SWAMI DAYANANDA
COLLEGE OF ARTS & SCIENCE,
MANJAKKUDI-612610

DEPERTMENT OF MATHEMATICS

Real Analysis(16SCCMM10)

Study Material

Class : III-B.Sc Mathematics

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CORE COURSE X
REAL ANALYSIS

Objectives: To enable the students to

1. Understand the real number system and countable concepts in real number system
2. Provide a Comprehensive idea about the real number system.
3. Understand the concepts of Continuity, Differentiation and Riemann Integrals
4. Learn Rolle's Theorem and apply the Rolle's theorem concepts.

UNIT I

- Real Number system – Field axioms –Order relation in R. Absolute value of a real number & its properties –Supremum & Infimum of a set – Order completeness property – Countable & uncountable sets.

UNIT II

- Continuous functions –Limit of a Function – Algebra of Limits – Continuity of a function –Types of discontinuities – Elementary properties of continuous functions – Uniform continuity of a function.

UNIT III

- Differentiability of a function –Derivability & Continuity –Algebra of derivatives – Inverse Function Theorem – Daurboux"s Theorem on derivatives.

UNIT IV

- Rolle's Theorem –Mean Value Theorems on derivatives- Taylor's Theorem with remainder- Power series expansion .

UNIT V

- Riemann integration –definition – Daurboux's theorem –conditions for integrability – Integrability of continuous & monotonic functions - Integral functions –Properties of Integrable functions - Continuity & derivability of integral functions – The Fundamental Theorem of Calculus and the First Mean Value Theorem.

TEXT BOOK(S)

1. M.K,Singhal & Asha Rani Singhal , A First Course in Real Analysis, R.Chand & Co., June 1997 Edition
2. Shanthi Narayan, A Course of Mathematical Analysis, S. Chand & Co., 1995

- UNIT – I - Chapter 1 of [1]
- UNIT – II - Chapter 5 of [1]
- UNIT – III - Chapter 6 – Sec 1 to 5 of [1]
- UNIT – IV - Chapter 8 – Sec 1 to 6 of [1]
- UNIT – V - Chapter 6 – Sec 6.2, 6.3, 6.5, 6.7, 6.9 of [2]

REFERENCE(S)

1. Goldberge, Richard R, Methods of Real Analysis, Oxford & IBHP Publishing Co., New Delhi, 1970.

closed, open interval

$A = \{1, 2, \dots\}$ $B = \{1, 2, 3\}$ $C = [-1, 1]$ $D = (-1, 1)$

C has -1 & 1 in the set   D has -1 & 1 are not in the set

$[a,b] = \{$ ~~φ~~ $x \mid a \le x \le b\}$ $(a, \infty) = +a < x < \infty$ (or) $x > a$

$(a,b) = \{x \mid a < x < b\}$ $(-\infty, a)$ $-\infty < x < a$ (or) $x < a$

$[a,b)$ $a \le x < b$

$(a,b]$ $a < x \le b$

Real Numbers

$W = \{0, 1, 2, 3 \dots\}$

$N = \{1, 2, 3, \dots\}$

$Q = \frac{p}{q}$ $q \ne 0$

$Z = \{-3, -2, -1, 0, 1, 2, 3\}$

$\mathbb{R} = \{-\infty \dots -1, 0, 1, \dots \infty\}$

$N \subset W \subset R$ $Q \subset R$, $Z \subset Q$, $N \subset Q$

R is a Superset of real number System.

Axioms

A_1 is closure (Additive)

$a, b \in \mathbb{R}$, $a+b \in R$

A_2 is Associative

$a, b, c \in \mathbb{R}$

$(a+b)+c = a+(b+c)$

A_3 is identity

$a + (0) = (0) + a = a$

A_4 is Inverse

$a + (-a) = (-a) + a = 0$

A_5 is commutative

$a, b \in \mathbb{R}$

~~a+b=~~ $a+b = b+a$

multiplicative

M_1 is closure

$a, b \in R$

$a \cdot b \in R$

M_2 is Associative

$a, b, c \in R$

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

M_3 is Identity

$a \cdot 1 = 1 \cdot a = a$

M_4 is Inverse

$a \cdot (a^{-1}) = (a^{-1}) \cdot a = 1$

M_5 is commutative

$a, b \in R$

$a \cdot b = b \cdot a$

Distributive law

D_1 $a, b, c \in R$

$a \cdot (b + c) = (a \cdot b + a \cdot c)$

A_1 to A_4 - additive group

A_1 to A_5 - additive Commutative group

M_1 to M_4 - Multiplicative group

M_1 to M_5 - multiplicative Commutative group

N satisfies A_1 & A_2

W satisfies A_1 to A_4

Z satisfies A_1 to A_5

Field

Examples $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

Rational Numbers, Real Numbers, Complex Numbers

$(\mathbb{R}, +, \cdot) \to$ abelian group

Ring $\to$ Satisfies A_1 to A_5, M_1, M_2 & D_1

Ring with ~~un~~ identity

A_1 to A_5, M_1, M_2, M_3 & D_1

Division Ring

A_1 to A_5, M_1, M_2, M_3, M_4 & D_1

Commutative division ring A_1 to A_6, M_1 to M_5 & D_1

$\therefore$ It satisfies the field axioms.

Theorem 1: There can exist atmost one identity element for addition 0 in $\mathbb{R}$.

(to prove 0 is unique)

Let $x \in \mathbb{R}$ $\quad x+0=x$

Suppose, Let us take $0'$ is another identity element.

$\Rightarrow x+0' = 0'+x = x$ $\qquad x+0' = 0'+x = x$

We know that $0'$ is additive identity $\Rightarrow 0+0' = 0'+0 = 0$ — ①

Similarly

We know that 0 is additive identity $\Rightarrow 0'+0 = 0+0' = 0'$ — ②

By Equation ① & ② we can conclude that

$0+0' = 0'+0$

$\Rightarrow 0 = 0'$ $\Rightarrow$ 0 is unique.

Hence proved.

Theorem 2

To each x in $\mathbb{R}$, There exist one and only real number y, such that $x+y = y+x = 0$ (i.e we have to prove that inverse of x is unique)

Let $x \in \mathbb{R}$

Let us take y_1, y_2 are additive inverses of x.

To prove : $y_1 = y_2$

We know that $x+y_1 = 0 = y_1+x$ & $x+y_2 = 0 = y_2+x$

$y_2 = y_2+0$ [by the property of 0]

$= y_2+x+y_1$ [by hypothesis y_1 = additive inverse of x]

$= (y_2+x)+y_1$ [by Associative law of Addition]

$= 0+y_1$ [Since y_2 = additive inverse of x]

$\boxed{y_2 = y_1}$ [by property of 0]

Hence proved.

Cancellation Law of addition

Theorem 3 If x, y, z be real numbers such that $x+z=y+z$ then $x=y$

proof: We have x, y, z such that $x+z=y+z$. To prove that $x=y$

Let $-z$ be the negative of z.

$x = x+0$ (by the property of 0)

$= x+(z+(-z))$ (by the inverse property)

$= (x+z)+(-z)$ (by Associative law of addition)

$= (y+z)+(-z)$ (Since we have $x+z=y+z$)

$= y+(z+(-z))$ (by associative law of addition)

$= y+0$ (by the inverse property)

$= y$ (by the property of 0)

$\therefore \boxed{x=y}$ Hence proved.

Theorem 4 For each real number x (i) $-(-x)=x$

(ii) $-(x+y) = (-x)+(-y)$

proof:

(i) Let the negative of $-x$ is denoted by z; then $-x+z=0$ & we have to prove that $x=z$

Now $x = x+0$ (by the property of 0)

$= x+(-x+z)$ (Since $-x+z=0$)

$= (x+(-x))+z$ (by associative law of addition)

$= 0+z$ (by property of negative)

$= z$ (by the prop. of 0)

$\Rightarrow x=z=-(-x)$

$\Rightarrow x=-(-x)$ Hence Proved.

(ii) To prove $-(x+y) = (-x)+(-y)$

Let us take $-(x+y) = -(x+y)+0$ (by the property of 0)

$= -(x+y)+0+0$

$= -(x+y)+(x+(-x))+(y+(-y))$

$= -(x+y)+(x+y)+((-x)+(-y))$ (by associative law)

$= 0+(-x)+(-y)$ (Since $-(x+y)+(x+y)=0$)

$-(x+y) = (-x)+(-y)$ (by the prop. of 0)

$\therefore$ Hence proved.

Theorem 5 (mulplicative identity)

There can exist at most one identity element for multiplication in R.

proof:

(we know that $\forall x \in R$ $x.1 = 1.x = x$)

Let u, u' be the multiplicative identities.

$\Rightarrow x.u = u.x = x$ & $x.u' = u'.x = x$

Let us take $x = u'$ $\Rightarrow u'.u = u.u' = u'$ —— ①

Let us take $x = u$ $\Rightarrow u.u' = u'.u = u$ —— ②

In second equation

from ②, ① we can conclude that $u' = u$. Hence proved.

Theorem 6 (multiplicative inverse) To each $x \in R$, $x \neq 0$ There Corresponds one & only real number y such that $xy = y.x = 1$

Proof:

Let $x \in R$ & y_1, y_2 be two multiplicative inverses of x in R.

$\Rightarrow xy_1 = y_1 x = 1$ & $xy_2 = y_2 x = 1$

Let $y_1 = y_1 . 1$ (by the property of 1)

$= y_1 . (xy_2)$ (Since $xy_2 = 1$)

$= (y_1 . x) . y_2$ (by associative law of multiplication)

$= 1 . y_2$ (by hypothesis)

$\therefore \boxed{y_1 = y_2}$ (by the property of 1)

Hence multiplicative inverse is unique.

Theorem 7 $x \cdot 0 = 0$ for all x in $\mathbb{R}$.

Proof:

$$x \cdot 0 = x(0+0) \quad \text{(By property of 0)}$$

$$= x \cdot 0 + x \cdot 0 \quad \text{(Distributive law)}$$

by using

$$\underbrace{x \cdot 0}_{z} + \underbrace{0}_{x} = \underbrace{x \cdot 0}_{y} + \underbrace{x \cdot 0}_{z} \quad \text{(cancellation law)}$$

$$\boxed{x \cdot 0 = 0}$$

Theorem 8 If x, y be real numbers such that $xy = 0$, then either $x = 0$ or $y = 0$.

proof:

we know that If $x = 0$ then $xy = 0$

If $x \neq 0$ then x^{-1} exist

$$xy = 0$$

$$x^{-1}(xy) = x^{-1}(0)$$

$$(x^{-1}x)y = 0 \quad \text{(by associative law of multiplication)}$$

$$1 \cdot y = 0$$

$$y = 0$$

Hence the proof.

Cancellation law of multiplication

Theorem 9 If x, y, z be real numbers such that $xz = yz$ and $z \neq 0$ Then $x = y$

Proof:

$[x+(-y)]z = xz+(-y)z$ (by Distributive law)

$= yz+(-y)z$ (Since $xz=yz$)

$= [y+(-y)]z$ (by Distributive law)

$= 0 \cdot z$ (by property of negative)

$= 0$ (by zero property)

But $z \neq 0$ $\therefore x+(-y)=0$ ($x=0$ or $y=0$ when $xy=0$)

$x+(-y) = y+(-y)$ (By cancellation law of addition)

$\boxed{x=y}$

For all x, y in R.
(i) $x(-y) = -(xy)$
(ii) $(-x)y = -(xy)$
(iii) $(-x)(-y) = xy$

proof: (1) $x(-y) = -(xy)$ (To Prove)

Let us take $x(-y)+xy = x[(-y)+y]$ (By Distributive law)

$= x \cdot 0$ (By the property of negative)

$= 0$

$x(-y)+xy = -(xy)+xy$ (By cancellation law of addition)

$x(-y) = -(xy)$ //

(ii) To prove $(-x)y = -(xy)$

$\Rightarrow (-x)y = y(-x)$ (By commutative law of multiplication)

$= -(yx)$

$= -(xy)$ //

Aliter $(-x)y + xy = y((-x)+x)$

$= y \cdot 0$

$(-x)y+xy = 0$

$(-x)y + xy = -(xy)+xy$

$(-x)y = -(xy)$ //

(iii) $(-x)(-y) = xy$

$= -[(-x)y]$

$= -[-(xy)]$

$= xy$

Subtraction & Division

Definition: The difference between two real numbers x & y is given by $x+(-y)$ and is denoted by $(x-y)$. The operation of finding the difference is called subtraction.

Definition: The quotient of a real number x by y $(y \neq 0)$ is given by xy^{-1} and is denoted by $\frac{x}{y}$ or $x \div y$

The operation of finding a quotient is called division.

By the property of 1 , It is true that for each nonzero real number y. $\frac{1}{y} = 1.y^{-1} = y^{-1}$

Order in R

Order1 (O_1) : Law of Trichotomy :

Given any two real numbers a, b one and only one of the following holds $a>b$, $a=b$, $b>a$

Order 2 (O_2) : Transitivity

For each triple of real numbers a, b, c if $a>b$, $b>c$ then $a>c$

Order 3 (O_3) : Monotone property for addition

For all real numbers a, b and c $a>b$ implies $a+c>b+c$

Order 4 (O_4) : Monotone property for multiplication :

For all real numbers a, b and c , $a>b$ and $c>0$ implies that $ac>bc$

If the field Satisfies O_1 to O_4 is called an ordered field.

Theorem 1   For each real number a , one and only one of the following holds

$a>0$, $a=0$, $-a>0$

Proof

In view of a : $a>0$, $a=0$, $0>a$

To prove that $0>a$ or $-a>0$

$0>a$

$\Rightarrow 0+(-a) > a+(-a)$

$\Rightarrow (-a) > 0$

$\Rightarrow -a > 0$

Conversely $-a>0$

$-a+a > 0+a$

$\Rightarrow 0 > a$ Hence Proved

Theorem 2   If a, b be positive real numbers , then a+b is a positive real number

Proof : $a>0$

$a+b > 0+b$ (by O_3)

$a+b > b > 0$ (Since $b>0$)

$a+b>0$. Hence Proved.

Theorem 3   If a, b be positive real numbers , then ab is positive real number

Proof : $a>0$, $b>0$ to prove $ab>0$

$a>0$

$ab > 0.b$ (by O_4)

But $0.b=0$

Therefore , we have $ab>0$

$\therefore$ $ab>0$ //

For less than $(<)$

O_1 $a<b$, $a=b$, $b<a$

O_2 $a<b$, $b<c \Rightarrow a<c$

O_3 $a<b \Rightarrow a+c < b+c$

O_4 $a<b, c>0 \Rightarrow ac<bc$

Theorem 4 For each real number a, one and only one of the following holds: $a<0$, $a=0$, $-a<0$

Proof (i) To prove that $0<a$ or $-a<0$

$0<a \Rightarrow -a<0$

$0+(-a) < a+(-a)$

$-a<0$

Conversely $-a<0$

$-a+a < 0+a$

$0<a$ Hence proved

(ii) To prove that $a<0$ or $-a>0$

$a<0$ $a+(-a) < 0+(-a)$

$0<-a$

$-a>0$

$-a+a > 0+a$

$0>a$

(iii) $a=0$ it is trivial.

Absolute value: If x be a real number, then its absolute value, denoted by $|x|$ is define by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$$

we may observe that $|x|$ is defined for every $x \in \mathbb{R}$.

Also $x_1 = x_2 \Rightarrow |x_1| = |x_2|$

Theorem 1 For every $x \in \mathbb{R}$, $|x| = \max\{-x, x\}$

Proof By the law of trichotomy, we know that

$x>0$ or $x=0$ or $x<0$

If $x \geq 0$ then $|x| = x$ and $x \geq -x$

If $x<0$ then $|x| = -x$ and $-x>x$.

Thus in either case, $|x|$ is the maximum of the two numbers x and $-x$, that is $|x| = \max\{x, -x\}$

Corollary For every $x \in \mathbb{R}$

$x \leq |x|$

Proof: $|x| = \max\{x, -x\} \geq x$

$|x| \geq x$

$\Rightarrow x \leq |x|$

Theorem 2

For every $x \in R$, $|x|^2 = x^2 = |-x|^2$

proof: By the definition $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

In either case $|x|^2 = x^2$

$|-x|^2 = (-x)^2$ $\{|x| = -x$ if $x < 0\}$

$= x^2$

$\therefore |x|^2 = x^2 = |-x|^2$ Hence the proof.

Theorem 3 For every $x \in R$, $|x| = |-x|$

proof $|x| = \max\{-x, x\}$

$|-x| = \max\{-(-x), +(-x)\}$

$= \max\{x, -x\} = |x|$ //

Theorem 4 For all $x, y \in R$, $|x.y| = |x|.|y|$

Proof: $|x.y|^2 = (xy)^2 = x^2y^2$ to prove $|x.y| = |x|.|y|$

$|xy|^2 = |x|^2.|y|^2$

It is nonnegative on bothsides $\therefore |xy| = |x|.|y|$

Theorem 5 State & prove triangle inequality.

For all real numbers x & y, $|x+y| \leq |x| + |y|$

solution

case(i) $x+y \geq 0$

$|x+y| = x+y$

we know that $x \leq |x|$

$y \leq |y|$

$\Rightarrow |x+y| \leq |x| + |y|$

case (ii) $x+y < 0$

$-(x+y) > 0$

$(-x) + (-y) > 0$

$|x+y| = |-(-x+y)|$

$= |(-x) + (-y)|$

$\leq |(-x)| + |(-y)|$ (By case(i))

We knowthat $|-x| = |x|$, $|-y| = |y|$

$|x+y| \leq |x| + |y|$

$\therefore$ By both case we proved the triangle inequality.

Absolute value: If x be a real number, then its absolute value, denoted by $|x|$ is define by
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

we may observe that $|x|$ is defined for every $x \in R$.

Also $x_1 = x_2 \Rightarrow |x_1| = |x_2|$

Theorem1 For every $x \in R$, $|x| = \max\{-x, x\}$

Proof By the law of trichotomy, we know that

$x > 0$ or $x = 0$ or $x < 0$

If $x \geq 0$ then $|x| = x$ and $x \geq -x$

If $x < 0$ then $|x| = -x$ and $-x > x$.

Thus in either case, $|x|$ is the maximum of the two numbers x and $-x$, that is $|x| = \max\{x, -x\}$

Corollary For every $x \in R$

$x \leq |x|$

Proof: $|x| = \max\{x, -x\} \geq x$

$|x| \geq x$

$\Rightarrow x \leq |x|$

Theorem 2

For every $x \in R$, $|x|^2 = x^2 = |-x|^2$

proof: By the definition $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

In either case $|x|^2 = x^2$

$|-x|^2 = (-x)^2$ $\{ |x| = -x \text{ if } x < 0 \}$

$= x^2$

$\therefore |x|^2 = x^2 = |-x|^2$ Hence the proof.

Theorem 3 For every $x \in R$, $|x| = |-x|$

proof $|x| = \max\{-x, x\}$

$|-x| = \max\{-(-x), +(-x)\}$

$= \max\{x, -x\} = |x|$ //

Theorem 4 For all $x, y \in R$, $|x \cdot y| = |x| \cdot |y|$

Proof: $|x \cdot y|^2 = (xy)^2 = x^2 y^2$ to prove $|x.y| = |x|.|y|$

$|xy|^2 = |x|^2 . |y|^2$

It is nonnegative on both sides $\therefore |xy| = |x|. |y|$

Theorem 5 State & prove triangle inequality.

For all real numbers x & y, $|x+y| \leq |x| + |y|$

solution

case(i) $x + y \geq 0$

$|x+y| = x+y$

We know that $x \leq |x|$

$y \leq |y|$

$\Rightarrow |x+y| \leq |x| + |y|$

case (ii) $x + y < 0$

$-(x+y) > 0$

$(-x) + (-y) > 0$

$|x+y| = |-(x+y)|$

$= |(-x) + (-y)|$

$\leq |(-x)| + |(-y)|$ (By case(i))

We know that $|-x| = |x|$, $|-y| = |y|$

$|x+y| \leq |x| + |y|$

$\therefore$ By both case we proved that triangle inequality.

Theorem 6 For all real number x & y, $|x-y| \geq ||x|-|y||$

Proof $|x| = |x+y-y|$

By triangle inequality $|x| \leq |x-y| + |y|$

$\Rightarrow |x|-|y| \leq |x-y|$ — ①

$|y| = |y-x+x|$

$|y| \leq |y-x|+|x|$

$|y|-|x| \leq |y-x|$

$-(|x|-|y|) \leq |y-x|$

$\Rightarrow -(|x|-|y|) \leq |x-y|$ — ②

By ① & ② $||x|-|y|| = \max\{(|x|-|y|), -(|x|-|y|)\}$

$||x|-|y|| \leq |x-y|$

$|x-y| \geq ||x|-|y||$

Hence the proof.

Example

1. If x, l, ξ be real numbers & $\xi>0$, ~~Showthat~~ show that $|x-l|<\xi \Leftrightarrow l-\xi < x < l+\xi$.

proof: to prove $|x-l|<\xi \Leftrightarrow l-\xi<x<l+\xi$

$|x-l|<\xi \Leftrightarrow \max\{(x-l), -(x-l)\} < \xi$

$\Leftrightarrow x-l<\xi$ and $-(x-l)<\xi$

$\Leftrightarrow x<l+\xi$ and $-x<\xi-l$

$\Leftrightarrow x<l+\xi$ and $x>l-\xi$

$|x-l|<\xi \Leftrightarrow l-\xi<x<l+\xi$

Hence proved.

2. If x, y be any two real numbers, show that $|x+y|^2+|x-y|^2 = 2(|x|^2+|y|^2)$

proof: We know that $|x|^2 = x^2 = |-x|^2$

$|x+y|^2 = (x+y)^2$ & $|x-y|^2 = (x-y)^2$

$(x+y)^2+(x-y)^2 = (x^2+y^2+2xy)+(x^2+y^2-2xy)$

$= 2x^2+2y^2 = 2(x^2+y^2)$

$|x+y|^2+|x-y|^2 = 2(|x|^2+|y|^2)$

Hence the proof.

Completeness

Definition 1: If for a set 'S' of real numbers; There exist a real number u, Such that:

$$x \in S \Rightarrow x \leq u.$$

Then u is called an upper bound of S. If there exist an upperbound for a Set S. Then S is said to be bounded above.

(R^+ is not bounded)

Definition 2 : If the set of all upper bounds of a set S of real numbers has a smallest number, Say W, then W is Said to be a least upper bound or a supremum of S.

C - order Completeness property

Every nonempty set of real numbers which is bounded above has a supremum.

The properties A_1 to A_5, M_1 to M_5, D, $O_1 - O_4$ and C listed above, is often expressed by saying that the Set of real numbers is a Complete ordered field.

Theorem 1 : If x, y be any positive real numbers Then there exists a positive integer n such that $ny > x$.

proof : Suppose $n \leq x$

Let $S = \{ y, 2y, 3y, \dots \}$

& x is an upper bound of S

We know that S is bounded above. By using ordered completeness property we can conclude that S has a supremum (say s)

Consequently $(n+1) y \leq S$

$ny + y \leq S$

$ny \leq S - y$

$\Rightarrow S y$ is an upper bound of S

Which is a Contradiction because S is the least upper bound.

$\therefore ny > x$

Hence the proof.

Corollary

1) If x be any real number, then there exist a positive integer n such that $n>x$

proof. take $y=1$ in $ny>x$ (since it is true for every x, y)

$\therefore n>x$

2) If x be any real number and y be any positive real number, then there exists a positive n such that $ny>x$

proof If $x>0$, $ny>x$ (By theorem)

If $x\leq 0$, $n=1$ then $1\cdot y = y > 0 \geq x$

$y \geq x$

3) If x be any real number, then there exists a positive integer n such that $n>x$

put $y=1$ in Corollary 2 $\Rightarrow n>x$

Theorem 2

Let S be any nonempty set of real numbers bounded above, then a real number s is the supremum of S iff the following conditions hold.

(i) $x \leq s$ for all $x\in S$

(ii) For each positive real number ϵ, there exist a real number $x\in S$, such that $x>s-\epsilon$

Proof Let $S\subset R$ is bounded above, s be a supremum of S

We have to prove the conditions holds.

s is LUB

$\Rightarrow s$ is an upper bound

$\Rightarrow \forall x\in S$, $x\leq s$

Let $\epsilon>0$ we know that $s-\epsilon<s$

$\therefore s-\epsilon$ is not an upper bound

$\Rightarrow \exists\, x\in S$ such that $x>s-\epsilon$

Converse part Assume that the conditions are holds.

to prove: s is supremum of S (ie to prove s is least upper bound)

Suppose $\exists$ another upper bound s' which is less than s

$\Rightarrow s'<s$

$s - s' > 0$

Let us say $\epsilon = s - s'$

By condition (ii) $x > s - \epsilon$

$x > s - (s - s')$

$x > s'$

Which is a contradiction. (Since s' is an upper bound by assumption)

$\Rightarrow$ s' is not an upper bound.

$\Rightarrow$ s is a least upper bound or Supremum.

Definition 3 If for a set S of real numbers, there exists a real number v such that $x \in S \Rightarrow x \geq v$ Then v is called a lower bound of S. If there exists a lower bound for the set S, then S is said to be bounded below.

Definition 4 If the set of all lower bounds of a set S of real numbers has a greatest number, say t, then t is said to be a greater lower bound or an infimum of S.

Theorem 3 Any non empty set of real numbers which is bounded below has an infimum

Proof: Let $S \subset R$ & S is bounded below

To prove: S has an infimum

say $T = \{-x \mid x \in S\}$

To prove T is above.

y is arbitrary element choosen

$y \in T$, $y = -x \Rightarrow x \in S$

We know that S is bounded below, v is lower bound of S

$\Rightarrow x \geq v$

$\Rightarrow -x \leq -v$

$\Rightarrow y \leq -v$

$\Rightarrow$ $-v$ is an upper bound of T.

$\Rightarrow$ T is bounded above

$\Rightarrow$ T has a Supremum (say t)

t is a Supremum of T

$\Rightarrow -t$ is an infimum of S (

It is enough to show that w is any lower bound of S then $-t \geq w$

We know that $-w$ is upper bound of T.

$\Rightarrow -x \leq -w$

Since t is a least upper bound of T

$\Rightarrow t \leq -w$

$\Rightarrow -t \geq w$

Hence Proved.

Theorem 4 Let S be a nonempty set of real numbers bounded below. A real number t is the infimum of S iff the following Conditions are hold

(i) $x \geq t$ for all $x \in S$

(ii) for each positive real number ε, there exist a real number $x \in S$ Such that $x < t+\varepsilon$

Proof : $S \subset R$ bounded below.

Let t is an infimum of S

To prove : the conditions are hold.

We know that t is a greatest lower bound.

$\Rightarrow t$ is a lower bound

$\Rightarrow \forall x \in S, x \geq t$

Let $\varepsilon > 0$ then $t+\varepsilon > t$

We have t - Greatest lower bound. $\Rightarrow t+\varepsilon$ will not be a lower bound.

$\Rightarrow \exists x$ suchthat $x < t+\varepsilon$

Conversepart

Assume that the Conditions are hold, To prove: t is infimum of S

to prove t is a gretest lower bound. Suppose not

let t' is a lower bound with $t' > t$

$\Rightarrow t' - t > 0$

Let $\varepsilon = t' - t$

By Condition (ii). we have $x < t+\varepsilon$

$\Rightarrow x < t + (t' - t)$

$\Rightarrow x < t'$

$\Rightarrow t'$ is not an lower bound. Which is a contradiction.

$\Rightarrow t$ is a infimum.

Some important Subsets of R

(1) Natural numbers: The set N is the Smallest subset of R which having the following properties.

(i) $1 \in N$

(ii) $m \in N \Rightarrow m+1 \in N$

The algebraic operations on N have the properties A_1, A_2, A_5 M_1, M_2, M_3, M_5 and D. It doesnot satisfy A_3, A_4, M_4

---

(2) Integers

The set Z of all integers is the smallest subset of R having following Properties

(i) $N \subset Z$

(ii) Z Contain an identity element (0) for addition

(iii) Z Contains the negative of each of its elements $(-N \subset Z)$

Algebraic operations on Z have the properties A_1 to A_5, M_1 to M_3, M_5 and D. They do not have M_4

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(3) Rational numbers ($\mathbb{Q}$)

The set $\mathbb{Q}$ of all rational numbers is the smallest subset of $\mathbb{R}$ having the following properties.

(i) $\mathbb{N} \subset \mathbb{Q}$ (ii) $\mathbb{Q}$ is a field.

The algebraic operations on $\mathbb{Q}$ have the properties A_1 to A_5, M_1 to M_6 & D

Theorem 1 : There is no rational number whose square is 2

Proof: To prove There is no p, q suchthat $\left(\frac{p}{q}\right)^2 = 2$

Suppose that there exist p, q suchthat $\left(\frac{p}{q}\right)^2 = 2$

Let $\gcd(p, q) = g$

$p = gn$

$q = gm$

$\frac{p}{q} = \frac{n}{m}$ suchthat $(n, m) = 1$

$\Rightarrow \left(\frac{n}{m}\right)^2 = \left(\frac{p}{q}\right)^2 = 2$

$n^2 = 2m^2$

$\Rightarrow n^2$ is even

$\Rightarrow$ n is even

$\Rightarrow$ $n = 2v$ for some v

$\Rightarrow n^2 = 4v^2$

$4v^2 = 2m^2$

$m^2 = 2v^2$

$\Rightarrow m^2$ is even

$\Rightarrow$ m is even

therefore $\gcd(n, m) = 2$ Which is a contradiction

(Since $\gcd(n, m) = 1$). $\Rightarrow$ our assumption is wrong.

Theorem 2 The set of rational number is not order complete.

Proof

Order complete: Every non empty set of real numbers which is bounded above has a supremum.

Let $S \subset \mathbb{Q}^+$

$S = \{x \mid x \in \mathbb{Q}^+, 0 < x^2 < 2\}$

$1 \in S$

therefore S is nonempty

We know that 2 is an upper bound of S

To prove: S has no supremum.

There is a rational number x; which is not the least upper bound of S.

we can prove this for 4 cases (i) $x \le 0$

(ii) $x > 0$ $\quad 0 < x^2 < 2$

(iii) $x > 0$, $x^2 = 0$

(iv) $x > 0$, $x^2 > 2$

(i) $x \le 0$

since $S \subset Q^+$

$\Rightarrow$ x is not an upper bound

(ii) $x > 0$, $0 < x^2 < 2$

Let $y = \frac{4+3x}{(3+2x)}$ ——— ①

$$y^2 - 2 = \frac{(4+3x)^2}{(3+2x)^2} - 2 = \frac{16+9x^2+24x}{9+4x^2+12x} - 2$$

$$= \frac{16+9x^2+24x-18-8x^2-24x}{(3+2x)^2}$$

$$y^2 - 2 = \frac{x^2-2}{(3+2x)^2} \text{ ——— ②}$$

Since $x^2 < 2$ $\Rightarrow x^2 - 2 < 0$

$\Rightarrow y^2 - 2 = \frac{x^2-2}{(3+2x)^2} < 0$

$\Rightarrow y^2 < 2$

$$y - x = \frac{4-3x}{3+2x} - x = \frac{4+3x-3x-2x^2}{3+2x}$$

$$= \frac{4-2x^2}{8+2x} = \frac{2(2-x^2)}{3+2x} \text{ ——— ③}$$

$y - x > 0$ (since $2 - x^2 > 0$)

$y > x$

$x < y$

$\Rightarrow x^2 < y^2 < 2$ (from ② we have $y^2 < 2$)

from ① we get $x > 0$

from ③ we can get $y > x \Rightarrow x < y$

$x > 0 \Rightarrow 0 < y$, $y^2 < 2$, $y \in S$

$0 < x < y$

$\Rightarrow$ x is not an upper bound.

(iii) $x > 0$, $x^2 = 0$

This case is not possible.

(iv) $x > 0$, $x^2 > 2$

from ① y is +ve

② $y^2 > 2 \Rightarrow 2 < y^2$

③ $y - x < 0$

$y < x$

$y^2 < x^2$

$p \in S$

$p^2 < 2 < y^2 < x^2$

$\Rightarrow p^2 < y^2 < x^2$

B.4

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$\Rightarrow$ y is an upperbound $\Rightarrow$ x is an upper bound.

$\Rightarrow$ x is not an least upper bound.

$\therefore$ for all cases x is not a LUB

$\therefore$ It is not order-complete.

Countable and uncountable Set

Definition 1

A set S is said to be finite, if either it is empty or for some natural number n, there exist a one to one mapping from the set $\{1, 2, \ldots n\}$ onto the set S. If a set is not finite. Then it is said to be Infinite.

Definition 2

A set S is said to be enumerable, if there exists a one to one mapping from the set N of all natural numbers onto the set S.

A Set is said to be countable if it is either finite or enumerable. If a set is not countable, then it is said to be uncountable

---

example $N \to Z$

$$f(n) = \frac{1}{2}(n-1) \quad \text{for } n = 1, 3, 5 \ldots$$

$$f(n) = -\frac{1}{2} n \quad \text{for } n = 2, 4, 6 \ldots$$

$$Z = \{0, -1, 1, 2, -2 \ldots\}$$

Theorem 1 Every subset of a countable set is countable.

proof

A is countable set

$B \subset A$

to prove : B is countable

suppose B is finite

We know that every finite set is countable.

$\therefore$ B is countable

Without loss of generality, we can choose B as an infinite subset of A. We have to prove that B is countable.

Let $A = \{a_1, a_2, a_3 \ldots\}$

Every element of B is a_i for some index i

Let n_1 - smallest element such that $a_{n_1} \in B$

$A - \{a_{n_1}\}$ $\quad a_{n_2} \in B$

n_2 - Smallest such that $a_{n_2} \in A - \{a_{n_1}\}$.

$A - \{a_{n_1}, a_{n_2}\}$

n_3 Such that $(\exists)$ $a_{n_3} \in B$

$a_{n_3} \in A - \{a_{n_1}, a_{n_2}\}$

We can do this Continuously, we ~~can~~ will get

$B = \{a_{n_1}, a_{n_2} \ldots\}$

We can get the mapping $k \longrightarrow a_{n_k}$

=> we can define a mapping $N \to B$ which is onto & oneone $f(K) = a_{nk}$

∴ B is countable.

Theorem 2

Every Superset of an uncountable set is uncountable

proof:

Let A is uncountable set & $A \subset B$ To prove that B is uncountable

Suppose that, B is countable

=> A is countable

But we know that A is uncountable

Which is a contradiction. => B is uncountable

Hence the proof.

Theorem 3

If $A_1, A_2 \dots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is uncountable.

proof

Let $A_1 = \{a_{11}, a_{12}, a_{13} \dots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, \dots\}$

$\vdots$

$A_i = \{a_{i1}, a_{i2} \dots\}$

$A_n = \{a_{n1}, a_{n2} \dots\}$

a_{ij} = j^{th} element of i^{th} set

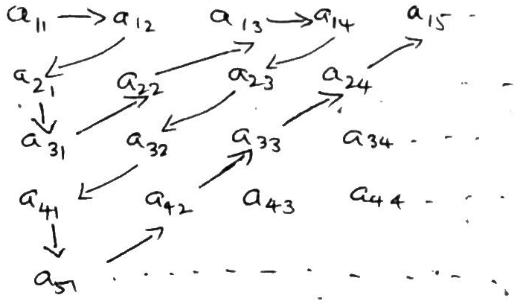
Height of (1+) a_{ij} = $i+j$.

