 = 15$

$-x_1 - x_2 + 10x_3 - 2x_4 = 27$

$-x_1 - x_2 - 2x_3 + 10x_4 = -9$

Soln:-

The matrix form of the given system of equation is

$$AX = B$$

$$\begin{bmatrix} 10 & -2 & -1 & -1 \\ -2 & 10 & -1 & -1 \\ -1 & -1 & 10 & -2 \\ -1 & -1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 27 \\ -9 \end{bmatrix}$$

$x_1 = \frac{1}{10}(3 + 2x_2 + x_3 + x_4) \rightarrow ①$

$x_2 = \frac{1}{10}(15 + 2x_1 + x_3 + x_4) \rightarrow ②$

$x_3 = \frac{1}{10}(27 + x_1 + x_2 + 2x_4) \rightarrow ③$

$x_4 = \frac{1}{10}(-9 + x_1 + x_2 + 2x_3) \rightarrow ④$

Iteration: 1

Put $x_2=0$; $x_3=0$; $x_4=0$ in eqn ①

$$x_1 = \frac{3}{10} = 0.3$$

② ⇒ $x_2 = \frac{1}{10}[15+2(0.3)] = 1.56$

③ ⇒ $x_3 = \frac{1}{10}[27+0.3+1.56] = 2.886$

④ ⇒ $x_4 = \frac{1}{10}[-9+1.56+2.886(2)+0.3] = -0.1368$

$\therefore x_1 = 0.3$; $x_2 = 1.56$; $x_3 = 2.886$; $x_4 = -0.1368$

Iteration: 2

Put $x_2 = 1.56$; $x_3 = 2.886$; $x_4 = -0.1368$ in eqn ①

$$x_1 = \frac{1}{10}[3+2(1.56)+2.886] = 0.8869$$

$$x_2 = \frac{1}{10}[15+2(0.8869)+2.886-0.1368] = 1.9523$$

$$x_3 = \frac{1}{10}[27+0.8869+1.9523+2(-0.1368)] = 2.9566$$

$$x_4 = \frac{1}{10}[-9+0.8869+1.9523+2(2.9566)] = -0.0248$$

$\therefore x_1 = 0.8869$; $x_2 = 1.9523$; $x_3 = 2.9566$; $x_4 = -0.0248$

Iteration: 3

Put $x_2 = 1.9523$; $x_3 = 2.9566$; $x_4 = -0.0248$. in eqn ①

$$x_1 = \frac{1}{10}[3+2(1.9523)+2.9566+(-0.0248)] = 0.9836$$

$$x_2 = \frac{1}{10}[15+2(0.9836)+2.9566+(-0.0248)] = 1.9899$$

$$x_3 = \frac{1}{10}[27+0.9836+1.9899+2(-0.0248)] = 2.9924$$

$$x_4 = \frac{1}{10}[-9+0.9836+1.9899+2(2.9924)] = -0.0042$$

$\therefore x_1 = 0.9836$; $x_2 = 1.9899$; $x_3 = 2.9924$;

$x_4 = -0.0042$

Iteration -4:-

Put $x_2 = 1.9899$; $x_3 = 2.9924$; $x_4 = -0.0042$

$$x_1 = \frac{1}{10}\left[3 + 2(1.9899) + 2.9924 + (-0.0042)\right]$$
$$\Rightarrow 0.9968.$$

$$x_2 = \frac{1}{10}\left[15 + 2(0.9968) + 2.9924 - 0.0042\right] = 1.998$$

$$x_3 = \frac{1}{10}\left[27 + 0.9968 + 1.9982 + 2(-0.0042)\right] = 2.9991$$

$$x_4 = \frac{1}{10}\left[-9 + 0.9968 + 1.9982 + 2(2.9991)\right] = -0.0007$$

Iteration -5:-

Put $x_2 = 1.9982$; $x_3 = 2.9991$; $x_4 = -0.0007$

$$x_1 = \frac{1}{10}\left[3 + 2(1.9982) + 2.9991 - 0.0007\right] = 0.9995$$

$$x_2 = \frac{1}{10}\left[15 + 2(0.9995) + 2.9991 + (-0.0007)\right] = 1.9997$$

$$x_3 = \frac{1}{10}\left[27 + 0.9995 + 1.9997 + 2(-0.0007)\right] = 2.9998$$

$$x_4 = \frac{1}{10}\left[-9 + 0.9995 + 1.9997 + 2(2.9998)\right] = -0.0001$$

Iteration -6:

Put $x_2 = 1.9997$; $x_3 = 2.9998$; $x_4 = -0.0001$

$$x_1 = \frac{1}{10}\left[3 + 2(1.9997) + 2.9998 - 0.0001\right] = 0.9999$$

$$x_2 = \frac{1}{10}\left[15 + 2(0.9999) + 2.9998 - 0.0001\right] = 2.9998$$

$$x_3 = \frac{1}{10}\left[27 + 0.9999 + 2.9998 + 2(-0.0001)\right] = 3$$

$$x_4 = \frac{1}{10}[-9 + 0.9999 + 2 + 2(3)] = 0$$

Iteration :7

Put $x_2 = 2$; $x_3 = 3$; $x_4 = 0$

$$x_1 = \frac{1}{10}[3 + 2(2) + 3 + 0] = 1$$

$$x_2 = \frac{1}{10}[15 + 2(1) + 3 + 0] = 2$$

$$x_3 = \frac{1}{10}[27 + 1 + 2 + 2(0)] = 3$$

$$x_4 = \frac{1}{10}[-9 + 1 + 2 + 2(3)] = 0$$

Iteration :8

Put $x_2 = 2$; $x_3 = 3$; $x_4 = 0$

$$x_1 = \frac{1}{10}[3 + 2(2) + 3 + 0] = 1$$

$$x_2 = \frac{1}{10}[15 + 2(1) + 3 + 0] = 2$$

$$x_3 = \frac{1}{10}[27 + 1 + 2 + 2(0)] = 3$$

$$x_4 = \frac{1}{10}[-9 + 1 + 2 + 2(3)] = 0$$

$\therefore x_1 = 1$; $x_2 = 2$; $x_3 = 3$; $x_4 = 0$

UNIT- V

29/8/19

FINITE DIFFERENCES :-

Let $y=f(x)$ be a given function of x and let $y_0, y_1, y_2, \dots$ be the values of y corresponding to $x_0, x_0+h, x_0+2h, \dots$ of the values of x

ie, $y_0 = f(x_0)$, $y_1 = f(x_0+h)$, $y_2 = f(x_0+2h) \dots$, $y_n = f(x_0+nh)$.

Here the independent variable x proceeds at equally spaced intervals and h (constant).

The difference between two consecutive values of x is called the interval of differencing.

Now $y_1 - y_0$; $y_2 - y_1$, $y_3 - y_2 \dots$ $y_n - y_{n-1}$ are called the first differences of the function y and differences of the y_n values are denoted by

$$\Delta y_n = y_{n+1} - y_n, \quad (n=0,1,2\dots)$$ forward difference operator

Here 'Δ' as an operator called Forward difference operator.

Thus,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

The differences of these first differences are called second differences

$$\Delta^2(y_0) = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$

$$= \Delta y_1 - \Delta y_0$$

$$= y_2 - y_1 - (y_1 - y_0)$$

$$\Delta^2(y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2(y_1) = \Delta(\Delta y_1)$$

$$= \Delta y_2 - \Delta y_1$$

$$= y_3 - y_2 - (y_2 - y_1)$$

$$\Delta^2(y_1) = y_3 - 2y_2 + y_1$$ and so on.

In general $\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$.

6/9/19 **Forward Difference Table :-**

X	Y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
				$\Delta^3 y_0$	
x_2	y_2	Δy_1			$\Delta^4 y_0$
			$\Delta^2 y_1$		
x_3	y_3	Δy_2		$\Delta^3 y_1$	
			$\Delta^2 y_2$		
x_4	y_4	Δy_3			

INTERPOLATION :-

Consider the table

x :	x_0	x_1	x_2	....	x_n
$f(x)$:	$f(x_0)$	$f(x_1)$	$f(x_2)$	----	$f(x_n)$

If the value of $f(y)$ is to be formed at some point y in the interval $[x_0, x_n]$ and y is not one of the tabulated points then the value of $f(y)$ is estimated by using the known values of $f(x)$ at the surrounding points.

This process are computing the value of the function inside the given range is called interpolation.

EXTRAPOLATION :-

If the point y lies outside the domain closed $[x_0, x_n]$ then the estimation of $f(y)$ is called extrapolation.

NEWTON'S FORWARD INTERPOLATION FORMULA :-

$$\Delta y_0 = y_1 - y_0 \Rightarrow y_1 = y_0 + \Delta y_0 = (1+\Delta) y_0.$$

$$\Delta y_1 = y_2 - y_1 \Rightarrow y_2 = y_1 + \Delta y_1 = (1+\Delta) y_1 = (1+\Delta)^2 y_0.$$

$$\Delta y_2 = y_3 - y_2 \Rightarrow y_3 = y_2 + \Delta y_2 = (1+\Delta) y_2 = (1+\Delta)^3 y_0.$$

In general, $y_n = (1+\Delta)^n y_0$.

By using Binomial Theorem,

$$y_n = \left\{1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + \cdots\right\} y_0$$

$$\therefore y_n = f(x_0+nh) = \left\{y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots\right\}$$

$(\because y_n = f(x_n)$; if $x_n = x_0 + nh)$

1. The following table gives the population of a town during the last six senses.

Estimate, using Newton's interpolation formula the increase in the population during the period 1946 to 1948

Year :	1911	1921	1931	1941	1951	1961
Population (in 1000's) :	12	13	20	27	39	52

Soln:-

Forward difference table is as follows.

Years (x)	Population (y)	Δ	Δ^2	Δ^3	Δ^4	Δ^5
(x_0) 1911	(y_0) 12					
		(Δy_0) 1				
1921	13		($\Delta^2 y_0$) 6			
		7		($\Delta^3 y_0$) -6		
1931	20		0		($\Delta^4 y_0$) 11	
		7		5		($\Delta^5 y_0$) -20
1941	27		5		-9	
		12		-4		
1951	39		1			
		13				
1961	52					

Here, $x_0 = 1911$; $h = 10$.

To find

By Newton's Forward difference formula we have

$$y_n = f(x_0+nh) = \left\{ y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots \right\}$$

To find $y(1946)$:-

$x_n = 1946$

ie, $x_n = x_0 + nh$

$\Rightarrow 1946 = 1911 + n(10)$

$10n = 1946 - 1911 = 35$

$n = 3.5$

$$y_n = \left\{12 + 3.5(1) + \frac{3.5(3.5-1)}{2}(6) + \frac{(3.5)(3.5-1)(3.5-2)}{6}(-6)\right.$$

$$+ \frac{(3.5)(3.5-1)(3.5-2)(3.5-3)}{24}(11) + \frac{(3.5)(3.5-1)(3.5-2)(3.5-3)(3.5-4)}{120}(-20)$$

$$= 12 + 3.5 + 26.2500 - 13.1250 + 3.0078 + 0.5469$$

$$\Rightarrow 32.1797 \Rightarrow 32.18$$

To find (1948):-

$$x_n = 1948$$

we $x_n = x_0 + nh$

$$\Rightarrow 1948 = 1911 + n(10)$$

$$10n = 1948 - 1911 = 37$$

$$n = 3.7$$

$$y_n = \left\{12 + 3.7(1) + \frac{3.7(3.7-1)}{2}(6) + \frac{(3.7)(3.7-1)(3.7-2)}{6}\right.$$

$$(-6) + \frac{(3.7)(3.7-1)(3.7-2)(3.7-3)}{24}(11) + \frac{(3.7)(3.7-1)(3.7-2)(3.7-3)(3.7-4)}{120}(-20)$$

$$= 12 + 3.7 + 29.9700 - 16.9830 + 5.4487 + 0.5944$$

$$= 34.7301 \Rightarrow 34.7301$$

7/9/19

∴ The population in the year 1946 is 32.18

And the population in the year 1948 is 34.73

Hence increase in the population during the period 1946 to 1948 = Population in 1948 - Population in 1946

$= 2.55$ (thousands)

BACKWARD DIFFERENCES:-

We use another operator called the backward difference operator ∇ and is denoted by

$$\nabla y_n = y_n - y_{n-1}$$

For $n = 0, 1, 2, \ldots$ we get

$$\nabla y_0 = y_0 - y_{-1}$$

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1 \text{ and so on.}$$

The second backward difference is,

$$\nabla^2 (y_n) = \nabla(\nabla y_n)$$

$$= \nabla(y_n - y_{n-1})$$

$$= \nabla y_n - \nabla y_{n-1}$$

$$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$$

$$= y_n - 2y_{n-1} + y_{n-2}$$

IIIly

$$\nabla^3 y_n = y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3} \text{ and so on.}$$

Backward difference table is as follows :-

x	y	∇	∇^2	∇^3	∇^4
(x_0-4h) x_{-4}	y_{-4}				
		∇y_{-3}			
(x_0-3h) x_{-3}	y_{-3}		$\nabla^2 y_{-2}$		
		∇y_{-2}		$\nabla^3 y_{-1}$	
(x_0-2h) x_{-2}	y_{-2}		$\nabla^2 y_{-1}$		$\nabla^4 y_0$
		∇y_{-1}		$\nabla^3 y_0$	
(x_0-h) x_{-1}	y_{-1}		$\nabla^2 y_0$		
		∇y_0			
x_0	y_0				

Newton's Backward Interpolation Formula

We know that

$$\nabla y_1 = y_1 - y_0 \quad (\text{or}) \quad (1-\nabla) y_1 = y_0$$

$$y_1 = (1-\nabla)^{-1} y_0 \rightarrow ①$$

Also we know that

$$y_1 = (1+\Delta) y_0 \rightarrow ②$$

From ① & ②

$$(1-\nabla)^{-1} \equiv (1+\Delta)$$

$$y_n = (1+\Delta)^n y_0 = (1-\nabla)^{-n} y_0$$

$$\therefore y_n = \left(1 + n\nabla + \frac{n(n+1)}{2!}\nabla^2 + \frac{n(n+1)(n+2)}{3!}\nabla^3 \ldots\right) y_0$$

$$\therefore y_n = f(x_0+nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_0 + \cdots$$

1. Given

x :	1	2	3	4	5	6	7	8
$f(x)$:	1	8	27	64	125	216	343	512

Estimate $f(7.5)$ use Newton's formula.