$H(a_{11}) = 1+1 = 2$ $\quad H(a_{12}) = 3$ $\quad H(a_{21}) = 3$

$H(a_{13}) = H(a_{22}) = H(a_{31}) = 4$

$H(a_{14}) = H(a_{23}) = H(a_{32}) = H(a_{41}) = 5$

$H(a_{1m}) = H(a_{2m-1}) = \dots = H(a_m) = m+1$



We can write in this order $a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, a_{51}, a_{42}, a_{33}, a_{24}, a_{15} \dots$ ∴ It is countable.

$\bigcup_{n=1}^{\alpha} A_i$ is countable.

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Theorem 4 :

The set $N \times N$ is Countable we may arrange the elements in the order indicated by the arrows

proof: We fix the index n

$$A_n = \{(n,1), (n,2), (n,3) \cdots\}$$

$$A_1 = \{(1,1), (1,2) \cdots\cdots\}$$

$$A_2 = \{(2,1), (2,2) \cdots\cdots\}$$

$$\vdots$$

$$A_n = \{(n,1), (n,2) \cdots\cdots\}$$

Let us define function $f : N \to A_i$

Such that $f(n) = (i, n)$

$\therefore A_1, A_2 \cdots\cdots A_n$ is clearly countable

$\Rightarrow \bigcup_{n=1}^{\infty} A_i$ is countable. $\therefore$ $N \times N$ is Countable.

Corollary 1 :

The set of all positive rational numbers is countable.

Proof. Let Q^+ is the set of all positive rational numbers.

p/q, $q \neq 0$, $p, q \in N$

Let $A = \left\{ p/q \mid p, q \in N \right\}$ & $B = \left\{ (p,q) \mid (p,q) \in N \times N \right\}$

~~(p,q)~~ Define $f : A \to B$

Let $a = p_1/q_1$ Such that $f(a) = (p_1, q_1)$

We have $B \subset N \times N$

$\therefore$ By theorem, we know that $N \times N$ is Countable.

& every subset of a countable set is countable.

$\Rightarrow Q^+$ is countable.

Corollary 2 :

The set of all negative rational numbers is countable.

Proof :

Let $A = \{ p/q \mid p, q \in N \}$ $B = \{ -p/q \mid p, q \in N \}$

Define $f : A \to B$

$f(p/q) = -p/q$

We know that the set of all positive rational numbers is countable. (i.e A is countable)

Since f is 1-1 & onto function from A to B

$\Rightarrow$ B is Countable. $\therefore$ the set of all negative rationals is Countable.

Corollary 3.

The set of all rational numbers is countable.

Proof:

We know that the set of all positive rational numbers is countable & the Set of all negative rational numbers is countable.

The union of countable sets is countable.

=> Union of +ve, -ve rational numbers is Countable.

Corollary 4:

The set of all rational numbers in [0,1] is countable.

proof:

~~Let A, B~~ We know that every subset of a Countable set is countable

Since Q is Countable. Let A = set of all rational numbers in [0,1]

Since $A \subset Q$ => Therefore A is countable.

∴ The set of all rational numbers in [0,1] is countable.

Theorem 5:

The set [0,1] is uncountable. The set of all real numbers is countable.

Proof:

To Prove : [0,1] is countable.

Suppose, [0,1] is Countable. => There exists a one one map between N to [0,1] Let us say $f : N \rightarrow [0,1]$

$$\Rightarrow f(1), f(2), f(3) \dots \in [0,1]$$

$$f(N) = [0,1]$$

$$f(1) = 0.a_{11}\, a_{12}\, a_{13} \dots$$

$$f(2) = 0.a_{21}\, a_{22}\, a_{23} \dots$$

$$\vdots$$

$$f(n) = 0.a_{n1}\, a_{n2}\, a_{n3} \dots$$

The number lies between 0 & 9 in decimal places

$$0 \leq a_{ij} \leq 9$$

$n \in N$, positive integer b_n

$$b_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{m} = 1 \end{cases} \qquad b_1 = \begin{cases} 1 & \text{if } a_{11} \neq 1 \\ 2 & \text{if } a_{11} = 1 \end{cases}$$

$$b_2 = \begin{cases} 1 & \text{if } a_{22} \neq 1 \\ 2 & \text{if } a_{22} = 1 \end{cases}$$ therefore $b_n \neq a_{nn}\ \forall n$

$$0.b_1 b_2 b_3 b_4 \dots$$

$$0.b_1 b_2 b_3 \dots \in [0,1]$$

but $0.b_1 b_2 b_3 \dots \neq f(n) \ \forall n$

∴ f is not onto

=> There is no 1-1 Correspondence between N & [0,1]

∴ [0,1] is uncountable.

Corollary 1: The set of real number is uncountable

proof. Toprove The set of real number is uncountable.

We know that $[0,1]$ is uncountable. & $[0,1] \subset R$

Since, Every superset of an uncountable set is uncountable

$R \supset [0,1]$ (i.e R is a superset of $[0,1]$)

$\therefore$ R is uncountable, Because $[0,1]$ is uncountable.

Theorem 6: Let P_n be the set of polynomial functions f of degree n defined by the relations of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

where n is a fixed nonnegative integer, the coefficients $a_0, a_1, a_2 \dots a_n$, are all integers and $a_0 \neq 0$, then then set is Countable.

proof $P_n = \{ f(x) \mid \deg f(x) = n \}$

proof by induction

Let $n=0$ $\quad P_0 : N \to Z - \{0\}$ Clearly it is countable

Assume that P_K is countable for positive degree $K > n$

Let $S_m = \{ f \mid f = mx^{K+1} + g \,,\ g \in P_K \}$

$S_{-m} = \{ f \mid f = -mx^{K+1} + g \,,\ g \in P_K \}$

$P_{K+1} = S_m \cup S_{-m}$

Since S_m is countable, S_{-m} is countable

$\bigcup_{m=1}^{\infty} T_m$ is countable

$P_{K+1} = \bigcup_{m=1}^{\infty} T_m$, P_{K+1} is countable.

Corollary The set of all polynomial functions with integer coefficients is countable.

proof: Let P_n be the set of polynomial functions of degree n with integer coefficients. We know that P_n is countable

Since $P = \bigcup_{n=0}^{\infty} P_n$

And since the union of countable sets is countable.

Therefore it follows that P is countable.

Corollary 2

set of all irrational numbers is uncountable

Proof:

Let S be the set of irrational numbers. To prove S is uncountable

Suppose S is countable.

We know that $S \cup Q = R$

Set of all irrational $\cup$ set of all rationals = Set of all real numbers

Since Q is countable. We have S as a Countable set by assumption. That implies $S \cup Q = R$ = Countable set which is a contradiction.

$\therefore$ S is uncountable.

Definition 3: A real number a is said to be algebraic if it is the root of some polynomial equation with rational coefficients $p(a) = 0 \Rightarrow a$ is algebraic.

Theorem 7

The set of all algebraic numbers is countable.

Proof:

To prove set of all algebraic numbers is countable.

Let n be a positive integer

Let Q_n be a polynomial with rational coefficients

$Q_n = \{ f(x) \mid \deg f(x) = n \}$

$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n x^0$

Where $a_0, a_1, \dots a_n$ are rational coefficients.

Let f_{nk} is the polynomial of degree n

& number of roots of $f_{nk} = n$

$\bigcup_{n=0}^{\infty} Q_n$

$\{ f_{n_1}, f_{n_2}, \dots f_{nk} \dots \}$

Let $A_{nk} = \{ a \mid f_{nk}(a) = 0 \}$

$\Rightarrow A_{n_1}, A_{n_2}, \dots A_{nk}$ is countable.

$\Rightarrow A_n = \bigcup_{n=1}^{\infty} A_{nk}$ is Countable

$\Rightarrow A = \bigcup_{n=1}^{\infty} A_n$ is Countable

$\therefore$ The set of all algebraic numbers is countable.

Definition 4

A real number is said to be transcendental if it is not an algebraic number.

Theorem 8

The set of all transcendental numbers is uncountable.

proof:

set of all transcendental number $\cup$ set of all algebraic numbers = set of all real numbers.

$$T \cup A = R$$

We know that, set of all real numbers is uncountable

& set of all algebraic numbers is [redacted] countable

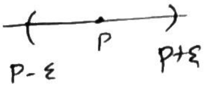
$\Rightarrow$ T must be uncountable

(because, if T is countable, then $T \cup A = R$ must be countable)
which is a contradiction.

UNIT 2

Neighbourhoods

A set $N \subset R$ is said to be a neighbourhood of a point $p \in R$, if there exist an $\varepsilon > 0$ such that $(p-\varepsilon, p+\varepsilon) \subset N$



Open sets

A set $G \subset R$ is said to be open if it is a neighbourhood of each of its points.

A set $G \subset R$ is said to be open if for each $p \in G$, There exists $\varepsilon > 0$ such that $(p-\varepsilon, p+\varepsilon) \subset G$.

closed sets:

A set $F \subset R$ is said to be closed if it complement $(R \setminus F)$ is open.

One sided limits

Definition 1

A function f defined on a set S containing (c, d) is said to approach a number l as x tends to (or approaches) c from the right, if given $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$c < x < c+\delta \Rightarrow |f(x) - l| < \varepsilon$$

$$\lim_{\substack{x \to c \\ x > c}} f(x) = l$$

(If f doesnot approches l as x tends to c then $c < x < c+\delta \to |f(x) - l| \geq x$)

$$\lim_{x \to c+0} f(x) = l$$

Definition 2

A function defined on a set S containing (b, c) is said to tends to (or approach) a number l as x tends to (or approaches) c from the left if given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$c-\delta < x < c \Rightarrow |f(x)-l| < \epsilon$$

In Symbols we write $\lim\limits_{x\to c-0} f(x) = l$ or $\lim\limits_{\substack{x\to c \\ x<c}} f(x) = l$

If there exists an $\epsilon > 0$ such that for each $\delta > 0$, there is some x for which $c-\delta < x < c$ and $|f(x)-l| \geq \epsilon$, then we can say that f(x) doesnot tends to l as x tends to c from the left.

Illustration

1) Let f be a function defined on $R-\{0\}$ by setting $f(x) = \frac{|x|}{x}$ where $x \neq 0$. Then $f(0+0) = 1$; $f(0-0) = -1$

For when $x > 0$

$$f(x) = \frac{|x|}{x} = \frac{x}{x} = 1$$

When $x < 0$ $\quad f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1$

$$\Rightarrow f(x) = \begin{cases} 1 & \text{when } x > 0 \\ -1 & \text{when } x < 0 \end{cases}$$

given any $\epsilon > 0$ taking $\delta > 0$ we have

$$c < x < c+\delta \Rightarrow |f(x)-l| = 0 < \epsilon$$

$$\epsilon > 0, \ \delta < 0$$

$$c-\delta < x < c \Rightarrow |f(x)-l| < \epsilon$$

Given $\epsilon > 0$, choose $\delta = \epsilon$

$0 < x < \delta$, $\quad |f(x)-1| = |1-1| = 0 < \epsilon$

$-\delta < x < 0$ $\quad |f(x)-(-1)| = |-1+1| = 0 < \epsilon$

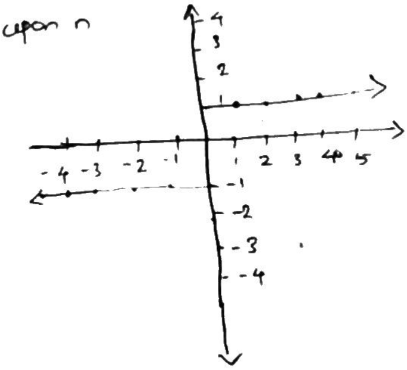
$\therefore \lim\limits_{x\to 0+0} f(x) = 1$

$f(0+0) = 1$

$\therefore \lim\limits_{x\to 0-0} f(x) = -1$

$f(0-0) = -1$

depending upon x



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2. Let f be the function defined on $\mathbb{R}$ by setting $f(x) = \lfloor x \rfloor$ where $\lfloor x]$ denotes the greatest integer not exceeding x.

If n be any integer

Then $f(n+0) = n$ $\quad f(n-0) = n-1$

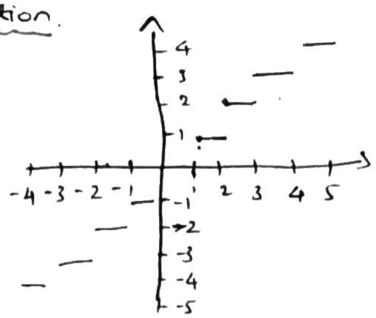
for every integer n, $f(x) = n$ when $n \leq x < n+1$

$\therefore f(n+0) = n$

Also, Since $f(x) = n-1$, when $n-1 \leq x < n$

Therefore $f(n-0) = n-1$

The function $f(x) = [x]$ considered in this illustration is called the greatest integer function.



3. Let f be the function defined on $\mathbb{R}$ as follows

$$f(x) = \begin{cases} 1-2x & \text{when } x<0 \\ 0 & \text{when } x=0 \\ 1+3x & \text{when } x>0 \end{cases}$$

Then $f(0+0) = f(0-0) = 1$

Case (i)

Let $x>0$

Given $\varepsilon>0$ $\quad |f(x)-1| = |-2x| = -2x < \varepsilon$

Whenever $-x < \varepsilon/2$

$x > -\varepsilon/2$

Taking $\delta = \frac{1}{2}\varepsilon$ we find that

$-\delta < x < 0 \Rightarrow |f(x)-1| < \varepsilon$

we have $\lim_{x \to 0-0} f(x) = 1$

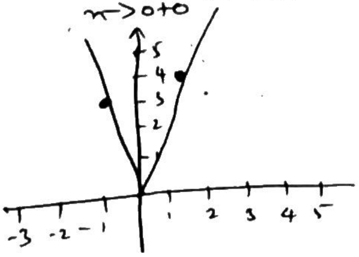
Case (ii) Let $x>0$

$|f(x)-1| = |1+3x-1| = |3x| = 3x < \varepsilon$

$\delta>0$, choose $\delta = \varepsilon/3$

$0<x<0+\delta \Rightarrow |f(x)-1| < \varepsilon$

we have $\lim_{x \to 0+0} f(x) = 1$



Theorem 1

Let f be defined on (c, d). Then $\lim_{x\to c+0} f(x)=l$ iff for every sequence $\langle x_n\rangle$, where $x_n>c$ for all $n\in N$, Converging to c, then sequence $\langle f(x_n)\rangle$ Converges to l.

Proof: Assume that

$\lim_{x\to c+0} f(x)=l$ and Sia $\langle x_n\rangle$ is a sequence with $x_n>c$ for $n\in N$ suchthat $x_n\to c$

Since $\lim_{x\to 0} f(x)=l$ therefore

given $\epsilon>0$, we can find a $\delta>0$ Such that

$$c<x<c+\delta \Rightarrow |f(x)-l|<\epsilon \longrightarrow ①$$

$\langle x_n\rangle\to c \quad x_n>c\ \forall\ n\geq m$.

There exists $m\in N$ suchthat $|x_n-c|<\delta\ \forall\ n\geq m$

$-\delta< x_n-c<\delta$ for all $n\geq m$

$(c-\delta)<x_n<(c+\delta)$ for all $n\geq m \longrightarrow ②$

from ① & ②

$|f(x)-l|<\epsilon\ \forall\ n\geq m$

$|f(x_n)|\to l$

Suppose, if $x\to c+0$ $f(x)\neq l$

$\epsilon>0$, $\exists\ \delta>0$

$c<x<c+\delta \Rightarrow |f(x)-l|\geq\epsilon$

$x_n\to c$, $x_n>c$, $\forall\ n\in N$

$|x_n-c|<\delta$, $\forall\ n\geq m$

$c-\delta<x_n<c+\delta$, $\forall\ n\geq m$

$|f(x_n)-l|\geq\epsilon\ \forall\ n\geq m$

$f(x_n)\not\to l$

Theorem 2 Let f be defined on (b, c). Then $\lim_{x\to c-0} f(x)=l$ iff for every sequence $\langle x_n\rangle$, where $x_n<c$ for all $n\in N$, Converging to c, the sequence $\langle f(x_n)\rangle$ Converges to l.

Proof:

Assume that $\lim_{x\to c-0} f(x)=l$ & $\langle x_n\rangle$ is a sequence with $x<c$ for all $n\in N$ Such that $x_n\to c$

Since $\lim_{x\to c-0} f(x)=l$, therefore Given $\epsilon>0$, we can find $\delta>0$ Such that $c-\delta<x<c \Rightarrow |f(x)-l|<\epsilon \longrightarrow ①$

$(x_n)\to c$, $x_n<c\ \forall\ n\geq m$

$\exists\ m \in N\ ,\ \ni\ \ |x_n - c| < \delta\ \ \forall\ \ n \geq m$.

$-\delta < (x_n - c) < \delta$ for all $n \geq m$

$c - \delta < x_n < c + \delta\ \ \forall\ n \geq m \longrightarrow$ ②

$\forall\ n \geq m$

from ①, ② we get $|f(x) - l| < \epsilon$

$|f(x_n)| \longrightarrow l$

Suppose if $x \to c - 0$ $f(x) \neq l$

Given $\epsilon > 0$ $\exists\ \delta > 0$

$c - \delta < x < \delta \Rightarrow |f(x) - l| \geq \epsilon$

$x_n \to c$, $x_n < c$ $\forall\ n \in W$

$|x_n - c| < \delta$ $\forall\ n \geq m$

$c - \delta < x_n < c + \delta$ $\forall\ n \geq m$

$|f(x_n) - l| \geq \epsilon$ $\forall\ n \geq m$

$f(x_n) \neq l$.

Limit as x approches c

Let f be a function defined on some neighbourhood N of c, except possible at $x = c$. f is said to approach a limit l as x approaches c if for every $\epsilon > 0$, there is some $\delta > 0$ suchthat $0 < |x - c| < \delta \Rightarrow |f(x) - l| < \epsilon$

Theorem 3

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = m$ then $l = m$

(ie limit is unique)

$\lim_{x \to a} f(x) = l$, $\epsilon > 0$, $\exists\ \delta_1 > 0$

suchthat $|x - a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon$

$\lim_{x \to a} f(x) = m$. $\epsilon > 0$ $\exists\ \delta_2 > 0$

suchthat $|x - a| < \delta_2 \Rightarrow |f(x) - m| < \epsilon$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$

$0 < |x - a| < \delta_0$

Suppose $l \neq m$

$l - m > 0$ So, we can choose $\epsilon = \frac{l - m}{2}$

$\therefore |f(x) - l| < \frac{l - m}{2}$, $|f(x) - m| < \frac{l - m}{2}$

When $0 < |x - a| < \delta_0$, then

$|l - m| = |f(x) - m - (f(x) - l)|$ (add & subtract $f(x)$)

$= |(f(x) - m) + (l - f(x))|$

$\leq |f(x) - m| + |l - f(x)|$

$\leq |f(x) - m| + |f(x) - l|$ (By the property of absolute value)

$$\leq \frac{l-m}{2} + \frac{l-m}{2}$$

$$\leq l-m$$

which is a contradiction. $\therefore l = m$.

Theorem 1

Let f be defined on (c,d). Then $\lim_{x\to c+0} f(x)=l$ iff for every sequence $\langle x_n\rangle$, where $x_n>c$ for all $n\in N$, Converging to c, Then sequence $\langle f(x_n)\rangle$ Converges to l.

$x_n\to c,\ x_n>c$

$x\to c+0$

$f(x)\to l$ $\quad f(x_n)\to l$

Proof: Assume that

$\lim_{x\to c+0} f(x)=l$ and ~~Sin~~ $\langle x_n\rangle$ is a sequence with $x_n>c$ for $n\in N$ suchthat $x_n\to c$

Since $\lim_{x\to c+0} f(x)=l$ therefore

given $\varepsilon>0$, we can find a $\delta>0$ such that

$c<x<c+\delta \Rightarrow |f(x)-l|<\varepsilon \longrightarrow ①$

$\langle x_n\rangle\to c \qquad x_n>c \ \forall\ n\geq m$.

There exists $m\in N$ suchthat $|x_n-c|<\delta \ \forall\ n\geq m$

(add c) $\quad -\delta<x_n-c<\delta$ for all $n\geq m$

$(c-\delta)<x_n<(c+\delta)$ for all $n\geq m \longrightarrow ②$

from ① & ②

$|f(x_n)-l|<\varepsilon \ \forall\ n\geq m$

$|f(x_n)|\longrightarrow l$

Suppose, if $x\to c+0$ $\quad f(x)\neq l$

$\varepsilon>0,\ \exists\ \delta>0$

$c<x<c+\delta \Rightarrow |f(x)-l|\geq\varepsilon$

$x_n\to c,\ x_n>c,\ \forall\ n\in N$

$|x_n-c|<\delta,\ \forall\ n\geq m$

$c-\delta<x_n<c+\delta,\ \forall\ n\geq m$

$|f(x_n)-l|\geq\varepsilon \ \forall\ n\geq m$

$f(x_n)\not\to l$

Theorem 2

Let f be defined on (b,c). Then $\lim_{x\to c-0} f(x)=l$ iff for every sequence $\langle x_n\rangle$, where $x_n<c$ for all $n\in N$, Converging to c, the sequence $\langle f(x_n)\rangle$ Converges to l.

Proof:

Assume that

$\lim_{x\to c-0} f(x)=l$ & $\langle x_n\rangle$ is a sequence with $x<c$ for all $n\in N$ Suchthat $x_n\to c$

Since $\lim_{x\to c-0} f(x)=l$, therefore Given $\varepsilon>0$, we can find $\delta>0$ Such that $c-\delta<x<c \Rightarrow |f(x)-l|<\varepsilon \longrightarrow ①$

$(x_n)\to c \quad ,x_n<c \ \forall\ n\geq m$

$\exists\ m \in N,\ \ni\ |x_n - c| < \delta\ \forall\ n \geq m$.

$-\delta < (x_n - c) < \delta$ for all $n \geq m$

$c - \delta < x_n < c + \delta\ \ \forall\ n \geq m \longrightarrow ②$

from ①, ② we get $|f(x_n) - l| < \varepsilon\ \ \forall\ n \geq m$

$|f(x_n)| \to l$

Suppose if $x \to c-0$ $f(x) \neq l$

Given $\varepsilon > 0$ $\exists\ \delta > 0$

$c - \delta < x < \delta \Rightarrow |f(x) - l| \geq \varepsilon$

$x_n \to c,\ x_n < c\ \ \forall\ n \in W$

$|x_n - c| < \delta\ \ \forall\ n \geq m$

$c - \delta < x_n < c + \delta\ \ \forall\ n \geq m$

$|f(x_n) - l| \geq \varepsilon\ \ \forall\ n \geq m$

$f(x_n) \neq l$.

Limit as x approches c

Let f be a function defined on some neighbourhood N of c, except possible at $x = c$. f is said to approach a limit l as x approaches c if for every $\varepsilon > 0$ there is some $\delta > 0$ suchthat $0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$

Theorem 3

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = m$ then $l = m$

(ie limit is unique)

$\lim_{x \to a} f(x) = l$, $\varepsilon > 0$, $\exists\ \delta_1 > 0$ suchthat $|x - a| < \delta_1 \Rightarrow |f(x) - l| < \varepsilon$

$\lim_{x \to a} f(x) = m$, $\varepsilon > 0$ $\exists\ \delta_2 > 0$ suchthat $|x - a| < \delta_2 \Rightarrow |f(x) - m| < \varepsilon$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$

$0 < |x - a| < \delta_0$

Suppose $l \neq m$

$l - m > 0$ so, we can choose $\varepsilon = \frac{l - m}{2}$

$\therefore |f(x) - l| < \frac{l - m}{2}$, $|f(x) - m| < \frac{l - m}{2}$

When $0 < |x - a| < \delta_0$, then

$|l - m| = |f(x) - m - (f(x) - l)|$ (add & subtract $f(x)$)

$= |(f(x) - m) + (l - f(x))|$

$\leq |f(x) - m| + |l - f(x)|$

$\leq |f(x) - m| + |f(x) - l|$ (By the property of absolute value)

$$\leq \frac{l-m}{2} + \frac{l-m}{2}$$

$$\leq l-m$$

which is a Contradiction $\therefore l = m$

Theorem 4

Let f be defined on a deleted neighbourhood N^* of c $\lim_{x\to c} f(x)$ exists and equals l iff $f(c+0)$, $f(c-0)$ both exists and are equal to l.

proof

If $\lim f(x)$ exists, then $\lim_{x\to c+0} f(x)$ and $\lim_{x\to c-0} f(x)$ both exists and are equal.

Then there exists a $\delta>0$ suchthat $0<|x-c|<\delta$ $\Rightarrow |f(x)-l|<\varepsilon \longrightarrow ①$

$c-\delta < x < c+\delta$

$c-\delta < x < c$ and $c < x < c+\delta$ implies $0<|x-c|<\delta$.

$|f(x)-l| < \varepsilon \Rightarrow c-\delta < x < c \longrightarrow ②$

$|f(x)-l| < \varepsilon \Rightarrow c < x < c+\delta \longrightarrow ③$

Since the equation ② implies that $\lim_{x\to c-0} f(x)$ exists and equals to l.

③ implies that $\lim_{x\to c+0} f(x)$ exists and equals to l.

Therefore we find that $\lim_{x\to c-0} f(x)$ and $\lim_{x\to c+0} f(x)$ both exists and are equal to l.

Converse part

Let us assume that $\lim_{x\to c+0} f(x) = l = \lim_{x\to c-0} f(x)$

let $\varepsilon>0$ be given. Since $\lim_{x\to c+0} f(x)=l$, there exists $\delta_1>0$ suchthat

$c<x<c+\delta_1 \Rightarrow |f(x)-l|<\varepsilon \longrightarrow ④$

Again, Since $\lim_{x\to c-0} f(x)=l$, there exists $\delta_2>0$ Suchthat

$c-\delta_2 < x < c \Rightarrow |f(x)-l| < \varepsilon \longrightarrow ⑤$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$

Then $c-\delta_0 < x < c \Rightarrow c-\delta_2 < x < c \longrightarrow ⑥$

$\Rightarrow |f(x)-l| < \varepsilon$ by ⑤

Also, $c < x < c+\delta_0 \Rightarrow c < x < c+\delta_1 \longrightarrow ⑦$

$\Rightarrow |f(x)-l| < \varepsilon$ by ④

From ⑥ & ⑦ we have

$0<|x-c|<\delta_0 \Rightarrow |f(x)-l|<\varepsilon$

$\therefore \lim_{x\to c} f(x) = l.$

Illustrations

1) ~~lim~~ let $f(x) = \frac{|x|}{x}$ whenever $x \neq 0$, Then $\lim_{x \to 0} f(x)$ doesnot exist.

2) Let $f(x) = [x]$ for all $x \in R$. Then $\lim_{x \to 0} f(x)$ doesnot exists.

3) Let $f(x) = \begin{cases} 1-2x & \text{when } x < 0 \\ 0 & \text{when } x = 0 \\ 1+3x & \text{when } x > 0 \end{cases}$

Then $\lim_{x \to 0} f(x)$ exists and equals 1.

Theorem 5

Let f be the function defined in some interval $(c-\delta, c+\delta)$ except possibly at $x = c$. Then $\lim_{x \to c} f(x)$ exists and equals l, iff for every sequence $\langle x_n \rangle$, where $0 < |x_n - c| < \delta$ for $n \in N$, Converging to c, the sequence $\langle f(x_n) \rangle$ converges to l.

Proof

Assume that $\lim_{x \to c} f(x) = l$ and $\langle x_n \rangle$ is a sequence with $0 < |x_n - c| < \delta$ for $n \in N$, $x_n \to c$

Since $\lim_{x \to c} f(x) = l$. $\Rightarrow$ Given $\varepsilon > 0$, we can find a $\delta > 0$ Such that $c - \delta < x < c + \delta \Rightarrow |f(x) - l| < \varepsilon \longrightarrow ①$

$|x_n - c| < \delta$ whenever $n \geq m$

$c - \delta < x_n < c + \delta$ whenever $n \geq m \longrightarrow ②$

$\therefore$ From ① & ② we will get $|f(x_n) - l| < \varepsilon \quad \forall\, n \geq m$

$\Rightarrow f(x_n) \longrightarrow l$

Converse part

to prove if $x \to c$ then $f(x) \to l$

Suppose if $x \to c \Rightarrow f(x) \neq l$

Given $\varepsilon > 0$, $\exists\, \delta > 0$

$c - \delta < x < c + \delta \Rightarrow |f(x) - l| \ngtr \varepsilon$

$x_n \to c \quad x_n = c$

$|x_n - c| < \delta \quad \forall\, n \geq m$

$c - \delta < x_n < c + \delta \quad \forall\, n \geq m$

$|f(x_n) - l| \geq \varepsilon \quad \forall\, n \geq m$

$f(x_n) \neq l$

Algebra of limits $(+, -, \times, \div)$

Let f and g be two functions with a common domain D and having ranges as subsets of R.

The sum of the functions f, g defined on D as follows

$$(f+g)(x) = f(x) + g(x) \quad \forall\ x \in D$$

Product of the functions f, g defined on D as follows

$$(f.g)(x) = f(x) \cdot g(x) \quad \text{for all } x \in D$$

~~Ag~~ scalar product of function f by a scalar c is defined as follows

$$(cf)(x) = c \cdot f(x) \quad \forall\ x \in D$$

If $g(x) \neq 0$ whenever $x \in D_1 \subset D$, then the reciprocal of g is the function $\frac{1}{g}$ defined on D_1 by setting

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} \quad \forall\ x \in D_1$$

Example: $\frac{1}{x}$ $\quad x \in R - \{0\} = D_1$

If $g(x) \neq 0$ whenever $x \in D_1 \subset D$ the quotient f/g is the function defined on D_1 by setting

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \forall\ x \in D_1$$

(provided $g(x) \neq 0$)

Theorem 6

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$ then $\lim_{x \to a}[f(x) \pm g(x)] = l \pm m$

proof

Let $\varepsilon > 0$ be given. Then in view of the given limits there exists $\delta_1 > 0$, $\delta_2 > 0$ such that

$$|f(x) - l| < \varepsilon/2 \text{ when } 0 < |x-a| < \delta_1$$

& $$|g(x) - m| < \varepsilon/2 \text{ when } 0 < |x-a| < \delta_2$$

Led $\delta = \min(\delta_1, \delta_2)$ Then we have

$$|f(x) - l| < \varepsilon/2 \text{ when } 0 < |x-a| < \delta$$

$$|g(x) - l| < \varepsilon/2 \text{ when } 0 < |x-a| < \delta$$

and now

$$\begin{aligned} |(f(x) \pm g(x)) - (l \pm m)| &= |(f(x) - l) \pm (g(x) - m)| \\ &\leq |f(x) - l| \pm |g(x) - m| \end{aligned}$$

(By triangle inequality)

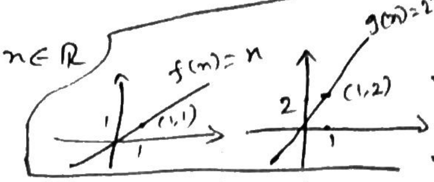
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ when } 0 < x-a| < \delta$$

$\therefore |(f(x) \pm g(x)) - (l \pm m)| < \varepsilon$ whenever $0 < |x-a| < \delta$

Hence $\lim_{x \to a} [f(x) \pm g(x)] = l \pm m$

Example

1) $f(x) = x$, $g(x) = 2x$ $\forall$ $x \in \mathbb{R}$



$\lim_{x \to 2} f(x) = 2$ $\lim_{x \to 2} g(x) = 4$

$\lim_{x \to 2} (f+g)(x) = \lim_{x \to 2} (x+2x) = \lim_{x \to 2} 3x = 6$ — L.H.S

$\lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 2+4 = 6$ — R.H.S

Theorem 7

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

$\lim_{x \to a} [f(x)\, g(x)] = l.m$

Proof:

~~W~~

$$|f(x)g(x) - lm| = |f(x)g(x) - lm + l.g(x) - l.g(x)|$$

$$= |g(x)(f(x)-l) + l(g(x)-m)|$$

$$\leq |g(x)(f(x)-l)| + |l(g(x)-m)|$$

$$\leq |g(x)||f(x)-l| + |l||g(x)-m| \quad \text{— ①}$$

(By triangle inequality & $|a.b| \leq |a|.|b|$)

Since $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = m$

for $0 < \varepsilon' < 1$, there exist some $\delta > 0$, Such that

$|f(x) - l| < \varepsilon'$ & $|g(x) - m| < \varepsilon'$ when $0 < |x-a| < \delta$ — ②

$$|g(x)| = |g(x) + m - m| \leq |m| + |g(x) - m|$$

$$|g(x)| < |m| + \varepsilon' \text{ when } 0 < |x-a| < \delta \quad \text{— ③}$$

From ①, ②, ③ we get

$$|f(x)g(x) - lm| < (|m| + \varepsilon')\varepsilon' + |l|\varepsilon' \text{ when } 0 < |x-a| < \delta$$

$$< (|m| + |l| + 1)\varepsilon' \quad (\text{Since } \varepsilon' < 1)$$

Let us choose ε' Such that $\varepsilon' < \dfrac{\varepsilon}{(|m|+|l|+1)}$ when $\varepsilon > 0$

$$|f(x)g(x) - lm| < (|m|+|l|+1) \cdot \frac{\varepsilon}{(|m|+|l|+1)}$$

$\Rightarrow |f(x)g(x) - lm| < \varepsilon$ when $0<|x-a|<\delta$

Hence $\lim\limits_{x\to a}[f(x)g(x)] = lm = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$

Theorem 8

If $\lim\limits_{x\to c} g(x)=m$ and $m\neq 0$, then $\lim\limits_{x\to c}\frac{1}{g(x)}=\frac{1}{m}$

Proof. Let $\varepsilon_0>0$, we have to find a $\delta>0$ such that

$$\left|\frac{1}{g(x)}-\frac{1}{m}\right| = \left|\frac{m-g(x)}{g(x)\cdot m}\right|$$

$$= \frac{|m-g(x)|}{|g(x)\cdot m|} = \frac{|g(x)-m|}{|g(x)|\cdot|m|} \quad ①$$

Since we have $\lim\limits_{x\to c} g(x)=m$

$\varepsilon>0,\ \delta>0 \quad |g(x)-m|<\varepsilon$ when $0<|x-c|<\delta$ ②

take $\varepsilon=\frac{|m|}{2} \Rightarrow |g(x)-m|<\frac{|m|}{2}$ when $0<|x-c|<\delta_0$

$$|m| = |m+g(x)-g(x)|$$

$$\leq |m-g(x)| + |g(x)|$$

$$|m| \leq \frac{m}{2} + |g(x)|$$

$$|m| - \frac{|m|}{2} < |g(x)|$$

$$\frac{|m|}{2} < |g(x)|$$

$$|g(x)| > \frac{|m|}{2}$$

$$\frac{1}{|g(x)|} < \frac{2}{|m|} \quad ③$$

From ①&③ $\left|\frac{1}{g(x)}-\frac{1}{m}\right| < \frac{|g(x)-m|}{|m|}\cdot\frac{2}{|m|}$

$$= \frac{2}{|m|^2}\cdot|g(x)-m|$$

$\delta_1>0 \quad \varepsilon>0$

$|g(x)-m| < \frac{\varepsilon|m|^2}{2}$ when $0<|x-c|<\delta_1$

$\delta_2 = \min\{\delta_1, \delta_0\}$

$\left|\frac{1}{g(x)}-\frac{1}{m}\right| < \frac{2}{|m|^2}\cdot\frac{\varepsilon|m|^2}{2} = \varepsilon$ when $0<|x-c|<\delta_2$

$\therefore \left|\frac{1}{g(x)}-\frac{1}{m}\right| < \varepsilon$

$\lim\limits_{x\to c}\frac{1}{g(x)} = \frac{1}{m}$ Hence Proved.

Theorem 9

If $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m$, then $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$

provided $m \neq 0$.

Proof $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \lim_{x \to c} \left\{ f(x) \cdot \frac{1}{g(x)} \right\}$

Since $\lim_{x \to c} g(x) = m \neq 0$ Therefore by theorem 8

We have $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$ exists.

Since $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$ therefore by theorem 7

we have $\lim_{x \to c} \left(f(x) \cdot \frac{1}{g(x)} \right) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} \frac{1}{g(x)}$

$$= l \cdot \frac{1}{m} = \frac{l}{m}$$

$$\Rightarrow \lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m} \quad //$$

Theorem 10

Let f be defined on D and let $f(x) \geq 0$ for all $x \in D$. If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} f(x) \geq 0$

Proof

Proof by Contradiction

Let $\lim_{x \to a} f(x) = l$, where $l < 0$ (if possible)

Then for given $\varepsilon = -\frac{l}{2} > 0$, we can find a $\delta > 0$ such that

$|f(x) - l| < \frac{-l}{2}$ whenever $0 < |x - a| < \delta$

$\Rightarrow -\left(\frac{-l}{2}\right) < f(x) - l < \left(-\frac{l}{2}\right)$ whenever $0 < |x - a| < \delta$

(add l) $l+\frac{l}{2} < f(x) < l-\frac{l}{2}$ Whenever $0<|x-a|<\delta$

$\frac{3l}{2} < f(x) < \frac{l}{2} < 0$ (since $\frac{l}{2} < 0$)

Which is a Contradiction Since $f(x) \geq 0$

$\therefore$ our assumption is wrong. l should be ≥ 0

$\Rightarrow \lim_{x \to a} f(x) = l \geq 0$

$\Rightarrow \lim_{x \to a} f(x) \geq 0$

Corollary

Let f be defined on D and let $f(x) > 0$ $\forall x \in D$

If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} f(x) \geq 0$.

Proof: $f(x) > 0 \Rightarrow f(x) \geq 0$ by theorem 10 we get

$\lim_{x \to a} f(x) \geq 0$.

Theorem 11 Let f, g be defined on D and let $f(x) \geq g(x)$ $\forall x \in D$. Then $\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$, provided these limits exists

proof

Let $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$

Let $h(x) = f(x) - g(x)$ $\forall x \in D$. Then we have

(i) $h(x) \geq 0$ $\forall x \in D$

(ii) $\lim_{x \to a} h(x) = l - m$ exists

(iii) $\lim_{x \to a} h(x) \geq 0 \Rightarrow l - m \geq 0$

$l - m \geq 0$

$l \geq m$

$\Rightarrow \lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$

Corollary Let f, g be defined on D and $f(x) > g(x)$ $\forall x \in D$

Then $\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$ provided these limits exists

Proof $f(x) > g(x)$

$f(x) - g(x) > 0 \Rightarrow f(x) - g(x) \geq 0$

$\therefore$ by theorem 11 $\lim_{x \to a} (f(x) - g(x)) \geq 0$

$\Rightarrow \lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$

Theorem 12 (Squeeze Principle)

Let f, g and h defined on D and let $f(x) \geq g(x) \geq h(x)$ for all x. Let $\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$. Then $\lim_{x \to c} g(x)$ exists and $\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$.

proof.

let $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$

Then for given $\varepsilon > 0$, There exists positive numbers δ_1, δ_2 Such that $|f(x) - l| < \varepsilon$ for $0 < |x-a| < \delta_1$

& $|h(x) - l| < \varepsilon$ for $0 < |x-a| < \delta_2$

$|f(x) - l| < \varepsilon \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$ Where $0 < |x-c| < \delta_1$ ——— ①

Simillarly

$|h(x) - l| < \varepsilon \Rightarrow l - \varepsilon < h(x) < l + \varepsilon$ Where $0 < |x-c| < \delta_2$ ——— ②

Let $\delta = \min\{\delta_1, \delta_2\}$

Then $0 < |x-c| < \delta \Rightarrow 0 < |x-c| < \delta_1$ & $0 < |x-c| < \delta_2$

Since $h(x) \leq g(x) \leq f(x)$

Then $l - \varepsilon < h(x) \leq g(x) \leq f(x) < l + \varepsilon$

When $0 < |x-c| < \delta$

Then $l - \varepsilon < g(x) < l + \varepsilon$ When $0 < |x-c| < \delta$

$\Rightarrow |g(x) - l| < \varepsilon$

$\therefore \lim_{x \to c} g(x) = l$

$\therefore \lim_{x \to c} g(x)$ exists and equals to $\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$

Theorem 13 $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| = |l|$ But The converse is not true $\left(\lim_{x \to a} |f(x)| = |l| \underset{\Longleftarrow}{\nRightarrow} \lim_{x \to a} f(x) = l \right)$

proof:

Since $\lim_{x \to a} f(x) = l$, for a given $\varepsilon > 0$, there exists a Positive number δ Such that $|f(x) - l| < \varepsilon$ whenever $0 < |x-a| < \delta$

Since $||a| - |b|| \leq |a-b|$

$\therefore ||f(x)| - |l|| \leq |f(x) - l| < \varepsilon$

When $0 < |x-a| < \delta$

$\Rightarrow ||f(x)| - |l|| < \varepsilon$ When $0 < |x-a| < \delta$

$a = 3$
$b = -2$
$||a| - |b|| = |3 - 2|$
$= 1$
$|a - b| = |3 - (-2)|$
$= 5$

$$\Rightarrow \lim_{x \to a} |f(x)| = |l| \quad \text{Hence proved}$$

$$\lim_{x \to a} |f(x)| = |l| \not\Rightarrow \lim_{x \to a} f(x) = l \quad \text{Counterexample}$$

Let $f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x \geq a \end{cases}$

Then $\lim_{x \to a+0} f(x) = \lim_{x \to a+0} 1 = 1$

$\lim_{x \to a-0} f(x) = \lim_{x \to a-0} (-1) = -1$

$\lim_{x \to a+0} f(x) \neq \lim_{x \to a-0} f(x)$, so $\lim_{x \to a} f(x)$ doesnot exists

But $|f(x)| = 1 \quad \forall \ x \in R$

So $\lim_{x \to a} |f(x)| = 1$ exists.

(p986)

Infinite Limits

Definition 1

A function f defined on a set S containing (c,d) $]c,d[$ is said to tends to $+\infty$ (respectively $-\infty$) as x tends to c from the right if given $K>0$, we can find a $\delta>0$ such that $c<x<c+\delta \Rightarrow f(x)>K$ (respectively $f(x)<-K$)

$$\lim_{\substack{x\to c \\ x>c}} f(x)=+\infty \text{ (respectively } -\infty)$$

Definition 2

A function f defined on S containing (b,c) is said to tends to $+\infty$ (respectively $-\infty$) as x tends to c from the left if given $K>0$, we can find a $\delta>0$ such that $c-\delta<x<c \Rightarrow f(x)>K$ (respectively) $f(x)<-k$

from Definitions 1,2

$$c<x<c+\delta \Rightarrow f(x)>K \;\&\; c-\delta<x<c \Rightarrow f(x)>k$$

$K>0$

$$0<|x-c|<\delta \Rightarrow f(x)>K$$

$$\lim_{x\to c} f(x)\to\infty$$

$K<0$

$$0<|x-c|<\delta \Rightarrow f(x)<K$$

$$\lim_{x\to c} f(x)\to -\infty$$

Definition 3 (limit as $x\to\infty$ or $x\to-\infty$)

Let f be a function defined in D which contains (x_0,∞) for some $x_0\in\mathbb{R}$. f is said to approach a real number l as x tends to $+\infty$ (positively ∞) (i.e $\lim_{x\to+\infty} f(x)=l$) if for given $\varepsilon>0$, $\exists$ $K\in\mathbb{R}$, such that $|f(x)-l|<\varepsilon$ whenever $x>K$

$$x>K \Rightarrow |f(x)-l|<\varepsilon$$

Definition 4

Let f be a function in D which contains $(-\infty,x_0)$ for some $x_0\in\mathbb{R}$. f is said to approach l as x becomes negatively infinite (i.e $\lim_{x\to-\infty} f(x)=l$) if for given $\varepsilon>0$, $\exists$ $K\in\mathbb{R}$ such that $|f(x)-l|<\varepsilon$ whenever $x<-K$

$$x<K \Rightarrow |f(x)-l|<\varepsilon$$

Definition 5

Let a function f defined in D which contains (x_0, ∞) for some $x_0 \in \mathbb{R}$. f is said to tends to $+\infty$ $(-\infty)$ i.e $\lim_{x \to \infty} f(x) = +\infty$ (respectively $\lim_{x \to \infty} f(x) = -\infty$) if for given $K > 0$, $\exists$ K^*, suchthat $f(x) > K$ (respectively $f(x) < -K$) whenever $x > K^*$.

Definition 6

Let f be a function defined in D which contains $(-\infty, x_0)$ for some $x_0 \in \mathbb{R}$. f is said to tends to $+\infty$ (resp. $-\infty$) as x tends to $-\infty$ (i.e) $\lim_{x \to -\infty} f(x) = +\infty$ (resp. $\lim_{x \to -\infty} f(x) = -\infty$), If given $K > 0$, $\exists$ K^* suchthat $f(x) > K$ (resp. $f(x) < -K$) whenever $x < K^*$.

One Sided limits $\lim_{x \to c} f(x) = l$	Infinite limits $\lim_{x \to c} f(x) = \infty$	Limits as $x \to \infty$ $(-\infty)$ $\lim_{x \to \infty} f(x) = l$	Infinite limits as $x \to \infty (-\infty)$ $\lim_{x \to \infty} f(x) \to \infty$
i) $x > c$ (c, d) given $\varepsilon > 0, \exists \delta > 0$ $c < x < c + \delta \Rightarrow$ $\lvert f(x) - l \rvert < \varepsilon$ $\lim_{\substack{x \to c \\ x > c}} \lvert f(x - l) \rvert = \varepsilon$	(i) $\lim_{x \to c} f(x) = \infty$ Given $K > 0$, $\exists \delta > 0$ $c < x < c + \delta \Rightarrow$ $f(x) > K$ $c - \delta < x < c \Rightarrow$ $f(x) > K$ $\therefore 0 < \lvert x - c \rvert < \delta$ $\Rightarrow f(x) > K$	$\lim_{x \to \infty} f(x) = l$ given $\varepsilon > 0, \exists K > 0$ $x > K \Rightarrow \lvert f(x) - l \rvert < \varepsilon$	(i) $\lim_{x \to \infty} f(x) = \infty$ Given $K > 0$, $\exists$ $K^* > 0$ $x > K^*$, $f(x) > K$
(ii) $x < c$ (b, c) given $\varepsilon > 0$ $\exists$ $\delta > 0$ $c - \delta < x < c \Rightarrow$ $\lvert f(x) - l \rvert < \varepsilon$ $\lim_{\substack{x \to c \\ x < c}} \lvert f(x) - l \rvert < \varepsilon$	(ii) $\lim_{x \to c} f(x) = -\infty$ $K < 0$, $\exists \delta > 0$ $c < x < c + \delta \Rightarrow$ $f(x) < K$ $c - \delta < x < c \Rightarrow$ $f(x) < K$ $\therefore 0 < \lvert x - c \rvert < \delta$ $\Rightarrow f(x) < K$	$\lim_{x \to -\infty} f(x) = l$ $\varepsilon > 0$, $\exists$ $K < 0$ $x < K \Rightarrow \lvert f(x) - l \rvert < \varepsilon$	(ii) $\lim_{x \to -\infty} f(x) = -\infty$ $K^* < 0$, $K < 0$ $x < K^*$, $f(x) < K$
			(iii) $\lim_{x \to -\infty} f(x) = \infty$ $K^* < 0$, $K > 0$ $(-\infty, x_0)$ $x < K^* \Rightarrow f(x) > K$
			(iv) $\lim_{x \to \infty} f(x) = -\infty$ $K^* > 0$, $K < 0$ $x > K^*$, $f(x) < K$

Examples

1) Let $f(x) = \frac{1}{x^2-1}$, $\forall\ x \in R - \{-1, 1\}$. Show that

$\lim_{x\to 1+0} f(x) = +\infty$, $\lim_{x\to 1-0} f(x) = -\infty$, $\lim_{x\to -1+0} f(x) = -\infty$, $\lim_{x\to -1-0} f(x) = +\infty$

Answer

(i) to prove $\lim_{x\to 1+0} f(x) = +\infty$

Let $k>0$ be given. We have to find $\delta > 0$ Suchthat

$< x < \delta \Rightarrow f(x) > K$

$0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow 1 < x^2 < (1+\delta)^2$

$\Rightarrow 0 < x^2-1 < 1+2\delta+\delta^2-1$

$\Rightarrow 0 < x^2-1 < 2\delta+\delta^2$

$\Rightarrow x^2-1 < 2\delta+\delta^2$

$\Rightarrow \frac{1}{x^2-1} > \frac{1}{2\delta+\delta^2}$ ——— ①

If $\frac{1}{2\delta+\delta^2} > K \Rightarrow \frac{1}{x^2-1} > K$ ——— ②

$\frac{1}{2\delta+\delta^2} > K \Rightarrow 2\delta+\delta^2 < \frac{1}{K}$

$\delta^2+2\delta+1-1 < \frac{1}{k}$

$\delta^2+2\delta+1 < \frac{1}{k}+1 \Rightarrow (\delta+1)^2 < (1+\frac{1}{k})$

$\Rightarrow -(1+1/k)^{1/2} < \delta+1 < (1+\frac{1}{k})^{1/2}$

$\Rightarrow -1-(1+1/k)^{1/2} < \delta < -1+(1+1/k)^{1/2}$

Thus if we choose $\delta = (1/2)\times(-1+(1+1/k)^{1/2})$ then

from ① & ② we have $f(x) > k$ whenever $1 < x < 1+\delta$

$\Rightarrow \lim_{x\to 1+0} f(x) = \infty$ i.e $\lim_{x\to 1+0} \frac{1}{x^2-1} = \infty$

(ii) to prove $\lim_{x\to 1-0} f(x) = -\infty$

Given $k>0$ we have to find $\delta > 0$

Suchthat $c-\delta < x < c$ $1-\delta < x < 1 \Rightarrow f(x) < -k$

$\lim_{x\to 1-0} f(x) = -\infty$

$f(x) < -k$

$\frac{1}{x^2-1} < -k \Rightarrow x^2-1 > \frac{1}{(-k)}$

$1-\delta < x < 1 \Rightarrow (1-\delta)^2 < x^2 < 1$

$1-2\delta+\delta^2 < x^2 < 1$

adding (-1) on both sides

$-2\delta+\delta^2 < x^2-1 < 0$

$\delta(\delta-2) < x^2-1 < 0$

$\frac{1}{\delta(\delta-2)} < \frac{1}{x^2-1} < -k$ whenever $\frac{1}{\delta^2-2\delta} < -k$

$\delta^2 - 2\delta + \frac{1}{k} < 0$

$\Rightarrow \delta^2 - 2\delta < -\frac{1}{k} \Rightarrow \delta^2 - 2\delta + 1 < 1 - 1/k$

$(\delta-1)^2 < 1 - 1/k$

$-(1-1/k)^{1/2} < \delta - 1 < (1-\frac{1}{k})^{1/2}$

(add 1) $1 - (1-\frac{1}{k})^{1/2} < \delta < 1 + (1+\frac{1}{k})^{1/2}$

if $\delta = \frac{1}{2} \times (1 + (1-\frac{1}{k})^{1/2})$

then $1-\delta < x < 1 \Rightarrow f(x) < -k \Rightarrow \lim_{x \to 1-0} f(x) = -\infty$

(iii) To prove $\lim_{x \to -1-0} f(x) = +\infty$

Given $k < 0$, to find $\delta > 0$ Such that $c-\delta < x < c$

$-1-\delta < x < -1 \Rightarrow f(x) > k$

$\frac{1}{x^2-1} > k$ $\quad x^2 - 1 < \frac{1}{k}$

$-1-\delta < x < -1 \Rightarrow -(1+\delta) < x < -1 \Rightarrow (1+\delta)^2 < x^2 < 1$

$\delta^2 - 2\delta + 1 < x^2 < 1 \quad \Rightarrow \delta^2 - 2\delta < x^2 - 1 < 0$

$\delta(\delta-2) < x^2 - 1$

$\delta(\delta-2) < x^2 - 1 < \frac{1}{k}$

$\delta^2 - 2\delta < \frac{1}{k} \Rightarrow \delta^2 - 2\delta + 1 < 1 + \frac{1}{k}$

$\Rightarrow (\delta-1)^2 < 1 + \frac{1}{k}$

$\Rightarrow -(1+\frac{1}{k})^{1/2} < \delta - 1 < (1+\frac{1}{k})^{1/2} \Rightarrow 1 - (1+\frac{1}{k})^{1/2} < \delta < 1 + (1+\frac{1}{k})^{1/2}$

if $\delta = \frac{1}{2} \times (1 + (1+\frac{1}{k})^{1/2})$

Then $-1-\delta < x < -1 \Rightarrow f(x) > k \Rightarrow \lim_{x \to -1-0} f(x) = +\infty$

(iv) $\lim_{x \to -1+0} f(x) = \infty$

Given $k > 0$ to find $\delta > 0$ Suchthat $c < x < c + \delta$

$-1 < x < -1 + \delta \Rightarrow f(x) < -k$

$\frac{1}{x^2-1} < -k$ $\quad x^2 - 1 > -\frac{1}{k}$

$-1 < x < -1 + \delta \Rightarrow 1 < x^2 < (\delta-1)^2$

$\Rightarrow 1 < x^2 < \delta^2 - 2\delta + 1$

(add -1) $\Rightarrow 0 < x^2 - 1 < \delta^2 - 2\delta \Rightarrow 0 < x^2 - 1 < \delta^2 - 2\delta$

Since $x^2 - 1 > -\frac{1}{k} \Rightarrow -\frac{1}{k} < x^2 - 1 < \delta^2 - 2\delta$

$\Rightarrow -\frac{1}{k} < \delta^2 - 2\delta$

$\Rightarrow 1 - \frac{1}{k} < \delta^2 - 2\delta + 1 \Rightarrow 1 - \frac{1}{k} < (\delta-1)^2$

$$-\left(1-\frac{1}{k}\right)^{1/2} < \delta-1 < \left(1-\frac{1}{k}\right)^{1/2} \Rightarrow 1-\left(1-1/k\right)^{1/2} < \delta < 1+\left(1-\frac{1}{k}\right)^{1/2}$$

if $\delta = \frac{1}{2} \times \left(1+\left(1-\frac{1}{k}\right)^{1/2}\right)$ then

$$-1 < x < -1+\delta \Rightarrow f(x) < -K \Rightarrow \lim_{x \to -1+0} f(x) = -\infty$$

---

2) $f(x) = \frac{1}{x^2}$ $\forall$ $x \in R-\{0\}$. Show that $\lim_{x \to 0+0} f(x) = +\infty$

$\lim_{x \to 0-0} f(x) = +\infty$.

__proof:__

To show $\lim_{x \to 0+0} f(x) = +\infty$

i.e to show , given $k>0$ we have to find $\delta>0$ such that

$c < x < c+\delta \Rightarrow f(x) > k$ $\quad$ $0 < x < \delta \Rightarrow f(x) > K$

$\frac{1}{x^2} > K \Rightarrow x^2 < \frac{1}{K} \Rightarrow x < \frac{1}{\sqrt{K}}$

$0 < x < \delta \Rightarrow 0 < x^2 < \delta^2 \Rightarrow$

$\Rightarrow$

if $\delta = \frac{1}{\sqrt{K}}$ $\quad$ $0 < x < \frac{1}{\sqrt{K}} \Rightarrow f(x) > K$

To show

$\lim_{x \to 0-0} f(x) = +\infty$ to find $\delta > 0$ such that

given $k > 0$

$c-\delta < x < c$ $\quad$ $-\delta < x < 0 \Rightarrow f(x) > K$

$\frac{1}{x^2} > k \Rightarrow x^2 < \frac{1}{K} \Rightarrow x < \frac{1}{\sqrt{K}}$

$\therefore$ $-\delta < x \Rightarrow -\delta < x < \frac{1}{\sqrt{K}} \Rightarrow -\delta < \frac{1}{\sqrt{K}}$

$\Rightarrow \delta > \frac{-1}{\sqrt{K}}$

if $\delta > -\frac{1}{\sqrt{K}}$, $-\delta < x < 0 \Rightarrow f(x) > K$

$x \to 0$

8.7. LIMITS AT INFINITY AND INFINITE LIMITS. DEFINITIONS

(Kanpur, 2001)

(*i*) A function f is said to tend to l as $x \to \infty$ if given $\varepsilon > 0$, there exists a positive number k such that $$| f(x) - l | < \varepsilon \text{ whenever } x > k$$

Also then we write $$\lim_{x \to \infty} f(x) = l \quad \text{or} \quad f(x) \to l \text{ as } x \to \infty.$$

(*ii*) A function f is said to tend to l as $x \to -\infty$ if given $\varepsilon > 0$, there exists a positive number k such that $$| f(x) - l | < \varepsilon \text{ whenever } x < -k.$$

Also then we write $$\lim_{x \to -\infty} f(x) = l \quad \text{or} \quad f(x) \to l \text{ as } x \to -\infty.$$

(*iii*) A function f is said to tend to ∞ as x tends to a, if given $k > 0$, however large, there exists a positive number δ such that $$f(x) > k \text{ whenever } 0 < | x - a | < \delta.$$

Also then we write $$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to a.$$

(*iv*) A function f is said to tend to $-\infty$ as x tends to a, if given $k > 0$, however large, there exists a positive number δ such that $$f(x) < -k \text{ whenever } 0 < | x - \infty | < \delta$$

Also then we write $$\lim_{x \to a} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to a.$$

(*v*) A function f is said to tend to ∞ as $x \to \infty$ if given $k > 0$, however large, there exists a positive number K such that $$f(x) > k \text{ whenever } x > K$$

Also then we write $$\lim_{x \to \infty} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to \infty.$$

(*vi*) A function f is said to tend to $-\infty$ as $x \to \infty$ if given $k > 0$, however large, there exists a positive number K such that $$f(x) < -k \text{ whenever } x > K$$

Also then we write $$\lim_{x \to \infty} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to \infty.$$

(*vii*) A function f is said to tend to ∞ as $x \to -\infty$, if given $k > 0$, however large, there exists a positive number K such that $$f(x) > k \text{ whenever } x < -K$$

Also then we write $$\lim_{x \to -\infty} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to -\infty.$$

(*viii*) A function f is said to tend to $-\infty$ as $x \to -\infty$, if given $k > 0$, however large, there exists a positive number K such that $$f(x) < -k \text{ whenever } x < -K.$$

Examples for limits as $x \to +\infty$ $(-\infty)$

1) Let $f(x) = \frac{1}{x^2+1}$ $\forall x \in \mathbb{R}$ Then Show that (i) $\lim_{x \to \infty} f(x) = 0$

(ii) $\lim_{x \to -\infty} f(x) = 0$

Proof

To prove $\lim_{x \to \infty} f(x) = 0$ (ie) Given $\varepsilon > 0$, $\exists K > 0$

Such that $x > K \Rightarrow |f(x) - l| < \varepsilon$

$$\left| \frac{1}{x^2+1} - 0 \right| < \varepsilon \Rightarrow \frac{1}{x^2+1} < \varepsilon \Rightarrow x^2+1 > \frac{1}{\varepsilon}$$

$$x^2 > \frac{1}{\varepsilon} - 1$$

$$x > (\varepsilon^{-1} - 1)^{1/2}$$

if $K < (\varepsilon^{-1} - 1)^{1/2}$ then $\because x > (\varepsilon^{-1} - 1)^{1/2} > K$

$\therefore x > K$, $|f(x) - 0| < \varepsilon$

To prove $\lim_{x \to -\infty} f(x) = 0$ Given $\varepsilon > 0$, $\exists K > 0$

Suchthat $x < -K \Rightarrow |f(x) - l| < \varepsilon$

$$\left| \frac{1}{x^2+1} - 0 \right| < \varepsilon \Rightarrow \frac{1}{x^2+1} < \varepsilon \Rightarrow x^2+1 > \frac{1}{\varepsilon}$$

$$x > \varepsilon^{-1} - 1$$

If $-K < \varepsilon^{-1} - 1$ then $x > \varepsilon^{-1} - 1 > K$

$\Rightarrow x > -K$

$\therefore$ $x > -K$, $|f(x) - 0| < \varepsilon$.

Hence proved.

2) Let $f(x) = x \cdot \sin\frac{1}{x}$ $\forall$ $x \in \mathbb{R} \sim \{0\}$ Then show that

i) $\lim_{x \to \infty} f(x) = 1$ ii) $\lim_{x \to -\infty} f(x) = 1$

<u>Proof:</u>

(i) Let $t = \frac{1}{x}$ $x \longrightarrow \infty \Rightarrow t = \frac{1}{x} \rightarrow 0$

$x = \frac{1}{t}$ $\lim_{x \to \infty} f(x) = \lim_{\substack{t \to 0 \\ t > 0}} \frac{\sin t}{t} = \lim_{\substack{t \to 0 \\ t > 0}} \frac{0}{0}$

[use L-hostpital rule]

$$\Rightarrow \lim_{\substack{t \to 0 \\ t > 0}} \frac{\cos t}{1} = \frac{\cos 0}{1} = 1$$

(ii) $\lim_{x \to -\infty} f(x) = \lim_{\substack{t \to 0 \\ t < 0}} \frac{\sin t}{t} = \lim_{\substack{t \to 0 \\ t < 0}} \frac{\cos 0}{1} = 1$ //

3) Let $f(x) = e^{1/x}$ for all $x \in R - \{0\}$, $f(0) = 0$ then Show that i) $\lim_{x \to \infty} f(x) = 1$ ii) $\lim_{x \to -\infty} f(x) = 1$

Proof.

let $t = \frac{1}{x}$ $x \to \infty \Rightarrow t \to 0$.

$$\lim_{x \to \infty} f(x) = \lim_{\substack{t \to 0 \\ t > 0}} f\left(\frac{1}{t}\right) = \lim_{\substack{t \to 0 \\ t > 0}} e^{\frac{1}{(1/t)}}.$$

$$= \lim_{\substack{t \to 0 \\ t > 0}} e^t = e^0 = 1$$

$$\lim_{x \to -\infty} f(x) = \lim_{\substack{t \to 0 \\ t < 0}} f\left(\frac{1}{t}\right) = \lim_{\substack{t \to 0 \\ t < 0}} e^t = e^0 = 1$$

4) Let $f(x) = \frac{x}{x^2+1}$ $\forall\, x \in \mathbb{R}$ Then show that

$$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$$

Proof: i) To prove Given $\varepsilon > 0$, $\exists\, K > 0$

Such that $x > K \Rightarrow |f(x) - 0| < \varepsilon$

$$x > K \Rightarrow \left| \frac{x}{x^2+1} - 0 \right| < \varepsilon$$

$$\left| \frac{x}{x^2+1} \right| < \varepsilon$$

$x > 0$ $\qquad \left| \frac{x}{x^2+1} \right| < \frac{1}{x} < \varepsilon$ (let us choose n) $\qquad \boxed{\frac{x}{x(x+\frac{1}{x})} = \frac{1}{x+\frac{1}{x}} < \frac{1}{x}}$

$$x > \frac{1}{\varepsilon}$$

$\therefore$ $\varepsilon > 0 \Rightarrow |f(x)| < \varepsilon$ whenever $x > \frac{1}{\varepsilon}$

we can choose $K = \frac{1}{\varepsilon}$ $\quad\therefore \lim_{x \to \infty} f(x) - 0$

(ii) $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x}{x^2+1}$

put $x = -t \Rightarrow \lim_{-t \to \infty} f(-t) = \lim_{-t \to \infty} \frac{-t}{t^2+1}$

$$\left| \frac{-t}{t^2+1} - 0 \right| < \varepsilon$$

$x < 0$ $\quad \frac{t}{t^2+1} < \frac{1}{t}$

$\varepsilon > 0 \Rightarrow |f(t)| < \varepsilon$

$$\varepsilon > \frac{1}{t}$$

$$t > \frac{1}{\varepsilon} \Rightarrow x < \frac{1}{\varepsilon}$$

We can choose $-k = \frac{1}{\varepsilon}$ $\quad \therefore \lim_{x \to -\infty} f(x) = 0$

$$\boxed{\begin{array}{l} \Rightarrow x < -k \\ |f(x) - l| < \varepsilon \end{array}}$$

Examples $x \to +\infty \, (-\infty)$ $f(x) \to +\infty \, (-\infty)$

1) Let $f(x) = x^2 \ \forall x \in \mathbb{R}$ Then $\lim\limits_{x \to \infty} f(x) = \infty$, $\lim\limits_{x \to -\infty} f(x) = \infty$

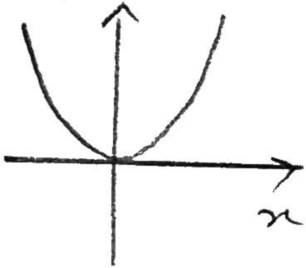
Answer

(i) to show for given $K > 0$ $\exists$ K^* suchthat $x > K^* \Rightarrow f(x) > K$

Let $f(x) > K$ i.e $x^2 > K \Rightarrow |x| > \sqrt{K} \Rightarrow x > \sqrt{K}$

choose $K^* = \sqrt{K} \Rightarrow x > K^* \Rightarrow f(x) > K$

$$\Rightarrow \lim_{x \to \infty} f(x) = +\infty$$



(ii) To show for given $K > 0$, $\exists$ $K_1^* < 0$ suchthat $x < K_1^* \Rightarrow f(x) > K$

let $f(x) > K \Rightarrow x^2 > K \Rightarrow |x| > \sqrt{K} \Rightarrow x > \sqrt{K}$

$x < -\sqrt{K} \Rightarrow x^2 < K$ So we can choose $K_1^* = -\sqrt{K} < 0$

$x < K_i^* \Rightarrow f(x) > K$

$$\Rightarrow \lim_{x \to -\infty} f(x) = +\infty$$

2) Let $f(x) = -x^2 \ \forall x \in \mathbb{R}$ Then $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = -\infty$

Answer:

(i) to show for given $K < 0$ $\exists K^*$ such that $x > K^* \Rightarrow f(x) < K$

$$-x^2 < K \quad x^2 > -K \quad x > \sqrt{-K}$$

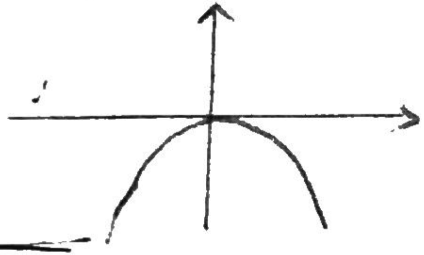
(Since $K < 0$, $-K > 0$, so $\sqrt{-K}$ is possible)

we can choose $K^* = -\sqrt{-K}$

$x > K^* \Rightarrow f(x) < K$

(ii) To show for given $K < 0$ $\exists K_1^* < 0$ such that $x < K_1^* \Rightarrow f(x) < K$

$$-x^2 < K$$
$$|-x^2| < |K|$$
$$|x^2| < |K|$$
$$x^2 < |K|$$
$$x < \sqrt{|K|}$$



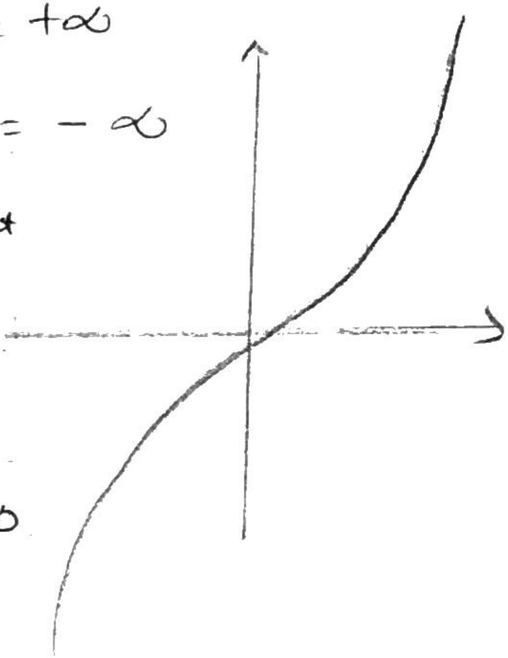
we can choose $K_1^* = \sqrt{|K|}$

$x < K_1^* \Rightarrow f(x) < K$

3) $f(x) = x^3 \;\; \forall x \in \mathbb{R}$, to show $\lim_{x \to +\infty} f(x) = +\infty$

$\lim_{x \to -\infty} f(x) = -\infty$

Answer



(i) To show that for given $K>0$, $\exists K^* > 0$ suchthat

$x > K^* \Rightarrow f(x) > K$

$x^3 > K \Rightarrow x > K^{1/3}$ (Taking cube root on both sides)

we can choose $K^* = K^{1/3}$

$\Rightarrow x > K^* \Rightarrow f(x) > K \quad \therefore \lim_{x \to \infty} f(x) = \infty$

(ii) To show that for given $K<0$, $\exists K_1^* < 0$

Suchthat $x < K_1^* \Rightarrow f(x) < K$

$f(x) < K \Rightarrow x^3 < K \Rightarrow x < K^{1/3}$

we choose if $0 > K_1^* > K^{1/3} \Rightarrow x < K^{1/3} < K_1^*$

$\Rightarrow x < K_1^*$

$\therefore x < K_1^* \Rightarrow f(x) < K \quad \therefore \lim_{x \to -\infty} f(x) = -\infty$

4) Let $f(x) = -x^3 \ \forall \ x \in \mathbb{R}$, Then Show that $\lim_{x \to \infty} f(x) = -\infty$

Answer $\lim_{x \to -\infty} f(x) = +\infty$

(i) To show that given $K < 0$, $\exists \ K^* > 0$ such that

$x > K^* \Rightarrow f(x) < K$

$f(x) < K \Rightarrow -x^3 < K \Rightarrow x^3 > -K \Rightarrow x > (-K)^{1/3}$

(Since $K < 0$, $-K > 0$)

$\therefore$ we can choose $K^* \leq (-K)^{1/3} \Rightarrow K^* < (-K)^{1/3} < x \Rightarrow x > K^*$

$\therefore \lim_{x \to \infty} f(x) = -\infty$

(ii) To show that, given $K > 0$, $\exists \ K_1^* < 0$ such that

$x \leq K_1^* \Rightarrow f(x) > K$

$f(x) > K \Rightarrow -x^3 > K \Rightarrow x^3 < -K \Rightarrow x^3 < K$

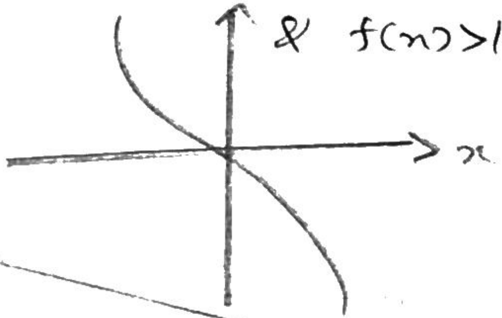
(Since $K > 0$)

$\Rightarrow x < K^{1/3}$

if we choose $K_1^* > K^{1/3} \Rightarrow K_1^* > K^{1/3} > x \Rightarrow x < K_1^*$

& $f(x) > K$

$\therefore \lim_{x \to -\infty} f(x) = \infty$



$f(x) = x^3 \ \forall \ x \in \mathbb{R}$ Then

5) Let $f(x) = \frac{x^3}{1+x^2}$ ∀ $x \in \mathbb{R}$ Then

(i) $\lim\limits_{x \to \infty} f(x) = \infty$ (ii) $\lim\limits_{x \to -\infty} f(x) = -\infty$

$$\begin{array}{r|l} & x \\ \hline 1+x^2 & \cancel{x^3} \\ & \cancel{x^3} + x \\ & (-)\quad (-) \\ \hline & -x \end{array}$$

$$\frac{x^3}{x^2+1} = x - \frac{x}{1+x^2}$$

$$\left(\frac{\text{Divident}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}} \right)$$

$\therefore$ $f(x) = x - \frac{x}{1+x^2}$

We know that $\frac{x}{1+x^2} < \frac{x^2}{1+x^2} < \frac{1}{2}$ $\Rightarrow \frac{x}{1+x^2} < \frac{1}{2}$

$\Rightarrow \frac{-x}{1+x^2} > -\frac{1}{2}$

$\therefore$ $f(x) = x - \frac{x}{1+x^2} > x - \frac{1}{2}$

$f(x) > x - 1/2 \Rightarrow f(x) + \frac{1}{2} > x$

if $x - \frac{1}{2} > K$ then $f(x) > x - \frac{1}{2} > K$

So $\Rightarrow$ $x > K + \frac{1}{2} \Rightarrow f(x) > K$

So we can choose $K^* = K + 1/2 \Rightarrow x > K^* \to f(x) > K$

$\therefore \lim\limits_{x \to \infty} f(x) = \infty$

. $\lim\limits_{x \to -\infty} f(x)$ Put $t = -x$ then $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{t \to \infty} f(-t) =$

$$= \lim_{t \to \infty} \frac{(-t)^3}{1+(-t)^2} = \lim_{t \to \infty} \frac{(-t)^3}{1+t^2}$$

$$= -\infty \quad //$$

We now begin the study of the most important class of functions that arises in real analysis: the class of continuous functions. The term "continuous" has been used since the time of Newton to refer to the motion of bodies or to describe an unbroken curve, but it was not made precise until the nineteenth century. Work of Bernhard Bolzano in 1817 and Augustin-Louis Cauchy in 1821 identified continuity as a very significant property of functions and proposed definitions, but since the concept is tied to that of limit, it was the careful work of Karl Weierstrass in the 1870s that brought proper understanding to the idea of continuity.