Soln:-

Backward difference table is as follows

x	y	∇	∇^2	∇^3	∇^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6 ($\nabla^3 y_0$)	
7	343		42 ($\nabla^2 y_0$)		($\nabla^4 y_0$)
		169 (∇y_0)			
8 (x_0)	512 (y_0)				

Here $x_0 = 8$, $h = 1$

By Newton's backward difference formula we have

$$y_n = f(x_0+nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_0 + \cdots$$

To find $f(7.5)$:

$$x_n = 7.5$$

i.e., $x_n = x_0 + nh$

$$7.5 = 8 + n(1)$$

$$n = -8 + 7.5$$

$$n = -0.5$$

$$y_n = 512 + (-0.5)169 + \frac{(-0.5)(-0.5+1)}{2} 42 + \frac{(-0.5)(0.5+1)(0.5+2)}{6}(6) + 0$$

$$= 512 - 84.5 + (-5.25) - 0.3750$$

$$\Rightarrow 421.8750$$

FORWARD DIFFERENCE :-

1) The function $f(x)$ is given by the following table.

Find $f(0.2)$ by a suitable formula

x	:	0	1	2	3	4	5	6
$f(x)$	:	176	185	194	203	212	220	229

Soln:-

Forward difference table is as follows.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
(x_0) 0	(y_0) 176	(Δy_0) 9					
			$(\Delta^2 y_0)$ 0				
1	185	9		$(\Delta^3 y_0)$ 0	$(\Delta^4 y_0)$		
			0		0	$(\Delta^5 y_0)$	
2	194	9		0		-1	$(\Delta^6 y_0)$
			0		-1		5
3	203	9		-1		4	
			0		3		
4	212	9		2			
			-1				
5	220	8					
			1				
6	229	9					

Here $x_0 = 0$; $h = 1$

By Newton's Forward difference formula

we have

$$y_n = f(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

To find $f(0.2)$:-

ie, $x_n = x_0 + nh$

$0.2 = 0 + n(1)$

$n = 0.2$

$$y_n = 176 + (0.2)9 + 0 + 0 + 0 + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)}{120}(-1) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)(0.2-5)}{720}(5).$$

$$y_n = 176 + 1.8 + (-0.0255) + (-0.1021)$$

$$= 177.7$$

$$y_n = 177.7$$

2. Find the value $e^{1.85}$ given $e^{1.7} = 5.4739$, $e^{1.8} = 6.0496$, $e^{1.9} = 6.6859$, $e^{2.0} = 7.3891$, $e^{2.1} = 8.1662$, $e^{2.2} = 9.0250$, $e^{2.3} = 9.9742$

The difference table as follows

x	Y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
(x_0) 1.7	(y_0) 5.4739						
		(Δy_0) 0.5757					
1.8	6.0496		$(\Delta^2 y_0)$ 0.0606				
		0.6363		$(\Delta^3 y_0)$ 0.0063			
1.9	6.6859		0.0669		$(\Delta^4 y_0)$ 0.0007		
		0.7032		0.0070		$(\Delta^5 y_0)$ 0.0001	
2.0	7.3891		0.0739		0.0008		$(\Delta^6 y_0)$ 0.0000
		0.7771		0.0078		0.0001	
2.1	8.1662		0.0817		0.0009		
		0.8588		0.0087			
2.2	9.0250		0.0904				
		0.9492					
2.3	9.9742						

$x_0 = 1.7 \qquad h = 0.1$

By Newton forward difference formula we have

$$y_n = f(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \cdots$$

To find $y(e^{1.85})$:- $x_n = x_0 + nh$

$$x_n = 1.85$$

$$x_n = x_0 + nh$$

$$1.85 = 1.7 + (0.1)n$$

$$\boxed{n = 1.5}$$

$$y(e^{1.85}) = 5.4739 + 1.5(0.5757) + \frac{1.5(0.5)}{2!}(0.0606) + \frac{1.5(0.5)(0.5)}{3!}(0.0063) + \frac{1.5(0.5)(-0.5)(-0.5)}{4!}(0.0001) - \frac{1.5(0.5)(-0.5)(-1.5)(-2.5)}{5!}(0.0001)$$

$= 5.4739 + 0.8636 + 0.0227 - 0.0004 + 0 + 0$

$= 6.3598$

3\. From the following data find θ at $x = 43$

X	: 40	50	60	70	80	90
θ	: 184	204	226	250	276	304

The difference table as follows:

X	Y	Δ	Δ^2	Δ^3
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
90	304			

$x_0 = 40, \quad h = 10$

By newton forward difference formula, we have.

$$y_n = f(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \cdots$$

To find $y(43)$:-

$x_n = 43$

$x_n = x_0 + nh$

$43 = 40 + 10h$

$10n = 3$

$n = 0.3$

$$y(43) = 184 + 0.3(20) + \frac{0.3(-0.7)}{2!}(2) + 0$$

$$= 184 + 6 + (-0.2100)$$

$$y(43) = 189.79$$

4. Using newton forward formula find $\sin(0.1604)$ from the following table:

x :	0.160	0.161	0.162
$\sin x$:	0.1593182066	0.1603053541	0.1612923412

The difference table as follows

x	y	Δ	Δ^2
0.160	0.1593182066	0.000987148 0.0010	-0.000000161
0.161	0.1603053541	0.000986987 0.0010	0
0.162	0.1612923412		

$x_0 = 0.160 \quad h = 0.0010$

By newton forward difference formula we have

$$y_n = f(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \cdots$$

To find $y(\sin(0.1604))$

$$x_n = 0.1604$$

$$x_n = x_0 + nh$$

$$0.1604 = 0.160 + n(0.0010)$$

$$0.0010n = 0.0004$$

$$n = 0.4$$

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$$\sin(0.1604) = 0.1593182066 + 0.4\,(0.0010)$$

(0.000987148)

0.000394859

$$= 0.1593182066 + 0.0004$$

$$= 0.159713066$$

5. x : 0.0 0.5 1.0 1.5 2.0

$f(x)$: 0.3989 0.3521 0.2420 0.1295 0.0540

find $f(1.8)$

The difference table as follows.

x	y	∇	∇^2	∇^3	∇^4
0.0	0.3989				
		-0.0468			
0.5	0.3521		-0.0633		
				0.0609	
1.0	0.2420	-0.1101			-0.0215
			-0.0024		
1.5	0.1295	-0.1125		0.0394	
			0.0370		
2.0	0.0540	-0.0755			

$x_0 = 2.0$ $h = 0.5$

By newton's backward difference formula we have

$$y_n = f(x_0+nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \cdots$$

To find $f(1.8)$

$$x_n = 1.8$$

$$x_n = x_0 + nh$$

$$1.8 = 2.0 + 0.5n$$

$$0.5n = -0.2$$

$$n = -0.4$$

$$y(1.8) = 0.0540 + \frac{(-0.4)(-0.0755)}{1} + \frac{(-0.4)(0.6)}{2}(0.0370)$$

$$+ \frac{(-0.4)(0.6)(1.6)}{3!}(0.0394) + \frac{(-0.4)(0.6)(1.6)(2.6)}{4!}(-0.0215)$$

0.0777

$$= 0.0540 + 0.0302 - 0.0044 - 0.0025 + 0.0009$$

$$= 0.0807 \quad 0.0782$$

6. x : 0 0.1 0.2 0.3 0.4

e^x : 1 1.1052 1.2214 1.3499 1.4918

Find the value of $y = e^x$ when $x = 0.38$

The difference table as follows

x	y	∇	∇^2	∇^3	∇^4
0	1				
		0.1052			
0.1	1.1052		0.0110		
		0.1162		0.0013	
0.2	1.2214		0.0123		−0.0002
		0.1285		0.0011	
0.3	1.3499		0.0134		
		0.1419			
0.4	1.4918				

$x_0 = 0.4 \qquad h = 0.1$

By newton backward difference formula we have,

$y_n = f(x_0+nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \ldots$

To find $f(0.38)$

$x_n = 0.38$

$x_n = x_0 + nh$

$0.38 = 0.4 + 0.1(n)$

$0.1n = -0.02$

$n = -0.2$

$y(0.38) = 1.4918 + (-0.2)(0.1419) + \frac{(-0.2)(0.8)}{2!}$

$0.0134 + \frac{(-0.2)(0.8)(1.8)}{3!}(0.0011) + \frac{(-0.2)(0.8)(2.8)(4.8)(-0.000)}{4!}$

$= 1.4918 - 0.0284 - 0.0011 - 0.0001 + 0$

$= 1.4622$

7.

x	:	0	10	20	30	40
sin x	:	0	0.17365	0.34202	0.50000	0.64279

Find sin 3?

The difference as follows

x	y	∇	∇^2	∇^3	∇^4
0	0				
		0.17365			
10	0.17365		-0.0053		
		0.1684		-0.0051	
20	0.34202		-0.0104		0.0003
		0.1580		-0.0048	
30	0.50000		-0.0152		
		0.1428			
40	0.64279				

$x_0 = 40 \quad h = 10$

$y_n = f(x_0+nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \cdots$

To find $f(38)$:-

$x_n = 38$

$x_n = x_0 + h \Rightarrow 38 = 40 + 10n$

$10n = -2.0$

$n = -0.2$

$y(38) = 0.64279 + (-0.2)(0.1428) + \frac{(-0.2)(0.8)(-0.0152)}{2!}$

$+ \frac{(-0.2)(0.8)(1.8)}{3!}(-0.0048) + \frac{(-0.2)(0.8)(1.8)(2.8)}{4!}(0.0003)$

$= 0.64279 - 0.0286 + 0.0012 + 0.0002 - 0$

$= 0.6156T$

8\. Year (x) : 1961 1971 1981 1991 2001

Population (1000's) : 46 66 81 93 101

Estimate the population in the year 1996. The difference as follows

x	y	∇	∇^2	∇^3	∇^4
1961	46				
		20			
1971	66		-5		
		15		2	
1981	81		-3		-3
		12		1	
1991	93		-4		
		8			
2001	101				

$x_0 = 2001, \quad h = 10$

By Newton's backward formula we have

$$y_n = f(x_0+nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 \ldots$$

To find $f(1996)$

$$x_n = 1996 \Rightarrow x_n = x_0 + nh$$

$$1996 = 2001 + 10n$$

$$10n = -5$$

$$n = -0.5$$

$$y(1996) = 101 + (-0.5)(8) + \frac{(-0.5)(0.5)}{2!}(-4) + \frac{(-0.5)(0.5)(1.5)(-1)}{3!} + \frac{(-0.5)(0.5)(1.5)(2.5)(-3)}{4!}$$

$$= 101 - 4 + 0.5 + 0.0625 + 0.1172$$

$$= 97.6797 \text{ (thousand's)}$$

16/9/19

LAGRANGE'S INTERPOLATION FORMULA FOR UNEQUAL INTERVALS.

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values of the function $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots x_n$, not necessarily equally spaced.

Let $f(x)$ be a polynomial in x of degree n.

Then we can represent $f(x)$ as

$$f(x) = a_0 (x-x_1)(x-x_2) \dots (x-x_n) + a_1 (x-x_0)(x-x_2) \dots (x-x_n) + \dots + a_n (x-x_0)(x-x_1) \dots (x-x_{n-1}) \rightarrow (1)$$

where $a_0, a_1, \dots a_n$ are constants.

Now, we have to determine the $(n+1)$ constants $a_0, a_1, \dots a_n$

Putting $x = x_0$ in (1), we get

$$f(x_0) = a_0 (x_0-x_1)(x_0-x_2) \dots (x_0-x_n)$$

ie, $$a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} \rightarrow (2)$$

Putting $x = x_1$ in (1) we get

$$f(x_1) = a_1 (x_1-x_0)(x_1-x_2) \dots (x_1-x_n)$$

ie, $$a_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} \rightarrow (3)$$

III ly

$$a_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\dots(x_2-x_n)} \rightarrow (4)$$

. . . . .

. . . . .

$$a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} \rightarrow (5)$$

Substituting (2), (3), (4), (5), in (1) we get

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

If we denote $f(x_0), f(x_1), \ldots f(x_n)$ by $y_0, y_1, \ldots y_n$ we get

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

which is Lagrange's Interpolation formula.

1) Use Lagrange's formula calculate $f(3)$ from the following table.

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Soln:-

$x_0 = 0$; $x_1 = 1$; $x_2 = 2$; $x_3 = 4$; $x_4 = 5$;

$x_5 = 6$

$y_0 = 1$; $y_1 = 14$; $y_2 = 15$; $y_3 = 5$; $y_4 = 6$; $y_5 = 19$

By Lagrange's interpolation formula, we have.

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_2-x_0)}$$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} y_4$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} y_5$$

$$f(x) = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \cdot 1$$

$$+ \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \cdot 14$$

$$+ \frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \cdot 15$$

$$+ \frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(3-0)(3-1)(3-2)(3-5)(3-6)} \cdot 5$$

$$+ \frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(4-0)(4-1)(4-2)(4-3)(4-6)} \cdot 6$$

$$+ \frac{(x-0)(x-1)(x-2)(x-3)(x-5)}{(5-0)(5-1)(5-2)(5-3)(5-5)} \cdot 19$$

Put $x = 3$.

$$= \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \cdot 1$$

$$+ \frac{(2)(1)(-1)(-2)(-3)}{(-1)(-2)(-4)(-5)(-6)} \cdot 1$$

$$+ \frac{(3)(1)(-1)(-2)(-3)}{(1)(-1)(-3)(-4)(-5)} \cdot 14$$

$$+\frac{(3)(\cancel{2})(-1)(\cancel{-2})(-3)}{(\cancel{2})(1)(\cancel{-2})(-3)(-4)} \cdot 15$$

$$+\frac{(3)(\cancel{2})(1)(\cancel{-2})(\cancel{-3})}{(4)(\cancel{2})(1)(\cancel{-2})(\cancel{3})} \cdot 5$$

$$+\frac{(\cancel{3})(\cancel{2})(1)(0)(-3)}{(5)(\cancel{4})(\cancel{3})(1)(-1)} \cdot 6^3$$

$$+\frac{(\cancel{3})(\cancel{2})(\cancel{1})(-1)(-2)}{(\cancel{6})(5)(4)(\cancel{2})(\cancel{1})} \cdot 19$$

$$= \frac{1}{20} - \frac{21}{5} + \frac{45}{4} + \frac{15}{4} - \frac{9}{5} + \frac{19}{20}$$

$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95$$

$f(3) = 10.00$

23/9/19

Numerical Integration :-

We know that $\int_a^b f(x)\,dx$ represents the area between $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. This integration is possible only if the $f(x)$ is explicity given and if it is integrable.

The problem of Numerical Integration can be stated as follows given a set of $(n+1)$ paired values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of a function $y = f(x)$, where $f(x)$ is not low explicitly it is required to compute integration $\int_{x_0}^{x_n} y\,dx$.