We will first define the notions of continuity at a point and continuity on a set, and then show that various combinations of continuous functions give rise to continuous functions. Then in Section 5.3 we establish the fundamental properties that make continuous functions so important. For instance, we will prove that a continuous function on a closed bounded interval must attain a maximum and a minimum value. We also prove that a continuous function must take on every value intermediate to any two values it attains. These properties and others are not possessed by general functions, as various examples illustrate, and thus they distinguish continuous functions as a very special class of functions.

In Section 5.4 we introduce the very important notion of uniform continuity. The distinction between continuity and uniform continuity is somewhat subtle and was not fully appreciated until the work of Weierstrass and the mathematicians of his era, but it proved to

Karl Weierstrass

Karl Weierstrass (=Weierstraß) (1815–1897) was born in Westphalia, Germany. His father, a customs officer in a salt works, insisted that he study law and public finance at the University of Bonn, but he had more interest in drinking and fencing, and left Bonn without receiving a diploma. He then enrolled in the Academy of Münster where he studied mathematics with Christoph Gudermann. From 1841–1854 he taught at various *gymnasia* in Prussia. Despite the fact that he had no contact with the mathematical world during this time, he worked hard on mathematical research and was able to publish a few papers, one of which attracted considerable attention. Indeed, the University of Königsberg gave him an honorary doctoral degree for this work in 1855. The next year, he secured positions at the Industrial Institute of Berlin and the University of Berlin. He remained at Berlin until his death.

A methodical and painstaking scholar, Weierstrass distrusted intuition and worked to put everything on a firm and logical foundation. He did fundamental work on the foundations of arithmetic and analysis, on complex analysis, the calculus of variations, and algebraic geometry. Due to his meticulous preparation, he was an extremely popular lecturer; it was not unusual for him to speak about advanced mathematical topics to audiences of more than 250. Among his auditors are counted Georg Cantor, Sonya Kovalevsky, Gösta Mittag-Leffler, Max Planck, Otto Hölder, David Hilbert, and Oskar Bolza (who had many American doctoral students). Through his writings and his lectures, Weierstrass had a profound influence on contemporary mathematics.

5.1.1 Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any number $\varepsilon > 0$ there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

If f fails to be continuous at c, then we say that f is **discontinuous at** c.

As with the definition of limit, the definition of continuity at a point can be formulated very nicely in terms of neighborhoods. This is done in the next result. We leave the verification as an important exercise for the reader. See Figure 5.1.1.

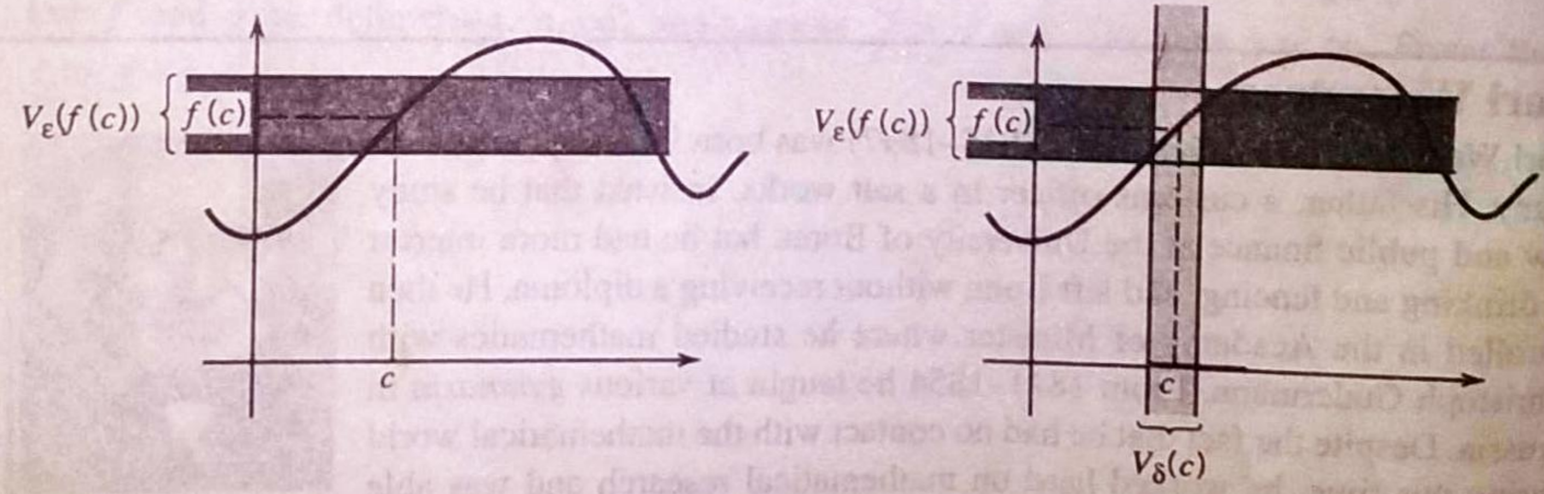


Figure 5.1.1 Given $V_\varepsilon(f(c))$, a neighborhood $V_\delta(c)$ is to be determined.

Continuous function

Definition 1: Let f be a function whose domain I is an open interval and whose range is contained in $\mathbb{R}$ and let $x_0 \in I$.

f is said to be Continuous at x_0, if

Given $\varepsilon > 0$ we can find a $\delta > 0$ such that

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

If f is Continuous for each $x_0 \in I$ Then we say that f is Continuous on I.

It can be easily Stated as follows

A function defined on an open interval I is said to be Continuous at $x_0 \in I$, if $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$

i.e $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition 2

A function f defined on an open interval I is said to be continuous from the left at $x_0 \in I$

If $\lim_{x \to x_0 - 0} f(x)$ exists and equals $f(x_0)$

f is said to be continuous from the right at $x_0 \in I$ if $\lim_{x \to x_0 + 0} f(x)$. exists and equals $f(x_0)$

Definition 3

A function defined on the closed interval $[a, b]$ is said to be continuous at a if it is continuous from the right at a. Also f is said to be continuous at b if it is continuous from the left at b.

f is said to be continuous on $[a, b]$ if

(i) f is continuous on (a, b)

(ii) Continuous from the right at a

(iii) Continuous from the left at b.

Theorem 1

A function f defined on $I \subset R$ is Continuous at $p \in I$ iff for a sequence p_n which converses to p, we have

$$\lim_{n \to \infty} f(p_n) = f(p)$$

Proof f is defined on $I \subset R$

Let f is Continuous at $p \in I$, $p_n \to p$

To prove that $f(p_n) \to p$

f is continuous at p.

$\Rightarrow \varepsilon > 0$, $\exists\ \delta > 0$ such that $|x-p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$ —— ①

since $p_n \to p \Rightarrow \exists\ m \in N$ such that $\varepsilon = \delta$

$|p_n - p| < \delta$ for $n \geq m$ —— ②

In ① put $x = p_n$

$|p_n - p| < \delta \Rightarrow |f(p_n) - f(p)| < \varepsilon$ —— ③

from ②, ③ $|f(p_n) - f(p)| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow f(p_n) \to f(p)$

<u>Conversely</u>

To prove $f(p_n)$ converges to $f(p)$. $\Rightarrow$ f is continuous at p.

proof by contradiction.

Suppose f is not continuous ~~at~~ p

$\Rightarrow |x-p| < \delta \Rightarrow |f(p_n) - f(p)| \geq \varepsilon$ —— ④

Since $p_n \to p \Rightarrow \exists\ m \in N$ $|p_n - p| < \delta$ for $n \geq m$

put $x = p_n$ in ④

$|p_n - p| < \delta \Rightarrow |f(p_n) - f(p)| \geq \varepsilon \quad \forall\ n \geq m$

$|f(p_n) - f(p)| \geq \varepsilon \quad \forall \quad n \geq m$

$\Rightarrow f(p_n) \not\to f(p)$ which is a contradiction.

$\therefore$ f must be continuous at p.

Theorem 2

A function f defined on $\mathbb{R}$ is continuous on $\mathbb{R}$ iff for each open set G in $\mathbb{R}$, $f^{-1}(G)$ is an open set in $\mathbb{R}$.

Proof

Let f be defined on $\mathbb{R}$ & Continuous on $\mathbb{R}$

Let G be an open set in $\mathbb{R}$. To prove $f^{-1}(G)$ is open

We take $f^{-1}(G)$ is nonempty. Let $x_0 \in f^{-1}(G)$

To prove $f^{-1}(G)$ is open. It is enough to show

$(x_0-\varepsilon, x_0+\varepsilon) \in f^{-1}(G)$

$f(x_0) \in G$ Since G is open

$\Rightarrow$ ~~G~~ $\exists\ \varepsilon > 0$ $(f(x_0)-\varepsilon, f(x_0)+\varepsilon) \in G$

Since f is continuous on $\mathbb{R}$

$\Rightarrow$ Given $\varepsilon > 0$, $\exists\ \delta > 0 \ni |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$x_0 - \delta < x < x_0 + \delta \Rightarrow f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$

$x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset G$

$\Rightarrow (x_0 - \delta, x_0 + \delta) \in f^{-1}(G)$

$\therefore f^{-1}(G)$ is open

Converse part

Let $f^{-1}(G)$ be open . To prove f is continuous on $\mathbb{R}$.
& G is open

$\varepsilon > 0$ $\quad f(x_0) \in \underset{\uparrow}{(f(x_0)-\varepsilon, f(x_0)+\varepsilon)}$ open set containing $f(x_0)$

$\Rightarrow f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ is also an open set containing a.

$\Rightarrow \exists\, \delta > 0$ suchthat $(x_0-\delta, x_0+\delta) \subset f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$

$\Rightarrow f((x_0-\delta, x_0+\delta)) \subset (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

Thus for given $\varepsilon > 0$ we have $\delta > 0$ Such that

$x \in (x_0-\delta, x_0+\delta) \Rightarrow f(x) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

$|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$

$\Rightarrow f$ is Continuous at x_0.

Since x_0 is an arbitrary point in $\mathbb{R}$. $\therefore f$ is continuous on $\mathbb{R}$.

Theorem 2

A function f defined on $\mathbb{R}$ is continuous on $\mathbb{R}$ iff for each open set G in $\mathbb{R}$, $f^{-1}(G)$ is an open set in $\mathbb{R}$.

Proof

Let f be defined on $\mathbb{R}$ & Continuous on $\mathbb{R}$

Let G be an open set in $\mathbb{R}$. To prove $f^{-1}(G)$ is open

We take $f^{-1}(G)$ is nonempty. Let $x_0 \in f^{-1}(G)$

To prove $f^{-1}(G)$ is open. It is enough to show

$(x_0-\varepsilon, x_0+\varepsilon) \in$ [redacted] $f^{-1}(G)$

$f(x_0) \in G$ Since G is open

$\Rightarrow$ ~~$G+$~~ $\exists\ \varepsilon > 0$ $(f(x_0)-\varepsilon, f(x_0)+\varepsilon) \in G$

Since f is continuous on $\mathbb{R}$

$\Rightarrow$ Given $\varepsilon > 0$, $\exists\ \delta > 0 \ni |x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$

$x_0-\delta < x < x_0+\delta \Rightarrow f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon$

$x \in (x_0-\delta, x_0+\delta) \Rightarrow f(x) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon) \subset G$

$\Rightarrow (x_0-\delta, x_0+\delta) \in f^{-1}(G)$

$\therefore f^{-1}(G)$ is open

Converse part

Let $f^{-1}(G)$ be open . To prove f is continuous on $\mathbb{R}$.
& G is open

$\varepsilon > 0$ $\quad f(x_0) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$
$\uparrow$ open set containing $f(x_0)$

$\Rightarrow f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ is also an open set containing a;

$\Rightarrow \exists\ \delta > 0$ suchthat $(x_0-\delta, x_0+\delta) \subset f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$

$\Rightarrow f((x_0-\delta, x_0+\delta)) \subset (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

Thus for given $\varepsilon > 0$ we have $\delta > 0$ Suchthat

$x \in (x_0-\delta, x_0+\delta) \Rightarrow f(x) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

$|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$

$\Rightarrow f$ is continuous at x_0.

Since x_0 is an arbitrary point in $\mathbb{R}$. $\therefore f$ is continuous on $\mathbb{R}$.

Theorem 3

A function f defined on $\mathbb{R}$ is Continuous on $\mathbb{R}$ iff for each closed Set F in R, $f^{-1}(F)$ is also a closed set in $\mathbb{R}$

Proof

$\Longrightarrow$

Let f be defined on $\mathbb{R}$. & continuous on R.

Let F be a closed set in $\mathbb{R}$. We have to prove that $f^{-1}(F)$ is closed in $\mathbb{R}$.

Since F is closed $\Rightarrow$ $R \sim F$ is open $\Rightarrow$ $f^{-1}(R \sim F)$ is open (Since f is continuous)

$\Rightarrow$ $R \sim f^{-1}(F)$ is open

$\Rightarrow$ $f^{-1}(F)$ is closed.

$\Longleftarrow$

Let F is closed in $\mathbb{R}$ & $f^{-1}(F)$ is closed in $\mathbb{R}$.

To prove that f is Continuous on $\mathbb{R}$

~~B~~ F Closed $\Rightarrow$ $R \sim F$ is open & $R \sim f^{-1}(F)$ is open

& $f^{-1}(F)$ closed $\Rightarrow$ $f^{-1}(R \sim F)$ is open

By previous theorem we can conclude that f is Continuous

Illustrating Examples.

1) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c \ \forall \ x \in \mathbb{R}$. (where c - any real number (constant))

Prove that f is Continuous on $\mathbb{R}$.

Proof: To prove f is Continuous.

$$f(x) = c$$

let us take $x_n \to x$ in $\mathbb{R}$

$f(x_n) = c \; \forall n$ (since f is constant)

& $f(x) = c$

$\Rightarrow f(x_n) \rightarrow f(x)$ in $\mathbb{R}$. $\therefore$ f is continuous on $\mathbb{R}$.

---

2) Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x \; \forall x \in \mathbb{R}$. (Identity function)

Prove that f is continuous.

Proof: To prove f is continuous.

$f(x) = x$ Let us take $x_n \rightarrow x$ in $\mathbb{R}$.

i.e $\lim_{n\to\infty} x_n = x$

Since $f(x_n) = x_n$

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x = f(x)$

$\therefore \lim_{n\to\infty} f(x_n) = f(x)$ (i.e) $f(x_n) \rightarrow f(x)$

---

3) Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2 \; \forall x \in \mathbb{R}$. prove that f is continuous.

Proof:

Let $x_n \rightarrow x$ in $\mathbb{R}$. we have to prove that $f(x_n) \rightarrow f(x)$

Since $f(x) = x^2$ & $\lim_{n\to\infty} x_n = x$

$\Rightarrow f(x_n) = x_n^2$

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n = x \cdot x = x^2$

$= f(x)$

$\therefore (f(x_n)) \longrightarrow f(x)$

4) **Dirichlet's function**

Let $f : \mathbb{R} \to \mathbb{R}$. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$

Prove that f is discontinuous at every point of $\mathbb{R}$.

proof

case (i) x is rational

Let x_n - irrational & x - rational.

$\exists$ +ve n, x_n - irrational number $\ni |x_n - x| < \frac{1}{n}$

we can choose $\varepsilon = \frac{1}{n}$ $\Rightarrow$ $x_n \to x$ (i.e) $\lim_{n\to\infty} x_n = x$

$f(x_n) = -1$ (Since x_n is irrational)

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (-1) = -1$ —— ①

$f(x) = 1$ (since x is rational) ②

From ① & ② $\lim_{n\to\infty} f(x_n) \neq f(x)$

$\therefore$ f is not continuous (discontinuous)

Case (ii)

Let x is irrational

$\exists$ +ve n, x_n - rational $\ni |x_n - x| < \frac{1}{n}$

We can choose $\varepsilon = \frac{1}{n}$ $\Rightarrow$ $x_n \to x$ $\lim_{n\to\infty} x_n = x$

$f(x_n) = 1$ (since x_n - rational)

$\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} 1 = 1$ ——— ③

$f(x) = -1$ (since x - irrational) ④

from ③ & ④ we will get $\lim\limits_{n\to\infty} f(x_n) \neq f(x)$

$\Rightarrow$ f is not continuous.

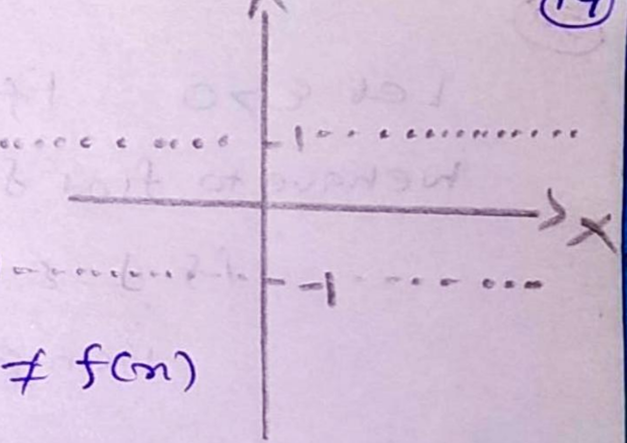
This is true for all $x \in \mathbb{R}$ (both rational, irrational)

$\therefore$ f is ~~dis~~ continuous at every point of $\mathbb{R}$.

$f(x_n) = 1$ (since x_n - rational)

$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 1 = 1$ ——— ③

$f(x) = -1$ (since x - irrational)
└ ④



from ③ & ④ we will get $\lim_{n \to \infty} f(x_n) \neq f(x)$

$\Rightarrow$ f is not continuous.

This is true for all $x \in \mathbb{R}$ (both rational, irrational)

$\therefore$ f is ~~dis~~ continuous at every point of $\mathbb{R}$.

5) Show that the function f defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

is continuous only at $x=0$

Answer: case(i)

Let $a \neq 0$ be any rational number so that $f(a) = a$

Then any neighbourhood $(a-\frac{1}{n}, a+\frac{1}{n})$ of a contains an irrational number a_n for each $n \in \mathbb{N}$, i.e

$$a_n \in (a - 1/n, a + 1/n) \Rightarrow |a_n - a| < \frac{1}{n}$$

$$\Rightarrow |a_n - a| \to 0 \text{ as } n \to \infty \quad \Rightarrow a_n \to a$$

$$f(a_n) = 0 \quad (\text{since } a_n \text{ is irrational})$$

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 0 = 0 \quad \text{— ①}$$

$f(a) = 1$ (Since a is ~~ir~~rational) — ②

from ① & ② $\quad \lim_{n\to\infty} f(a_n) \neq f(a) \quad \therefore f(a_n) \not\to f(a)$

$\therefore$ f is discontinuous at every rational $\neq 0$

Case (ii) Let $b \neq 0$ be any irrational. $f(b) = 0$

Then any neighbourhood $(b - 1/n, b + 1/n)$ of b contains an rational number b_n for each $n \in \mathbb{N}$.

$$b_n \in \left(b - \frac{1}{n}, b + \frac{1}{n}\right) \Rightarrow |b_n - b| < \frac{1}{n}$$

$$\Rightarrow |b_n - b| \to 0 \text{ as } n \to \infty \Rightarrow b_n \to b$$

$f(b_n) = b_n \ \forall n \in \mathbb{N}$ (Since b_n is rational)

$$\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} b_n = b \quad \text{——} ①$$

$$f(b) = 0 \quad \text{——} ②$$

from ① & ② $\lim_{n \to \infty} f(b_n) \neq f(b)$ $\therefore f(b_n) \not\to f(b)$

$\therefore$ f is discontinuous at every irrational $\neq 0$

Now we shall prove that f is continuous at $x = 0$

We have $f(0) = 0$ & $|f(x) - f(0)| = |f(x)| = \begin{cases} |x| & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

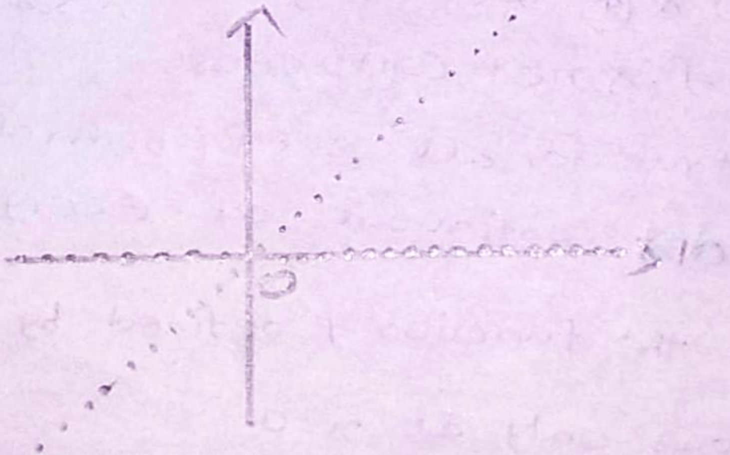
(50)

Let $\varepsilon > 0$ $|f(x) - f(0)| < \varepsilon$ for $|x-0| < \delta$

wehave to find δ

if $d = \varepsilon$ then $|x-0| < \varepsilon \Rightarrow |x| < \varepsilon \Rightarrow |f(x) - f(0)| < \varepsilon$

Hence f is continuous at $x = 0$



5) Let f be a function defined on (0,1) by setting $f(x)=0$ if x is irrational, $f(x)=\frac{1}{q}$ if $x=\frac{p}{q}$ (ie) x is rational where p, q are positive integers having no factor in common. Show that f is continuous at each irrational point & discontinuous at each rational point.

Proof:

Case (i) Let a is rational in (0,1)

$a=\frac{p}{q}$, gcd $(p,q)=1$ (no common divisor)

for any positive n $\exists$ x_n is irrational & $|x_n-a|<\frac{1}{n}$

$\Rightarrow x_n \to a$

$f(x_n)=0$ (since x_n is irrational) $\forall n$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \quad \text{—} ①$$

$$f(a) = f\left(\frac{p}{q}\right)=\frac{1}{q} \quad \text{—} ②$$

from ① & ② $\lim_{n\to\infty} f(x_n) \neq f(a)$

$\therefore$ f is not continuous at $x=a$.

Case (ii) Let b is irrational in $(0,1)$ $f(b) = 0$

To prove f is continuous at each irrational

To prove f is continuous at b

Given $\epsilon > 0$ $\exists$ positive n suchthat $\frac{1}{n} < \epsilon$

Given $\delta > 0$ $\exists$ rational number in $(b-\delta, b+\delta) \subset (0,1)$

$$|x-b| < \delta \Rightarrow |f(x) - f(b)| = |f(x) - 0|$$

$$= |f(x)| = 0 < \epsilon$$ if x is irratio

$$|x-b| < \delta \Rightarrow |f(x) - f(b)| = |f(x)| < \frac{1}{n} < \epsilon$$ if x is rational

$\therefore$ f is Continuous at $x = b$.

$\therefore$ f is Continuous at each irrational point.

Let f be the function defined on $\mathbb{R}$ by setting $f(x) = |x| \;\forall x \in \mathbb{R}$ then f is cts on $\mathbb{R}$.

Answer:

x is any real number

$\langle x_n \rangle$ be the seq $\ni$ $x_n \to x$

$$\lim_{n\to\infty} x_n = x$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} |x_n| = |x| = f(x)$$

$$\therefore \lim_{n\to\infty} f(x_n) = f(x)$$ f is cts on $\mathbb{R}$.

$$f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} \;\forall x \geq 0$$

Let x_n be a sequence in $(0,1)$ $\lim_{n\to\infty} x_n = 0$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$$

Let f be the function defined on $[0, \infty)$ by setting $f(x) = [x]$. the greatest integer not exceeding x $\forall$ $x \geq 0$

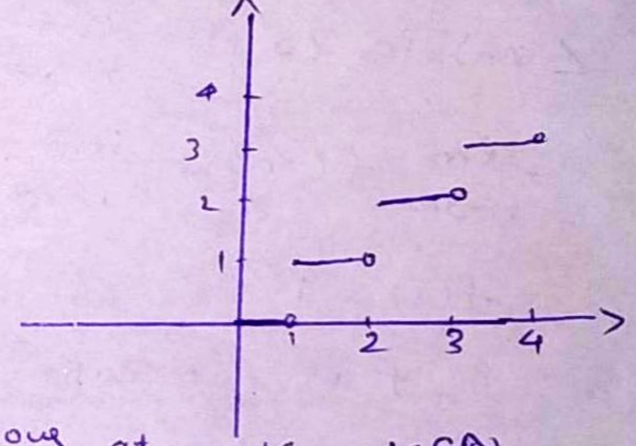
The function f is not continuous at $x = 1, 2, 3 \dots$ and is Continuous elsewhere.

Answer : $f(x) = 0$ $0 \leq x < 1$

$f(x) = 1$ $1 \leq x < 2$

$f(x) = 2$ $2 \leq x < 3$

$f(x) = k$ $k \leq x < k+1$



We have to show that f is not continuous at $x = k$ $k \in \mathbb{N}$.

let $x_n = k - \frac{1}{n}$

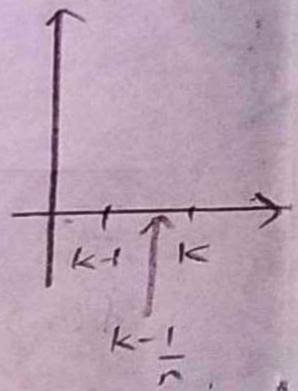
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} k - \lim_{n\to\infty} \frac{1}{n} = k - 0 = k \Rightarrow x_n \to k$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} [x_n] = \lim_{n\to\infty} \left[k - \frac{1}{n}\right] = k - 1$$

$f(k) = [k] = k$ $\therefore \lim_{n\to\infty} f(x_n) \neq f(k)$

$x_n \to k$ $f(x_n) \not\to f(k)$

$\therefore$ f is not continuous at $x = k$



$\left[k - \frac{1}{n}\right] = k - 1$

$[k] = k$

Let f be the function defined on $\mathbb{R}$ by setting $f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

then f is continuous on $\mathbb{R}$.

Answer: Let $\varepsilon > 0$ choose $\delta = \varepsilon/2$

$$|x - 0| < \delta \Rightarrow |f(x) - f(0)| = |x \sin(\tfrac{1}{x}) - 0|$$

$$= |x \sin \tfrac{1}{x}| \leq |x| |\sin \tfrac{1}{x}| \leq |x| < \delta = \tfrac{\varepsilon}{2} < \varepsilon$$

$$\Rightarrow |f(x) - f(0)| < \varepsilon$$

f is continuous at 0. Obviously f is continuous at all points other than 0. $\therefore$ f is continuous on $\mathbb{R}$.

Types of Discontinuity

Let f be a function defined on the interval I. If f is discontinuous at $p \in I$, then we say that

f has a removable discontinuity at p

if $\lim_{x \to p} f(x)$ exists but it is not equal to $f(p)$

f has a **discontinuity of the first kind from the left** at p if $\lim_{x \to p-0} f(x)$ exists but is not equal to f(p). $\left(\lim_{x \to p-0} f(x) \neq f(p) = \lim_{x \to p+0} f(x)\right)$

f has a **discontinuity of the first kind from the right** at p if $\lim_{x \to p+0} f(x)$ exists but not equal to f(p) $\left(\lim_{x \to p+0} f(x) \neq f(p) = \lim_{x \to p-0} f(x)\right)$

f has a **discontinuity of the first kind** at p if $\lim_{x \to p-0} f(x)$ and $\lim_{x \to p+0} f(x)$ exists but are unequal.

f has a **discontinuity of the second kind from left** at p if $\lim_{x \to p-0} f(x)$ doesnot exist.

f has a **discontinuity of the second kind from the right at p** if $\lim_{x \to p+0} f(x)$ does not exist

f has a **discontinuity of the second kind** at p if neither $\lim_{x \to p-0} f(x)$ & $\lim_{x \to p+0} f(x)$ exists.

Illustrations

Let f be the function defined on $\mathbb{R}$ by setting $f(x) = \frac{\sin x}{x}$ if $x \neq 0$

$f(0) = 0$

Here we have $\lim_{x \to 0} f(x) = 1$ $\quad \lim_{x \to 0} \frac{\sin x}{x} = 1$

but $f(0) = 0$

$\therefore \lim_{x \to 0} f(x) \neq f(0)$ $\quad \therefore f$ is not continuous at $x = 0$

$\therefore f$ has removable discontinuity at $x = 0$

Theorem2

If $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$, then $\lim_{x \to a} [f(x)g(x)] = f(a)g(a)$

Proof:

$$|f(x)g(x) - f(a)g(a)| = |f(x)g(x) - g(x)f(a) + g(x)f(a) - f(a)g(a)|$$

$$= |g(x)(f(x) - f(a)) + f(a)(g(x) - g(a))|$$

$$\leq |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)| \quad \text{— ①}$$

Since $\lim_{x \to a} f(x) = f(a)$ & $\lim_{x \to a} g(x) = g(a)$ choose $0 < \varepsilon' < 1$

Given $\varepsilon' > 0$, $\exists\ \delta > 0 \rightarrow |f(x) - f(a)| < \varepsilon'$ & $|g(x) - g(a)| < \varepsilon'$ whenever $0 < |x - a| < \delta$ — ②

$$|g(x)| = |g(a) + g(x) - g(a)| \leq |g(a)| + |g(x) - g(a)|$$

$\Rightarrow |g(x)| < |g(a)| + \varepsilon'$ whenever $0 < |x - a| < \delta$ — ③

$|f(x)g(x) - f(a)g(a)| < (|g(a)| + \varepsilon')\varepsilon' + |f(a)|\varepsilon'$ when $0 < |x - a| < \delta$

$< (|g(a)| + 1)\varepsilon' + |f(a)|\varepsilon'$ Since $\varepsilon' < 1$

$\therefore |f(x)g(x) - f(a)g(a)| < (|g(a)| + |f(a)| + 1)\varepsilon'$ $0 < x - a| < \delta$

if we choose $\varepsilon > 0$ Suchthat $\varepsilon' < \dfrac{\varepsilon}{(|f(a)| + |g(a)| + 1)}$

then $|f(x)g(x) - f(a)g(a)| < (|g(a)| + |f(a)| + 1)\dfrac{\varepsilon}{(|g(a)| + |f(a)| + 1)}$.

$\Rightarrow |f(x)g(x) - f(a)g(a)| < \varepsilon$ $0 < |x - a| < \delta$

$$\therefore \lim_{x \to a} f(x)g(x) = f(a)g(a) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

theorem 3

Let f is continuous at p. Then cf is Continuous at p

Proof:

Let f is continuous at p. $\Rightarrow$ $P_n \to P \Rightarrow \lim_{n\to\infty} f(P_n) = f(P)$

To prove cf is continuous at p.

$$\lim_{n\to\infty} (C(f))(P_n) = \lim_{n\to\infty} c\, f(P_n) = C \cdot \lim_{n\to\infty} f(P_n)$$

$$= C \cdot f(P)$$

$$= cf(P)$$

$\therefore$ cf is continuous at p.

Theorem 4

Let f and g be defined on an interval I and let $g(p) \neq 0$. If f, g are continuous at $p \in I$, Then f/g is continuous at p.

Proof:

Let $\langle p_n \rangle$ be any sequence converges to p. Let g is Continuous at P. Therefore $\lim_{n \to \infty} g(p_n) = g(p)$

Since $g(p) \neq 0$ $\Rightarrow$ $\exists$ positive integer m such that $g(p_n) \neq 0$ whenever $n > m$.

Also, since f is Continuous at p (i.e) $\lim_{n \to \infty} f(p_n) = f(p)$

$$\lim_{n \to \infty} \left(\frac{f}{g}\right)(p_n) = \lim_{n \to \infty} \left(\frac{f(p_n)}{g(p_n)}\right)$$

$$= \lim_{n \to \infty} f(p_n) \Big/ \lim_{n \to \infty} g(p_n)$$

$$= f(p) / g(p)$$

$$= \left(\frac{f}{g}\right)(p)$$

$\therefore \lim_{n \to \infty} \left(f/g\right)(p_n) = \left(\frac{f}{g}\right)(p)$ $\therefore$ $\frac{f}{g}$ is continuous at p.

Theorem 5

If f is continuous, Then $|f|$ is continuous.

Proof:

Let p be any point, $\langle p_n \rangle$ be a sequence converging to p.

$$\lim_{n\to\infty} |f|(p_n) = \lim_{n\to\infty} |f(p_n)|$$

$$= \left|\lim_{n\to\infty} f(p_n)\right| \quad \text{(Since f is Continuous)}$$

$$= |f(p)|$$

$$= |f|(p)$$

$\therefore$ $|f|$ is continuous at p.

Illustrations

1) Let f be the function defined on $\mathbb{R}$ by setting $f(x) = \frac{\sin x}{x}$ if $x \neq 0$

$f(0) = 0$

Here we have $\lim_{x \to 0} f(x) = 1$ $\quad \lim_{x \to 0} \frac{\sin x}{x} = 1$

but $f(0) = 0$

$\therefore \lim_{x \to 0} f(x) \neq f(0)$

$\therefore f$ is not continuous at $x = 0$

$\therefore f$ has removable discontinuity at $x = 0$

---

2) Let f be the function defined on $\mathbb{R}$ by setting $f(x) = \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ if $x \neq 0$

$f(0) = 1$

Answer:

$$f(0+0) = \lim_{\substack{h \to 0 \\ h > 0}} f(0+h) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{e^{1/h}\left(1 - e^{-1/h} \times \frac{1}{e^{1/h}}\right)}{e^{1/h}\left(1 + e^{-1/h} \times \frac{1}{e^{1/h}}\right)}$$

$$= \lim_{\substack{h\to 0 \\ h>0}} \frac{\cancel{e^{1/h}}\,(1-e^{-2/h})}{\cancel{e^{1/h}}\,(1+e^{-2/h})} = \frac{1-e^{-\infty}}{1+e^{-\infty}} = \frac{1-0}{1+0} = 1 \; //$$

$$f(0-0) = \lim_{\substack{h\to 0 \\ h<0}} f(0-h) = \lim_{\substack{h\to 0 \\ h>0}} \frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}} = \lim_{\substack{h\to 0 \\ h>0}} \frac{\cancel{e^{1/h}}\left(e^{-1/h}\times\frac{1}{e^{1/h}}-1\right)}{\cancel{e^{1/h}}\left(e^{-1/h}\times\frac{1}{e^{1/h}}+1\right)}$$

$$= \lim_{\substack{h\to 0 \\ h>0}} \frac{e^{-2/h}-1}{e^{-2/h}+1} = \frac{0-1}{0+1} = -1 \; //$$

$f(0) = f(0+0) \neq f(0-0)$

$\therefore$ f is continuous from the right at $x=0$ and f has a discontinuity of first kind from the left at $x=0$.

3) Let f be the function defined on $\mathbb{R}$ by setting

$$f(x) = \frac{e^{1/x}}{1+e^{1/x}} \text{ if } x \neq 0, \quad f(0)=0$$

Answer Here

$$f(0+0) = \lim_{\substack{h\to 0\\ h>0}} \frac{e^{1/h}}{1+e^{1/h}} = \lim_{\substack{h\to 0\\ h>0}} \frac{\cancel{e^{1/h}}}{\cancel{e^{1/h}}\left(\frac{1}{e^{1/h}}+1\right)}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$$f(0-0) = \lim_{\substack{h\to 0\\ h<0}} \frac{e^{1/h}}{1+e^{1/h}} = \lim_{\substack{h\to 0\\ h>0}} \frac{e^{-1/h}}{1+e^{-1/h}} = \lim_{\substack{h\to 0\\ h>0}} \frac{e^{-1/h}(1)}{e^{-1/h}(e^{1/h}+1)}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{1}{1+e^{1/h}} = \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$$

$$\therefore \quad f(0-0) = f(0) \neq f(0+0)$$

∴ f is continuous from the left at $x=0$ and it has discontinuity of the first kind from the right at $x=0$

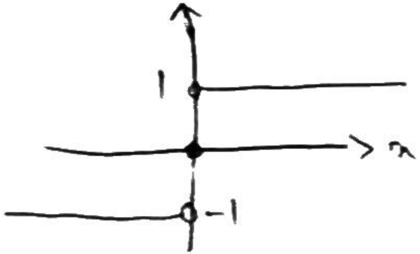
4) Let f be function defined on $\mathbb{R}$ by setting $f(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases}$

Answer $f(0+0) = \lim\limits_{\substack{x\to 0 \\ x>0}} f(x) = 1$

$f(0-0) = \lim\limits_{\substack{x\to 0 \\ x<0}} f(x) = -1$ $\quad f(0) = 0$

$f(0+0) \neq f(0-0) \neq f(0)$

$\therefore$ f has a discontinuity of the first kind from both sides at $x=0$

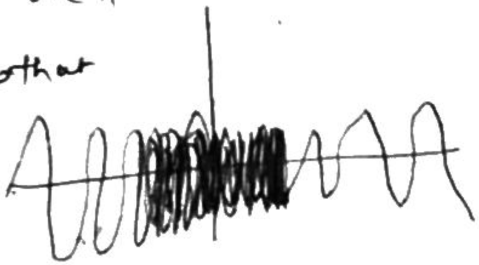


5) Let f be the function defined on $\mathbb{R}$ by setting $f(x) = \sin\frac{1}{x}$ when $x \neq 0$
$f(0) = 0$

Examine the continuity at $x=0$

Answer: We need to find $f(0+0)$, $f(0-0)$

We know that $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \ \forall \ x \in \mathbb{R}$



$\left|\sin\frac{1}{x}\right| \leq 1 \ \forall \ x \in \mathbb{R}$. We have to prove that $f(0+0)$, $f(0-0)$ does not exist

Suppose if $f(0+0)$ exists, it lies b/w -1 & 1

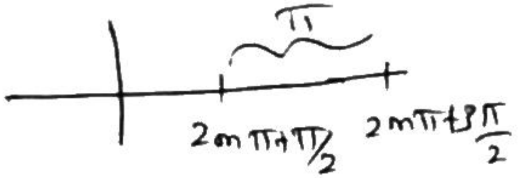
$$|f(x) - l| < 1$$

$$\lim_{x \to 0+0} f(x) = l$$

$\varepsilon = 1 \quad \delta = \delta_0, \quad 0 < x < \delta_0 \Rightarrow |f(x) - l| < \varepsilon$

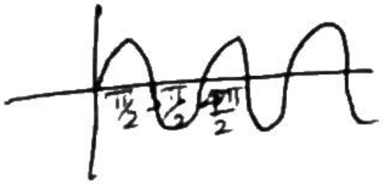
$(c < x < c+\delta \Rightarrow |f(x) - l| < \varepsilon)$

$\exists \ m \in \mathbb{N} \rightarrow 2m\pi + \frac{\pi}{2} > \frac{1}{\delta_0}$



$x_1 = \frac{1}{2m\pi + \pi/2} \qquad x_2 = \frac{1}{2m\pi + 3\pi/2}$

$x_0 < x_1 < \delta_0, \quad 0 < x_2 < \delta_0$



$\left|\sin\frac{1}{x_1} - l\right| < 1 \qquad \left|\sin\left(\frac{1}{x_2}\right) - l\right| < 1$

$$\left|\sin\frac{1}{x_1} - \sin\left(\frac{1}{x_2}\right)\right| = \left|\sin\left(\frac{1}{x_1}\right) - l - \frac{1}{\sin}\left(\frac{1}{x_2}\right) + l\right|$$

$$\leq \left|\sin\left(\frac{1}{x_1}\right) - l\right| + \left|\sin\left(\frac{1}{x_2}\right) - l\right|$$

$$< 1 + 1 = 2 \quad \text{———} \ ①$$

$\sin \frac{1}{x_1} = \sin (2m\pi + \pi/2) = 1$

$\sin \frac{1}{x_2} = \sin (2m\pi + 3\pi/2) = -1$

$\left| \sin \frac{1}{x_1} - \sin \frac{1}{x_2} \right| = | 1 - (-1) | = 2$ ——— ②

from ① & ② we get $2 < 2$ which is not possible

$\therefore$ f(0+0) doesnot exist.

Similary we can prove that f(0-0) doesnot exists. Hence f has a

$\therefore$ f(0+0), f(0,-0) doesnot exists. on bothsides at $x=0$

discontinuity of the second kind

Let f defined on $\mathbb{R}$ by setting $f(x)=1$ when x is irrational $f(x)=-1$ when x is rational . Show that f is discontinuous at every point of $\mathbb{R}$.

Proof Let $x=p$ be a rational number. Suppose f is Continuous at p

$\varepsilon=1$ $\exists$ $\delta>0 \ni |x-p|<\delta \Rightarrow |f(x)-f(p)|<1$

Let x_0 is a irrational number

$x_0 \in (p-\delta, p+\delta)$

$|x_0-p| < \delta$

$|f(x_0)-f(p)| = |1-(-1)| = 1+1=2 \not< 1$

Which is a Contradiction $\therefore$ f is not continuous at $x=p$ (rational)

Let $x=q$ is a irrational number. Suppose f is Continuous at $x=q$ Let $\varepsilon=1$ $\exists$ $\delta^*>0 \ni |x-q|<\delta^* \Rightarrow |f(x)-f(q)|<1$

$\exists$ rational number $x^* \in (x-\delta, x+\delta)$

$\therefore |x^*-q|<\delta^* \Rightarrow |f(x^*)-f(q)| = |(-1)-1| = |-2|$

$= 2 \not< 1$

Which is a Contradiction. Hence f is discontinuous at $x=q$ (irrational). This results holds forall rationals and irrationals. $\therefore$ f is discontinuous at every point of $\mathbb{R}$.

Theorem 1 **Algebra [redacted]**

Let f,g be defined on an interval I. If f and g are continuous at $p \in I$. Then f+g is continuous at P.

Proof:

Let f,g are continuous at $p \in I$. If $\langle P_n \rangle$ be any sequence converges to p. Then $\lim_{n\to\infty} f(P_n) = f(P)$

& $\lim_{n\to\infty} g(P_n) = g(P)$ (Since f,g are both continuous at p)

$$\lim_{n\to\infty} (f+g)(P_n) = \lim_{n\to\infty} \left(f(P_n) + g(P_n)\right) = \lim_{n\to\infty} f(P_n) + \lim_{n\to\infty} g(P_n)$$

$$= f(p) + g(P)$$

$$= (f+g)(P)$$

$\therefore$ $P_n \to P \Rightarrow (f+g)(P_n) \to (f+g)(P)$

$\therefore$ $(f+g)$ is continuous at $x = P$

Theorem2

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, Then $\lim_{x \to a} [f(x)g(x)] = lm$

Proof:

$$|f(x)g(x) - lm| = |f(x)g(x) - g(x)l + g(x)l - lm|$$

$$= |g(x)(f(x)-l) + l(g(x)-m)|$$

$$\le |g(x)|\,|f(x)-l| + |l|\,|g(x)-m| \quad ①$$

Since $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$ choose $0 < \varepsilon' < 1$

Given $\varepsilon' > 0$, $\exists\ \delta > 0 \ni |f(x) - l| < \varepsilon'$ & $|g(x) - m| < \varepsilon'$ whenever $0 < |x-a| < \delta$ ②

$$|g(x)| = |m + g(x) - m| \le |m| + |g(x) - m|$$

$\Rightarrow |g(x)| < |m| + \varepsilon'$ whenever $0 < |x-a| < \delta$ ③

$|f(x)g(x) - lm| < (|m| + \varepsilon')\varepsilon' + |l|\varepsilon'$ when $0 < |x-a| < \delta$

$< (|m|+1)\varepsilon' + |l|\varepsilon'$ Since $\varepsilon' < 1$

$\therefore\ |f(x)g(x) - lm| < (|m| + |l| + 1)\varepsilon'$ $\quad 0 < x-a| < \delta$

if we choose $\varepsilon > 0$ Suchthat $\varepsilon' < \dfrac{\varepsilon}{(|m|+|l|+1)}$

then $|f(x)g(x) - lm| < (|m|+|l|+1)\dfrac{\varepsilon}{(|m|+|l|+1)}$

$\Rightarrow |f(x)g(x) - lm| < \varepsilon$ $\quad 0 < |x-a| < \delta$

$\therefore \lim_{x \to a} f(x)g(x) = lm = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

Theorem 5

If f is continuous, Then $|f|$ is continuous.

Proof:

Let p be any point, $\langle p_n \rangle$ be a sequence converging to p.

$$\lim_{n\to\infty} |f|(p_n) = \lim_{n\to\infty} |f(p_n)|$$

$$= \left|\lim_{n\to\infty} f(p_n)\right| \quad \text{(since f is Continuous)}$$

$$= |f(p)|$$

$$= |f|(p)$$

$\therefore$ $|f|$ is continuous at p.

Result

f continuous $\Rightarrow$ $|f|$ is continuous

but converse is not true $|f|$ continuous $\not\Rightarrow$ f is continuous

Example
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

$|f(x)| = 1 \ \forall x \in \mathbb{R}$ which is continuous.

But f is not continuous.

Theorem 6 Let f, g be defined on an interval I. If f, g are both continuous at $p \in I$. Then the functions $\max\{f, g\}$ & $\min\{f, g\}$ are both continuous at p.

Proof: Let f, g are continuous at $p \in I$

We know that $\max\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$

$$\min\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

$\max\{2, 4\} = 4$

$\frac{1}{2}(2+4) + \frac{1}{2}|(2-4)|$

$= \frac{1}{2}(6) + \frac{1}{2}|(-2)|$

$= 3 + \frac{1}{2}(2) = 3+1 = 4$

We know that $(f+g)$ is continuous at $p \in I$

$(f-g)$ is continuous at $p \in I$

$\frac{1}{2}(f+g)$ is continuous at $p \in I$ (cf is continuous, if f is continuous
here $c = \frac{1}{2}$ $f = f+g$)

we have

$|f-g|$ is continuous at $p \in I$. ($|f|$ is continuous if f is continuous)

$$\therefore \max\{f, g\} = \underbrace{\frac{1}{2}(f+g)}_{\text{continuous}} + \underbrace{\frac{1}{2}|f-g|}_{\text{continuous.}}$$

$\therefore \max\{f, g\}$ is continuous at $p \in I$

Similary $\min\{f, g\} = \underbrace{\frac{1}{2}(f+g)}_{\text{continuous}} - \underbrace{\frac{1}{2}|f-g|}_{\text{continuous.}}$

$\therefore \min\{f, g\}$ is continuous at $p \in I$.

Th

Theorem 7

Let f, g be defined on intervals I, J respectively and let $f(I) \subset J$. If f is continuous at $p \in I$ and g is continuous at $f(p)$, Then $g \circ f$ is continuous at p.

proof: Let f defined on I. & f is continuous at $p \in I$

$\therefore p_n \to p \Rightarrow f(p_n) \to f(p)$

$\lim_{n \to \infty} f(p_n) = f(p)$

$g(f(p_n)) \to g(f(p))$ (Since g is continuous at $f(p)$)

$\lim_{n \to \infty} g(f(p_n)) = g(f(p))$

$\therefore \lim_{n \to \infty} g \circ f(p_n) = g \circ f(p)$

$g \circ f$ is continuous at p.

also continuous at $x = 0$.

8.13. FUNCTION OF A FUNCTION. COMPOSITES OF FUNCTIONS

Let f and g be two functions such that

$$\text{Domain } f = [a, b] \text{ and domain of } g = [\alpha, \beta]$$

We suppose that the range of the function g is a sub-set of the domain of the function f,

$$i.e., \text{ Range } g \subset \text{domain } f.$$

Now $\quad t \in [\alpha, \beta]$

$\Rightarrow\ g(t) \in \text{Range } g \quad \Rightarrow \quad g(t) \in \text{Domain } f \quad \Rightarrow \quad f(g(t))$ has a meaning.

We have thus a new real valued function with $[\alpha, \beta]$ as its domain.

This new function is called a *function of function* and is also denoted as $f \circ g$ and called the *composite* of f and g. Thus, we have

$$(fog)(t) = f(g(t)).$$

It may be emphasized that the composite function $f \circ g$ *has a meaning if and only if the range of the function* g *is a sub-set of the domain of the function* f.

EXERCISE

1. Let f, g be two functions defined as follows :

$$f(x) = \sqrt{x} \ \ \forall\ x \geq 0, \qquad g(x) = x^2 + 1 \ \ \forall\ x \in \mathbf{R}.$$

Show that
$$(f \circ g)(x) = \sqrt{(x^2 + 1)}.$$

What is the domain of the function $f \circ g$?

2. If $\qquad f(x) = x^3 + x - 2, \qquad g(x) = 1/(x + 1)$

give explicit definitions of $f \circ g$ and $g \circ f$ giving also their domains.

3. Let $\qquad f(x) = x^2 + 1, \qquad g(x) = x^4.$

Show that $f \circ g \neq g \circ f$.

4. Let $\qquad f(x) = \sqrt{x}, \qquad g(x) = 1/(x^2 - 1),$

determine $g \circ f$ and $f \circ g$ with their domains.

Theorem 1

Every function defined & continuous on a closed interval is bounded above therein. That is, if f is Continuous on $[a,b]$, then $\exists$ a real number $u \ni f(I) \leq u$

f is Continuous on $I = [a,b]$ Then $\exists\, u \in \mathbb{R} \ni$

$f(x) \leq u \;\forall\, x \in I$

~~fr~~ $f : I \to \mathbb{R}$ $\quad (I = [a,b])$

$f(I) \leq u$

proof:

Let $f: I \to \mathbb{R}$ which is Continuous

To prove f is bounded above.

(i.e $\exists\, u \in \mathbb{R} \ni f(x) \leq u \;\forall\, x \in I$)

We can prove by Contradiction method. Suppose f is not bdd above.

$\exists\, n \ni x_n \in I \quad f(x_n) > n \;\forall\, n$

now $\langle x_n \rangle \subseteq I$

$\exists\, x_{n_k} \to x_0 \quad x_0 \in I$

$f(x_{n_k}) \not\to f(x_0)$

$\lim_{n \to 0} f(x_{n_k}) \neq f(x_0)$

$\Rightarrow$ f is not Continuous. Which is a contradiction. $\therefore$ f must be a bounded above function.

Theorem 2

If f is continuous on $[a,b]$, Then it is bounded below on $[a,b]$.

Theorem 4

Every function defined and continuous on a closed interval attains (i.e) if f is continuous on a closed interval $I = [a,b]$ and u be the supremum of f on I, then there exists a point $x_0 \in I$ such that $f(x_0) = u$

proof: $f(x_0) = u$

$f(x_0) = u$ if possible let $f(x) < u \;\forall\; x \in I$

$f(x) - u < 0$

$u - f(x) > 0 \;\forall\; x \in I$

Let $g(x) = \frac{1}{u-f(x)} \;\forall\; x \in I$

$\therefore$ $g(x)$ is continuous. $\exists$ positive real $k \ni g(x) \leq k$

$$\frac{1}{u-f(x)} \leq k$$

$$u - f(x) \geq \frac{1}{k}$$

$$-f(x) \geq \frac{1}{k} - u$$

$$f(x) \leq u - \frac{1}{k} \;\forall\; x \in I$$

$\Rightarrow u - \frac{1}{k}$ is an upperbound of $f(x)$

$\Rightarrow u - \varepsilon$ is ~~not~~ an upperbound of $f(x)$

Which is a contradiction. Since u is least upper bound.

Theorem 5

Every function defined and continuous on closed interval attains its infimum (i.e) f is Continuous on $[a,b]$ and l is infimum of f. Then $\exists\ x_0 \in [a,b]$ such that $f(x_0)=l$

$f(x_0)=l$

Suppose $f(x) > l \quad \forall x \in I$

$f(x) - l > 0$

$g(x) = \frac{1}{l - f(x)} \quad \forall x \in I$ $\therefore f$ is continuous

$\exists$ ve real number $k \ni g(x) \geq K$

$$\frac{1}{l - f(x)} \geq K$$

$$l - f(x) \leq \frac{1}{K}$$

$$-f(x) \leq \frac{1}{K} - l$$

$$f(x) \geq l - \frac{1}{K} \quad \forall x \in I$$

But $\Rightarrow l - 1/K$ is not an lower bound which is a Contradiction. Since l is the greatest lower bound.

Remarks & Examples

1) ~~f~~ be the function defined by setting $f(x)=x$ ∀ x in $[0,1)$

$[0,1[$

f is continuous in $[0,1)$ $\Rightarrow$ It is bounded above in $[0,1)$

$\Rightarrow$ f attains its supremum

$f(x)=x$ ∀ x in $[0,1)$

$\sup\{f(x)\} = 1$

2) Let f be the function defined by $f(x)=\frac{1}{1+x^2}$ ∀ x in $\mathbb{R}$.

$x=0 \Rightarrow f(x)=\frac{1}{1+0}=1$

$x=1 \Rightarrow f(x)=\frac{1}{1+1^2}=\frac{1}{2}$

upper bound $=1$ & lower bound $=0$

$\Rightarrow \exists\, x_0 \in \mathbb{R}\ f(x_0)=u$, $\exists\, x_0 \in \mathbb{R}$, $f(x_0)=l$

f attains its supremum . doesnot attains infimum

Intermediate value theorem

Intermediate Value theorem

Theorem 1.

Let f be continuous on $[a,b]$ and let $a < x_0 < b$

Then (i) $f(a) < 0$ implies that $\exists\ \delta_0 > 0$ such that $f(x) < 0$ $\forall\ x \in [a, a+\delta_0)$

(ii) $f(x_0) < 0$ implies that there exists $\delta_0 > 0$ such that $f(x) < 0$ $\forall$ x in $(x_0 - \delta_0, x_0 + \delta_0)$

(iii) $f(b) < 0$ implies that there exists $\delta_0 > 0$ such that $f(x) < 0$ $\forall$ x in $(b-\delta_0, b]$

Proof :

$f(a) < 0 \Rightarrow -f(a) > 0$

Since f is Continuous at a. ($\Rightarrow$ f is right continuous at a)

$\exists\ \delta_0 \Rightarrow a < x < a+\delta_0 \Rightarrow |f(x) - f(a)| < \varepsilon$

$a < x < a+\delta_0 \Rightarrow f(a) - \varepsilon < f(x) < f(a) + \varepsilon$

$\varepsilon > 0$ we can choose $\varepsilon = -f(a)$

$f(a) - (-f(a)) < f(x) < f(a) - f(a)$

$2f(a) < f(x) < 0$

$\therefore f(x) < 0$

(ii)

Boundedness of Continuous functions

Theorem 1

Every function defined & continuous on a closed interval is bounded above therein. That is , if f is continuous on $[a,b]$, then $\exists$ a realnumber u $\ni$ $f(I) \leq u$

- f is Continuous on $I = [a,b]$ Then $\exists\, u \in \mathbb{R} \ni$ $f(x) \leq u \;\forall\, x \in I$

~~ff~~ $f : I \to \mathbb{R}$ $\quad (I = [a,b])$

$$f(I) \leq u$$

proof:

Let $f: I \to \mathbb{R}$ which is Continuous

To prove f is bounded above. (i.e $\exists\, u \in \mathbb{R} \ni f(x) \leq u \;\forall\, x \in I$)

We can prove by Contradiction method. Suppose f is not bdd above

$$\exists\, n \ni x_n \in I \quad f(x_n) > n \;\forall\, n$$

$$\text{now } \langle x_n \rangle \in I$$

$$\exists\, x_{n_k} \to x_0 \quad x_0 \in I$$

$$f(x_{n_k}) \not\to f(x_0)$$

$$\lim_{n \to \infty} f(x_{n_k}) \neq f(x_0)$$

$\Rightarrow$ f is not Continuous. Which is a contradiction. $\therefore$ f must be a bounded above function.

Theorem 2

If f is continuous on [a,b], Then it is bounded below on [a,b].

i.e f is continuous on $I = [a,b]$, then

$\exists\, u \in \mathbb{R} \ni f(x) \geq u \;\forall\, x \in I$.

$f : I \to \mathbb{R}$ $\quad f(I) \geq u \Rightarrow$ [redacted]

Proof f is defined and continuous on I.

to prove f is bounded below.

(Contradiction method) Suppose f is not bounded below

$\exists\, n \ni x_n \in I$ - $f(x) < n \;\forall\, n$

$\langle x_n \rangle \in I \quad \exists\, x_{n_k} \to x_0,\; x_0 \in I$

$f(x_{n_k}) \not\to f(x_0)$

$\lim_{h \to 0} f(x_{n_k}) \neq f(x_0)$

Theorem 3

If f is continuous on $[a,b]$, then it is bounded on $[a,b]$

f is continuous on $[a,b]$ $\Rightarrow$ f is bounded above

f is continuous on $[a,b]$ $\Rightarrow$ f is bounded below

$\therefore$ f is continuous on $[a,b]$ $\Rightarrow$ f is bounded above & bounded below

$\Rightarrow$ f is bounded.

Let F and g are both continuous at $p \in I$.

To P.T $F+g$ is continuous at p.

In P_n be any sequence converging to p.

Then,
$$\lim_{n\to\infty} f(P_n) = f(P)$$
$$\lim_{n\to\infty} g(P_n) = g(P).$$

Since, F and g are continuous at p.

Now,
$$\lim_{n\to\infty} (f+g)(P_n) = \lim_{n\to\infty} [f(P_n)+g(P_n)].$$
$$= \lim_{n\to\infty} f(P_n) + \lim_{n\to\infty} g(P_n).$$
$$= f(P)+g(P).$$
$$=(f+g)P.$$

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$$\therefore \lim_{n\to\infty} (f+g)(P_n) = (F+g)(P)$$

$\therefore (F+g)$ is continuous at P.

Theorem If F is continuous then $|F|$ is continuous.

Proof: Let F be a continuous function.

Let P_n be any sequence convergence to P.

Then,
$$\lim_{n\to\infty} |F|(P_n) = \lim_{n\to\infty} |f(P_n)|.$$
$$= |\lim_{n\to\infty} f(P_n)|$$
$$= |f(P)|$$
$$\therefore \lim_{n\to\infty} |F|P_n = |F|P.$$

$\therefore |F|$ is continuous at P.

Intermediate value theorem.

Theorem

Let F be a continuous function on $a[a,b]$

and let x_0 be $\ni a < x_0 < b$ then,

[illegible] implies that there exits a

[a, a+δ_0].

$\ni f(x) < 0 \;\forall\; x$ in

(ii) $F(x_0)<0$ implies that there exist a $\delta_0>0 \ni f(x)<0 \ \forall \ x$ in $(x_0-\delta_0, x_0+\delta_0)$.

(iii) $F(b)<0$ implies that there exists a $\delta_0>0 \ni f(x)<0 \ \forall$ in $x \ [b-\delta_0, b]$.

Proof :-

i) Since $f(a)<0$

$\therefore -f(a)>0$.

$\because$ F is continous at $x=a$.

$\therefore$ corresponding to any $\epsilon>0, \delta>0 \ni$,

$a \leq x < a+\delta \Rightarrow f(a)-\epsilon < f(x) < f(a)+\epsilon$.

Let $\epsilon = -f(x)$ and $\delta = \delta_0$.

Then, $a \leq x < a+\delta_0 \Longrightarrow 2f(a) < f(x) < 0$.

$\Longrightarrow f(x)<0 \ \forall . \ x$ in $[a, a+\delta_0]$

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ii) Since, $f(x_0)<0$.

$\therefore -f(x_0)>0$

$\because$ f is continous at $x=x_0$.

$\therefore$ corresponding to any $\epsilon>0, \delta>0 \ni$,

$x_0-\delta < x < x_0+\delta \Rightarrow f(x_0)-\epsilon < f(x) < f(x_0)+\epsilon$.

Then,

$x_0-\delta_0 < x < x_0+\delta_0 \Rightarrow 2f(x_0) < f(x) < 0$

$\Longrightarrow F(x)<0 \ \forall \ x$ in $(x_0-\delta_0, x_0+\delta_0)$.

iii) Since $f(b)<0$

$\therefore -f(b)>0$.

$\because$ f is continuous at $x=b$.

$\therefore$ corresponding to any $\epsilon>0, \delta>0 \ni$

$b-\delta < x \leq b \Rightarrow F(b)-\epsilon < f(x) < f(b)+\epsilon$.

let $\epsilon = -f(b)$ and $\delta = \delta_0$.

Then, $b-\delta_0 < x < b \Rightarrow 2f(b) < f(x) < 0$.

Inverse function theorem.

Theorem

If f be a continuous one-to-one function on the closed interval $[a,b]$, then f^{-1} is also continuous.

Proof:-

Let F be the function defined on continuous of $[a,b]$.

P.T F^{-1} is continuous.

$\because$ F is 1-1

$\therefore$ $a \neq b$ and $f(a) \neq f(b)$. 36

case(i) Let $f(a) < f(b)$.

Hence $f(a) = c$ and $f(b) = d$.

Let therefore the image of closed interval $[a,b]$ under F is the closed interval $[c,d]$.

$f([a,b]) = [c,d]$.

$\therefore$ F^{-1} is 1-1 function

$[a,b]$ on to $[c,d]$.

F is 1-1 and on to function.

$\therefore$ F^{-1} is exists.

$g = F^{-1}$.

PT g is continuous.

Step 1 PT If y_1 & y_2 be any two real numbers such that $c < y_1 < y_2 < d$. Then.

$\exists$ real number x_1 and x_2 $\ni$

$a < x_1 < x_2 < b$ and $f(x_1) = y_1$, $f(x_2) = y_2$.

$\therefore$ f is 1-1.

$\exists$ y_1 and y_2 $\ni$ $c < y_1 < y_2 < d$.

$\exists$ The unique real no x_1 and x_2 in $[a,b] \ni$

$f(x_1) = y_1$, $f(x_2) = y_2$.

$\because$ f is 1-1

$\therefore y_1 \neq y_2$.

$f(x_1) \neq f(x_2)$.

$x_1 \neq x_2$.

PT $a < x_1 < x_2 < b$.

$x_1 \in (a,b)$

f is continuous on $[a,b]$.

f is continuous on $[x_1,b]$.

$\exists$ is $y \ni f(x_1) < y < f(b)$.

Then, $\exists$ $x \in (x_1,b) \ni$

$f(x_1) = y$.

$y = y_2$ it follows that

$f(x_1) < y_2 < f(b)$.

$x_1 < x_2 < b$.

llly $a < x_1 < x_2$.

Hence the result.

<u>Step 2</u>:- PT if x_1 and x_2 be any two real number such that,

$a < x_1 < x_2 < b$ then,

$f(a) < f(x_1) < f(x_2) < f(b)$.

$\therefore a < x_1 < x_2 < b$.

$\therefore$ $f(x_1)$ and $f(x_2)$ lies b/w $f(a)$ & $(f(b)$.

$x_1 \neq x_2$.

$f(x_1) \neq f(x_2)$

$\therefore f(x_2) < f(x_1) \Rightarrow x_2 < x_1$.

$\therefore f(x_2) \not< f(x_1)$.

Hence $f(x_1) < f(x_2)$.

<u>step3</u> PT g is continuous.

Let y_0 be any point $\ni c < y_0 < d$ $\exists$ $x \ni f(x_0) = y_0$ and $a < x_0 < b$.

Let ϵ be any +ve number.

$y_1 = f(x_0 - \epsilon)$; $y_2 = f(x_0 + \epsilon)$.

$\delta = \min[y_0 - y_1, y_2 - y_0]$.

$(y_0 - \delta, y_0 + \delta) \subset (y_1, y_2)$.

$|y - y_0| < \delta \Rightarrow y_0 - \delta < y < y_0 + \delta$.

$\Rightarrow y_1 < y < y_2$.

$\Rightarrow g(y_1) - g(y) < g(y_2)$.

$\Rightarrow x_0 - \epsilon < g(y) < x_0 + \epsilon$

$\Rightarrow g(y_0) - \epsilon < g(y) < g(y_0) + \epsilon$.

$|g(y) - g(y_0)| < \epsilon$.

$\therefore g$ is continuous on $[c, d]$.

<u>case(ii)</u> Let $f(a) > f(b)$.

Define h to be the function $-f$, so that h is 1-1 and continuous on $[a, b]$ and $h(a) < h(b)$.

by case (i) h^{-1} exist and also continuous $\therefore f^{-1}$ is exists.

$f^{-1} = -(h^{-1})$ is continuous.

ST $f(0)$ and $F(\pi/2)$ are different sign and explain still why f does not vanish on $[0,$

$$0 \le x < 1 \quad x^{2n} \to 0 \text{ as } n \to \infty .$$

$$\text{if } x > 1 \; x^{-2n} \to 0 \; n \to \infty .$$

If $0 \le x < 1$ then

$$F(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

$$= \log(2+x) \longrightarrow ①$$

If $x=1$,

$$f(x) = \lim_{n\to\infty} \frac{\log 3 - \sin 1}{1+1} = \frac{\log 3 - \sin 1}{2} \rightarrow (2)$$

If $x>1$,

$$f(x) = \lim_{n\to\infty} \frac{x^{-2n}\log(2+x) - \sin x}{x^{-2n}+1}$$

$$= \frac{-\sin x}{1} = -\sin x \longrightarrow (3)$$

$$f(x) = \begin{cases} \log(2+x & \text{if } 0 \le x < 1 . \\ 1/2(\log 3 \sin 1) & \text{if } x = 1 \\ -\sin x & \text{if } x > 1 \end{cases}$$

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$$f(1-0) = \lim_{\substack{n\to 0 \\ n \ge 0}} f(1-h)$$

$$= \lim_{\substack{n\to 0 \\ n \ge 0}} \log(2+1-h)$$

$$= \lim_{\substack{n\to 0 \\ n \ge 0}} \log(3-h) = \log 3$$

$$f(1+0) = \lim_{\substack{n\to 0 \\ n \ge 0}} F(1+h)$$

$$\lim_{\substack{n\to 0 \\ n \ge 0}} -\sin(1+h) = -\sin 1 .$$

$\therefore$ $f(1+0)$ exist and $f(1-0)$ exist . both are $\neq f(1)$.

$\therefore$ f is discontinuity of First kind .

$f(0) = \log(2+0) = \log 2 > 0$.

$f(\pi/2) = -\sin \pi/2 = -1 < 0$

Hence $F(0)$ and $F(\pi/2)$ does not vanish $[0, \pi/2]$

uniform continuity :- (continuous is not uniformly continuous (ex) $f(x)=x^2$ $\forall x \in R$) (uniform continuous is continuous) }

A function f defined on an interval I is said to be uniformly continuous on I. if given $\epsilon > 0$ there exist a $\delta > 0$ such that If x, y are in I and $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$

Theorem

If f be uniformly continuous on interval I then it is continuous on I .

Proof :- Let f be uniformly continuous on I . Let x_0 be any point of I and Let $\epsilon > 0$ we given

since, f is uniformly continuous on I

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Let $x_0 \in I$.

$\epsilon > 0$, $\delta > 0$, $x, y \in I$.

$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$.

$y = x_0$

$|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \epsilon$

$\rightarrow$ f is continuous .

Theorem

f is continuous on the closed and bounded interval I . Then f is uniformly continuous on I .

Proof :- If f is continuous on the closed and bounded $I = [a, b]$.

P.T f is uniformly continuous.

Suppose f is not uniformly continuous.