A general quadrature formula for equidistant ordinates (or) Newton's formula.

For equally spaced interval, we have Newton's forward differences formula as.

$$y(x) = y(x_0+nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 y_0 + \dots \quad \text{--}(1)$$

Now, instant of $f(x)$, we will replace it by this interpolating formula of Newton. Here, $n = \frac{x-x_0}{h}$, where h interval of differencing.

$\therefore$ we have

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_0+nh} f(x)\,dx$$

where $P_n(x)$ is interpolating polynomial of degree n.

$$= \int_{x_0}^{x_0+nh} \left(y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 y_0 + \dots \right) h\,dn$$

$$= h\int_0^n \left(y_0 + n\Delta y_0 + \frac{n^2-n}{2}\Delta^2 y_0 + \frac{n^3-3n^2+2n}{6}\Delta^3 y_0 + \frac{n^4-6n^3+11n^2-6n}{24}\Delta^4 y_0 \cdots\right) dn.$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\left(\frac{n^3}{3}-\frac{n^2}{2}\right)}{2}\Delta^2 y_0 + \frac{\left(\frac{n^4}{4}-\frac{3n^3}{3}+\frac{2n^2}{2}\right)}{6}\Delta^3 y_0 + \frac{\left(\frac{n^5}{5}-\frac{6n^4}{4}+\frac{11n^3}{3}-\frac{6n^2}{2}\right)}{24}\Delta^4 y_0 + \cdots\right]_0^n$$

$n(n-1)(n-2)$

$\Rightarrow (n^2-n)(n-2) \Rightarrow n^3-2n^2-n^2+2n$

$\Rightarrow n^3-3n^2+2n$

Integrant value at the lower limit is zero.

$n(n-1)(n-2)(n-3)$

$(n^2-n)(n^2-3n-2n+6) \Rightarrow n^4-3n^3-2n^3+6n^2-n^3+3n^2+2n^2-6n$

$\therefore$ The integration.

$$\int_{x_0}^{x_n} y(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{2n^3-3n^2}{12}\Delta^2 y_0 + \frac{3n^4-12n^3+12n^2}{72}\Delta^3 y_0 + \frac{12n^5-90n^4+220n^3-180n^2}{1440}\Delta^4 y_0 + \cdots\right]$$

$n(n-1)(n-2)$

$n(n^2-2n-n+3)$

$n^3-2n^2-n^2+3$

n^3-3n^2

It is called Newton's cote quadrature formula.

Trapezoidal Rule:-

By putting $n=1$ in cote formula we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$ ($\because$ the other differences does not exist).

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] \quad (\because \Delta y_0 = y_1 - y_0)$$

$$= h\left(\frac{y_0 + y_1}{2}\right)$$

$$= \frac{h}{2}(y_0 + y_1) \quad ②$$

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left[y_1 + \frac{1}{2}(\Delta y_1)\right]$$

$$= h\left[y_1 + \frac{1}{2}(y_2 - y_1)\right].$$

$$= \frac{h}{2}(y_1 + y_2) \text{ and so on.}$$

Now,

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_0+2h} f(x)dx$$

$$+ \int_{x_0+2h}^{x_0+3h} f(x)dx + \dots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx$$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \dots$$

$$+ \frac{h}{2}(y_n + y_{n-1}).$$

$$= \frac{h}{2}\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \dots + (y_n + y_{n-1})\right]$$

$$= \frac{h}{2}\left[y_0 + (y_1 + y_1) + (y_2 + y_2) + \dots + (y_{n-1} + y_{n-1}) + y_n\right]$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})\right]$$

It is called Trapezoidal rule.

By putting $n=3$ in cote's formula we get,

$$\int_{x_0}^{x_0+3h} f(x)dx = \int_{x_0}^{x_3} f(x)dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4}-18\right)\Delta^3 y_0 + \cdots\right] \quad (3)$$

$$= h\left[3y_0 + \frac{9}{2}(y_1-y_0) + \frac{9}{4}(y_2-2y_1+y_0) + \frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{3h}{8}(y_3 + 3y_2 + 3y_1 + y_0).$$

If n is multiples of 3 we get,

$$\int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \int_{x_0+6h}^{x_0+9h} f(x)dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)dx$$

$$= \frac{3h}{8}[y_3+3y_2+3y_1+y_0] + \frac{3h}{8}(y_6+3y_5+3y_4+y_3).$$

$$+ \frac{3h}{8}[y_9 + 3y_8 + 3y_7 + y_6) + \cdots + \frac{3h}{8}[y_n + 3y_{n-1} + 3y_{n-2} + y_{n-3}]$$

$$= \frac{3h}{8}[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\cdots) + 2(y_3+y_6+y_9+\cdots)]$$

It is called Newton's $\left(\frac{3}{8}\right)^{th}$ rule

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24/9/19

Putting $n=2$ in cote formula we get,

$$\int_{x_0}^{x_0+2h} f(x)dx = \int_{x_0}^{x_2} f(x)dx = h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3}-\frac{4}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1-y_0) + \frac{1}{3}(y_2-2y_1+y_0)\right]$$

$$= \frac{h}{3}\left[y_2+4y_1+y_0\right]$$

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}\left[y_2+4y_3+y_4\right]$$

$$\int_{x_4}^{x_6} f(x)dx = \frac{h}{3}\left[y_4+4y_5+y_6\right] \ldots$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}(y_2 + 4y_1 + y_0) + \frac{h}{3}(y_2 + 4y_3 + y_4) \ldots$$

$$+ \frac{h}{3}(y_n + 4y_{n-1} + y_{n-2})$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\right]$$

is known as Newton $\left(\frac{1}{3}\right)^{rd}$ Rule.

1) Lagrange's Interpolation formula find the value corresponding to $x=10$ from the following table:

x :	5	6	9	11
y :	12	13	14	16.

Soln:-

$x_0 = 5$; $x_1 = 6$; $x_2 = 9$; $x_3 = 11$

$y_0 = 12$; $y_1 = 13$; $y_2 = 14$; $y_3 = 16$.

By Lagrange's Interpolation formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \cdot 12$$

$$+ \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \cdot 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \cdot 14$$

$$+ \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \cdot 16$$

Put $x = 10$

$$= \frac{(10-6)(10-9)(10-11)}{(-1)(-4)(-6)} \cdot 12$$

$$+ \frac{(10-5)(10-9)(10-11)}{(1)(-3)(-5)} \cdot 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(4)(3)(-2)} \cdot 14$$

$$+ \frac{(10-5)(10-6)(10-9)}{(6)(5)(2)} \cdot 16.$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)} \cdot \cancel{12}^{\,2}$$

$$+ \frac{(5)(1)(-1)}{(1)(-3)(-5)} \cdot 13$$

$$+ \frac{(5)(4)(-1)}{(4)(3)(-2)} \cdot \cancel{14}^{\,7}$$

$$+ \frac{(5)(4)^{2}(+1)}{(6)_{3}(5)(2)} \cdot 16$$

$$= 2 + 4.3333 + 11.6667 + 5.333$$

$$= 14.6667$$

2) Use Lagrange's Interpolation formula find polynomial to the data

x :	0	1	3	4
y :	-12	0	6	12

Soln:-

$x_0 = 0$; $x_1 = 1$; $x_2 = 3$; $x_3 = 4$

$y_0 = -12$; $y_1 = 0$; $y_2 = 6$; $y_3 = 12$.

By Lagrange's Interpolation formula.

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_2)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \cdot 12$$

$$+ \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} \cdot 0$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \cdot 6$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \cdot 12$$

$$f(x) = \frac{(x-1)(x-3)(x-4)}{-\cancel{12}} \cdot \cancel{12} + 0 +$$

$$\frac{(x-0)(x-1)(x-4)}{-\cancel{6}} \cdot \cancel{6} + \frac{(x-0)(x-1)(x-3)}{\cancel{12}} \cdot \cancel{12}$$

$$= (x-1)(x-3)(x-4) - (x-0)(x-1)(x-4) + (x-0)(x-1)(x-3)$$

$$= (x^2 - x - 3x + 3)(x-4) - (x^2 - x)(x-4) + (x^2 - x)(x-3)$$

$$= (x^3 - x^2 - 3x^2 + 3x - 4x^2 + 4x + 12x - 12) - (x^3 - 4x^2 - x^2 + 4x) + (x^3 - 3x^2 - x^2 + 3x)$$

$$= (x^3 - 8x^2 + 19x - 12) - (x^3 - 5x^2 + 4x) + (x^3 - 3x^2 - x^2 + 3x)$$

$$= x^3 - 8x^2 + 19x - 12 - x^3 - 5x^2 + 4x + x^3 - 4x^2 + 3x$$

$$= x^3 - 7x^2 + 18x - 12$$

$f(x) = x^3 - 7x^2 + 18x - 12$

Put $x = 2$

$f(2) = 2^3 - 7(2)^2 + 18(2) - 12 = 4$

$x = 6$

$f(6) = 6^3 - 7(6)^2 + 18(6) - 12 = 60.$

2. Use Lagrange's interpolation formula find polynomial through (0,0), (1,1) and (2,2)

<u>Soln:-</u>

X :	0	1	2
Y :	0	1	2

$x_0 = 0$; $x_1 = 1$; $x_2 = 2$

$y_0 = 0$; $y_1 = 1$; $y_2 = 2.$

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)} \cdot 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \cdot 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \cdot 2$$

$$= 0 + \frac{(x)(x-2)}{-1} \cdot 1 + \frac{(x)(x-1)}{\cancel{2}} \cdot \cancel{2}$$

$$= -(x^2-2x) + x^2 - x$$

$$= -x^2 + 2x + x^2 - x$$

$$= 2x - x$$

$$= x$$

$$f(x) = x$$

$$y = x.$$

4. Using Lagrange's formula find cubic curve passing through the points $(-1,-8)$, $(0,3)$, $(2,1)$ and $(3,2)$

Soln:-

x	:	-1	0	2	3
y	:	-8	3	1	2

$x_0 = -1$; $x_1 = 0$; $x_2 = 2$; $x_3 = 3$

$y_0 = -8$; $y_1 = 3$; $y_2 = 1$; $y_3 = 2$

By Lagrange's formula.

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$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$= \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)} \cdot -8$$

$$+ \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)} \cdot 3$$

$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)} \cdot 1$$

$$+ \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)} \cdot 2$$

$$= \frac{(x-0)(x-2)(x-3)}{(-1)(-3)(-4)} \cdot -8 + \frac{(x+1)(x-2)(x-3)}{(1)(-2)(-3)} \cdot 3$$

$$+ \frac{(x+1)(x-0)(x-3)}{(3)(2)(-1)} \cdot 1 + \frac{(x+1)(x-0)(x-2)}{(4)(3)(1)} \cdot 2$$

$$= \frac{(x)(x-2)(x-3)}{-126} \cdot \cancel{-8}\,4 + \frac{(x+1)(x-2)(x-3)}{6} \cdot 3 +$$

$$\frac{(x+1)(x)(x-3)}{-6} \cdot 1 + \frac{(x+1)(x)(x-2)}{\cancel{12}\,6} \cdot 2!$$

$$= \frac{x^2-2x(x-3)}{6} \cdot 4 + \frac{(x^2-2x+x-2)(x-3)}{6} \cdot 3 +$$

$$\frac{(x^2+x)(x-3)}{-6} \cdot 1 + \frac{(x^2+x)(x-2)}{6} \cdot 1$$

$$= \frac{(x^3-3x^2-2x^2+6x)}{6} \cdot 4 + \frac{[x^3-2x^2+x^2-2x-3x^2+6x-3x+6]}{6} \cdot 3$$

$$+ \frac{(x^3+x^2-3x^2-3x)}{-6} + \frac{(x^3-2x^2+x^2-2x)}{6}$$

$$6y = 4x^3-12x^2-8x^2+24x+3x^3-6x^2+3x^2-6x-9x^2+18x-9x+18-x^3-x^2+3x^2+3x+x^3-2x^2+x^2-2x.$$

$$6y = 7x^3-31x^2+28x-18.$$

5) Find the eqn of the parabola passing through the points (0,0), (1,1), (2,20) using Lagrange's formula.

Soln:-

x :	0	1	2

$x_0 = 0$; $x_1 = 1$; $x_2 = 2$

$y_0 = 0$; $y_1 = 1$; $y_2 = 20$.

By using Lagrange's formula.

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)} \cdot 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \cdot 1$$

$$+ \frac{(x-0)(x-1)}{(2-0)(2-1)} \cdot 20$$

$$= 0 + \frac{x(x-2)}{(1)(-1)} \cdot 1 + \frac{x(x-1)}{(2)(1)} \cdot 20$$

$$= 0 + \frac{x^2-2x}{-1} \cdot 1 + \frac{x^2-x}{2} \cdot \cancel{20}^{10}$$

$$= 0 - x^2 + 2x + 10x^2 - 10x$$

$$f(x) = 9x^2 - 8x$$

$$y = 9x^2 - 8x.$$

Simpson's $\frac{1}{3}$ Rule:-

Putting $n=2$ in the above relation and neglecting all differences above the second we get,

$$\int_{x_0}^{x_0+2h} y(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[\frac{6y_0 + 6\Delta y_0 + \Delta^2 y_0}{6}\right]$$

$$= 2h\left[\frac{6y_0 + 6(y_1 - y_0) + y_2 - 2y_1 + y_0}{6}\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\int_{x_0}^{x_0+2h} y(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \rightarrow (1)$$

Similarly for the next two intervals x_0+2h to x_0+4h we get

$$\int_{x_0+2h}^{x_0+4h} y(x)\,dx = \frac{h}{3}\left[y_2+4y_3+y_4\right] \rightarrow (2)$$

In general,

$$\int_{x_0+(n-2)h}^{x_0+nh} y(x)\,dx = \frac{h}{3}\left[y_{n-2}+4\,y_{n-1}+y_n\right] \rightarrow (3)$$

Adding all the above integrals (1), (2), (3)

we get,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_0+4(y_1+y_3+\cdots)+2(y_2+y_4+\cdots)+y_n\right].$$

$$= \frac{h}{3}\left[(y_0+y_n)+4\,(\underbrace{\text{sum of odd ordinates}}_{y_1+y_3+y_5+\cdots})+2\,(\underbrace{\text{sum of even ordinates}}_{y_2+y_4+y_6+\cdots})\right].$$

This is called Simpson's one third rule or Simpson's $\frac{1}{3}$ rule.