$\epsilon > 0$, There exist $\delta > 0$, x, y in $I \ni$

$|x-y| < \delta \Rightarrow |f(x) - f(y)| \geq \epsilon$.

Inparticular, for each +ive integer n.

$|x_n - y_n| < 1/n \Rightarrow |f(x_n) - f(y_n)| \geq \epsilon$. $\longrightarrow (1)$

$\langle x_n \rangle$ & $\langle y_n \rangle$ are sequences in I.

Every sequence is closed interval has a convergent subsequence.

There exist $\langle x_{nk} \rangle$ of $\langle x_n \rangle$.

$\langle y_{nk} \rangle$ of $\langle y_n \rangle$

$\langle x_{nk} \rangle \longrightarrow x_0$ ..

$\langle y_{nk} \rangle \rightarrow y_0$.

$(1) \Rightarrow |x_{nk} - y_{nk}| < 1/n_k$ and

$|f(x_{nk}) - f(y_{nk})| \geq \epsilon$. for all k.

$\lim_{k\to\infty} x_{nk} = \lim_{k\to\infty} y_{nk}$.

$x_0 = y_0$.

$\langle f(x_{nk}) \rangle$ and

$\langle f(y_{nk}) \rangle$ continuous.

but their exist and f is not continuous at x_0.

$\therefore$ f is uniformly continuous on I.

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UNIT-III

DERIVABILITY ON AN OPEN INTERVAL:-

Let f be a real valued function defined on an open interval $I \subset R$ If $x_0 \in I$ then we define g with domain $I - \{x_0\}$ by setting

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0} \quad \text{for all } x \in I - \{x_0\}$$

It $\lim_{x \to x_0} g(x)$ exists and is finite we denote it by $f'(x_0)$ and say that f is derivable at x_0, or that 'f' has a derivative at x_0 or simply that $f'(x_0)$ exist $f'(x_0)$ is called the derivative of f at x_0

If $\lim_{x \to x_0+0} g(x)$ exists and is finite we denote it by $R\,f'(x_0)$ and say that f is derivable from the right at no. $R\,f'(x_0)$ is called derivative of f from the right at x_0

If $\lim_{x \to x_0-0} g(x)$ exists and is finite we denote it by $Lf'(x_0)$ & say that f is derivable from the left at x_0.

$Lf'(x_0)$ is called the derivative of from the left at no.

f is derivable at x_0 if f$Lf'(x_0)$ and $Rf'($

both exist and are equal.

1) $f(x) = x$ for all $x \in R$.

Soln:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1.$$

$$= 1$$

2. $f(x) = c$ for all $x \in R$.

Soln:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{c - c}{x - x_0}$$

$$= \lim_{x \to x_0} 0 = 0$$

3) $f(x) = |x|$ for all $x \in R$. Find the derivability of f at $x = 0$.

Soln:

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$Rf'(0) = \lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \to 0+0} \frac{x - 0}{x + 0} = \lim_{x \to 0^+} 1. = 1$$

$$Lf'(x) = \lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0-0} \frac{-x - 0}{x - 0} = \lim_{x \to 0-0} \frac{-x}{x} = -1$$

Hence $Lf'(0) \neq Rf'(0)$

$\therefore$ f is not derivative at $x=0$

Example:

? $f(x) = x^n$ for all $\in R$ where n is positive integer.

Soln:

Let $x_0 \in R$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(x - x_0)(x^{n-1} + x^{n-2} x_0 + x^{n-3} x_0^2 + \dots + x_0^{n-1}}{x - x_0}$$

$$= \lim_{x \to x_0}$$ [redacted] $$x^{n-1} + x^{n-2} x_0^1 + x^{n-3} x_0^2 + \dots + x_0^{n-1}$$

$$= x_0^{n-1} + x_0^{n-2} x_0^1 + x_0^{n-3} x_0^2 + \dots + x_0^{n-1}$$

$$= x_0^{n-1} + x_0^{n-1} + x_0^{n-1} + \dots + x_0^{n-1}$$

$$= n x_0^{n-1}$$

Inverse function theorem.

Theorem

If f be a continuous one-to-one function on the closed interval $[a,b]$, then f^{-1} is also continuous.

Proof:-

Let F be the function defined on continuous of $[a,b]$.

P.T F^{-1} is continuous.

$\because$ F is 1-1

$\therefore a \neq b$ and $f(a) \neq f(b)$. 36

case(i) Let $f(a) < f(b)$.

Hence $f(a) = c$ and $f(b) = d$.

Let therefore the image of closed interval $[a,b]$ under F is the closed interval $[c,d]$.

$$f([a,b]) = [c,d].$$

$\therefore$ F [illegible] is 1-1 function

$[a,b]$ on to $[c,d]$.

F is 1-1 and on to function.

$\therefore F^{-1}$ is exists.

$$g = F^{-1}.$$

PT g is continuous.

Step 1 PT If y_1 & y_2 be any two real numbers Such that $c < y_1 < y_2 < d$. Then

$\exists$ real number x_1 and $x_2 \ni$

$a < x_1 < x_2 < b$ and $f(x_1) = y_1$, $f(x_2) = y_2$.

$\therefore$ f is 1-1 .

$\exists$ y_1 and y_2 $\ni c < y_1 < y_2 < d$.

$\exists$ The unique real no x_1 and x_2 in $[a,b] \ni$

$f(x_1) = y_1$, $f(x_2) = y_2$.

$\because$ f is 1-1

$\therefore y_1 \neq y_2$.

$f(x_1) \neq f(x_2)$.

$x_1 \neq x_2$.

PT $a < x_1 < x_2 < b$.

$x_1 \in (a,b)$

f is continuous on $[a,b]$.

f is continuous on $[x_1, b]$.

$\exists$ is $y \ni f(x_1) < y < f(b)$.

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Then, $\exists$ $x \in (x_1, b) \ni$

$f(x_1) = y$.

$y = y_2$ it follows that

$f(x_1) < y_2 < f(b)$.

$x_1 < x_2 < b$.

|||ly $a < x_1 < x_2$.

Hence the result .

Step 2 :- PT if x_1 and x_2 be any two real number such that,

$a < x_1 < x_2 < b$ then,

$f(a) < f(x_1) < f(x_2) < f(b)$.

$\therefore a < x_1 < x_2 < b$.

$\therefore$ $f(x_1)$ and $f(x_2)$ lies b/w $f(a)$ & $(f(b)$.

$x_1 \neq x_2$.

$f(x_1) \neq f(x_2)$

$\therefore f(x_2) < f(x_1) \Rightarrow x_2 < x_1$.

$\therefore f(x_2) \nless f(x_1)$.

Hence $f(x_1) < f(x_2)$.

step 3 PT g is continuous.

Let y_0 be any point $\ni c < y_0 < d$ ∃

$x \ni f(x_0) = y_0$ and $a < x_0 < b$.

Let ϵ be any +ve number.

$y_1 = f(x_0 - \epsilon)$; $y_2 = f(x_0 + \epsilon)$.

$\delta = \min[y_0 - y_1, y_2 - y_0]$.

$(y_0 - \delta, y_0 + \delta) \subset (y_1, y_2)$.

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$|y - y_0| < \delta \Rightarrow y_0 - \delta < y < y_0 + \delta$.

$\Rightarrow y_1 < y < y_2$.

$\Rightarrow g(y_1) < g(y) < g(y_2)$.

$\Rightarrow x_0 - \epsilon < g(y) < x_0 + \epsilon$.

$\Rightarrow g(y_0) - \epsilon < g(y) < g(y_0) + \epsilon$.

$|g(y) - g(y_0)| < \epsilon$.

$\therefore g$ is continuous on $[c, d]$.

case (ii) Let $f(a) > f(b)$.

Define h to be the function $-f$, so that h is 1-1 and continuous on $[a,b]$ and $h(a) < h(b)$.

by case (i) h^{-1} exist and also continuous. $\therefore f^{-1}$ is exists.

$f^{-1} = -(h^{-1})$ is continuous.

ST $f(0)$ and $F(\pi/2)$ are different sign and explain still why f does not vanish in $[0,1)$

$0 \leq x < 1 \quad x^{2n} \to 0$ as $n \to \infty$.

if $x > 1$ $x^{-2n} \to 0$ $n \to \infty$.

If $0 \leq x < 1$ then

$$F(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

$$= \log(2+x) \longrightarrow ①$$

If $x = 1$,

$$f(x) = \lim_{n\to\infty} \frac{\log 3 - \sin 1}{1+1} = \frac{\log 3 - \sin 1}{2} \longrightarrow (2)$$

If $x > 1$,

$$f(x) = \lim_{n\to\infty} \frac{x^{-2n}\log(2+x) - \sin x}{x^{-2n}+1}$$

$$= \frac{-\sin x}{1} = -\sin x \longrightarrow (3)$$

$$f(x) = \begin{cases} \log(2+x & \text{if } 0 \leq x < 1. \\ 1/2(\log 3 \sin 1) & \text{if } x = 1 \\ -\sin x & \text{if } x > 1 \end{cases}$$

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$$f(1-0) = \lim_{\substack{n\to 0 \\ n \geq 0}} f(1-h)$$

$$= \lim_{\substack{n\to 0 \\ n \geq 0}} \log(2+1-h)$$

$$= \lim_{\substack{n\to 0 \\ n \geq 0}} \log(3-h) = \log 3$$

$$f(1+0) = \lim_{\substack{n\to 0 \\ n \geq 0}} F(1+h)$$

$$\lim_{\substack{n\to 0 \\ n \geq 0}} -\sin(1+h) = -\sin 1.$$

4/11/19

UNIT-III

DERIVABILITY ON AN OPEN INTERVAL:-

Let f be a real valued function defined on an open interval $I \subset R$ If $x_0 \in I$ then we define g with domain $I - \{x_0\}$ by setting

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0} \quad \text{for all } x \in I - \{x_0\}$$

If $\lim_{x \to x_0} g(x)$ exists and is finite we denote it by $f'(x_0)$ and say that f is derivable at x_0, or that 'f' has a derivative at x_0 or simply that $f'(x_0)$ exist $f'(x_0)$ is called the derivative of f at x_0

If $\lim_{x \to x_0+0} g(x)$ exists and is finite we denote it by $Rf'(x_0)$ and say that f is derivable from the right at x_0. $Rf'(x_0)$ is called derivative of f from the right at x_0

If $\lim_{x \to x_0-0} g(x)$ exists and is finite we denote it by $Lf'(x_0)$ & say that f is derivable from the left at x_0.

$Lf'(x_0)$ is called the derivative of from the left at x_0.

f is derivable at x_0 if $Lf'(x_0)$ and $Rf'(x_0)$ both exist and are equal.

1) $f(x) = x$ for all $x \in R$.

Soln:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1.$$

$$= 1$$

2. $f(x) = c$ for all $x \in R$.

Soln:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{c - c}{x - x_0}$$

$$= \lim_{x \to x_0} 0 = 0$$

3) $f(x) = |x|$ for all $x \in R$. Find the derivability of f at $x = 0$.

Soln:

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$Rf'(0) = \lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \to 0+0} \frac{x - 0}{x - 0} = \lim_{x \to 0^+} 1. = 1$$

$$Lf'(x) = \lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0-0} \frac{-x - 0}{x - 0} = \lim_{x \to 0-0} \frac{-x}{x} = -1$$

Hence $Lf'(0) \neq Rf'(0)$

$\therefore$ f is not derivative at $x=0$

Example :

?) $f(x) = x^n$ for all $\in R$ where n is positive integer.

Soln :

Let $x_0 \in R$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(x - x_0)(x^{n-1} + x^{n-2} x_0 + x^{n-3} x_0^2 + \dots + x_0^{n-1})}{x - x_0}$$

$= \lim_{x \to x_0}$ [redacted] $x^{n-1} + x^{n-2} x_0^1 + x^{n-3} x_0^2 + \dots + x_0^{n-1}$

$$= x_0^{n-1} + x_0^{n-2} + x_0^1 + x_0^{n-3} x_0^2 + \dots + x_0^{n-1}$$

$$= x_0^{n-1} + x_0^{n-1} + x_0^{n-1} + \dots + x_0^{n-1}$$

$$= n x_0^{n-1}$$

2) Let f be the function defined on R by $f(x) = 0$ $x \le 0$, $f(x) = x$ if $x > 0$. Find the derivability of f at $x = 0$.

Soln:

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$$

$$Rf'(0) = \lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x - 0}{x - 0} = 1$$

$$Lf'(0) = \lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \to 0^{-0}} \frac{0 - 0}{x - 0} = 0$$

Since $Lf'(0) \neq Rf'(0)$

$\therefore f$ is not derivable at x_0

DERIVABILITY AND CONTINUITY

Theorem :-

Let f be defined on an interval I. If f be derivable at a point $x_0 \in I$, then it is continuous at x_0.

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} (x - x_0)$$

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \times \lim_{x \to x_0} (x - x_0)$$

Since $\left(\lim_{x \to x_0} (x - x_0) = 0 \right)$

$\Rightarrow f'(x_0)(0) = 0$

If f be derivable at a

$$\lim_{x \to x_0} f(x) - f(x_0) = 0$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0) \Rightarrow f \text{ is continuous at } x = x_0$$

Derivability ⇒ continuity but converse is not true

Example

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$$

$f(x) = |x|$

is continuous at 0 but not derivable at $x=0$

ALGEBRA OF DERIVATIVES

$(f+g)'(x_0) = f'(x_0) + g'(x_0)$ where f and g are derivable at $x = x_0$

Proof:-

Let f, g are derivable at x_0

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0)$$

$$\lim_{x \to x_0} \left(\frac{(f(x)+g(x)) - (f(x_0)+g(x_0))}{x - x_0} \right) = (f+g)' x_0$$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}+\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0)}=(f+g)'_{x_0}$$

$$f'(x_0)+g'(x_0)=(f+g)'x_0$$

Hence proved.

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① $f(x)=x\left(\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right)$ $x\neq 0$, $f(0)=0$ find the derivability of f at $x=0$

Soln:

$$Rf'(x)=\lim_{\substack{h\to 0\\ h>0}}\frac{f(x+h)-f(x)}{h}$$

$$Rf'(0)=\lim_{\substack{h\to 0\\ h>0}}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{\substack{h\to 0\\ h>0}}\frac{h\left(\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}\right)}{h}$$

$$=\lim_{\substack{h\to 0\\ h>0}}\frac{e^{1/h}(1-e^{-2/h})}{e^{1/h}(1+e^{-2/h})}$$

Rough

$e^{-1/h}\times\frac{e^{1/h}}{e^{1/h}}$

$=e^{-2/h}$

Since $e^{-1/h}$ tends to 0 as $h\to 0$

$\Rightarrow e^{-2/h}\to 0$ as $h\to 0$

$$\lim_{\substack{h\to 0\\ h>0}}\frac{1-0}{1+0}=1$$

$$Lf'(0)=\lim_{\substack{h\to 0\\ h>0}}\frac{f(x-h)-f(x)}{-h}$$

$$Lf'(0) = \lim_{\substack{h\to 0\\ h>0}} \frac{(-h)\left(\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}\right)}{(-h)} \Rightarrow \lim_{\substack{h\to 0\\ h>0}} \frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{e^{1/h}(e^{-2/h}-1)}{e^{1/h}(e^{-2/h}+1)} \Rightarrow \lim_{\substack{h\to 0\\ h>0}} \frac{0-1}{0+1} = -1$$

Since $Lf'(0) \neq Rf'(0)$, therefore f is not derivable at the point $x=0$

2) Let f be the function defined on R by setting $f(x) = |x-1| + |x+1|$ for all $x \in R$ then f is derivable at $x=1$ and $x-1$

Soln :

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ -(x-1) & \text{if } x-1 < 0 \end{cases}$$

$$|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1 < 0 \end{cases}$$

Case (i) $x+1<0,\ x-1<0$

Case (ii) ~~$x+1\geq 0,\ x+1\geq 0$~~ $x+1 \geq 0,\ x-1<0$

Case (iii) $x-1 \geq 0,\ x+1 \geq 0$

Case (i) :

$x+1<0,\ x-1<0$

$$f(x) = |x-1| + |x+1|$$

$$= -(x-1) + (-(x+1))$$

$$= -x+1-x-1$$

$$= -2x.$$

Case (ii)

$$x+1 \geq 0, \quad x-1 < 0$$

$$f(x) = |x-1| + |x+1|$$

$$= -(x-1) + (x+1) = 1+1 = 2$$

Case (iii)

$$x+1 \geq 0, \quad x+1 \geq 0$$

$$f(x) = (x-1) + (x+1)$$

$$= 2x$$

$$f(x) = \begin{cases} -2x & \text{if } x+1<0 ; x-1<0 \\ 2 & \text{if } x+1 \geq 0 ; x-1<0 \\ 2x & \text{if } x-1 \geq 0 ; x+1 \geq 0 \end{cases} \qquad \begin{matrix} x < -1 , & x < 1 \\ x \geq -1 , & x < 1 \\ x \geq 1 , & x \geq -1 \end{matrix}$$

$$= \begin{cases} -2x & \text{if } x < -1 \\ 2 & \text{if } -1 \leq x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$$

$$Rf'(-1) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{f((-1+h)) - f(-1)}{h}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-1) - f(-1)}{h} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{2-2}{h} = 0$$

$$Lf'(-1) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{-2(-1-h) - 2}{h} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{2h}{-h} = -2.$$

Since $Lf'(-1) \neq Rf'(-1)$ therefore f is not derivable at the point $x=-1$

ALGEBRA OF DERIVATIVES

Theorem:

5m (or) 10m

P.T $(fg)'(x_0) = f'(x_0)g'(x_0)$. let f & g be defined on an interval I. If f and g are derivable at $x_0 \in I$ then so also is fg.

Proof:

$$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = g(x)\left(\frac{f(x)-f(x_0)}{x-x_0}\right)+ f(x_0)\left(\frac{g(x)-g(x_0)}{x-x_0}\right)$$

$$\lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \lim_{x\to x_0} g(x) \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\right) + \lim_{x\to x_0} f(x_0) \lim_{x\to x_0}\left(\frac{g(x)-g(x_0)}{x-x_0}\right)$$

$$(fg)'(x_0) = g(x_0)f'(x_0)+f(x_0)g'(x_0)$$

Hence proved.

Theorem:

Let f be derivable at x_0 and let $f(x_0) \neq 0$, then the function $1/f$ is derivable at x_0 and $(1/f)'(x_0) = -f(x_0) / \{f(x_0)\}^2$.

Proof:

$$\frac{1/f(x) - 1/f(x_0)}{x-x_0} = \frac{f(x_0) - f(x)}{f(x).f(x_0)} \cdot \frac{1}{x-x_0}$$

$$= -\left(\frac{f(x) - f(x_0)}{x-x_0}\right) \cdot \frac{1}{f(x)f(x_0)}$$

$$\lim_{x\to x_0} \frac{\frac{1}{f(x)} - \frac{1}{f(x_0)}}{x-x_0} = \lim_{x\to x_0} -\left(\frac{f(x)-f(x_0)}{x-x_0}\right) \cdot \lim_{x-x_0} \frac{1}{f(x)f(x_0)}$$

$$= -f'(x_0) \frac{1}{(f(x_0))^2}$$

$$= \frac{-f'(x_0)}{(f(x_0))^2}$$

Hence Proved.

CHAIN RULE OF DERIVABILITY

Theorem:

Let f and g be functions such that the range of f is contained in the domain of g. If f is derivable at x_0 and g is derivable at $f(x_0)$, then gof is derivable at x_0 and $(gof)'(x_0) = g'(f(x_0)).f'(x_0)$

Proof:

Since the range of f is contained in the domain of g.

∴ gof has the same domain as that f we required to show that

$$\lim_{h\to 0} \frac{(gof)(x_0+h)-(gof)(x_0)}{h} \text{ exists and}$$

equals $g'(f(x_0)) \cdot f(x_0)$

Let us define
$$F(h) = \begin{cases} \dfrac{g(f(x_0+h))-g(f(x_0))}{f(x_0+h)-f(x_0)} & f(x_0+h) \neq f(x_0) \\ g'(f(x_0)) \text{ if } f(x_0+h) = f(x_0) \end{cases}$$

$$\text{Let } G(h) = \frac{gof(x_0+h)-gof(x_0)}{h}$$

$$\frac{gof(x_0+h)-gof(x_0)}{h} \times \frac{f(x_0+h)-f(x_0)}{f(x_0+h)-f(x_0)}$$

$$\frac{gof(x_0+h) - gof(x_0)}{f(x_0+h) - f(x_0)} \times \frac{f(x_0+h) - f(x_0)}{h}$$

$$= F(h) \cdot \frac{f(x_0+h) - f(x_0)}{h} \text{———} ①$$

whenever $h \neq 0$.

Since f is derivable at x_0, $f'(x_0)$ exists

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h}$$

It is enough to show that $\lim_{h \to 0} F(h)$ exists and equals $g'(f(x_0))$ we have to find that $\lim_{h \to 0} F(h)$ exists and equals $g'(f(x))$. To S.T we proceed as follows.

$$\lim_{k \to 0} \frac{g(f(x_0')+k) - g(f(x_0))}{k}$$ exists & equals

$g'(f(x_0))$. This means that given $\epsilon < 0$ $\exists$ $\delta >\ni$ if $0 < |k| < \delta$ then

$$\left| \frac{g(f(x_0)+k) - g(f(x_0))}{k} - g'(f(x_0)) \right| < \epsilon \quad ②$$

Also since f is derivable at x_0

f is continuous at x_0

$\therefore$ we can find $\delta' > 0 \ni |h| < \delta'$,

then $|f(x_0 + h) - f(x_0)| < \delta \longrightarrow$ ③

Choose any number h satisfies $|h| < \delta'$.

If for this h, $f(x_0+h) = f(x_0)$ then,

$|F(h) - g'(f(x_0))| = 0 < \epsilon \longrightarrow$ ④

From the definition of F.

If $f(x_0+h) \neq f(x_0)$. then writing

$f(x_0+h) - f(x_0) = k$.

we have,

$$F(h) = \frac{g(f(x_0+h)) - g(f(x_0))}{f(x_0+h) - f(x_0)}$$

$$= \frac{g(f(x_0)+k) - g(f(x_0))}{k}$$

So that by ②, we find that

$|F(h) - g'(f(x_0))| < \epsilon$, Provided $|k| < \delta$ — ⑤

(ie) Provides $|f(x_0+h) - f(x_0)| < \delta$, which is true by (3)

Thus from ④ & ⑤ we find that

$|h| < \delta'$, then, $|F(h) - g'(f(x_0))| < \epsilon$

ie, $\lim\limits_{h \to 0} F(h)$ exists and equals $g'(f(x_0))$

This completes the Proof.

Theorem:-

Let f be the function defined on R by setting

$$f(x) = \begin{cases} x^2 \sin 1/x, & \text{when } x \neq 0 \\ 0; & \text{when } x = 0 \end{cases}$$

We shall S.T f is derivable $\forall x \in R$, and that f' is not continuous at $x=0$

Soln:

Since the function $f(x) = 1/x$, $f_2(x) = \sin x$, are both derivable, when $x \neq 0$.

By using chain rule $(f_2 o f_1)$ also derivable when $x \neq 0 \Rightarrow \sin 1/x$ is derivable when $x \neq 0$

Since the function x^2 & $\sin 1/x$ are both derivable when $x \neq 0$.

$\therefore$ The product of product of two functions $x^2 \sin 1/x$ is derivable when $x \neq 0$.

$$f(x) = x^2 \sin 1/x$$

$$f'(x) = x^2 \frac{d}{dx}(\sin 1/x) + (\sin 1/x) \cdot 2x$$

$$= x^2 \left(-1/x^2 \cos 1/x\right) + \sin 1/x \cdot 2x$$

$$= -\cos 1/x + 2x \sin 1/x$$

$$Rf'(0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(0+h)-f(0)}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{h^2 \sin 1/h - 0}{h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} h \cdot \sin 1/h = 0$$

Since $\sin 1/h$ is bounded $\forall\, h \in R$.

$$Lf'(0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(0-h)-f(0)}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{(-h)^2 \cdot \sin(1/-h) - 0}{-h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{h^2 \sin(-1/h)}{-h} = \lim_{\substack{h\to 0\\ h>0}} -h \cdot \sin -1/h = 0$$

$\therefore$ Since $Lf'(0) = Rf'(0)$, so f is derivable at $x=0$ and $f'(0) = Lf'(0) = Rf'(0)$

$$= 0$$

Next we have to S.T f' is not continuous at $x=0$ (ie $\lim_{x\to 0} f'(x)$ not exist) (or) (exist and it is not equal to $f'(0)$)

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} (2x \cdot \sin 1/x - \cos 1/x) = \lim_{x\to 0} 2x \sin 1/x - \lim_{x\to 0} \cos 1/x$$

Since $\lim_{x\to 0} x \cdot \sin 1/x$ exists, but $\lim_{x\to 0} \cos 1/x$ does not exist

Therefore $f(x)$ is not continuous at $x=0$

Inverse function theorem for derivatives

Let $f: X \to Y$ be a function and g be a inverse function of f $g = f^{-1}$. If f is continuous at x_0. Then g is continuous at $f(x_0)$. If f is derivable at x_0, then must g be derivable at $f(x_0)$

We know that f is derivable at x_0 & g is derivable at $f(x_0)$. Then by chain rule $g \circ f$ must be derivable at x_0, & $(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$

Since g, f are inverse to each other $\Rightarrow (g \circ f)'(x) = 1 \; \forall x$ in X.

This means that $g'(f(x_0))\, f'(x_0) = 1$.

$$g'(f(x_0)) = \frac{1}{f(x_0)}$$ is exist when $f'(x_0) \neq 0$

but g is not derivable at $f(x_0)$

This provides a necessary condition for the differentiability of inverse of f

(ie) $f'(x_0) \neq 0$.

10m) Rom

9.

INVERSE FUNCTION THEOREM

Let f be a continuous one-to-one function defined on an interval I and let f be derivable at x_0 with $f'(x) \neq 0$. Then the inverse of a function f is derivable at $f(x_0)$ and its derivatives at $f(x_0)$ is $\frac{1}{f'(x_0)}$

Proof:

Let $f: X \to Y$ be a continuous one-one function. It g be the inverse of f, then g is a function of domain Y and range X (ie $g: Y \to X$) Such that $f(x) = y \Rightarrow g(y) = x$

Let now $f(x_0) = y_0$, So that $g(y_0) = x_0$ and Choose y_0+k be any Point in Y different from y_0. Since f is 1-1, therefore, the exists a unique Point (pre image of y_0+k) Say, x_0+h, different from x_0, Such that $f(x_0+h) = y_0+k$,

we also have $g(y_0+k) = x_0+h$

we thus have,

$$f(x_0) = y_0 \quad f(x_0+h) = y_0+k$$

$$g(y_0) = x_0, \; g(y_0+k) = x_0+h \quad \textcircled{1}$$

$$k \neq 0, \; h \neq 0$$

It can be easily seen that if $k \to 0$, then $h \to 0$. Infact f is derivable at x_0.

$\Rightarrow$ f is continuous at x_0 $\Rightarrow$ g is continuous at y_0 & consequently.

$$\lim_{k \to 0} [g(y_0+k) - g(y_0)] = 0 \Rightarrow \lim_{k \to 0} [(x_0+h) - x_0]$$

$$= \lim_{k \to 0} h = 0$$

Now, let $k \neq 0$ then,

$$\frac{g(y_0+k) - g(y_0)}{k} = \frac{[(x_0+h) - x_0]}{k} = \frac{h}{y_0+k-y_0}$$

$$= \frac{1}{\frac{f(x_0+h) - f(x_0)}{h}}$$

This is permissible since $h \neq 0$

Let $k \to 0$, we have $h \to 0$, which implies that

$$\lim_{k \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= f'(x_0)$$

ie) $$\lim_{k\to 0} \frac{g(y_0+k)-g(y_0)}{k} = 1/f'(x_0)$$

Thus, $g'(y_0)$ exists and equals $\frac{1}{f'(x_0)}$

DAURBOUX'S THEOREM

Let f be defined and derivable on $[a,b]$. If $f'(a)$ $f'(b)<0$, then there exist a real number c between a,b, Such that $f'(c)=0$.

Proof:

Case(1):-

Step-1:- let $f'(b)>0$

Since $f'(a)<0$. Therefore, $\exists$ $h_1>0$ Such that $f(x)<f(a)$ $\forall$ $x \in (a, a+h_1)$

Infact, since f is derivable at a.

Therefore,

$$\lim_{x\to a+0} \frac{f(x)-f(a)}{x-a} = f'(a)$$

Taking $f'(a) = -\epsilon$ $(\epsilon = -f'(a))$ (This being permissible. Since we have $f'(a)<0$)

we can find $h_1>0$ Such that if $a<x<a+h_1$, then $\left|\frac{f(x)-f(a)}{x-a} f'(a)\right| < \epsilon$

From the Second part of the inequality we find that since $f'(a) + \epsilon = 0$. &

$x > a$, therefore, $f(x) < f(a)$

Step-2:-

Since $f'(b) > 0$; Therefore there exists, $h_2 > 0$ such that $f(x) < f(b) \forall x \in (b-h_2, b)$

Infact, since f is derivable at b,

therefore $\lim_{x \to b-0} \frac{f(x) - f(b)}{x-b} = f'(b)$

taking $\epsilon = f'(b)$ (This being permissible since $f'(b) > 0$). we can find $h_2 > 0 \ni$ if $b-h_2 < x < b$ then $\left| \frac{f(x) - f(b)}{x-b} - f'(b) \right| < \epsilon$,

that is $f'(b) - \epsilon < \frac{f(x) - f(b)}{x-b} < f'(b) + \epsilon$

From the first part of the above inequality. we find that since

$f'(b) - \epsilon = 0$ & $x < b$

Therefore, $f(x) < f(b)$

Step-3:-

Since f is derivable on $[a, b]$ = Continuous on $[a, b]$ $\Rightarrow$ attains supremum and infimum in $[a, b]$. Now by Step 1. $\inf f \neq f(a)$, & by step2 $\inf f \neq f(b)$

This means that f does not have its infimum at the end points a, b.

Therefore, there exists, a real number c in (a, b) such that inf f is attained at c.

Step4:-

$f'(c) \ngtr 0$. For if $f'(c) > 0$, then $Lf'(c) > 0$ and by step2, we can find $h_3 > 0$ such that $f(x) < f(c)$ $\forall x \in (c-h_3, c)$ & this contradicts the fact that $f(c)$ is infimum of f on $[a, b]$

Hence $f'(c) \ngtr 0$

Step-5:-

$f'(c) \nless 0$. For if $f'(c) < 0$ then, $Rf'(c) < 0$ & as in step1. we can find $h_4 > 0$ such that $f(x) < f(c)$ $\forall x \in (c, c+h_4)$ & this contradicts the fact that f attains infimum at c. Hence $f'(c) \nless 0$

We have proved that $f'(c) \nless 0$, $f'(c) \ngtr 0$. by the law of trichitomy we have $f'(c) = 0$

Case-(ii)

Let $f'(a) > 0$ & $f'(b) < 0$. If g be a function of (ie, $g(x) = -f(x)$) then g is derivable on $[a, b]$, $g'(a) < 0$, $g'(b) > 0$.

So that by case (i) $\exists$ a real number $d \in (a, b)$ such that $g'(d) = 0$

Now $f'(d) = -g'(d) = 0$

CHAIN RULE OF DERIVABILITY

Theorem:

Let f and g be functions such that the range of f is contained in the domain of g. If f is derivable at x_0 and g is derivable at $f(x_0)$, then gof is derivable at x_0 and $(gof)'(x_0) = g'(f(x_0)).f'(x_0)$

Proof:

Since the range of f is contained in the domain of g.

∴ gof has the same domain as that f we required to show that

$$\lim_{h \to 0} \frac{(gof)(x_0+h) - (gof)(x_0)}{h} \quad \text{exists and}$$

equals $g'(f(x_0)) . f(x_0)$

Let us define
$$F(h) = \begin{cases} \dfrac{g(f(x_0+h)) - g(f(x_0))}{f(x_0+h) - f(x_0)} & f(x_0+h) \neq f(x_0) \\ g'(f(x_0)) & \text{if } f(x_0+h) = f(x_0) \end{cases}$$

Let
$$G(h) = \frac{gof(x_0+h) - gof(x_0)}{h}$$

$$\frac{gof(x_0+h) - gof(x_0)}{h} \times \frac{f(x_0+h) - f(x_0)}{f(x_0+h) - f(x_0)}$$

$$\frac{g\circ f(x_0+h) - g\circ f(x_0)}{f(x_0+h) - f(x_0)} \times \frac{f(x_0+h) - f(x_0)}{h}$$

$$= F(h) \cdot \frac{f(x_0+h) - f(x_0)}{h} \quad \text{———} ①$$

whenever $h \neq 0$

Since f is derivable at x_0, $f'(x_0)$ exists

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x)}{h}$$

It is enough to show that $\lim_{h\to 0} F(h)$ exists and equals $g'(f(x_0))$ we have to find that $\lim_{h\to 0} F(h)$ exists and equals $g'(f(x_0))$. To S.T we proceed as follows.

$$\lim_{k\to 0} \frac{g(f(x_0')+k) - g(f(x_0))}{k}$$ exists & equals

$g'(f(x_0))$. This means that given $\epsilon < 0 \ \exists \ \delta > 0$

if $0 < |k| < \delta$ then $\left| \frac{g(f(x_0)+k) - g(f(x_0))}{k} - g'(f(x_0)) \right| < \epsilon$ ②

Also since f is derivable at x_0

f is continuous at x_0

∴ we can find $\delta' > 0 \ni |h| < \delta'$,

then $|f(x_0+h) - f(x_0)| < \delta \longrightarrow ③$

Choose any number h satisfies $|h| < \delta'$.

If for this h, $f(x_0+h) = f(x_0)$ then,

$|F(h) - g'(f(x_0))| = 0 < \epsilon \longrightarrow ④$

From the definition of F.

If $f(x_0+h) \neq f(x_0)$. then writing

$f(x_0+h) - f(x_0) = k$.

We have,

$$F(h) = \frac{g(f(x_0+h)) - g(f(x_0))}{f(x_0+h) - f(x_0)}$$

$$= \frac{g(f(x_0)+k) - g(f(x_0))}{k}$$

So that by ②, we find that

$|F(h) - g'(f(x_0))| < \epsilon$, Provided $|k| < \delta$ — ⑤

(ie) Provides $|f(x_0+h) - f(x_0)| < \delta$, which is true by (3)

Thus from ④ & ⑤ we find that

$|h| < \delta'$, then, $|F(h) - g'(f(x_0))| < \epsilon$

ie, $\lim_{h \to 0} F(h)$ exists and equals $g'(f(x_0))$

This completes the Proof.

Theorem:-

Let f be the function defined on R by setting

$$f(x) = \begin{cases} x^2 \sin 1/x, & \text{when } x \neq 0 \\ 0; & \text{when } x = 0 \end{cases}.$$

We shall S.T f is derivable $\forall x \in R$. and that f' is not Continuous at $x=0$

Soln:

Since the function $f_1(x) = 1/x$, $f_2(x) = \sin x$, are both derivable, when $x \neq 0$.

By using chain rule $(f_2 o f_1)$ also derivable when $x \neq 0 \Rightarrow \sin 1/x$ is derivable when $x \neq 0$

Since the function x^2 & $\sin 1/x$ are both derivable when $x \neq 0$.

∴ The product of product of two functions $x^2 \sin 1/x$ is derivable when $x \neq 0$.

$$f(x) = x^2 \sin 1/x$$

$$f'(x) = x^2 \frac{d}{dx}(\sin 1/x) + (\sin 1/x) \cdot 2x$$

$$= x^2 \left(-1/x^2 \cos 1/x\right) + \sin 1/x \cdot 2x$$

$$= -\cos 1/x + 2x \sin 1/x$$

$$Rf'(0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(0+h)-f(0)}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{h^2 \sin 1/h - 0}{h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} h \cdot \sin 1/h = 0$$

Since $\sin 1/h$ is bounded $\forall\, h \in R$.

$$Lf'(0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(0-h)-f(0)}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{(-h)^2 \cdot \sin(-1/h) - 0}{-h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{h^2 \sin(-1/h)}{-h} = \lim_{\substack{h\to 0\\ h>0}} -h \cdot \sin -1/h = 0$$

∴ Since $Lf'(0) = Rf'(0)$, so f is derivable at $x=0$ and $f'(0) = Lf'(0) = Rf'(0)$

$= 0$

Next we have to S.T f' is not continuous at $x=0$ (ie $\lim_{x\to 0} f'(x)$ not exist) (or) (exist and it is not equal to $f'(0)$)

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} (2x \cdot \sin 1/x - \cos 1/x) = \lim_{x\to 0} 2x \sin 1/x - \lim_{x\to 0} \cos 1/x$$

Since $\lim_{x\to 0} x \cdot \sin 1/x$ exists, but $\lim_{x\to 0} \cos 1/x$ does not exist

Therefore $f'(x)$ is not continuous at $x=0$

Inverse function theorem for derivatives

Let $f : X \to Y$ be a function and g be a inverse function of f $g = f^{-1}$. If f is continuous at x_0. Then g is continuous at $f(x_0)$. If f is derivable at x_0, then must g be derivable at $f(x_0)$?

We know that f is derivable at x_0 & g is derivable at $f(x_0)$. Then by chain rule $g \circ f$ must be derivable at x_0, & $(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$

Since g, f are inverse to each other $\Rightarrow (g \circ f)'(x) = 1 \ \forall \ x$ in X.

This means that $g'(f(x_0))\, f'(x_0) = 1$.

$$g'(f(x_0)) = \frac{1}{f'(x_0)}$$ is exist when $f'(x_0) \neq 0$

but g' is not derivable at $f(x_0)$

This provides a necessary condition for the differentiability of inverse of f

(ie) $f'(x_0) \neq 0$.

10m) Rom

9.

INVERSE FUNCTION THEOREM

Let f be a continuous one-to-one function defined on an interval I and let f be derivable at x_0 with $f'(x_0) \neq 0$. Then the inverse of a function f is derivable at $f(x_0)$ and its derivatives at $f(x_0)$ is $\frac{1}{f'(x_0)}$

Proof:

Let $f : X \rightarrow Y$ be a continuous one-one function. If g be the inverse of f, then g is a function of domain Y and range X (ie $g : Y \rightarrow X$) such that

$f(x) = y \Rightarrow g(y) = x$

Let now $f(x_0) = y_0$, so that $g(y_0) = x_0$ and choose $y_0 + k$ be any point in Y different from y_0. Since f is 1-1, therefore, there exists a unique point (pre image of y_0+k) say, $x_0 + h$, different from x_0, such that $f(x_0+h) = y_0+k$,

we also have $g(y_0+k) = x_0+h$

we thus have,

$$f(x_0) = y_0 \quad f(x_0+h) = y_0+k$$

$$g(y_0) = x_0, \; g(y_0+k) = x_0+h \;\; \&$$

$$k \neq 0, \; h \neq 0$$

It can be easily seen that if $k \to 0$, then $h \to 0$. Infact f is derivable at x_0.

$\Rightarrow$ f is continuous at x_0 $\Rightarrow$ g is continuous at y_0 & consequently.

$$\lim_{k \to 0} [g(y_0+k) - g(y_0)] = 0 \Rightarrow \lim_{k \to 0} [(x_0+h) - x_0]$$

$$= \lim_{k \to 0} h = 0$$

Now, let $k \neq 0$ then,

$$\frac{g(y_0+k) - g(y_0)}{k} = \frac{[(x_0+h) - x_0]}{k} = \frac{h}{y_0+k-y_0}$$

$$= \frac{1}{\frac{f(x_0+h) - f(x_0)}{h}}$$

This is permissible since $h \neq 0$

Let $k \to 0$, we have $h \to 0$, which implies that

$$\lim_{k \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= f'(x_0)$$ (Since $f'(x_0) \neq 0$)

ie) $\lim_{k \to 0} \frac{g(y_0+k)-g(y_0)}{k} = 1/f'(x_0)$

Thus, $g'(y_0)$ exists and equals $\frac{1}{f'(x_0)}$

DAURBOUX'S THEOREM

Let f be defined and derivable on $[a,b]$. If $f'(a)$ $f'(B) < 0$, then there exist a real number c between a, b, Such that $f'(c) = 0$.

Proof :

Case(1):-

Step-1:- let $f'(b) > 0$

Since $f'(a) < 0$. Therefore, $\exists\ h_1 > 0$ Such that $f(x) < f(a)\ \forall\ x \in (a, a+h_1)$

Infact, Since f is derivable at a.

Therefore,

$$\lim_{x \to a+0} \frac{f(x)-f(a)}{x-a} = f'(a)$$

Taking $f'(a) = -(\epsilon = -f'(a))$ (This being permissible. Since we have $f'(a) < 0$)

We can find $h_1 > 0$ Such that if $a < x < a+h_1$, then $\left| \frac{f(x)-f(a)}{x-a} f'(a) \right| < \epsilon$

From the Second part of the inequality we find that Since $f'(a) + \epsilon = 0$. G

DAURBOUX'S THEOREM

Let f be defined and derivable on $[a, b]$. If $f'(a)$ $f'(b) < 0$, then there exist a real number c between a, b, such that $f'(c) = 0$.

Proof:

Case(1):-

Step-1:- let $f'(b) > 0$

Since $f'(a) < 0$. Therefore, ∃ $h_1 > 0$ such that $f(x) < f(a)\ \forall\ x \in (a, a+h_1)$

Infact, since f is derivable at a.

Therefore,

$$\lim_{x \to a+0} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Taking $f'(a) = -(\epsilon = -f'(a))$ (This being permissible. Since we have $f'(a) < 0$)

We can find $h_1 > 0$ such that if $a < x < a + h_1$, then $\left| \frac{f(x) - f(a)}{x - a} f'(a) \right| < \epsilon$

From the second part of the inequality we find that since $f'(a) + \epsilon = 0$. ϵ_1

$x > a$, therefore, $f(x) < f(a)$

Step-2:-

Since $f'(b) > 0$, Therefore there exists $h_2 > 0$ such that $f(x) < f(b)$ ∀ $x \in (b-h_2, b)$

Infact, since f is derivable at b,

therefore $\lim_{x \to b-0} \frac{f(x) - f(b)}{x - b} = f'(b)$

taking $\epsilon = f'(b)$ (This being permissible since $f'(b) > 0$). we can find $h_2 > 0$ ∋

if $b - h_2 < x < b$ then $\left| \frac{f(x) - f(b)}{x - b} - f'(b) \right| < \epsilon$,

that is $f'(b) - \epsilon < \frac{f(x) - f(b)}{x - b} < f'(b) + \epsilon$

From the first part of the above inequality. we find that since

$f'(b) - \epsilon = 0$ & $x < b$

$$0 < \frac{f(x) - f(b)}{x - b}$$

$$0 < \frac{f(b) - f(x)}{(b - x)}$$

$$f(x) < f(b)$$

Step - 3 :-

Since f is derivable on $[a,b]$ = continuous on $[a,b]$ ⇒ attains supremum and infimum in $[a,b]$. Now by Step 1. $\inf f \neq f(a)$, & by step2 $\inf f \neq f(b)$

This means that f does not have its infimum at the end points a, b.

Therefore, there exists. a real number c in (a, b) such that inf f is attained at c.

Step4:-

$f'(c) \ngtr 0$. For if $f'(c) > 0$, then $Lf'(c) > 0$ (may) and by step2, we can find $h_3 > 0$ such that $f(x) < f(c)$ $\forall x \in (c-h_3, c)$ & this contradicts the fact that f(c) ~~attains~~ is infimum of f on [a, b]

Hence $f'(c) \ngtr 0$

Step-5:-

$f'(c) \nless 0$. For if $f'(c) < 0$ then, $Rf'(c) < 0$ & as in step1. we can find $h_4 > 0$ such that $f(x) < f(c)$ $\forall x \in (c, c+h_4)$ & this contradicts the fact that f attains infimum at c. Hence $f'(c) < 0$

we have proved that $f'(c) \nless 0$, $f'(c) \ngtr 0$.

by the law of trichitomy we have $f'(c) = 0$

Case-(ii)

Let $f'(a) > 0$ & $f'(b) < 0$. If g be a function f (ie, $g(x) = -f(x)$) then g is derivable on [a, b], $g'(a) < 0$, $g'(b) > 0$.

So that by case (i) ∃ a real number $d \in (a, b)$ such that $g'(d) = 0$

rollaries

1) If f be defined and derivable on $[a,b]$ and if k be any number between $f'(a)$ and $f'(b)$. Then there exists a real number c between a and b, such that $f'(c)=k$

proof:

Let f be derivable on $[a,b]$

define $g(x)= f(x)-kx \ \forall \ x\in [a,b]$

Now g is derivable on $[a,b]$

$g'(x)= f'(x)-k$

$g'(a)= f'(a)-k \qquad g'(b)=f'(b)-k$

Since k lies between $f'(a)$ & $f'(b)$

$\Rightarrow \ f'(a)-k<0$ & $f'(b)-k>0$

(or) $f'(a)-k>0$ & $f'(b)-k<0$

$\Rightarrow$ $g'(a)$ and $g'(b)$ are of opposite signs.

Since g is derivable on $[a,b]$ & $g'(a)g'(b)<0$. Therefore $\exists$ a real number c b/w a & b such that $g'(c)=0$.

$\therefore \ f'(c)-k=0$

$\Rightarrow f'(c)=k$ Hence Proved.

2) If f is defined and derivable on an interval, The range of f' is an interval

Proof:

Let f defined & derivable on ~~[a,b]~~ X

to prove range of f' is an interval.

et range of f' is y. To prove Y is an interval.

Let p, q be two distinct points of Y.

$\therefore$ $\exists$ some $a, b \in X$ such that $f'(a) = p$

and $f'(b) = q$

We can assume that $a < b$.

Since X is an interval & $a \in X$, $b \in X$

Therefore $[a,b] \subset X$

f derivable on $X \Rightarrow f$ derivable on $[a,b]$

If r be any real number between

p, q (i.e r lies b/w $f'(a), f'(b)$)

Then $\exists$ c between a, b such that

$f'(c) = r \quad \Rightarrow \quad r \in Y$

$\therefore$ Thus we find that if p and q are in Y. Then every number between p, q is in Y. $\Rightarrow Y$ is an interval.

1. ROLLE'S THEOREM

The following theorem, known as Rolle's theorem, is one of the simplest yet one of the most important theorems of real analysis. It is at the root of all mean value theorems, Taylor's theorem and Maclaruin's theorem which we propose to discuss in the present chapter.

Theorem 1-1. *Let f be a function defined on $[a, b]$ such that*

(*i*) *f is continuous on $[a, b]$;*

(*ii*) *f is derivable on $]a, b[$;*

(*iii*) *$f(a) = f(b)$.*

Then there exists a real number c between a and b such that $f'(c) = 0$.

Proof. Since f is continuous on $[a, b]$, and since every function that is continuous on a closed interval is bounded therein, therefore, f must be bounded on $[a, b]$. Let $\sup f = M$, $\inf f = m$.

Two different cases arise :

(1) $M = m$. Then f is constant over $[a, b]$ and consequently, $f'(x) = 0$, for all x in $[a, b]$.

(2) $M \neq m$. Since $f(a) = f(b)$, therefore, at least one of the numbers M and m is different from $f(a)$ and therefore, also from $f(b)$. For the sake of definiteness, assume that $M \neq f(a)$.

Since every function that is continuous on a closed interval attains its supremum, therefore, there exists some real number c in $[a, b]$, such that $f(c) = M$. Further, since $f(a) \neq M \neq f(b)$, therefore, c is different from both a and b. This means that c lies in the open interval $]a, b[$.

Since $f(c)$ is the supremum of f on $[a, b]$, therefore,

$$f(x) \leq f(c) \text{ for all } x \text{ in } [a, b]. \qquad \ldots(i)$$

In particular,

$$f(c-h) \leq f(c),$$

for all positive real numbers h such that $c - h$ lies in $[a, b]$.

This means that

$$\frac{f(c-h)-f(c)}{-h} \geq 0,$$

for all positive real numbers h such that $c - h$ lies in $[a, b]$.

Taking limits as $h \to 0$ and observing that since $f'(x)$ exists at each point of $]a, b[$, and therefore, in particular at $x = c$, we have

$$f(c-h) \leq f(c)$$

$$\frac{f(c-h)-f(c)}{h} \leq 0$$

$$\frac{f(c-h)-f(c)}{-h} \geq 0$$

$$\lim_{h \to 0} \frac{f(c-h)-f(c)}{-h} = Lf'(c)$$

$h>0$

$$\Rightarrow Lf'(c) \geq 0 \quad \text{——} \ (ii)$$

From (i) we have

$$f(c+h) \leq f(c)$$

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

where h be a positive number

& $c+h$ lies in $[a, b]$.

$$\Rightarrow \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\Rightarrow \quad R f'(c) \leq 0. \qquad \text{——— (iii)}$$

Since $f'(x)$ exists at $x = c$

$$L f'(c) = f'(c) = R f'(c) \qquad \text{——— (iv)}$$

$$\underset{\geq 0}{\downarrow} \qquad\qquad \underset{\leq 0}{\downarrow} \quad \text{From (ii), (iii) \& (iv)}$$

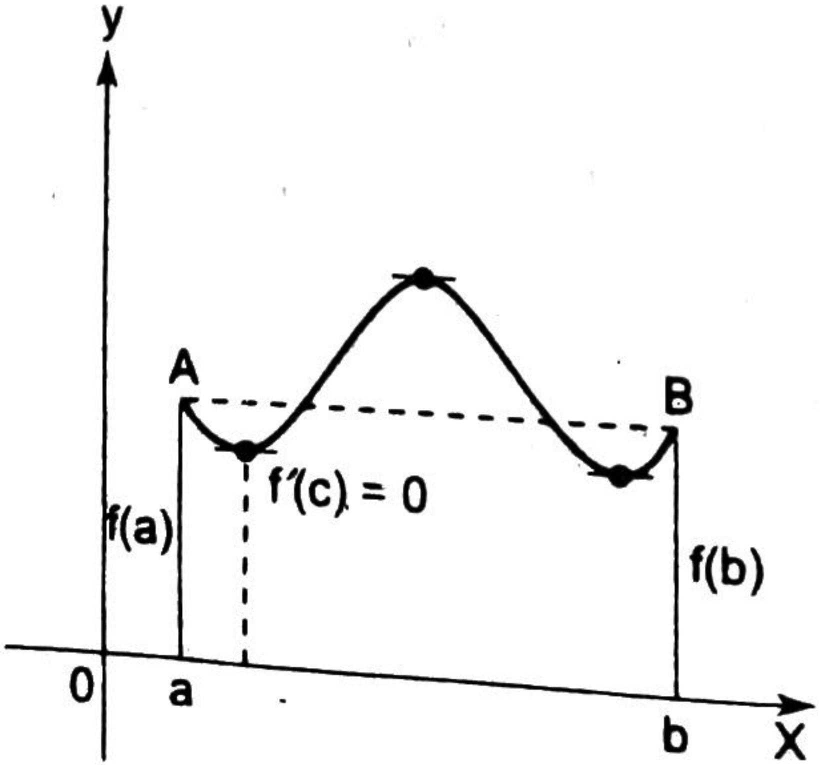
$$\Rightarrow f'(c) \geq 0 \quad \& \; f'(c) \leq 0$$

$$\Rightarrow f'(c) = 0$$

From (*ii*), (*iii*) and (*iv*), we find that $f'(c) = 0$.

The case $M = f(a) \neq m$ can be disposed of in the same manner as above. ☐

Remark. Rolle's theorem ensures us about the existence of *at least one* real number c such that $f'(c) = 0$. It does not say anything about the existence or otherwise of more than ne such number. As we shall see in problems, for a given f, there may exist several numbers c such that $f'(c) = 0$.



2. LAGRANGE'S MEAN VALUE THEOREM

Theorem 2-1. *Let f be a function defined on $[a, b]$, such that*

(i) *f is continuous on $[a, b]$,*

and (ii) *f is derivable on $]a, b[$.*

Then there exists a real number $c \in]a, b[$ such that

$$f(b) - f(a) = (b - a)\ f'(c).$$

Proof. Let F be a function defined on $[a, b]$ by setting

$$F(x) = f(x) + Ax, \text{ for all } x \text{ in } [a, b], \quad \text{...}(i)$$

where A is a constant to be suitably chosen.

Now,

(1) Since f is continuous on $[a, b]$ and the function $x \to Ax$ is continuous on $[a, b]$, therefore, F is continuous on $[a, b]$.

(2) Also, since f is derivable on $]a, b[$ and the function $x \to Ax$ is derivable on $]a, b[$, therefore, F is derivable on $]a, b[$.

(3) Let us choose A so that $F(a) = F(b)$. This gives us

$$-A = \frac{f(b) - f(a)}{b - a}. \quad \text{...}(ii)$$

From (1), (2) and (3) above, we find that F satisfies all the conditions of Rolle's theorem on $[a, b]$, and consequently, there exists a real number c in $]a, b[$ such that $F'(c) = 0$. From (i), this gives

$$f'(c) + A = 0. \quad \text{...}(iii)$$

From (ii) and (iii), we have (on equating the values of A)

$$\frac{f(b) - f(a)}{b - a} = f'(c),$$

i.e.,

$$f(b) - f(a) = (b - a)\ f'(c).$$

☐

Remark. If in the above theorem, we take $b = a + h$, then c can be written as $a + \theta h$, where θ is some real number such that $0 < \theta < 1$. Lagrange's theorem then reads as follows :

Let f be defined and continuous on $[a, a + h]$ and derivable on $]a, a + h[$. Then for some real number θ $(0 < \theta < 1)$,

$$f(a + h) - f(a) = h f'(a + \theta h).$$

Corollary. *If f is defined and continuous on $[a, b]$ and is derivable on $]a, b[$, and if $f'(x) = 0$ for all x in $]a, b[$, then $f(x)$ has a constant value throughout $[a, b]$.*

Proof. Let c be any point of $]a, b]$. Then (*i*) f is continuous on $[a, c]$;

(*ii*) f is derivable on $]a, c[$.

Since f satisfies all the conditions of Lagrange's mean value theorem on $[a, c]$, therefore, there exists a real number d between a and c such that

$$f(c) - f(a) = (c - a) f'(d).$$

Since $f'(x) = 0$ for all x in $]a, b[$, therefore,in particular, $f'(d) = 0$, and consequently $f(c) - f(a) = 0$. Since c is any point of $]a, b]$, therefore, it follows that $f(x) = f(a)$ for all x in $]a, b]$.

Hence $f(x)$ has a constant value throughout $[a, b]$. □

Remark. The hypothesis of Lagrange's theorem cannot be weakened. To see this, consider the following examples :

1. Let f be the function defined on $[1, 2]$ by setting

$$f(1) = 2,$$

$$f(x) = x^2, \text{ whenever } 1 < x < 2,$$

$$f(2) = 1.$$

It can be easily seen that f is continuous as well as derivable on $]1, 2[$ but is not continuous at $x = 1$ and at $x = 2$. That is, the first of the two conditions is violated.

Also, $$\frac{f(2) - f(1)}{2 - 1} = -1,$$

$$f'(x) = 2x, \text{ whenever } 1 < x < 2,$$

so that $f'(x)$ is positive for all x in $]1, 2[$.

Thus $\dfrac{f(2) - f(1)}{2 - 1} \neq f'(x)$ for any x in $]1, 2[$.

2. Let f be the function defined on $[-1, 2]$ by setting

$$f(x) = |x|, \text{ for all } x \text{ in } [-1, 2].$$

Here f is continuous on $[-1, 2]$, and derivable at all points of $[-1, 2[$ except at $x = 0$ (so that second of the two conditions is violated).

Now $$f'(x) = -1, \text{ if } x \in]-1, 0[,$$

$$= 1, \text{ if } x \in]0, 2[.$$

Also, $$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{1}{3},$$

so that $\dfrac{f(2)-f(-1)}{2-(-1)} \neq f'(x)$ for any x in $]-1, 2[$.

2·1. Geometrical interpretation of Lagrange's theorem

Interpreted geometrically, Lagrange's mean value theorem says (Fig. 7-2) that the tangent to the graph of f at some suitable point between a and b is parallel to the chord joining the points on the graph with abscissae a and b.

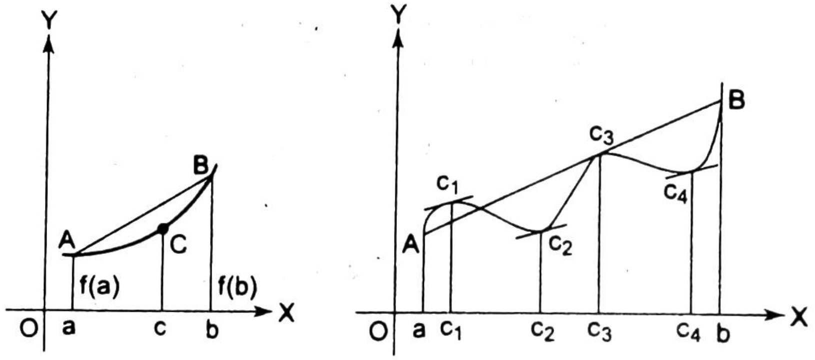


Fig. 7–2.

Example 1. *Find a 'c' of Lagrange's mean value theorem if*

$$f(x) = x(x-1)(x-2);\ a = 0,\ b = \tfrac{1}{2}.$$

Solution. $$f(x) = x(x-1)(x-2).$$

$\therefore$ $$f(0) = 0,\ f\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right) = \tfrac{3}{8}. \qquad \text{...(1)}$$

Also, $$f'(x) = 3x^2 - 6x + 2. \qquad \text{...(2)}$$

Putting $a = 0, b = \frac{1}{2}$ in

$$f(b) - f(a) = (b-a)\, f'(c),$$

we have from (1) and (2),

$$\tfrac{3}{8} = \tfrac{1}{2}(3c^2 - 6c + 2),$$

or $$12c^2 - 24c + 5 = 0.$$

$$\therefore \quad c = \frac{6 \pm \sqrt{21}}{6}.$$

Since $\frac{6+\sqrt{21}}{6}$ lies outside $]0, \frac{1}{2}[$, therefore, this value of c has to be discarded.

Hence the required value of c is $\left(6-\sqrt{21}\right)/6$.

Example 2. *Let f be defined and continuous on $[a - h, a + h]$, and derivable on $]a - h, a + h[$. Prove that there is a real number θ between 0 and 1 for which*

$$f(a + h) - 2f(a) + f(a - h) = h\{ f'(a + \theta h) - f'(a - \theta h)\}.$$

Solution. Let F be the function defined on [0, 1] by setting

$$F(t) = f(a + ht) + f(a - ht), \text{ for all } t \in [0, 1].$$

Then F is continuous on [0, 1] and derivable on [0, 1[. By Lagrange's mean value theorem, there exists a number θ between 0 and 1 such that

$$F(1) - F(0) = (1 - 0)\ F'(\theta),$$

i.e., $$f(a + h) + f(a - h) - 2f(a) = h\{ f'(a + \theta h) - f'(a - \theta h)\}.$$

PROBLEMS

1. Verify the hypotheses and the conclusion of Lagrange's mean value theorem for the function f defined on $[a, b]$ in each of the following cases :
   (a) $f(x) = x^3$; $a = -2, b = 1$.
   (b) $f(x) = 1/x$; $a = 1, b = 4$.
   (c) $f(x) = x^n$ (n being a positive integer) ; $a = -1, b = 1$.
   (d) $f(x) = \cos x$; $a = 0, b = \pi/2$.
2. Examine the validity of the hypotheses and the conclusion of Lagrange's mean value theorem for the function f defined on $[a, b]$ in each of the following cases :
   (a) $f(x) = |x|$; $a = -2, b = 1$.
   (b) $f(x) = 1/x$; $a = -1, b = 2$.
   (c) $f(x) = x^{1/3}$; $a = -1, b = 1$.
   (d) $f(x) = 1 + x^{2/3}$; $a = -8, b = 1$.
3. Find the number (numbers) θ that appears in the conclusion of Lagrange's mean value theorem in each of the following cases :

3. CAUCHY'S MEAN VALUE THEOREM

Theorem 3-1. *Let f and g be functions defined on $[a, b]$ such that*

(i) f and g are continuous on $[a, b]$,

(ii) f and g are derivable on $]a, b[$, and

(iii) $g'(x)$ does not vanish at any point of $]a, b[$.

Then there exists a real number $c \in]a, b[$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Let us first observe that as a consequence of condition (*iii*), $g(a) \neq g(b)$. For, if $g(a)$ were equal to $g(b)$, then the function g would satisfy all the conditions of Rolle's theorem, and consequently for some x in $]a, b[$ we would have $g'(x) = 0$.

Consider the function F defined on $[a, b]$ by setting

$$F(x) = f(x) + Ag(x), \text{ for all } x \text{ in } [a, b], \qquad \ldots(i)$$

where A is a constant to be suitably chosen. Now,

(1) Since f and g are continuous on $[a, b]$, therefore, F is continuous on $[a, b]$.

(2) Also, since f and g are derivable on $]a,b[$, therefore, F is derivable on $]a, b[$.

(3) Let us choose A so that $F(a) = F(b)$. This gives us

$$-A = \frac{f(b)-f(a)}{g(b)-g(a)}, \qquad \ldots(ii)$$

division by $g(b) - g(a)$ being permissible since we have already shown that $g(b) \neq g(a)$.

From (1), (2) and (3), we find that F satisfies all the conditions of Rolle's theorem on $[a, b]$, and consequently, there exists a real number c in $]a, b[$ such that $F'(c) = 0$. From (i), this gives

$$f'(c) + A\,g'(c) = 0,$$

or

$$-A = \frac{f'(c)}{g'(c)}, \qquad \ldots(iii)$$

division by $g'(c)$ being permissible since $g'(x)$ is not zero for any x in $]a, b[$.

From (ii) and (iii), we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}. \qquad \square$$

Remarks. 1. If we put $b = a + h$, then c can be written as $a + \theta h$ where θ is some real number such that $0 < \theta < 1$. The above theorem then reads as follows :

If f and g are continuous on $[a, a + h]$ and are derivable on $]a, a + h[$, and if $g'(x)$ does not vanish for any x in $]a, a + h[$, then there exists a real number θ between 0 and 1, such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

2. If we take $g(x) = x$, for all x in $[a, b]$, then Cauchy's mean value theorem yields Lagrange's mean value theorem as a particular case.

3. The reader might be tempted to prove Cauchy's mean value theorem by applying Lagrange's mean value theorem to the functions f and g. It can be easily seen that the desired result cannot be obtained in this manner. In fact, we would thus obtain that

LMVT problems

1) find a 'c' of LMVT if $f(x) = x(x-1)(x-2)$

$a=0 \quad b=1/2$

Soln: $f(x) = x(x-1)(x-2)$ or $(x^2-x)(x-2)$

$$= x^3-2x^2-x^2+2x$$

$$= x^3-3x^2+2x$$

$f(a)=f(0)=0 \qquad f(b)=f(\frac{1}{2}) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)$

$$= \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)$$

$$= \frac{3}{8}$$

$f'(x) = 3x^2-6x+2$

put $a=0$, $b=\frac{1}{2}$ in LMVT

$$f(b)-f(a) = (b-a)f'(c)$$

$$\frac{3}{8} - 0 = \left(\frac{1}{2}-0\right)f'(c)$$

$$\frac{3}{8} = \frac{1}{2}(3c^2-6c+2)$$

$$3 = \frac{8}{2}(3c^2-6c+2)$$

$$3 = 4(3c^2-6c+2)$$

$$12c^2-24c+8-3=0$$

$$12c^2-24c+5=0$$

$$\begin{array}{r} 24\times24 \\ \hline 96 \\ 48 \\ \hline 576 \end{array}$$

So~~luti~~ $\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{24\pm\sqrt{576-4(12)(5)}}{2(12)}$

$= \frac{24\pm\sqrt{576-240}}{24} = \frac{24\pm\sqrt{336}}{24} = \frac{24\pm\sqrt{21\times16}}{24}$

$= \frac{24\pm4\sqrt{21}}{24} = \frac{6\pm\sqrt{21}}{6}$

Since $\frac{6+\sqrt{21}}{6}$ lies outside $(0,\frac{1}{2})$ $(\sqrt{21}\approx 4.\ldots)$

$\therefore$ c has to be $(6-\sqrt{21})/6$

Example 2. *Let f be defined and continuous on $[a - h, a + h]$, and derivable on $]a - h, a + h[$. Prove that there is a real number θ between 0 and 1 for which*

$$f(a + h) - 2f(a) + f(a - h) = h\{f'(a + \theta h) - f'(a - \theta h)\}.$$

Solution. Let F be the function defined on [0, 1] by setting

$$F(t) = f(a + ht) + f(a - ht), \text{ for all } t \in [0, 1].$$

Then F is continuous on [0, 1] and derivable on [0, 1[. By Lagrange's mean value theorem, there exists a number θ between 0 and 1 such that

$$F(1) - F(0) = (1 - 0)\ F'(\theta),$$

$$F(1) - F(0) = (1-0)\, f'(\theta) \quad \text{——} \quad ①$$

Since $F(1) = f(a+h) + f(a-h)$

$F(0) = f(a) + f(a) = 2f(a)$

~~$\therefore F(1) - F(0) = (1-0)\, F'(\theta)$~~

$$F'(t) = f'(a+ht)(h) + f'(a-ht)(-h)$$

$$= h\left[f'(a+ht) - f'(a-ht)\right]$$

$$F'(\theta) = h\left[f'(a+h\theta) - f'(a-h\theta)\right]$$

$\therefore F(1) - F(0) = (1-0)\, F'(\theta)$ will becomes

$$f(a+h) + f(a-h) - 2f(a) = h\left[f'(a+\theta h) - f'(a-\theta h)\right]$$

3. CAUCHY'S MEAN VALUE THEOREM

Theorem 3-1. *Let f and g be functions defined on* $[a, b]$ *such that*

(i) f and g are continuous on $[a, b]$,

(ii) f and g are derivable on $]a, b[$, *and*

(iii) $g'(x)$ *does not vanish at any point of* $]a, b[$.

Then there exists a real number $c \in \,]a, b[$ *such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Let us first observe that as a consequence of condition (*iii*), $g(a) \neq g(b)$. For, if $g(a)$ were equal to $g(b)$, then the function g would satisfy all the conditions of Rolle's theorem, and consequently for some x in $]a, b[$ we would have $g'(x) = 0$.

Consider the function F defined on $[a, b]$ by setting

$$F(x) = f(x) + Ag(x), \text{ for all } x \text{ in } [a, b], \qquad \ldots(i)$$

where A is a constant to be suitably chosen. Now,

(1) Since f and g are continuous on $[a, b]$, therefore, F is continuous on $[a, b]$.

(2) Also, since f and g are derivable on $]a,b[$, therefore, F is derivable on $]a, b[$.

(3) Let us choose A so that $F(a) = F(b)$. This gives us

$$-A = \frac{f(b) - f(a)}{g(b) - g(a)}, \qquad \text{...(ii)}$$

division by $g(b) - g(a)$ being permissible since we have already shown that $g(b) \neq g(a)$.

From (1), (2) and (3), we find that F satisfies all the conditions of Rolle's theorem on $[a, b]$, and consequently, there exists a real number c in $]a, b[$ such that $F'(c) = 0$. From (*i*), this gives

$$f'(c) + A\,g'(c) = 0,$$

or

$$-A = \frac{f'(c)}{g'(c)}, \qquad \text{...(iii)}$$

division by $g'(c)$ being permissible since $g'(x)$ is not zero for any x in $]a, b[$.

From (*ii*) and (*iii*), we have

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \qquad \square$$

Remarks. 1. If we put $b = a + h$, then c can be written as $a + \theta h$ where θ is some real number such that $0 < \theta < 1$. The above theorem then reads as follows :

If f and g are continuous on $[a, a + h]$ and are derivable on $]a, a + h[$, and if $g'(x)$ does not vanish for any x in $]a, a + h[$, then there exists a real number θ between 0 and 1, such that

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

2. If we take $g(x) = x$, for all x in $[a, b]$, then Cauchy's mean value theorem yields Lagrange's mean value theorem as a particular case.

3. The reader might be tempted to prove Cauchy's mean value theorem by applying Lagrange's mean value theorem to the functions f and g. It can be easily seen that the desired result cannot be obtained in this manner. In fact, we would thus obtain that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c_1)}{g'(c_2)},$$

where $a < c_1 < b$, $a < c_2 < b$. Note that here c_1 is not necessarily equal to c_2.

Theorem 3-2. *(Generalised Mean Value Theorem). If f, g and h are continuous on* $[a, b]$ *and derivable on* $]a, b[$*, then there exists a number c in* $]a, b[$ *such that*

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof. Consider the function F defined by setting

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \qquad \text{...(1)}$$

for all x in $[a, b]$.

Since each of the functions, f, g and h is continuous on $[a, b]$ and derivable on $]a, b[$, therefore, F is also continuous on $[a, b]$ and derivable on $]a, b[$. Also, $F(a) = F(b) = 0$. Thus F satisfies all the conditions of Rolle's theorem on $[a, b]$. Consequently, there exists c in $]a, b[$ such that $F'(c) = 0$.

Since $$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix},$$

therefore, the result follows. □

Cauchy MVT

Let f, g be functions defined on $[a, b]$ such that

(i) f, g are continuous on $[a, b]$

(ii) f, g are derivable on (a, b) and

(iii) $g'(x)$ does not vanish at any point of (a, b)

Then there exists a real number $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof:

From the consequence of condition (iii) we can concludo $g(a) \neq g(b)$ (if $g(a) = g(a)$ then $g'(x)$ will vanish)

If $g(a) = g(b)$, Then g satisfies all the conditions of Rolle's theorem, and consequently for some x in (a, b) we would have $g'(x) = 0$.

Let. F be defined on $[a, b]$ by setting

$$F(x) = f(x) + Ag(x) \quad \forall \; x \in [a, b] \quad ①$$

where A is a constant to be suitably chosen.

(i) Since f, g are continuous on $[a, b]$.

$\therefore$ F is continuous on $[a, b]$

(ii) Since f, g are derivable on (a, b)

$\therefore$ F is derivable on (a, b)

(iii) Let us choose A so that $F(a) = F(b)$

$$f(a) + Ag(a) = f(b) + Ag(b)$$

$$Ag(a) - Ag(b) = f(b) - f(a)$$

$$-A(g(b) - g(a)) = f(b) - f(a)$$

$$-A = \frac{f(b)-f(a)}{g(b)-g(a)} \quad ②$$

This is permissible since $g(a) \neq g(b)$.