Euler's Method:-

Consider the first order differential equation.

$$\frac{dy}{dx} = f(x,y) \rightarrow (1)$$

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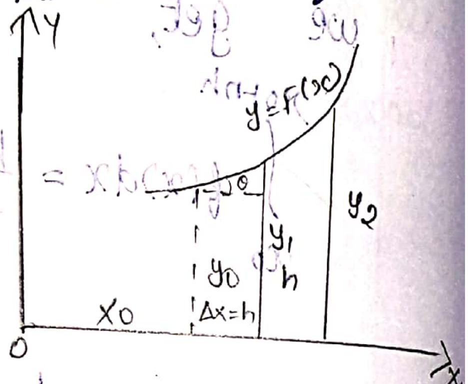
Let us solve this differential equation under the condition $y(x_0) = y_0$. The solution of ① gives y as a function x, which may be written symbollically as $y = F(x) \rightarrow$ ②.

The graph of ② is a curve in the XY plane, and since a smooth curve is practically straight for a short distance from any point on it we have from figure.



$$\tan\theta \approx \frac{\Delta y}{\Delta x}$$

(i.e) $\Delta y \approx \Delta x \tan\theta$

$$= \Delta x_0 \left(\frac{dy}{dx}\right)_0 \quad \left[\because \text{slope at } (x_0, y_0) = \left(\frac{dy}{dx}\right)_0 = \tan\theta\right]$$

$$\therefore y_1 = y_0 + \Delta y$$

$$y_1 = y_0 + \left(\frac{dy}{dx}\right)_0 \Delta x$$

$$y_1 \approx y_0 + f(x_0, y_0) h \quad [\text{Assuming } \Delta x = h]$$

$$\left[\because \frac{dy}{dx} = f(x, y) \text{ from ①} \right.$$

$$\left. \therefore \left(\frac{dy}{dx}\right)_0 = f(x_0, y_0)\right]$$

The next value of y corresponding to $x = x_2 (= x_1 + h)$ is

$$y_2 \approx y_1 + \left(\frac{dy}{dx}\right)_1 h$$

$$y_2 \approx y_1 + f(x_1, y_1) h, \quad \left[\because \left(\frac{dy}{dx}\right)_1 = f(x_1, y_1)\right]$$

$$y_3 \approx y_2 + f(x_2, y_2) h, \text{ etc}$$

In general, $y_{n+1} \approx y_n + f(x_n, y_n) h.$

By taking h small enough and proceeding in this manner, we could tabulate the expression (2) as a set of corresponding values of x & y.

This is method is given by Euler.

This is method is either too slow (in case of h is small) or two inaccurate (in case h is not small) for practical use.

25/9/19 IMPROVED EULER'S METHOD :-

Let the given first order differential equation of be

$$\frac{dy}{dx} = f(x, y) \rightarrow (1)$$

Let us solve this eqn under condition

$$y(x_0) = y_0$$

Starting with the initial value y_0, an approximate value of y_1 is computed from the relation as.

$$y_1 \approx y_0 + f(x_0, y_0)h \rightarrow (2)$$

sub, this approximate value of y_1 in (1) we get an approximate value of $\frac{dy}{dx}$ (x_1, y_1)

i.e, $\left(\frac{dy}{dx}\right)_1^{(1)} = f[x_1, y_1^{(1)}]$

Now an improved value of Δy_1 is found by multiply h with the mean value of $\frac{dy}{dx}$ at x_0 and x_1.

i.e., $$\Delta y = \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2} h$$

$$= \left\{\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right\} h$$

$$= \frac{h}{2}\left\{f(x_0, y_0) + f[x_0 + h, y_0 + h f(x_0, y_0)]\right\}$$

Now $y_1 = y_0 + \Delta y$

$$= y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f[x_0 + h, y_0 + h f(x_0, y_0)]\right\}$$

In general.

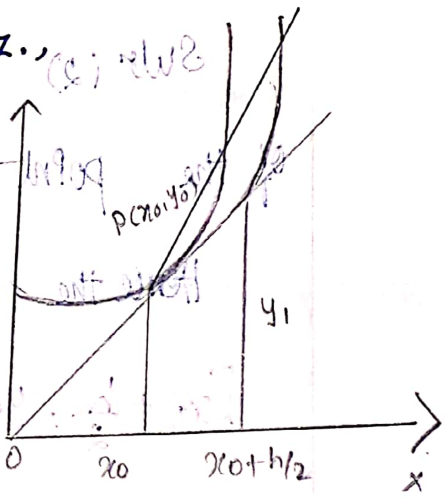
$$y_{m+1} = y_m + \frac{h}{2}\left\{ f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)] \right\}$$

This formula is called <u>Improved Euler's formula</u>.

MODIFIED EULER'S METHOD:-

In Improved Euler's method the solution curve is approximated in the interval $[x_0, x_0+h]$ by a straight line. This line is the passing through (x_0, y_0) whose slope is the averages of the slope viz.,

$$\frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1}{2}$$



But in modified Euler's method the curve is approximated by averaging the points.

Let $P(x_0, y_0)$ be the point on the solution curve. Let PA be the tangent at (x_0, y_0) to the curve.

Let this tangent meet the co-ordinates at $Q(x_0 + h/2)$ at P_1.

The y co-ordinates of the point P_1 is $y_0 + \Delta y$ .... (1) where Δy is the small increment along QP_1.

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Now, considering the triangle PP_1M, we have

$$\tan\theta = \frac{\Delta y}{\Delta x}$$

i.e., $\Delta y = (\tan\theta).\Delta x = \left(\frac{dy}{dx}\right).\frac{h}{2}$

$$\Delta y = \frac{h}{2} f(x_0, y_0)$$

$\{$ Here $\Delta x = \frac{h}{2}\}$

Sub (2) in (1) we get the y coordinates of the point P_1 is $y_0 + \frac{h}{2} f(x_0, y_0)$

Hence the co-ordinates of the point P_1 is

$\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right] \rightarrow (3)$

The slope at P_1 is $\left(\frac{dy}{dx}\right)$ at P_1

But $\left(\frac{dy}{dx}\right) = f(x, y)$ (Given diff-equation)

$\therefore \left(\frac{dy}{dx}\right)$ at $P_1 = f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$ (4)

$\{\because$ Replacing in (4): x by $x_0 + \frac{h}{2}$ & y by $y_0 + \frac{h}{2} f(x_0, y_0)\}$

Now, draw a line P_1B with this slope (slope at P_1). Then draw a line through

$P(x_0, y_0)$ and parallel to the line P_1B.

This line is taken to the approximate to the curve in the interval (x_0, x_0+h). The equation of this line viz., PC is,

$$y - y_0 = f\left[x_0+\frac{h}{2}, y_0+\frac{h}{2} f(x_0, y_0)\right](x-x_0) \rightarrow (5)$$

{using the formula $y-y_0 = m(x-x_0)$ which is the equation of the straight line passing through $h(x_0, y_0)$ and having slope m}.

Let this line (5) meets the ordinate $x = x_1 = (x_0+h)$ at the point (x_1, y_1).

Since (x_1, y_1) lies on the line (5) we have

$$y_1 - y_0 = (x_1 - x_0) f\left[x_0+\frac{h}{2}, y_0+\frac{h}{2} f(x_0, y_0)\right]$$

$$= h f\left[x_0+\frac{h}{2}, y_0+\frac{h}{2} f(x_0, y_0)\right]$$

ie., $y_1 = y_0 + h f\left[x_0+\frac{h}{2}, y_0+\frac{h}{2} f(x_0, y_0)\right]$.

In general.

$$y_{n+1} = y_n + h f\left[x_n+\frac{h}{2}, y_n+\frac{h}{2} f(x_n, y_n)\right].$$

(or)

$$y(x+h) = y(x) + h f\left[x+\frac{h}{2}, y+\frac{h}{2} f(x, y)\right]$$

This formula is called modified Euler's formula.

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03/10/19

1. Evaluate $\int_{0}^{1} \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = 0.2$ hence obtain an approximate value of π

0.785

Soln:-

$$f(x) = \frac{1}{1+x^2}$$

Here $h = 0.2$

$x_n = x_0 + nh$

$\frac{x_n - x_0}{h} = n$

$$n = \frac{x_n - x_0}{h} = \frac{1-0}{0.2} = 5$$

$x_0 = 0$

$x_1 = x_0 + h = 0.2$

$x_n = x_0 + nh$

$x_2 = x_0 + 2h = 0.4$

$x_0 = 0 + 0(0.2) = 0$

$x_3 = x_0 + 3h = 0.6$

$x_1 = 0 + 1(0.2) = 0.2$

$x_4 = x_0 + 4h = 0.8$

$x_5 = x_0 + 5h = 1.0$

		x_0	x_1	x_2	x_3	x_4	x_5
x	:	0	0.2	0.4	0.6	0.8	1.0
		y_0	y_1	y_2	y_3	y_4	y_5
$y = \frac{1}{1+x^2}$	:	1 ~~0.9615~~	0.9615	0.8621	0.7353	0.6098	0.5

$1 + 0.2^2 = 1.04$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots+y_{n-1})\right]$$

$$= \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_4+y_4)\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.2}{2}\left[(1+0.5)+2(0.9615+0.8621+0.7353+0.6098)\right]$$

$$= \frac{0.2}{2}\left[1.5+6.3374\right]$$

$$= 0.1\left[7.8374\right]$$

$$= 0.78374$$

Deduction :-

$$\int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} = 0.78374$$

$$\therefore \pi = 3.135$$

2. Evaluate $\int_0^1 e^{-x^2}\,dx$ by dividing the range of integration into 4 equal parts using Trapezoidal rule

Soln:-

Let $f(x) = e^{-x^2}$

$$h = \frac{x_n - x_0}{n} = \frac{1-0}{4} = 0.25$$

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$h = 0.25$

$e^{(-x^2)}$

	x_0	x_1	x_2	x_3	x_4
x :	0	0.25	0.50	0.75	1.00
	y_0	y_1	y_2	y_3	y_4
y :	1	0.9394	0.7788	0.5698	0.3679

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2+y_3)\right]$$

$$\int_0^1 e^{-x^2} dx = \frac{0.25}{2}\left[(1+0.3679) + 2(0.9394 + 0.7788 + 0.5698)\right]$$

$$= \frac{0.25}{2}\left[1.3679 + 4.5760\right]$$

$$= 0.1250\left[5.9439\right]$$

$$= 0.7430$$

3) Compute the value of definite integral

0.1 → 4 equal
1.2 → 6 equal.

$\int_4^{5.2} \log_e x \, dx$ (or) $\int_4^{5.2} \ln x \, dx$ using Trapezoidal rule.

Soln:-

$\ln(x)$

$$f(x) = \ln x$$

Let us divide the interval of the integral into 6 equal parts.

$x_0 = 4$

$h = \frac{5.2 - 4}{6} = 0.2$ $\qquad x_1 = x_0 + nh = 4 + 0.2 = 4.2$

x :	4	4.2	4.4	4.6	4.8	5.0	5.2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
y :	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

$$\int_4^{5.2} \ln x\,dx = \frac{0.2}{2}\left[(y_0 + y_6) + 2\left[(y_1 + y_2 + y_3 + y_4 + y_5\right]\right.$$

$$= \frac{0.2}{2}\left[(1.3863 + 1.6487) + 2\,(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)\right]$$

$$= \frac{0.2}{2}\left[(3.0350) + 2\,(7.6208)\right]$$

$$= \frac{0.2}{2}\left[(3.0350) + 15.2416\right]$$

$$= \frac{0.2}{2}\left[18.2766\right]$$

$$= 0.1\left[18.2766\right]$$

$$= 1.8277.$$

1) Use Simpson's $\frac{1}{3}$ rule to estimate the value of

$$\int_1^5 f(x)\,dx.$$

x :	1	2	3	4	5
	y_0	y_1	y_2	y_3	y_4
y $f(x)$:	13	50	70	80	100

<u>Soln:-</u> $\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\{(y_0+y_n)+3(y_1+y_2+y_4+y_5+y_7 \ldots)+2(y_3+y_6+y_9+\ldots)\}$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\{(y_0+y_n)+4\{y_1+y_3+y_5+\ldots\}+2\{y_2+y_4+y_6+\ldots\}\}$$

Here, $h=1$

$$\int_1^5 f(x)\,dx = \frac{1}{3}\{(y_0+y_4)+4(y_1+y_3)+2(y_2)\}$$

$$= \frac{1}{3}\{(13+100)+4(50+80)+2(70)\}$$

$$= \frac{1}{3}\{(113)+520+140\}$$

$$= \frac{1}{3}\{773\}$$

$$= 257.67$$

2) $\int_0^1 \frac{x^2}{1+x^3}\,dx$.

x : 1 2 3 4 5

y f(x) :

Soln:-

$f(x) = \frac{x^2}{1+x^3}$

$x_n = x_0 + nh$.

$h = \frac{x_n - x_0}{n}$

$n = \frac{x_n - x_0}{h}$

$\boxed{h = \frac{x_n - x_0}{n}} = \frac{1-0}{4} = 0.25$

$h = 0.25$

x :	0	0.25	0.50	0.75	1.00
y :	y_0 0	y_1 0.0615	y_2 0.2222	y_3 0.3956	y_4 0.5

$\int_0^1 \frac{x^2}{1+x^2}\,dx = \frac{0.25}{3}\{(y_0+y_4)+4(y_1+y_3)+2(y_2)\}$

$= \frac{0.25}{3}\{(0+0.5)+4(0.0615+0.3956)+2(0.2222)\}$

$= \frac{0.25}{3}\{(0.5)+1.8284+0.4444\}$

$= \frac{0.25}{3}\{2.7728\} = 0.2311$

3) $\int_0^4 e^x\,dx$

Soln:-

$f(x) = e^x$

$h = \frac{4-0}{4} = 1$

$h = 1$

x :	0	1	2	3	4
y :	1	2.7183	7.3891	y_3 20.0855	y_4 54.5982

$$\int_0^4 e^x dx = \frac{h}{3}\{(y_0+y_n)+4(y_1+y_3+y_5+\ldots)+2(y_2+y_4+\ldots)\}$$

$$= \frac{1}{3}\{(1+54.5982)+4(2.7183+20.0855)+2(\cancel{54.5982}\ 7.3891)\}$$

$$= \frac{1}{3}\{55.5982+91.2152+109.1964\}$$

$$= \frac{1}{3}\{55.5982+91.2152+14.7782\}$$

$$= 53.8639$$

4) $\int_0^{\pi/2} \sin x\, dx$.

Soln:-

$f(x) = \sin x$

$$h = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$$

x	:	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$
		y_0	y_1	y_2	y_3	y_4	y_5	y_6
y	:	0	0.2586	0.5	0.7071	0.8660	0.9659	1

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\{(y_0+y_n)+4(y_1+y_3+\ldots)+2(y_2+y_4+\ldots)\}$$

$$\int_0^{\pi/2} \sin x\, dx = \frac{h}{3}\{(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\}$$

$$= \frac{\pi}{3(12)}\{(0+1)+4(0.2586+0.7071+0.9659)+2(0.5+0.8660)\}$$

$$= \frac{\pi}{3(12)} \{(1) + 7.7260 + 2.7320\}$$

$$= \frac{\pi}{3(12)} \{11.4580\}$$

$$= 0.0873 \{11.4580\}$$

$$= 1.0003.$$

5) $\int_0^1 \frac{dx}{1+x^2}$

Soln:-

$$f(x) = \frac{1}{1+x^2}$$

$\cancel{h = \frac{1-0}{4} = \frac{1}{4} = 0.2}$

$$h = 0.2$$

$$n = \frac{1-0}{0.2} = 5$$

x	:	0	0.2	0.4	0.6	0.8	1.0
		y_0	y_1	y_2	y_3	y_4	y_5
y	:	1	0.9615	0.8621	0.7353	0.6098	0.5

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3} \{(y_0+y_n) + 4(y_1+y_3+y_5+\ldots) + 2(y_2+y_4+y_6+\ldots)\}$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{3} \{(y_0+y_5) + 4(y_1+y_3) + 2(y_2+y_4)\}$$

$$= \frac{0.2}{3} \{(1+0.5) + 4(0.9615 + 0.7353) + 2(0.8621 + 0.6098)\}$$

6) $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into a equal points using Simpson's rule.

Soln:-

$$f(x) = e^{-x^2}$$

$e^{(-x^2)}$

Here $n = 4$

$$h = \frac{x_0 - x_n}{4} = \frac{1-0}{4} = 0.25$$

x :	0	0.25	0.50	0.75	1.00
	y_0	y_1	y_2	y_3	y_4
y :	1	0.9394	0.7788	0.5698	0.3679

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\{(y_0+y_n) + 4(y_1+y_3+\ldots) + 2(y_2+y_4+\ldots)\}$$

$$\int_{x_0}^{x_n} e^{-x^2} dx = \frac{0.25}{3}\{(1+0.3679) + 4(0.9394+0.5698) + 2(0.7788)\}$$

$= 0.0833 \left[1.3679 + 6.0368 + 1.5576 \right]$

$= 0.7466$

7) $\int_{4}^{5.2} \ln x \, dx$

Soln:-

$f(x) = \ln(x)$

Let us divided the integral of the interval into 6 parts.

$$h = \frac{5.2-4}{6} = 0.2$$

x :	4	4.2	4.4	4.6	4.8	5.0	5.2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
y :	1.3868	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3} \left[(y_0+y_n) + 4(y_1+y_3+\cdots) + 2(y_2+y_4+\cdots) \right]$$

$$\int_{4}^{5.2} \ln x \, dx = \frac{0.2}{3} \left[(1.3868+1.6487) + 4(1.4351+1.5261+1.6094) + 2(1.4816+1.5686) \right]$$

$= 0.0667 \left[(3.0355) + 18.2824 + 6.1004 \right]$

$= 1.8288$

Trapezoidal rule:-

4) To estimate the value of $\int_1^5 f(x)\, dx$ curves

x :	1	2	3	4	5
	y_0	y_1	y_2	y_3	y_4
y :	13	50	70	80	100

Soln:-

Here $h = 1$

$$\int_{x_0}^{x_n} f(x)\, dx = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$\int_1^5 f(x)\, dx = \frac{1}{2}\left[(13+100)+2(50+70+80)\right]$$

$$= \frac{1}{2}\left[(113)+400\right]$$

$$= \frac{1}{2}\left[513\right]$$

$$= 256.50.$$

5) $\int_0^1 \frac{x^2}{1+x^3}\, dx$

Soln:-

$$f(x) = \frac{x^2}{1+x^3}$$

$$\frac{x^2}{(1+x^3)}$$

$$h = \frac{1-0}{4} = 0.25$$

x :	0	0.25	0.50	0.75	1.00
	y_0	y_1	y_2	y_3	y_4
y :	0	0.0615	0.2222	0.3956	0.5

$$\int_{x_0}^{x_n} f(x)\, dx = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{0.25}{2}\Big[(0+0.5)+2(0.0615+0.2222+0.3956)\Big]$$

$$=0.1250\,[0.5+1.3586]$$

$$=0.1250\,[1.8586]$$

$$=0.2323$$

6) $\int_0^{\pi/2} \sin x\,dx$

Soln:-

$f(x) = \sin x$ $\qquad\qquad$ $\sin(x)$

$$h = \frac{\pi/2 - 0}{6} = \frac{\pi}{12} = 0.2618$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\Big[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\Big]$$

x_0 :	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
y :	0	0.2586	0.5	0.7071	0.8660	0.9659	1

$$\int_0^{\pi/12} \sin x\,dx = \frac{0.2618}{2}\Big[(0+1)+2(0.2588+0.5+0.7071+0.8660+0.9659)\Big]$$

$$=0.0873\,[1+6.5952]$$

$$=0.6631$$

7) $\int_0^4 e^x\,dx.$

Soln:-

Let $f(x) = e^x$

$$h = \frac{4-0}{4} = 1$$

x :	0	1	2	3	4
	y_0	y_1	y_2	y_3	y_4
y :	1	2.7183	7.3891	20.0855	54.5982

$$\int_{x_0}^{x_n} f(x)dx = \frac{1}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$= \frac{1}{2}\left[(1+54.5982)+2(2.7183+7.3891+20.0855)\right]$$

$$= \frac{1}{2}\left[55.5982+60.0358\right]$$

$$= \frac{1}{2}\left[115.9840\right]$$

$$= 57.9920.$$

1) Euler's Method Problem:-

Given $\frac{dy}{dx} = -y$ and $y(0)=1$, determine the value of y at $x = 0.01, 0.02$ & 0.03. By Euler's method and compose with exact value of 'y'.

Soln:-

we have $\frac{dy}{dx} = -y = f(x,y)$ (say)

$x_0 = 0$; $y_0 = 1$

$y_0 = 1$	$y_1 = ?$	$y_2 = ?$	$y_3 = ?$
$x_0 = 0$	$x_1 = 0.01$	$x_2 = 0.02$	$x_3 = 0.03$

$h = 0.01$

By Euler's formula,

$$\boxed{y_1 = y_0 + h\, f(x_0, y_0)}$$

$= 1 + (0.01)\, f(0,1)$

$= 1 + (0.01)(-1)$

$= 1 - 0.01$

$= 0.99$

$$\boxed{y_2 = y_1 + h f(x_1, y_1)}$$

$= 0.99 + (0.01) f(0.01, 0.99)$ $f(x,y) = -y$

$= 0.99 + (0.01)(-0.99)$

$= 0.99 - 0.0099$

$= 0.9801$

$$\boxed{y_3 = y_2 + h f(x_2, y_2)}$$

$= 0.9801 + (0.01) f(0.02, 0.9801)$

$= 0.9801 + (0.01)(-0.9801)$

$= 0.9801 - 0.009801$

$= 0.9703$

By Exact solution

By variable separable

$$\frac{dy}{dx} = -y$$

$$\int \frac{dy}{y} = -\int dx$$

$$\log y = -x + c$$

$$y = e^{-x} \cdot e^{c}$$

$$y = c e^{-x}$$

at $x = 0,\ y = 1$

$$1 = c e^{-0}$$

$\therefore \boxed{c = 1}$

$\therefore y = e^{-x}$

at $x = 0.01$

$y = e^{-0.01} = 0.990049 = 0.99005$

$x = 0.02$

$y = e^{-0.02} = 0.980199 = 0.9801$

$x = 0.03$

$y = e^{-0.03} = 0.9704$

	at $x=0.01$	at $x=0.02$	at $x=0.03$
Euler's Method	0.99	0.9801	0.9703
Exact value of y	0.99005	0.9801	0.9704

2) Complete y at $x = 0.25$ and 0.50 by modified Euler method given that $y' = 2xy$ and $y(0) = 1$

Soln:-

We have $\frac{dy}{dx} = 2xy = f(x,y)(2xy)$

$x_0 = 0$; $y_0 = 1$

$y_0 = 1$ $\quad$ $y_1 = ?$ $\quad$ $y_2 = ?$

$x_0 = 0$ $\quad$ $x_1 = 0.25$ $\quad$ $x_2 = 0.50$

$h = 0.25$

By Modified Euler's Method:- $\quad f(x_0, y_0) = 2(0)(1) = 0$

$y_1 = y_0 + h\left[f\left(x_0 + h/2,\ y_0 + h/2\, f(x_0, y_0)\right)\right]$

$y_0 + \frac{h}{2} f[x_0, y_0] = 1 + \frac{0.25}{2} + (0, 1)$

$= 1 + \frac{0.25}{2}\,(2(0)(1)) = 1$

$f(x_0 + h/2, y_0 + h/2\, f(x_0, y_0)) = f(0 + \frac{0.25}{2}, 1)$

$= f(0.125, 1)$

$= 2(0.125)(1)$

$= 0.25$

$y_1 = y_0 + h\,[f(x_0 + h/2, y_0 + h/2\, f(x_0, y_0)]$

$= 1 + (0.25)(0.25)$

$= 1 + 0.0625$

$= 1.0625.$

$y_2 = y_1 + h\,[f(x_1 + h/2, y_1 + h/2\, f(x_1, y_1))]$

$y_1 + h/2\, f(x_1, y_1) = 1.0625 + \frac{0.25}{2} f(0.25, 1.0625)$

$= 1.0625 + (0.125)(2(0.25)(1.0625))$

$= 1.1289.$

$f(x_1 + h/2, y_1 + h/2\, f(x_1, y_1)) = f(0.25 + \frac{0.25}{2}, 1.1289)$

$= f(0.375, 1.1289)$

$= 2(0.375)(1.1289)$

$= 0.8467.$

$y_2 = y_1 + h\,[f(x_1 + h/2, y_1 + h/2\, f(x_1, y_1))]$

$= 1.0625 + (0.25)(0.8467)$

$= 1.2742.$

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3) Using Improved Euler's method find 'y' at $x=0.1$ and 0.2 given that $\frac{dy}{dx} = y - \frac{2x}{y}$ and $y(0)=1$

Soln:-

we have $\frac{dy}{dx} = y - \frac{2x}{y} = f(x,y)$ (say)

$x_0 = 0$; $y_0(0) = 1$

$y_0 = 1$ $\quad$ $y_1 = ?$ $\quad$ $y_2 = ?$

$x_0 = 0$ $\quad$ $x_1 = 0.1$ $\quad$ $x_2 = 0.2$

$h = 0.1$

By Improved Euler's Method:-

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h, hf(x_0,y_0))\right]$$

$$f(x_0+h, hf(x_0,y_0)) = f\left[0+0.1, (0.1)f(0,1)\right]$$

$\frac{dy}{dx} = y - \frac{2x}{y}$

$$= f\left[0.1, (0.1)\left(1-\frac{2(0)}{1}\right)\right]$$

$$= f\left[0.1, (0.1)\right]$$

$$= f\left[0.1, 0.1\right]$$

$$= 0.1 - \frac{2(0.1)}{0.1}$$

$\frac{dy}{dx} = y - \frac{2x}{y}$

$$= 0.1 - 2$$

$$= -1.9$$

$$y_1 = y_0 + h/2\left[f(x_0,y_0) + f(x_0+h, hf(x_0,y_0))\right]$$

$$= 1 + \frac{0.1}{2}\left[f(0,1) + (-1.9)\right]$$

$= 1+0.05[1-1.9]$

$= 1+(0.05)(-0.9)$

$= 1-0.045$

$= 0.955$

$$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_1+h, hf(x_1, y_1))\right]$$

$$= 0.955 + \frac{0.1}{2}\left[f(0.1, 0.955) + f(0.1+0.1, hf(0.1, 0.955))\right]$$

$$= 0.955 + \frac{0.1}{2}\left[f(0.1, 0.955) + f(0.1+0.1, hf(0.1, 0.9855))\right]$$

$$f(0.1+0.1, hf(0.1, 0.955)) = f(0.1+0.1, (0.1)f(0.1, 0.955))$$

$\frac{dy}{dx} = y - \frac{2x}{y}$

$$= f\left(0.2, (0.1)\left(0.955 - \frac{2(0.1)}{(0.955)}\right)\right)$$

$$= f(0.2, 0.1)(0.955 - 0.2094)$$

$$= f(0.2, 0.1[0.7456])$$

$$= f(0.2, 0.07456)$$

$$= 0.07456 - \frac{2(0.2)}{0.07456} \qquad \frac{dy}{dx} = y - \frac{2x}{y}$$

$$= 0.07456 - 5.36481$$

$$= -5.29025.$$

$$y_2 = 0.9855 + \frac{0.1}{2}\left[f(0.1, 0.955) + (-5.29025)\right]$$

$$= 0.955 + 0.05\left[0.955 - \frac{2(0.1)}{0.955}\right] - (5.29025)$$

$$= 0.955 + 0.05(0.955 - 0.20942) - 5.29025$$

$= 0.955 + 0.05\,(0.74558) - 5.29025$

$= 0.955 + 0.037279 - 5.29025\,(0.05)$

$= 0.955 + 0.037279 - 0.2645$

$y_2 = 0.7278.$

4) Using Improved Euler's method find $y(0.2)$ and $y(0.4)$ from $y' = x + y,\ y(0) = 1$ with $h = 0.2$

Soln:-

The Improved Euler's is.

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f(x_m + h, y_m + h f(x_m, y_m))$$

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_0 + h, y_0 + h f(x_0, y_0)) \rightarrow (1)$$

Putting $m = 0$ in (1) we get

$$\boxed{y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + h f(x_0, y_0)]\}} \rightarrow (2)$$

Here $x_0 = 0,\ y_0 = 1,\ f(x, y) = x + y,\ h = 0.2.$

$\therefore f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1 \rightarrow (3)$

Sub (3) in (1) we get

$$y_1 = y_0 + \frac{h}{2}\{[1 + f(x_0 + h, y_0 + h.1]\}$$

$$= 1 + 0.1\{1 + f[x_0 + h, y_0 + h]\}.$$

$$= 1 + 0.1\{1 + f(0 + 0.2, 1 + 0.2)\}$$

$$= 1 + 0.1\{1 + f(0.2, 1.2)\} \rightarrow (4)$$

Now $f(0.2, 1.2) = 0.2 + 1.2 = 1.4 \rightarrow (5)$

sub (5) in (4) we get.

$$y_1 = 1 + 0.1\{1+1.4\}$$
$$= 1 + (0.1)(2.4) = 1 + 0.24$$
$$y_1 = 1.24$$
$$\therefore y(0.2) = 1.24$$

Putting $m=1$ in (1) we get

$$y_2 = y_1 + \frac{h}{2}\{f(x_1,y_1) + f(x_1+h, y_1 + h f(x_1,y_1))] \rightarrow (6)$$

Here $x_1 = 0.2$, $y_1 = 1.24$, $h = 0.2$

Now, $f(x_1,y_1) = f(0.2, 1.24) = 0.2 + 1.24$

$$f(x_1,y_1) = 1.44 \rightarrow (7)$$

sub (7) in (6) we get

$$y_2 = 1.24 + \frac{0.2}{2}\{1.44 + f[0.2+0.2, 1.24 + 0.2(1.44)]\}$$

$$y_2 = 1.24 + 0.1\{1.44 + f(0.4, 1.528)\} \rightarrow (8)$$

Now $f(0.4, 1.528) = 0.4 + 1.528$

$$f(0.4, 1.528) = 1.928 \rightarrow (9)$$

sub (9) in (8) we get

$$y_2 = 1.24 + 0.1(1.44 + 1.928)$$
$$= 1.24 + 0.1(3.368)$$
$$y_2 = 1.24 + 0.3368 = 1.5768$$
$$\therefore y(0.4) = 1.5768.$$

5) Use Improved Euler's method solve $y' = x+y+xy$, $y(0)=1$ compute y at $x=0.1$, by taking $h=0.1$

Soln:-

Given $y' = x+y+xy$, $x_0=0$, $y_0=1$ and $h=0.1$

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\left[f(x_m, y_m) + f(x_m+h, y_m+hf(x_m, y_m))\right] \rightarrow (1)$$

Putting $m=0$ in (1) we get.

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f[x_0+h, y_0+hf(x_0, y_0)]\right] \rightarrow (2)$$

Here $x_0=0$, $y_0=1$, $f(x,y) = x+y+xy$

$$\therefore f(x_0, y_0) = x_0+y_0+x_0y_0 = 0+1+0(1) = 1 \rightarrow (3)$$

Sub (3) in (1) we get

$$y_1 = 1 + \frac{0.1}{2}\{1 + f(0+0.1 ; 1+0.1(1))\} \rightarrow (4)$$

$$y_1 = 1+0.05\{1+f(0.1, 1.1)\}$$

Now $f(0.1, 1.1) = 0.1 + 1.1 + (0.1)(1.1) = 1.31 \rightarrow (5)$

Sub (5) in (4) we get

$$y_1 = 1+0.05(1+1.31)$$

$$y(0.1) = 1+(0.05)(2.31) = 1+0.1155$$

$$y(0.1) = 1.1155$$

6) Using Improved Euler's method find y at $x=0.1$ and $x=0.2$ given $\frac{dy}{dx}=y-\frac{2x}{y}$, $y(0)=1$

Soln:-

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m+h, y_m+h\, f(x_m, y_m)]\} \rightarrow (1)$$

Putting $m=0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f[x_0+h, y_0+h\, f(x_0, y_0)]] \rightarrow (2)$$

Here $x_0=0$, $y_0=1$, $f(x,y) = y-\frac{2x}{y}$

$$\therefore f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1-0=1 \rightarrow (3)$$

sub (3) in (2) we get

$$y_1 = y_0 + \frac{h}{2}\{[1+f(x_0+h, y_0+h\cdot 1)]\}$$

$$= 1+\frac{0.1}{2}\{1+f(0+0.1, 1+0.1)\}$$

$$= 1+\frac{0.1}{2}\{1+f(0.1, 1.1)\} \rightarrow (4)$$

Now $f(0.1, 1.1) = 1.1 - \frac{2(0.1)}{1.1} = 0.9182 \rightarrow (5)$

sub (5) in (4) we get

$$y_1 = 1+\frac{0.1}{2}(1+0.9182)$$

$$\therefore y(0.1) = 1.0959$$

Putting $m=1$ in (1) we get

$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f[x_1+h, y_1+h\, f(x_1, y_1)]] \rightarrow (6)$$

Here $x_1 = 0.1$, $y_1 = 1.0959$

Now $f(x_1, y_1) = f(0.1, 1.0959)$

$$= 1.0959 - \frac{2(0.1)}{1.0959} \rightarrow 7$$

$$f(x_1, y_1) = 0.9135$$

Sub 7 in 6 we get

$f(x_0 + h, y_1 + h f(x_0$

$$y_2 = y_1 + \frac{h}{2}\{0.9135 + f[0.2, y_1 + h(0.9135)]$$

$$= 1.0959 + \frac{0.1}{2}\{0.9135 + f(0.2, 1.0959 + 0.1(0.9135))]\}$$

$x_0 + h$ $y_1 + h$

$$y_2 = 1.0959 + 0.05\{0.9135 + f(0.2, 1.1872)\} \rightarrow 8$$

Now $f(0.2, 1.1872) = 1.1872 - \frac{2(0.2)}{1.1872}$ $\rightarrow 8$

$$= 0.8503$$

Sub 9 in 8 we get

$$y_2 = 1.0959 + 0.05\{0.9135 + 0.8503\}$$

$$y(0.2) = 1.1841$$

x	0	0.1	0.2
y	1	1.0959	1.1841

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7) solve $\frac{dy}{dx} = y + e^x$, $y(0) = 0$ for $x = 0.2; 0.4$ by using Improved Euler's method.

<u>soln:-</u>

Given $\frac{dy}{dx} = y + e^x$; $x_0 = 0$, $y_0 = 0$ and $h = 0.2$

The Improved Euler's algorithm is:

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \rightarrow (1)$$

Putting $m = 0$ in (1) we get,

$$y_1 = y_0 + h/2\{f(x_0, y_0) + f[x_0 + h, y_0 + h f(x_0, y_0)]\} \rightarrow (2)$$

Here $x_0 = 0$, $y_0 = 0$

$$f(x, y) = y + e^x$$

$$f(x_0, y_0) = 0 + e^0 = 1 \rightarrow (3)$$

Sub (3) in (2), we get

$$y_1 = y_0 + \frac{h}{2}\{1 + f(x_0 + h, y_0 + h)\}$$

$$= 0 + \frac{0.2}{2}\{1 + f(0 + 0.2, 0 + 0.2)\} \rightarrow (4)$$

Now $f(0.2, 0.2) = 0.2 + e^{0.2}$

Sub (5) in (4), we get

$$y_1 = y(0.2) = 0.1[1 + 0.2 + e^{0.2}]$$

$$= 0.1[1.2 + 1.2214]$$

$$y_1 = 0.24214$$

Putting $m = 1$ in (1) we get

$$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f[x_1 + h, y_1 + h f(x_1, y_1)]\} \rightarrow (6)$$

Here $x_1 = 0.2$, $y_1 = 0.24214$ and $h = 0.2$.

Now $f(x_1, y_1) = y_1 + e^{x_1}$

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$$f(0.2, 0.24214) = 0.24214 + e^{0.2}$$

$$= 1.46354 \rightarrow (7)$$

$$y_1 + h f(x_1, y_1) = 0.24214 + (0.2)(1.46354)$$

$$= 0.53485 \rightarrow (8)$$

$$f[x_1+h, y_1 + h f(x_1, y_1)] = f(0.4, 0.53485)$$

$$= 0.53485 + e^{0.4}$$

$$= 2.02667 \rightarrow (9)$$

Sub (7) and (9) in (6) we get

$$y_2 = y(0.4) = 0.24214 + \frac{0.2}{2}[1.46354 + 2.02667]$$

$$= 0.059116$$

$$\therefore y(0.4) = 0.59116$$

x	0	0.2	0.4
y	0	0.24214	0.59116

8) Given $y' = x^2 - y$, $y(0) = 1$, find correct to four decimal places the value of $y(0.1)$, by using Improved Euler's Method.

Soln:-

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\}$$

→(1)

Putting $m=0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0,y_0) + f[x_0+h, y_0+hf(x_0,y_0)]\} \rightarrow (2)$$

Here $x_0=0$, $y_0=1$ and $h=0.1$; $f(x,y)=x^2-y$

$$\therefore f(x_0,y_0) = x_0^2 - y_0 = (0)^2 - 1 = -1 \rightarrow (3)$$

sub (3) in (2), we get

$$y_1 = 1+\frac{0.1}{2}\{-1+f[0+0.1, 1+0.1(-1)]\}$$

$$= 1+0.05\{(-1)+f(0.1,0.9)\} \rightarrow (4)$$

Now $f(0.1, 0.9) = (0.1)^2 - 0.9 = 0.01-0.9 = -0.89 \rightarrow (5)$

sub (5) in (4) we get.

$$y_1 = 1+0.05\{(-1)-0.89\} = 0.9055$$

$$\therefore y(0.1) = 0.9055$$

9) Find the values of $y(1.2)$ and (1.4) using Improved Euler's method with $h=0.2$, given that $\frac{dy}{dx} = \frac{2y}{x} + x^3$; $y(1) = 0.5$,

Soln:-

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m+h, y_m+hf(x_m,y_m)]\} \rightarrow (1)$$

Putting $m=0$ in (1), we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0,y_0) + f[x_0+h, y_0+h f(x_0,y_0)]\} \rightarrow (2).$$

Here $x_0=1$, $y_0=0.5$ and $h=0.2$.

Now $f(x_0, y_0) = \frac{2y_0}{x_0} + x_0^3 = 1+1 = 2 \rightarrow ③$

Sub ③ in ②, we get

$$y_1 = y_0 + \frac{h}{2} = \frac{2y_0}{x_0} \{2 + f(x_0+h, y_0+2h)\}$$

$$= 0.5 + 0.1\,[2 + f(1.2, 0.9)] \rightarrow ④$$

Now,

$$f(1.2, 0.9) = \frac{2(0.9)}{1.2} + (1.2)^3 = 3.228 \rightarrow ⑤$$

Sub ⑤ in ④, we get

$$y_1 = y(1.2) = 0.5 + 0.1[2 + 3.228]$$

$$= 1.0228$$

$$y(1.2) = 1.0228.$$

Putting $m=1$ in (1) we get.

$$y_2 = y_1 + \frac{h}{2}\{f(x_1,y_1) + f[x_1+h, y_1+hf(x_1,y_1)]\} \rightarrow ⑥$$

Here $x_1 = 1.2$; $y_1 = 1.0228$ and $h = 0.2$

$$f(x_1,y_1) = f(1.2, 1.0228) = \frac{2(1.0228)}{1.2} + (1.2)^3$$

$$= 1.70466667 + 1.728 = 3.43267$$

$$y_1 + hf(x_1,y_1) = 1.0228 + 0.2(3.43267).$$

$$= 1.70933 \rightarrow ⑦.$$

Sub ⑦ in ⑥ we get,

$$y_2 = 1.0228 + 0.1\,[3.43267 + f(1.4, 1.70933)] \quad \to (8)$$

Now $f(1.4, 1.70933) = \frac{2(1.70932)}{1.4} + (1.4)^3 \to (9)$

sub (9) in (8), we get

$$y_2 = y(1.4) = 1.0228 + 0.1\,[3.43267 + 5.18590]$$

$$= 1.884657$$

$$\therefore y(1.4) = 1.8847$$

x	0	1.2	1.4
y	0	1.0228	1.8847

10) Using Modified Euler Method, find the solution of initial value problem $\frac{dy}{dx} = \log(x+y)$, $y(0)=2$ at $x=0.2$ by assuming $h=0.2$.

Soln:-

Given $\frac{dy}{dx} = \log(x+y)$, $x_0=0$; $y_0=2$, $h=0.2$

The modified Euler's algorithm is

$$y_{n+1} = y_n + h f\left[x_n+\frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right] \quad \to (1)$$

Putting $n=0$ in (1) we get

$$y_1 = y_0 + h f\left[x_0+\frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$y_1 = 2 + (0.2) f\left[0+\frac{0.2}{2}, 2 + \frac{0.2}{2} f(x_0, y_0)\right] \quad \to (2)$$

Now, $f(x,y) = \log(x+y)$

$f(x_0, y_0) = \log(x_0+y_0) = \log(0+2) = \log 2$.

$f(x_0, y_0) = 0.3010 \to (3)$

sub (3) in (2), we get

$$y_1 = 2 + 0.2\, f\left[0.1,\, \underline{2 + 0.1\,(0.3010)}\right] \to ④$$

Now $f(0.1, 2.0301) = \log(0.1 + 2.0301)$

$$= \log(2.1301)$$

$$f(0.1, 2.0301) = 0.3284 \quad \to ⑤$$

Sub ⑤ in ④, we get

$$y_1 = 2 + 0.2\,(0.3284)$$

$$y_1 = 2 + 0.0657$$

$$= 2.0657.$$

$$\therefore y(0.2) = 2.0657.$$

11) Solve the equation $\frac{dy}{dx} = 1 - y$, given $y(0) = 0$ using Modified Euler's Method and tabulate the solutions at $x = 0.1, 0.2$ and 0.3

Soln:-

The modified Euler's algorithm is

$$y_{n+1} = y_n + h\, f\left[x_n + \frac{h}{2},\, y_n + \frac{h}{2} f(x_n, y_n)\right] \to ①$$

Putting $n = 0$ in (1), we get

$$\boxed{y_1 = y_0 + h\, f\left[x_0 + \frac{h}{2},\, y_0 + \frac{h}{2} f(x_0, y_0)\right]} \to ②$$

Here $x_0 = 0$; $y_0 = 0$ and $h = 0.1$

$$f(x, y) = \frac{dy}{dx} = 1 - y$$

$$f(x_0, y_0) = 1 - y_0 = 1 - 0 = 1 \quad \to ③$$

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$$y_1 = 0 + (0.1)\, f\left[0 + \frac{0.1}{2},\ 0 + \frac{0.1}{2}(1)\right]$$

$$y_1 = (0.1)\, f(0.05, 0.05) \rightarrow (4)$$

Now $f(0.05, 0.05) = 1 - 0.05 = 0.95 \rightarrow (5)$

Sub (5) in (4), we get

$$y_1 = (0.1)(0.95) = 0.095$$

$$y(0.1) = 0.095$$

Putting $n=1$ in (1), we get

$$y_2 = y_1 + h f\left[x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

Here $x_1 = 0.1$; $y_1 = 0.095$ and $h = 0.1$

$$y_2 = 0.095 + (0.1)\, f\left[0.1 + \frac{0.1}{2},\ 0.095 + \frac{0.1}{2} f(0.1, 0.095)\right]$$

Now $f(0.1, 0.095) = 1 - y_1 = 1 - 0.095 = 0.905 \rightarrow (6)$

$$y_2 = 0.095 + 0.1\, f[0.15,\ 0.095 + 0.05(0.905)]$$

$$= 0.095 + 0.1\, f[0.15,\ 0.095 + 0.04525]$$

$$= 0.095 + (0.1)\, f[0.15,\ 0.14025] \rightarrow (7)$$

$$f(0.15, 0.14025) = 1 - 0.14025$$

$$= 0.85975 \rightarrow (8)$$

Sub (8) in (7) we get

$$y_2 = 0.095 + 0.1(0.85975)$$

$$= 0.095 + 0.085975 = 0.180975$$

$$y(0.2) = 0.1809$$

Putting $n=2$ in (1), we get

$$y_3 = y_2 + hf\left[x_2+\frac{h}{2},\ y_2+\frac{h}{2}f(x_2,y_2)\right]$$

Here $x_2 = 0.2$; $y_2 = 0.1809$ and $h = 0.1$

$$y_3 = 0.1809 + (0.1) f\left[0.2+\frac{0.1}{2},\ 0.1809+\frac{0.1}{2} f(0.2,\ 0.1809)\right]$$

$$y_3 = 0.1809 + (0.1) f\left[0.25,\ 0.1809 + 0.05 f(0.2,\ 0.1809)\right] \to (9)$$

Now $f(0.2,\ 0.1809) = 1 - 0.1809 = 0.8191 \to (10)$

sub (10) in (9), we get

$$y_3 = 0.1809 + 0.1 f\left[0.25,\ 0.1809 + 0.05(0.8191)\right]$$

$$= 0.1809 + 0.1 f\left[0.25,\ 0.221855\right]$$

Now,

$$f\left[0.25,\ 0.221855\right] = 1 - 0.221855$$

$$f\left[0.25,\ 0.221855\right] = 0.778145 \quad \to (11)$$

Sub (11) in (9), we get

$$y_3 = 0.1809 + 0.1(0.778145)$$

$$= 0.1809 + 0.0778$$

$$= 0.2587.$$

THE RUNGE - KUTTA METHOD:-

This method was derived by Runge. Therefore we call this method as Runge. Kutta Method.

Here the set of formulae are given without proof for solving the differential equation of the form.

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0 .$$

Let h denotes the length of interval b/w the equidistant values of x.

Runge-Kutta Second Order:-

If their initial values are x_0, y_0 for the differential equation

$$\frac{dy}{dx} = f(x,y)$$

Then the 1^{st} increment in y, Δy is computed form.

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$$

$$\Delta y = K_2$$

Now $x_1 = x_0 + h$; $y_1 = y_0 + \Delta y$, the increment in y for the second interval is computed by

$$K_1 = hf(x_1, y_1)$$

$$K_2 = hf(x_1 + h/2, y_1 + k_1/2)$$

$\Delta y = K_2$ and so on.

Runge-Kutta third Order:

The third order Runge-Kutta method is designed by the following formulae.

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$$

$$K_3 = hf(x_0 + h, y_0 + 2K_2 - K_1)$$

Now the first increment in y, Δy is computed form.

$$\Delta y = \frac{1}{6}(K_1 + 4K_2 + K_3)$$

Now $x_1 = x_0 + h$; $y_1 = y_0 + \Delta y$, the increment in y for the 2nd interval is computed in a similar manner.

$$K_1 = hf(x_1, y_1)$$

$$K_2 = hf(x_1 + h/2, y_1 + K_1/2)$$

$$K_3 = hf(x_1 + h, y_1 + 2K_2 - K_1)$$

and $\Delta y = \frac{1}{6}(K_1 + 4K_2 + K_3)$

Runge-kutta fourth Order:-

The fourth order runge-kutta method is designed by the following formulae.

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left(x_0 + h/2,\ y_0 + K_1/2\right)$$

$$K_3 = hf\left(x_0 + h/2,\ y_0 + K_2/2\right)$$

$$K_4 = hf\left(x_0 + h/2,\ y_0 + K_3\right)$$

$$\Delta y = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right).$$

Now $x_1 = x_0 + h$; $y_1 = y_0 + \Delta y$

The increment in y for the second, third and so on intervals is computed in the similar manner.

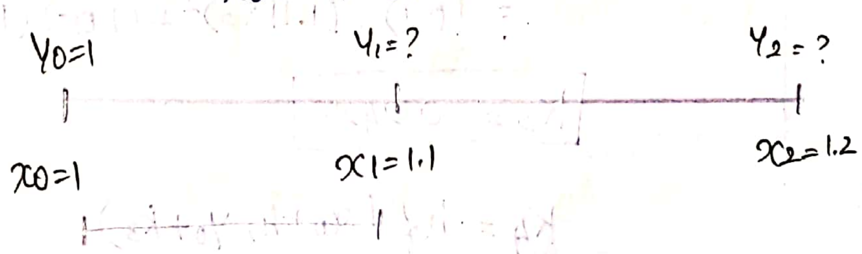
Problem:-

1) Find the value of $y(1.1)$ and $y(1.2)$ using the Runge-kutta method of the fourth Order given that $\frac{dy}{dx} = y^2 + xy$ and $y(1) = 1$

Soln:-

We have, $\frac{dy}{dx} = y^2 + xy = f(x, y)$

$x_0 = 1,\ y_0 = 1$



$h = 0.1$

$x_0 = 1, y_0 = 1$

$K_1 = hf(x_0, y_0)$

$= (0.1) f(1,1)$ $\quad f(x,y) = y^2 + xy$

$= (0.1)(1^2 + (1)(1))$ $\quad f(x_0, y_0) = 1^2 + (1)(1)$

$= (0.1)(2)$ $\quad = 2$

$\boxed{K_1 = 0.2}$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= (0.1) f\left(1 + \frac{0.1}{2}, 1 + \frac{0.2}{2}\right)$ $\quad f(x,y) = y^2 + xy$

$= (0.1) f(1.05, 1.1)$

$= (0.1)((1.1)^2 + (1.05)(1.1))$

$= (0.1)(1.21 + 1.155)$

$= (0.1)(2.365)$

$= (0.1)(2.365)$

$\boxed{K_2 = 0.2365}$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= (0.1) f\left(1 + \frac{0.1}{2}, 1 + \frac{0.2365}{2}\right)$

$= (0.1) f(1.05, 1.1183)$

$= (0.1)((1.1183)^2 + (1.05)(1.1183))$

$\boxed{K_3 = 0.2425}$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.1) f(1 + 0.1, 1 + 0.2425)$

$= (0.1) f\ (1.1, 1.2425)$

$= (0.1)\ ((1.2425)^2 + (1.1)\ (1.2425))$

$= 0.2911$

$\therefore \Delta y = \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}\ [0.2 + 2\ (0.2365) + 2\ (0.2425) +$
$0.2911)$

$= \frac{1}{6}\ [0.2 + 0.473 + 0.485 + 0.2911)$

$\boxed{\Delta y = 0.2415.}$

$\therefore \underline{y_1 = y_0 + \Delta y}$

$= 1 + 0.2415$

$\boxed{y_1 = 1.2415}$

$K_1 = hf\ (x_1, y_1)$

$= (0.1) f\ (1.1, 1.2455)$

$= (0.1)\ ((1.2415)^2 + (1.1)\ (1.2415))$

$= 0.2907$

$K_2 = hf\ (x_1 + h/2,\ y_1 + K_1/2)$

$= (0.1) f\ \left(1.1 + \frac{0.1}{2},\ 1.2415 + \frac{0.2907}{2}\right)$

$= (0.1) f\ (1.15, 1.3869)$

$= (0.1)\ ((1.3869)^2 + (1.15)\ (1.3869))$

$= 0.3518.$

$K_3 = hf\ (x_1 + h/2,\ y_1 + K_2/2)$

$= (0.1) f\ \left(1.1 + \frac{0.1}{2},\ 1.2415 + \frac{0.3518}{2}\right)$

$= (0.1) f\ (1.15, 1.4174)$

$= (0.1) f\ ((1.4174)^2 + (1.15)(1.4174))$

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$= 0.3639$

$K_4 = hf(x_1+h, y_1+k_3)$

$= (0.1) f(1.1+0.1, 1.2415+0.3639)$

$= (0.1) f(1.2, 1.6054)$

$= (0.1) ((1.6054)^2 + (1.2)(1.6054))$

$K_4 = 0.4504$

$\Delta y = \frac{1}{6}(K_1+2K_2+2k_3+K_4)$

$= \frac{1}{6}(0.2907+2(0.3518)+2(0.3639)+0.4504)$

$= \frac{1}{6}(0.2907+0.7036+0.7278+0.4504)$

$\Delta y = 0.3621$

$y_2 = y(1.2) = y_1+\Delta y$

$= 1.2415+0.3621$

$y_2 = 1.6036.$

2) Find the values of $y(0.2)$ & $y(0.4)$ using R-K fourth order method with $h=0.2$, given that $\frac{dy}{dx} = \sqrt{x^2+y}$ & $y(0)=0.8$

Soln:-

We have $\frac{dy}{dx} = f(x,y) = \sqrt{x^2+y}$

$x_0=0$; $y_0=0.8$; $h=0.2$

$y_0 = 0.8$ $\quad$ $y_1 = ?$ $\quad$ $y_2 = ?$

x_0 $\quad$ $x_1 = 0.2$ $\quad$ $x_2 = 0.4$

$$K_1 = hf(x_0, y_0)$$
$$= (0.2)\, f(0, 0.8)$$
$$= (0.2)\sqrt{0^2 + 0.8}$$
$$= (0.2)\sqrt{0.8}$$
$$= 0.1789$$

$$K_2 = hf\left(x_0 + h/2,\ y_0 + k_1/2\right)$$
$$= (0.2)\, f\left(0 + \frac{0.2}{2},\ 0.8 + \frac{0.1719}{2}\right)$$
$$= (0.2)\, f(0.1,\ 0.8 + 0.0894)$$
$$= (0.2)\, f(0.1,\ 0.8894)$$
$$= (0.2)\sqrt{(0.1)^2 + 0.8894}$$
$$= (0.2)\sqrt{0.01 + 0.8894}$$
$$= 0.1897.$$

$$K_3 = hf\left(x_0 + h/2,\ y_0 + k_2/2\right)$$
$$= (0.2)\, f\left(0 + \frac{0.2}{2},\ 0.8 + \frac{0.1897}{2}\right)$$
$$= (0.2)\, f(0.1,\ 0.8949)$$
$$= (0.2)\sqrt{(0.1)^2 + 0.08949}$$
$$= 0.1903$$

$$K_4 = hf(x_0 + h,\ y_0 + k_3)$$
$$= (0.2)\, f(0 + 0.2,\ 0.8 + 0.1903)$$

$= (0.2)\ f\ (0.2, 0.9903)$

$= (0.2)\sqrt{(0.2)^2 + 0.9903}$

$= (0.2)\sqrt{1.0303}$

$= 0.2030.$

$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}(0.1789 + 2(0.1897) + 2(0.1903) + 0.2030)$

$= \frac{1}{6}(0.1789 + 0.3794 + 0.3806 + 0.2030)$

$\Delta y = 0.1903$

$y_1 = y(0.2) = y_0 + \Delta y$

$= 0.8 + 0.1903$

$= 0.9903$

For y_2:-

$K_1 = hf(x_1, y_1)$

$= (0.2)\ f\ (0.2, 0.9903)$

$= (0.2)\sqrt{(0.2)^2 + 0.9903}$

$K_1 = 0.2030$

$K_2 = hf(x_1 + h/2, y_1 + K_1/2)$

$= (0.2) f\left(0.2 + \frac{0.2}{2}, 0.9903 + \frac{0.2030}{2}\right)$

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$= (0.2)\, f\,(0.3,\ 0.9903 + 0.1015)$

$= (0.2)\, f\,(0.3,\ 1.0918)$

$= (0.2)\sqrt{(0.3)^2 + 1.0918}$

$K_2 = 0.2174$

$K_3 = h f\,(x_1 + h/2,\ y_1 + K_2/2)$

$= (0.2)\, f\left(0.2 + \frac{0.2}{2},\ 0.9903 + \frac{0.2174}{2}\right)$

$= (0.2)\, f\,(0.3,\ 0.9903 + 0.1087)$

$= (0.2)\, f\,(0.3,\ 1.0990)$

$= (0.2)\sqrt{(0.3)^2 + 1.0990}$

$K_3 = 0.2181$ 4) 1) $\tan\theta = \frac{2\mu}{(1-\mu)^2}$

$K_4 = h f\,(x_1 + h,\ y_1 + K_3)$

$= (0.2)\, f\,(0.2 + 0.2,\ 0.9903 + 0.2181)$

$= (0.2)\, f\,(0.4,\ 1.2084)$ 2) $P\cos A + Q\cos B + R\cos C = 0$

$= (0.2)\, f\,(0.4,\ 1.2084)$

$= (0.2)\sqrt{(0.4)^2 + 1.2084}$

5) 3) $\frac{\log(n+1)}{n-1}$

$= 0.2340$

$\Delta y = \frac{1}{6}\,(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}\,(0.2030 + 2\,(0.2174) + 2\,(0.2181) + 0.2340)$

$= \frac{1}{6}\,(0.2030 + 0.4348 + 0.4362 + 0.2340)$

$= 0.2180$.

$$\therefore\ y_2 = y(0.4) = y_1 + \Delta y$$

$$= 0.9903 + 0.2180$$

$$= 1.2083$$

RK METHOD FOR SOLVING THE SIMULTANEOUS DIFFERENTIAL EQUATION:

Consider the differential equation

$$\frac{dy}{dx} = f(x, y, z)$$

$$\frac{dz}{dx} = g(x, y, z)$$

$$y(x_0) = y_0 \ \& \ z(x_0) = z_0$$

To solve this system the system of differential equation at an interval of h, the increment in y and z for the 1st increment in x computed by.

1) Use Runge-Kutta method to approximate y, when $x = 0.1, 0.2, 0.3$, $h = 0.1$ given $x = 0$ when $y = 1$ and $\frac{dy}{dx} = x+y$.

Soln:-

Given $y' = x+y$

ie, $f(x,y) = x+y$

And also given that $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

To find $y(0.1)$ using third order Runge-Kutta Method:-

Now $k_1 = hf(x_0, y_0)$

$= h[x_0 + y_0]$

$k_1 = (0.1)[0+1] = 0.1$

$k_2 = hf[x_0 + \frac{h}{2}, y_0 + k_1/2]$

$= hf[0. \frac{0.1}{2} + 1 + \frac{0.1}{2}]$

$= (0.1)(0.05 + 1.05)$

$\boxed{k_2 = 0.11}$

$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$

$= (0.1)(0 + 0.1 + 1 + 2(0.11) - 0.1)$

$= (0.1)(1.22)$

$\boxed{k_3 = 0.122}$

$\Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$

$= \frac{1}{6}[0.1 + 4(0.11) + 0.122]$

$= \frac{1}{6}(0.662)$

$\Delta y = 0.1103$

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$$y_1 = y_0 + \Delta y$$

$$= 1 + 0.1103$$

$$\therefore y(0.1) = 1.1103$$

To find $y(0.2)$ using third order Runge-kutta method:-

Here $x_0 = 0.1$, $y_0 = 1.1103$, $h = 0.1$

Now $k_1 = hf(x_0, y_0) = h(x_0 + y_0)$

$$= (0.1)(0.1 + 1.1103)$$

$$k_1 = 0.12103$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$

$$= (0.1)\left[0.1 + \frac{0.1}{2} + 1.1103 + \frac{0.12103}{2}\right]$$

$$= (0.1)(1.3208)$$

$$k_2 = 0.13208$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

$$= h(x_0 + h + y_0 + 2k_2 - k_1)$$

$$= (0.1)(0.1 + 0.1 + 1.1103 + 2(0.13208) - 0.12103)$$

$$= (0.1)(1.4534)$$

$$k_3 = 0.14534$$

$$\Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$= \frac{1}{6}[0.12103 + 4(0.132089) + 0.14534]$$

$$= \frac{1}{6}(0.7947)$$

$$\therefore \Delta y = 0.1324$$

$$y_2 = y_1 + \Delta y$$

$$y(0.2) = y(0.1) + \Delta y$$

$$= 1.1103 + 0.1324$$

$$\therefore y(0.2) = 1.2427$$

To find $y(0.3)$:-

Here $x_0 = 0.2$, $y_0 = 1.2427$, $h = 0.1$

Now $K_1 = hf(x_0, y_0) = h(x_0 + y_0)$

$$= (0.1)[0.2 + 1.2427] = (0.1)(1.4427)$$

$$K_1 = 0.14427.$$

$$K_2 = hf[x_0 + h/2, y_0 + K_1/2]$$

$$= h[x_0 + h/2 + y_0 + K_1/2]$$

$$= (0.1)\left[0.2 + \frac{0.1}{2} + 1.2427 + \frac{0.14427}{2}\right]$$

$$= (0.1)(1.5648)$$

$$K_2 = 0.15648$$

$$K_3 = hf(x_0 + h, y_0 + 2K_2 - K_1)$$

$$= h(x_0 + h + y_0 + 2K_2 - K_1)$$

$$= (0.1)(0.2 + 0.1 + 1.2427 + 2(0.15648) - 0.14427)$$

$$= (0.1)(1.7114)$$

$k_3 = 0.17114$

$\therefore \Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$

$= \frac{1}{6}[0.14427 + 4(0.15648) + 0.17114]$

$= \frac{1}{6}(0.9413).$

$\therefore \Delta y = 0.1569$

$y_3 = y_2 + \Delta y$

i.e., $y(0.3) = y(0.2) + \Delta y$

$= y\ 1.2427 + 0.1569$

$\therefore y(0.3) = 1.3996.$

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3996

2) Find the values of $y(1.1)$ using Runge-kutta method of the third order and (ii) Runge-kutta method of the fourth order given that $\frac{dy}{dx} = y^2 + xy$; $y(1) = 1$.

Soln:-

Given $y' = y^2 + xy$

i.e., $f(x,y) = y^2 + xy$

And also given $x_0 = 1$, $y_0 = 1$ and $h = 0.1$

To find $y(1.1)$ using third order Runge-Kutta Method:-

Now. $k_1 = hf(x_0, y_0)$

$$= (0.1)\left[(1)^2 + (1)(1)\right] = (0.1)(2)$$

$$k_1 = 0.2$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[\left(y_0 + \frac{k_1}{2}\right)^2 + \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)\right]$$

$$= (0.1)\left[\left(1 + \frac{0.2}{2}\right)^2 + \left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.2}{2}\right)\right]$$

$$= (0.1)\left[(1+0.1)^2 + (1+0.05)(1+0.1)\right]$$

$$k_2 = 0.2365$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

$$= h\left[(y_0 + 2k_2 - k_1)^2 + (x_0 + h)(y_0 + 2k_2 - k_1)\right]$$

$$= (0.1)\left\{\left[(1 + 2(0.2365) - 0.2\right]^2 + (1 + 0.1)\left[1 + 2(0.2365) - 0.2\right]\right\}$$

$$= (0.1)\left[(1.273)^2 + (1.1)(1.273)\right]$$

$$= (0.1)\left[3.0208\right]$$

$$k_3 = 0.30208.$$

$$\therefore \Delta y = \frac{1}{6}\left[k_1 + 4k_2 + k_3\right].$$

$$= \frac{1}{6}\left[0.2 + 4(0.2365) + 0.30208\right]$$

$$= \frac{1}{6}\left[1.44808\right]$$

$$\therefore \Delta y = 0.24135$$

$$y_1 = y_0 + \Delta y$$

To find $y(1.1)$ using Runge Kutta Method of Fourth order:-

Here $x_0 = 1$, $y_0 = 1$ and $h = 0.1$

Now $k_1 = hf(x_0, y_0)$

$= h(y_0^2 + x_0 y_0)$

$= h(y_0^2 + x_0 y_0)$

$= (0.1)[1^2 + (1)(1)] = (0.1)(2)$

$k_1 = 0.2$

$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$

$= h\left[\left(y_0 + \frac{k_1}{2}\right)^2 + \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)\right]$

$= (0.1)\left[\left(1 + \frac{0.2}{2}\right)^2 + \left(1 + \frac{0.1}{2}\right)\left(1 + \frac{(0.2)}{2}\right)\right]$

$= (0.1)[(1.1)^2 + (1.05)(1.1)]$

$k_2 = 0.2365$

$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$

$= h\left[\left(y_0 + \frac{k_2}{2}\right)^2 + \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_2}{2}\right)\right]$

$= (0.1)\left[\left(1 + \frac{0.2}{2}\right)^2 + \left(1 + \frac{0.1}{2}\right)\left(1 + \frac{(0.2)}{2}\right)\right]$

$= (0.1)[(1.1)^2 + (1.05)(1.1)]$

$k_2 = 0.2365$

$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$

$= h\left[\left(y_0 + \frac{k_2}{2}\right)^2 + \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_2}{2}\right)\right]$

$$= (0.1)\left[\left(1+\frac{0.2365}{2}\right)^2+\left(1+\frac{0.1}{2}\right)\left(1+\frac{0.2365}{2}\right)\right]$$

$$= (0.1)[(1.11825)^2 + (1+0.05)(1.11825)]$$

$$= (0.1)[2.4246]$$

$k_3 = 0.24246.$

$$k_4 = hf(x_0+h, y_0+k_3)$$

$$= (0.1)[(y_0+k_3)^2 + (x_0+h)(y_0+k_3)]$$

$$= (0.1)[(1.24246)^2 + (1.1)(1.24246)]$$

$$= (0.1)[1.5437 + 1.366706]$$

$$= (0.1)[2.91041]$$

$k_4 = 0.29104$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.2 + 2(0.2365) + 2(0.24246) + 0.29104]$$

$$= \frac{1}{6}[1.44896]$$

$$= 0.24149$$

$$\therefore y_1 = y_0 + \Delta y = 1 + 0.24149 = 1.24149$$

$$y(0.1) = 1.24149.$$

3) By applying the fourth order Runge-kutta Method find $y(0.2)$ from $y' = y - xy$ $(0) = 2$ taking $h = 0.1$

Soln :-

Given $y' = y - x$

ie, $f(x, y) = y - x$

and $y(0) = 2$ ie., $x_0 = 0$, $y_0 = 2$ and $h = 0.1$ we know that the fourth order Runge-kutta formula for finding the first increment in y viz Δy is given by

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$f(x_0, y_0) = (y - x$

$= 2 - 0$

$$\therefore k_1 = (0.1)(y_0 - x_0) = 0.1(2-0) = 0.2$$

$$k_2 = (0.1)\left[\left(y_0 + \frac{k_1}{2}\right) - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2 + \frac{0.2}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.1 - 0.05] = 0.205$$

$$k_3 = (0.1)\left[\left(y_0 + \frac{k_2}{2}\right) - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2 + \frac{0.205}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.1025 - 0.05]$$

$$= 0.20525$$

$$k_4 = (0.1)[(y_0 + k_3) - (x_0 + h)]$$

$$= (0.1)[2 + 0.20525 - 0 - 0.1]$$

$$= 0.210525$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.2 + 2(0.205) + 2(0.20525) + 0.210525]$$

$$= \frac{1}{6}[0.2 + 0.41 + 0.4105 + 0.210525]$$

$$= 0.20517$$

$$\therefore y(0.1) = y_1 = y_0 + \Delta y$$

$$= 2 + 0.20517$$

$$\therefore y(0.1) = 2.20517.$$

Next we have to find $y(0.2) = y_2 = y_1 + \Delta y$

where $\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

Now $k_1 = hf(x_1, y_1) = h[y_1 - x_1]$

$$= (0.1)[2.20517 - 0.1] = 0.210517.$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= h\left[\left(y_1 + \frac{k_1}{2}\right) - \left(x_1 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2.20517 + \frac{0.2105}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.31042 - 0.15] = 0.21604$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= h\left[\left(y_1 + \frac{k_2}{2}\right) - \left(x_1 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2.20517 + \frac{0.2105}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.31042 - 0.15] = 0.21604.$$

$$= (0.1)\left[\left(2.20517 + \frac{0.21604}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)\,[2.31349 - 0.15] = 0.21632$$

$$k_4 = hf(x_1 + h,\, y_1 + k_3)$$

$$= h[(y_1 + k_3) - (x_1 + h)]$$

$$= (0.1)[(2.20517 + 0.21632) - (0.1 + 0.1)]$$

$$= (0.1)[2.142149 - 0.2] = 0.22214$$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.2105 + 2(0.21604) + 2(0.21632) + 0.22214]$$

$$= \frac{1}{6}[0.2105 + 0.43208 + 0.43264 + 0.22214]$$

$$= 0.21622$$

$$\therefore y_2 = y_1 + \Delta y$$

$$= 2.20517 + 0.21622$$

$$y(0.2) = 2.42139$$

Hence we have the following table.

x	0	0.1	0.2
y	2	2.20517	2.42139

4) Find the values of $y(0.2)$ & $y(0.4)$ using Runge-Kutta method of fourth order with $h=0.2$, given that $\frac{dy}{dx} = \sqrt{x^2+y}$; $y(0)=0.8$

Soln:-

Given $y' = \sqrt{x^2+y}$

i.e., $f(x,y) = \sqrt{x^2+y}$

And also given that $x_0=0$, $y_0=0.8$ and $h=0.2$

To find $y(0.2)$:-

We know that the fourth order Runge-Kutta formula to find the first increment in y viz, Δy is given by

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

When $k_1 = hf(x_0, y_0) = h[\sqrt{x_0^2+y_0}]$

$$= (0.2)[\sqrt{0+0.8}]$$

$$k_1 = 0.17889.$$

$$k_2 = hf\left[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right]$$

$$= h\left[\sqrt{\left(x_0+\frac{h}{2}\right)^2 + \left(y_0+\frac{k_1}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{\left(0+\frac{0.2}{2}\right)^2+\left(0.8+\frac{0.17889}{2}\right)}\right]$$

$$= (0.2)\sqrt{(0.1)^2+0.8+0.08944}$$

$$k_2 = 0.18968$$

$$k_3 = hf\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right]$$

$$= h\left[\sqrt{\left(x_0+\frac{h}{2}\right)^2 + \left(y_0+\frac{k_2}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{\left(0+\frac{0.2}{2}\right)^2+\left(0.8+\frac{0.18968}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{(0.1)^2+0.8+0.09484}\right]$$

$$k_3 = 0.19025$$

$$k_4 = hf(x_0+h,\ y_0+k_3)$$

$$= h\sqrt{(x_0+h)^2+(y_0+k_3)}$$

$$= h\sqrt{(0+0.2)^2+(0.8+0.19025)}$$

$$= (0.2)\sqrt{0.04+0.99025}$$

$$k_4 = 0.20300$$

$$\therefore \Delta y = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= \frac{1}{6}[0.17889+2(0.18968)+2(0.19025)+0.20300]$$

$$= \frac{1}{6}[1.14175]$$

$$\Delta y = 0.19029$$

$$\therefore y(0.2) = y_0+\Delta y$$

$$= 0.8+0.19029$$

$$y(0.2) = 0.99029$$

To find $y(0.4)$:-

Here $x_1 = 0.2$, $y_1 = 0.99029$ and $h = 0.2$

Now $k_1 = hf(x_1, y_1) = h[\sqrt{x_1^2+y_1}]$.

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$$= (0.2)\left[\sqrt{(0.2)^2+0.99029}\right]$$

$$K_1 = 0.20301$$

$$K_2 = hf\left[x_1+\frac{h}{2}, y_1+\frac{K_1}{2}\right]$$

$$= h\left[\sqrt{(x_1+h/2)^2+(y_1+K_1/2)}\right]$$

$$= (0.2)\left[\sqrt{\left(0.2+\frac{0.2}{2}\right)^2+\left(0.99029+\frac{0.20301}{2}\right)}\right]$$

$$= (0.2)\sqrt{(0.3)^2+0.99029+0.10150}$$

$$K_2 = 0.21742$$

$$K_3 = hf\left[x_1+\frac{h}{2}, y_1+\frac{K_2}{2}\right]$$

$$= h\left[\sqrt{\left(x_1+\frac{h}{2}\right)^2+\left(y_1+\frac{K_2}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{\left(0.2+\frac{0.2}{2}\right)^2+\left(0.99029+\frac{0.21742}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{(0.3)^2+0.99029+0.10871}\right]$$

$$K_3 = 0.21808$$

$$K_4 = hf(x_1+h, y_1+K_3)$$

$$= h\sqrt{(x_1+h)^2+(y_1+K_3)}$$

$$= h\sqrt{(0.2+0.2)^2+(0.99029+0.21808)}$$

$$= (0.2)\sqrt{1.36837} = 0.23396$$

$$\therefore \Delta y = \frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$$

$$= \frac{1}{6}\left[0.20301+2(0.21742)+2(0.21808)+0.23396\right]$$

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$$= \frac{1}{6}\left[1.30797\right] = 0.217996$$

$$\therefore y_2 = y_1 + \Delta y$$

$$= 0.99029 + 0.217996$$

$$\therefore y(0.4) = 1.20828$$

x	0	0.2	0.4
y	0.8	0.99029	1.20828