From (i), (ii) (iii) we find that F satisfies all the conditions of Rolle's theorem on $[a, b]$, Therefore there exists a real number $c \in (a, b)$

such that $F'(c) = 0$ from (1) we have

$$F(x) = f(x) + A\,g(x)$$

$$F'(x) = f'(x) + A\,g'(x)$$

$$F'(c) = f'(c) + A\,g'(c) = 0$$

$$\Rightarrow \quad -A = \frac{f'(c)}{g'(c)} \quad \text{—— ③}$$

This is permissible since $g'(c) \neq 0$

From (2), (3) we have

$$\frac{f(b) - f(a)}{g(b) - g(a)} = -A = \frac{f'(c)}{g'(c)}$$

$$\therefore \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Hence proved.

Remarks. 1. If we put $b = a + h$, then c can be written as $a + \theta h$ where θ is some real number such that $0 < \theta < 1$. The above theorem then reads as follows :

If f and g are continuous on $[a, a + h]$ and are derivable on $]a, a + h[$, and if $g'(x)$ does not vanish for any x in $]a, a + h[$, then there exists a real number θ between 0 and 1, such that

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

Hence proved.

Remarks

2) If we take $g(x) = x \ \forall \ x \in [a,b]$

Then Cauchy's MVT yields Lagrange's MVT as a particular case.

Soln $g(x) = x \qquad g'(x) = 1 \ \forall \ x \in [a,b]$

~~Let g, f~~ Let f be any function defined on $[a,b]$ which is continuous & derivable.

By ~~Lagr~~ Cauchy MVT $\exists \ c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Here $g(b) = b$, $g(a) = a$, $g'(c) = 1$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1}$$

$$\therefore f(b) - f(a) = (b-a) f'(c)$$

which gives us Lagrange's MVT.

3) Proving Cauchy's MVT by using Lagrange's MVT. Desired result cannot be obtained in this manner.

f continuous on $[a,b]$, f derivable on (a,b) ——— (i)

g continuous on $[a,b]$, g derivable on (a,b) ——— (ii)

By (i) we get some $c_1 \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c_1)$

By (ii) we get some $c_2 \in (a, b)$ such that $\dfrac{g(b) - g(a)}{b - a} = g'(c_2)$

$$\frac{\dfrac{f(b) - f(a)}{b - a}}{\dfrac{g(b) - g(a)}{b - a}} = \frac{f'(c_1)}{g'(c_2)}$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)}$$

Where $a < c_1 < b$, $a < c_2 < b$. Note that here c_1 is not necessarily equal to c_2

Theorem 3-2. *(Generalised Mean Value Theorem). If f, g and h are continuous on [a, b] and derivable on]a, b[, then there exists a number c in]a, b[such that*

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof. Consider the function F defined by setting

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \qquad \text{...(1)}$$

for all x in $[a, b]$.

Here f, g, h is cts on [a, b] and derivable on (a, b). ∴ F is also Continuous on [a, b] and derivable on (a, b)

$$F(a) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$\therefore F(a) = 0$

$$F(b) = \begin{vmatrix} f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$\therefore F(b) = 0$

(i) F cts on [a, b] (ii) F derivable on (a, b)

(iii) $F(a) = F(b)$ (Conditions of Rolles theorem was satisfied)

By Rolles theorem there exist $c \in (a, b)$ such that $F'(c) = 0$

Since
$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\therefore F'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

∴

Hence Proved.

Corollaries. 1. *Taking the function h to be the constant function defined by setting* $h(x) = 1$, *for all x in* $[a, b]$, *the above theorem reduces to Cauchy's mean value theorem.*

2. *Taking the functions g and h as defined by* $g(x) = x$, $h(x) = 1$, *for all x in* $[a, b]$, *the above theorem reduces to Lagrange's mean value theorem.*

Example 3. *Assuming that* $f''(x)$ *exists for all x in* $[a, b]$, *show that*

$$f(c) - f(a)\,\frac{b-c}{b-a} - f(b)\,\frac{b-a}{b-a} - \frac{1}{2}\,(c-a)\,(c-b)\,f''(\xi) = 0,$$

where c and ξ *both lies in* $[a, b]$.

Solution. Let us first observe that if the result to be proved be multiplied throughout by $(b - a)$ and the terms be re-arranged, then it can be shown to be equivalent to

$$(b-a)f(c) - f(a)(b-c) - f(b)(c-a) + \frac{1}{2}(c-a)(c-b)(b-a)f''(\xi) = 0$$

equivalent to

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2}f''(\xi)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0 \quad ①$$

&

$$\Big[(b f(c) - c f(b)) - 1(a f(c) - c f(a)) + 1(a f(b) - b f(a))\Big]$$

$$-\frac{1}{2}f''(\xi)\Big[(bc^2 - cb^2) - 1(ac^2 - ca^2) + 1(ab^2 - ba^2)\Big] = 0$$

$$\overset{①}{b} f(c) - c\overset{③}{f}(b) - \overset{①}{a} f(c) + c\overset{②}{f}(a) + a\overset{③}{f}(b) - b\overset{②}{f}(a)$$

$$-\frac{1}{2}f''(\xi)\Big[bc^2 - cb^2 - ac^2 + ca^2 + ab^2 - ba^2\Big] = 0$$

$$\Rightarrow (b-a)f(c) - (b-c)f(a) - (c-a)f(b)$$

$$-\frac{1}{2}f''(\xi)\Big[c^2 - bc - ac + ab\Big] = 0$$

$$(b-a)f(c) - (b-c)f(a) - (c-a)f(b)$$

$$-\frac{1}{2}f''(\xi)\Big[(c-a)(c-b)\Big] = 0$$

∴ equivalent to

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2}f''(\xi)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

(1) suggests that we must consider the function F defined by setting

$$F(x) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & x \\ f(a) & f(b) & f(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 1 \\ a & b & x \\ a^2 & b^2 & x^2 \end{vmatrix}, \quad \text{...(2)}$$

where A is a constant to be so chosen that

$$F(c) = 0. \quad \text{...(3)}$$

$$F(a) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & a \\ f(a) & f(b) & f(a) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 1 \\ a & b & a \\ a^2 & b^2 & a^2 \end{vmatrix}$$

$$= 0$$

$$F(b) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & b \\ f(a) & f(b) & f(b) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 1 \\ a & b & b \\ a^2 & b^2 & b^2 \end{vmatrix}$$

$$= 0$$

$\therefore F(a) = F(b) = 0$

Since $F(a) = F(b) = 0$, and since $F'(x)$ exists in $[a, b]$, therefore, F satisfies the conditions of Rolle's theorem in each of the intervals $[a, c]$and $[c, b]$. Consequently, there exists real numbers ξ_1 and ξ_2 such that $a < \xi_1 < c < \xi_2 < b$, and $F'(\xi_1) = 0,\ F'(\xi_2) = 0$.

$$\text{Now, } F'(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ f(a) & f(b) & f'(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ a^2 & b^2 & 2x \end{vmatrix}, \qquad \ldots(4)$$

$$F'(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ f(a) & f(b) & f'(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ a^2 & b^2 & 2x \end{vmatrix}$$

④

F' is continuous on $[\xi_1, \xi_2]$

F' is derivable on (ξ_1, ξ_2)

$F'(\xi_1) = F'(\xi_2) = 0$

$\therefore$ F' satisfies all the conditions of Rolle's theorem in $[\xi_1, \xi_2]$. Consequently there exist a real number $\xi \in (\xi_1, \xi_2)$ Such that $F''(\xi) = 0$

Now
$$F''(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ f(a) & f(b) & f''(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 2 \end{vmatrix}$$

$F''(\xi) = 0 \Rightarrow$

$$\begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ f(a) & f(b) & f''(\xi) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 2 \end{vmatrix} = 0$$

$$1(b f''(\xi) - 0) - 1(a f''(\xi) - 0) - A[1(2b) - 1(2a)] = 0$$

$$(b-a) f''(\xi) - 2A(b-a) = 0$$

$$2A(b-a) = (b-a) f''(\xi)$$

⑤

Again, since by (3), $F(c) = 0$, therefore,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2} f''(\xi) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0, \ldots (6)$$

where we have substituted the value of A as obtained in (5).

We have thus shown that (1) holds and this is equivalent to the relation desired to be proved.

Example 4. *If f'' be continuous on $[a, b]$ and derivable on $]a, b[$, then prove that*

$$f(b) - f(a) - (b - a)\{f'(a) + f'(b)\} = -\frac{(b-a)^3}{12} f'''(d),$$

for some real number d between a and b.

Solution. Let g be the function defined on $[a, b]$ by setting

$$g(x) = f(x) - f(a) - \tfrac{1}{2}(x - a)\{f'(a) + f'(x)\} + A(x - a)^3,$$

for all x in $[a, b]$, where A is a constant to be suitably chosen.

Now (1) g is continuous on $[a, b]$;

(2) g is derivable on $]a, b[$;

(3) Let A be so chosen that $g(a) = g(b)$. Since $g(a) = 0$, therefore, this means that A is such that

$$f(b) - f(a) - \tfrac{1}{2}(b - a)\{f'(a) + f'(b)\} + A(b - a)^3 = 0 \quad ...(i)$$

The function g now satisfies all the conditions of Rolle's theorem in $[a, b]$ and therefore, there exists a real number c between a and b, such that

$$g'(c) = 0,$$

i.e.,
$$\tfrac{1}{2}\{f'(c) - f'(a)\} - \tfrac{1}{2}(c - a) f''(c) + 3A(c - a)^2 = 0. \quad ...(ii)$$

Let h be the function defined on $[a, c]$ by setting

$$h(x) = \tfrac{1}{2}\{f'(x) - f'(a)\} - \tfrac{1}{2}(x - a) f''(x) + 3A(x - a)^2, \quad ...(iii)$$

for all x in $[a, c]$. Now

(1′) h is continuous on $[a, c]$;

(2′) h is derivable on $]a, c[$;

(3′) $h(c) = 0$ by (*ii*), so that $h(c) = h(a)$.

The function h now satisfies all the conditions of Rolle's theorem in $[a, c]$ and therefore, there exists a real number $d (a < d < c < b)$, such that

$$h'(d) = 0$$

From (*iii*), this gives

$$h'(d) = \tfrac{1}{2} f''(d) - \tfrac{1}{2} f''(d) - \tfrac{1}{2}(d - a) f'''(d) + 6A(d - a) = 0,$$

i.e.,
$$A = f'''(d)/12, \text{ since } d - a \neq 0. \quad ...(iv)$$

From (*i*) and (*iv*), we have

$g(a)=0 \Rightarrow g(a)=g(b)=0$

$$g(b)=f(b)-f(a)-\frac{1}{2}(b-a)\left(f'(a)+f'(b)\right)+A(b-a)^3=0 \quad \text{—(i)}$$

~~$g'(x)=f'(x)-1$~~

$$g(x)=f(x)-f(a)-\frac{1}{2}(x-a)f'(a)-\frac{1}{2}(x-a)f'(x)+A(x-a)^3$$

$$g'(x)=f'(x)-\frac{1}{2}f'(a)-\frac{1}{2}f'(x)-\frac{1}{2}(x-a)f''(x)+3A(x-a)^2$$

$g'(c)=0$

$$\Rightarrow \frac{1}{2}\left(f'(c)-f'(a)\right)-\frac{1}{2}(c-a)f''(c)+3A(c-a)^2=0$$

Example 4. *If f'' be continuous on $[a, b]$ and derivable on $]a, b[$, then prove that*

$$f(b) - f(a) - (b - a)\{f'(a) + f'(b)\} = -\frac{(b-a)^3}{12} f'''(d),$$

for some real number d between a and b.

Solution. Let g be the function defined on $[a, b]$ by setting

$$g(x) = f(x) - f(a) - \tfrac{1}{2}(x - a)\{f'{}_(a) + f'(x)\} + A(x - a)^3,$$

for all x in $[a, b]$, where A is a constant to be suitably chosen.

$g(a)=0 \quad =) \quad g(a)=g(b)=0$

$$g(b)=f(b)-f(a)-\frac{1}{2}(b-a)\left(f'(a)+f'(b)\right)$$

$$+A(b-a)^3=0 \text{ --- } (i)$$

Now (1) g is continuous on $[a, b]$;

(2) g is derivable on $]a, b[$;

(3) Let A be so chosen that $g(a) = g(b)$. Since $g(a) = 0$, therefore, this means that A is such that

$$f(b) - f(a) - \frac{1}{2}(b - a)\{f'(a) + f'(b)\} + A(b - a)^3 = 0 \text{ ...(i)}$$

The function g now satisfies all the conditions of Rolle's theorem in $[a, b]$ and therefore, there exists a real number c between a and b, such that

$$g'(c) = 0,$$

$$g(x) = f(x) - f(a) - \frac{1}{2}(x-a) f'(a)$$

$$- \frac{1}{2}(x-a) f'(x) + A(x-a)^3$$

$$g'(x) = f'(x) - \frac{1}{2} f'(a) - \frac{1}{2} f'(x)$$

$$- \frac{1}{2}(x-a) f''(x) + 3A(x-a)^2$$

$$g'(c) = 0$$

$$\Rightarrow \frac{1}{2}\left(f'(c) - f'(a)\right) - \frac{1}{2}(c-a) f''(c)$$

$$+ 3A(c-a)^2 = 0$$

(ii)

Let h be the function defined on $[a, c]$ by setting

$$h(x) = \frac{1}{2} \{ f'(x) - f'(a) \} - \frac{1}{2} (x - a) f''(x) + 3A (x - a)^2, \quad \text{...(iii)}$$

for all x in $[a, c]$. Now

$h(c) = 0$ by (ii) ~~$h(a) =$~~

$$h(a) = \frac{1}{2}\left\{ f'(a) - f'(a) \right\} - \frac{1}{2}(a-a) f''(a) + 3A(a-a)^2$$

$$= 0$$

$$\therefore h(c) = h(a)$$

$$h'(x) = \frac{1}{2} f''(x) - \frac{1}{2} f''(x) - \frac{1}{2}(x-a) f'''(x) + 6A(x-a)$$

(1′) h is continuous on $[a, c]$;

(2′) h is derivable on $]a, c[$;

(3′) $h(c) = 0$ by (ii), so that $h(c) = h(a)$.

The function h now satisfies all the conditions of Rolle's theorem in $[a, c]$ and therefore, there exists a real number $d(a < d < c < b)$, such that

$$h'(d) = 0$$

$$h'(d) = 0$$

$$\Rightarrow h'(d) = \frac{1}{2} f''(d) - \frac{1}{2} f''(d)$$

$$- \frac{1}{2}(d-a) f'''(d) + 6A(d-a) = 0$$

$$A = f'''(d)/12 \quad \text{Since } d-a \neq 0$$

(iv)

from (i) & (iv) we have

$$f(b) - f(a) - \frac{1}{2}(b-a)\{f'(a) + f'(b)\}$$

$$= -(b-a)^3 f'''(d)/12$$

$f'''(d)/12.$

Example 5. *If $f(0) = 0$ and $f''(x)$ exists on $[0, \infty[$, show that*

$$f'(x) - \frac{f(x)}{x} = \tfrac{1}{2} x f''(\xi),\ 0 < \xi < x, \qquad \text{...(1)}$$

and deduce that if $f''(x)$ is positive for positive values of x, then $f(x)/x$ strictly increases in $]0, \infty[$.

Solution. The relation (1) can be re-arranged in the form

$$f(x) - x f'(x) + \tfrac{1}{2} x^2 f''(\xi) = 0. \qquad \text{...(2)}$$

We may, therefore, consider the function F, defined by setting

$$F(x) = f(x) - x f'(x) + \tfrac{1}{2} A x^2, \qquad \text{...(3)}$$

where A is a constant to be suitably chosen.

Let c be any positive real number. Choosing A in (3) so that

$$F(c) = 0,$$

we find that F satisfies the hypothesis of Rolle's theorem on $[0, c]$.

Therefore, there exists x such that $0 < \xi < c$ and $F'(\xi) = 0$.

Since $\quad F'(x) = -x f''(x) + Ax,$

therefore, $\quad F'(\xi) = 0$ yields

$$A = f''(\xi). \qquad \text{...(4)}$$

Also, $\quad F(c) = 0$ yields

$$f(c) - c f'(c) + \frac{1}{2} A c^2 = 0. \qquad \text{...(5)}$$

From (4) and (5), we have

$$f(c) - c f'(c) + \frac{1}{2} c^2 f''(\xi) = 0,$$

or

$$f'(c) - \frac{f(c)}{c} = \frac{1}{2} c f''(\xi). \qquad \text{...(6)}$$

Since (6) is true for each $c > 0$, therefore, (1) is established. Also, if G be defined by setting

$G(x) = f(x)/x$, whenever $x > 0$, then

$$G'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{1}{2} f''(\xi), \text{ by (1).}$$

Assuming that $f''(x) > 0$ whenever $x > 0$, it follows that $G'(x) > 0$, whenever $x > 0$.

If x_1 and x_2 be any two positive real numbers such that $x_2 < x_1$, then by applying the mean value theorem to G in $[x_1, x_2]$ it follows that

$$G(x_2) - G(x_1) = (x_2 - x_1)\, G'(\eta),$$

where η is some real number in $]x_1, x_2[$.

Since $G'(\eta) > 0$, therefore, it follows that

$$G(x_2) < G(x_1).$$

Hence $f(x)/x$ is strictly increasing in $]\,0, \infty\,[$.

PROBLEMS

1. Calculate a value of c for which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

for each of the following pairs of functions :

(*a*) $f(x) = \sin x$, $g(x) = \cos x$; $a = -\pi/2$, $b = 0$.

(*b*) $f(x) = e^x$, $g(x) = e^{-x}$; $a = 0$, $b = 1$.

(*c*) $f(x) = x^2$, $g(x) = x$; $a = 0$, $b = 1$.

(*d*) $f(x) = x^2$, $g(x) = x^4$; $a = 2$, $b = 4$.

2. Give a geometrical interpretation of Cauchy's mean value theorem.

3. If $f'(x)$ and $g'(x)$ exist for all x in $[a, b]$, and if $g'(x)$ does not vanish anywhere in $]a, b[$, then prove that for some c between a and b,

$$\frac{f(c) - f(a)}{f(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$

[**Hint.** Apply Rolle's theorem to the function $fg - f(a)g - g(b)f$.]

4. Show that

$$f(b) - f(a) - (b - a) f'(a) \qquad f''(x$$

5. TAYLOR'S SERIES

Having seen that under certain conditions, the value of a function f at a point x can be approximated by polynomials of the form

$$f(a) + (x - a)\ f'(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a),$$

a very natural questions arises as to whether we can express $f(x)$ as an infinite series in the form

$$f(a) + (x-a)\ f'(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \dots, \quad \dots(1)$$

and if so, under what conditions. This questions may be split up into the following :

(*i*) Under what conditions is each term of the series (1) defined ?

(*ii*) Under what conditions does the series (1) converge ?

(*iii*) Under what conditions is the sum of the series (1) equal to $f(x)$?

We shall now examine each of the above questions.

(*i*) Each term of the series (1) is defined iff $f^n(a)$ exists for each positive integer n.

(*ii*) Assuming that $f^n(a)$ exists for each positive integer n, let us write

$$S_n = f(a) + (x-a)\ f'(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a). \quad \dots(2)$$

Suppose that f satisfies the conditions of Taylor's theorem in an interval $[a-h, a+h]$, so that for each $x \in [a-h, a+h]$,

$$f(x) = f(a) + (x-a)\ f'(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n, \dots(3)$$

where R_n is the remainder after n terms.

From (2) and (3), we have

$$S_n = f(x) - R_n. \quad \dots(4)$$

From (4), we find that $<S_n>$ converges iff $\lim\limits_{n\to\infty} R_n$ exists and consequently, the series (1) converges iff $\lim\limits_{n\to\infty} R_n$ exists.

(*iii*) Assuming that the series (1) converges, we find from (4) that its sum is $f(x) - \lim\limits_{n\to\infty} R_n$.

Now $f(x) - \lim\limits_{n\to\infty} R_n = f(x)$ iff $\lim\limits_{n\to\infty} R_n = 0$, showing that the series (1) converges to $f(x)$ provided $\lim\limits_{n\to\infty} R_n = 0$.

Summing up the above discussion, we have the following result :

If (i) a function f be defined on an interval $[a-h, a+h]$,

(ii) for each positive integer n, $f^n(c)$ *exists for all c in* $]a-h, a+h[$,

(iii) $\lim\limits_{n\to\infty} R_n(x) = 0$ *for each x in* $[a-h, a+h]$,

then for each $x \in [a - h, a + h]$,

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!}\, f''(a) + \dots$$

$$+ \frac{(x-a)^{n-1}}{(n-1)!}\, f^{n-1}(a) + \dots \qquad \dots(5)$$

Also, we then say that the series is the expansion of $f(x)$ in a Taylor's series around the point a. We also sometimes say that (5) is the *expression for $f(x)$ as a power series in $(x - a)$.*

If we put $a = 0$ in the above result, then we have the following result :

If (*i*) f be defined on an interval $[-h, h]$,

(*ii*) for each positive integer n, $f^n(c)$ exists for all c in $]-h, h[$,

(*iii*) $\lim\limits_{n \to \infty} R_n(x) = 0$, for each x in $[-h, h]$,

then for each x in $[-h, h]$,

$$f(x) = f(0) + x f'(0) + \dots + \frac{x^{n-1}}{(n-1)!}\, f^{n-1}(0) + \dots \qquad \dots(6)$$

The series (6) is called *Maclaurin's expansion* of $f(x)$.

Remark. In the above discussion, one may consider, *any* form of remainder R_n, that is, either Lagrange's form or Cauchy's form.

6. POWER SERIES EXPANSIONS OF SOME STANDARD FUNCTIONS

We shall now consider Maclaurin's series expansions of the functions e^x, $\sin x$, $\cos x$, $(1 + x)^m$ and $\log(1 + x)$.

(a) e^x. Let $f(x) = e^x$, for all $x \in \mathbf{R}$.

Then $$f^n(x) = e^x, \text{ for all } x \in \mathbf{R}.$$

Thus for each positive integer n, f^n is defined in the interval $[-h, h]$, whatever positive real number h may be. Also, writing Lagrange's remainder after n terms, we have

$$R_n(x) = \frac{x^n}{n!}\, f^n(\theta x), \text{ where } \theta \text{ is a real number between 0 and 1,}$$

$$= \frac{x^n}{n!}\, e^{\theta x}.$$

We shall now show that whatever x may be, $\lim\limits_{n\to\infty} R_n(x) = 0$.

For this purpose, it is enough to show that $e^{\theta x}$ is bounded in $[-h, h]$,

and $$\lim_{n\to\infty} \frac{x^n}{n!} = 0,$$

whatever x may be.

Since $0 < \theta < 1$ and $x \in [-h, h]$, therefore $|\theta x| < h$, and consequently, $0 < e^{\theta x} < e^h$, whence $e^{\theta x}$ is bounded.

Let us write $$a_n = \frac{x^n}{n!} \cdot \text{ for all } n \in \mathbf{N}.$$

Then $$\frac{a_{n+1}}{a_n} = \frac{x}{n+1},$$

so that $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}, = 0.$$

From above, it follows that $\lim\limits_{n\to\infty} a_n$ exists and equals zero.

Now $$\lim_{n\to\infty} R_n(x) = e^{\theta x}\left(\lim_{n\to\infty} \frac{x^n}{n!}\right) = 0.$$

Thus we find that whatever h may be, the function f has a Maclaurin's series expansion for each x in $[-h, h]$. This implies that for the given function, we have

$$f(x) = f(0) + x f'(0) + .. + \frac{x^{n-1}}{(x-1)!} f^{n-1}(0) + \ldots \qquad \ldots(1)$$

for all $x \in \mathbf{R}$.

Substituting $f(x) = e^x, f^n(x) = e^x$, we have

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \ldots \text{ for all } x \in \mathbf{R}}$$

(b) *sin x*. Let $f(x) = \sin x$, for all $x \in \mathbf{R}$.

Then $$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right), \text{ for all } x \in \mathbf{R}.$$

Thus for each $n \in \mathbf{N}$, f^n is defined in every interval $[-h, h]$. Writing agrange's remainder after n terms, we have

$$R_n(x) = \frac{x^n}{n!} f^n(\theta x), \text{ where } 0 < \theta < 1,$$

$$= \frac{x^n}{n\,!} \sin\left(\theta x + \frac{n\pi}{2}\right).$$

Now for all $x \in \mathbf{R}$,

$$|\, R_n(x) \,| \leq \left|\frac{x^n}{n!}\right|,$$

and $$\lim_{n \to \infty} \frac{x^n}{n!} = 0, \text{ as in } (a),$$

next class

Therefore, $\lim\limits_{n \to \infty} R_n(x) = 0$.

Thus we find the whatever h may be, the function f has a Maclaurin's series expansion for each x in $[-h, h]$. This implies that for the given function, we have

$$f(x) = f(0) + x f'(0) + \ldots + \frac{x^{n-1}}{(n-1)\,!} f^{n-1}(0) + \ldots \qquad \ldots(2)$$

for all $x \in \mathbf{R}$. Substituting $f(x) = \sin x$, $f^n(0) = \sin \frac{n\pi}{2}$ in (2), we have

$$\boxed{\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots, \text{ for all } x \in \mathbf{R}}$$

(c) *cos x*. Let $f(x) = \cos x$, for all $x \in \mathbf{R}$.

Then $f^n(x) = \cos\left(x + \frac{n\pi}{2}\right)$, for all $x \in \mathbf{R}$.

Thus for each $n \in \mathbf{N}$, f^n is defined in every interval $[-h, h]$. Writing Lagrange's remainder after n terms, we have

$$R_n(x) = \frac{x^n}{n\,!} f^n(\theta x), \text{ where } 0 < \theta < 1,$$

$$= \frac{x^n}{n\,!} \cos\left(\theta x + \frac{n\pi}{2}\right).$$

Now for all $x \in \mathbf{R}$

$$|\, R_n(x) \,| \leq \left|\frac{x^n}{n\,!}\right|.$$

and $\lim\limits_{n \to \infty} \frac{x^n}{n\,!} = 0$, as in (a).

Therefore $\lim_{n \to \infty} R_n(x) = 0$.

Thus we find that whatever h may be, the function f has a Maclaruin's series expansion for each x in $[-h, h]$. This implies that for the given function, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots \quad \ldots(2)$$

for all $x \in \mathbf{R}$.

Substituting $f(x) = \cos x, f^n(0) = \cos \frac{n\pi}{2}$ in (2), we have

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots, \text{ for all } x \in \mathbf{R}.}$$

6. POWER SERIES EXPANSIONS OF SOME STANDARD FUNCTIONS

We shall now consider Maclaurin's series expansions of the functions e^x, $\sin x$, $\cos x$, $(1+x)^m$ and $\log(1+x)$.

(a) e^x. Let $f(x) = e^x$, for all $x \in \mathbf{R}$.

Then $$f^n(x) = e^x, \text{ for all } x \in \mathbf{R}.$$

Thus for each positive integer n, f^n is defined in the interval $[-h, h]$, whatever positive real number h may be. Also, writing Lagrange's remainder after n terms, we have

$$R_n(x) = \frac{x^n}{n!} f^n(\theta x), \text{ where } \theta \text{ is a real number between 0 and 1,}$$

$$= \frac{x^n}{n!} e^{\theta x}.$$

We shall now show that whatever x may be, $\lim_{n\to\infty} R_n(x) = 0$.

For this purpose, it is enough to show that $e^{\theta x}$ is bounded in $[-h, h]$,

and
$$\lim_{n\to\infty} \frac{x^n}{n!} = 0,$$

whatever x may be.

Since $0 < \theta < 1$ and $x \in [-h, h]$, therefore $|\theta x| < h$, and consequently, $0 < e^{\theta x} < e^h$, whence $e^{\theta x}$ is bounded.

Let us write
$$a_n = \frac{x^n}{n!} \text{ for all } n \in \mathbf{N}.$$

Then
$$\frac{a_{n+1}}{a_n} = \frac{x}{n+1},$$

$a_n \to l$

$\frac{a_{n+1}}{a_n} \to l$

so that
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}, = 0.$$

From above, it follows that $\lim_{n\to\infty} a_n$ exists and equals zero.

Now
$$\lim_{n\to\infty} R_n(x) = e^{\theta x}\left(\lim_{n\to\infty} \frac{x^n}{n!}\right) = 0.$$

Thus we find that whatever h may be, the function f has a Maclaurin's series expansion for each x in $[-h, h]$. This implies that for the given function, we have

$$f(x) = f(0) + xf'(0) + .. + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \ldots \qquad \ldots(1)$$

for all $x \in \mathbf{R}$.

Substituting $f(x) = e^x, f^n(x) = e^x$, we have

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \ldots \text{ for all } x \in \mathbf{R}}$$

(b) *sin x*. Let $f(x) = \sin x$, for all $x \in \mathbf{R}$.

Then
$$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right), \text{ for all } x \in \mathbf{R}.$$

Thus for each $n \in \mathbf{N}$, f^n is defined in every interval $[-h, h]$. Writing Lagrange's remainder after n terms, we have

$$R_n(x) = \frac{x^n}{n!} f^n(\theta x), \text{ where } 0 < \theta < 1,$$

$$= \frac{x^n}{n\,!} \sin\left(\theta x + \frac{n\pi}{2}\right).$$

Now for all $x \in \mathbf{R}$,

$$| R_n (x) | \leq \left|\frac{x^n}{n!}\right|,$$

and
$$\lim_{n\to\infty} \frac{x^n}{n!} = 0, \text{ as in } (a),$$

Therefore, $\lim\limits_{n\to\infty} R_n (x) = 0.$

Thus we find the whatever h may be, the function f has a Maclaurin's series expansion for each x in $[-h, h]$. This implies that for the given function, we have

$$f(x) = f(0) + x f'(0) + \ldots + \frac{x^{n-1}}{(n-1)\,!} f^{n-1} (0) + \ldots \qquad \ldots(2)$$

for all $x \in \mathbf{R}$. Substituting $f(x) = \sin x$, $f^n(0) = \sin \dfrac{n\pi}{2}$ in (2), we have

$$\boxed{\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots, \text{ for all } x \in \mathbf{R}}$$

$f'(0) =$ 1
0
-1

(c) *cos x*. Let $f(x) = \cos x$, for all $x \in \mathbf{R}$.

Then $\quad f^n(x) = \cos\left(x + \dfrac{n\pi}{2}\right)$, for all $x \in \mathbf{R}$.

Thus for each $n \in \mathbf{N}$, f^n is defined in every interval $[-h, h]$. Writing Lagrange's remainder after n terms, we have

$$R_n(x) = \frac{x^n}{n\,!} f^n(\theta x), \text{ where } 0 < \theta < 1,$$

$$= \frac{x^n}{n\,!} \cos\left(\theta x + \frac{n\pi}{2}\right).$$

Now for all $x \in \mathbf{R}$

$$| R_n(x) | \leq \left|\frac{x^n}{n\,!}\right|.$$

and $\lim\limits_{n\to\infty} \dfrac{x^n}{n\,!} = 0$, as in (a).

Therefore $\lim_{n\to\infty} R_n(x) = 0.$

Thus we find that whatever h may be, the function f has a Maclaruin's series expansion for each x in $[-h, h]$. This implies that for the given function, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots \quad \dots(2)$$

for all $x \in \mathbf{R}$.

Substituting $f(x) = \cos x, f^n(0) = \cos \frac{n\pi}{2}$ in (2), we have

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \text{ for all } x \in \mathbf{R}.}$$

(*d*) $(1 + x)^m$, *distinguishing between the cases when m is a positive integer and is not a positive integer*. If m is a positive integer, then letting

$$f(x) = (1 + x)^m,$$

for all $x \in \mathbf{R}$, we find that for each $n \in \mathbf{N}, f^n(x)$ exists for all $x \in \mathbf{R}$, and that whenever $n > m, f^n(x) = 0$ for all $x \in \mathbf{R}$.

Thus $R_n(x) = 0$, whenever $n > m$.

Hence $\lim_{n\to\infty} R_n(x) = 0$, and for all $x \in \mathbf{R}$, we have

$$f(x) = f(0) + xf'(0) + \dots + \frac{x^m}{m!} f^m(0),$$

since the other terms all vanish.

Substituting the values of $f(x), f(0), f^m(0)$, we have

$$\boxed{\begin{array}{l}\textit{If } m \textit{ is a positive integer, then} \\ (1 + x)^m = 1 + mx + \dfrac{m(m-1)}{2!} x^2 + \dots + x^m, \\ \text{for all } x \in \mathbf{R}\end{array}}$$

Let us now consider the case when m is not a positive integer and $|x| < 1$. In this case, we find that if we write

$$f(x) = (1 + x)^m, \text{ whenever } x \neq = -1,$$

then

$$f^n(x) = m(m-1)(m-n+1)(1+x)^{m-n},$$

whenever $x \neq = -1$.

From the above we find that for each positive integer n, f^n is defined in $[-h, h]$ for each h between 0 and 1.

Writing Cauchy's remainder after n terms, we have

$$R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x), \text{ where } 0 < \theta < 1,$$

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} m(m-1) \ldots (m-n+1)(1+\theta x)^{m-n},$$

$$= \frac{m(m-1)\ldots.(m-n+1)}{(n-1)!} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1}$$

We shall show that if $|x| < 1$, then $R_n(x) \to 0$ as $n \to \infty$. Let us assume that for the rest of the discussion, $|x| < 1$.

Let us observe that

$$(i) \quad \lim_{n\to\infty} \frac{m(m-1)\ldots.(m-n+1)}{(n-1)!} x^n = 0.$$

In fact, writing

$$a_n = \frac{m(m-1)\ldots.(m-n+1)}{(n-1)!} x^n,$$

we have
$$\frac{a_{n+1}}{a_n} = \frac{(m-n)x}{n},$$

so that
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -x.$$

Since $|x| < 1$, therefore, from the above, it follows that

$$\lim_{n\to\infty} a_n = 0.$$

$$(ii) \lim_{n\to\infty} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$$

In fact, since $0 < \theta < 1$ and $-1 < x < 1$, therefore,

$$0 < \frac{1-\theta}{1+\theta x} < 1,$$

and consequently,

$$\lim_{n\to\infty} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$$

(*iii*) If $m > 1$, then

$$(1 + \theta x)^{m-1} < (1 - |x|)^{m-1}.$$

From (*i*), (*ii*) and (*iii*), we find that for all x in $]-1, 1[$,

$$\lim_{n\to\infty} R_n(x) = 0.$$

Thus we find that for each h between 0 and 1, the function f has Maclaurin's series expansion for all $x \in [-h, h]$. This implies that for the given function,

$$f(x) = f(0) + xf'(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \dots,$$

for all x in $]-1, 1[$.

Substituting the values of $f(x), f(0), f'(0), \dots, f^{n-1}(0)$, we have

> *If m is not a positive integer, then*
>
> $$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \dots$$
>
> $$+ \frac{m(m-1)\dots(m-n+1)}{n!} x^n + \dots,$$
>
> *whenever* $-1 < x < 1$.

(*e*) $\log(1 + x)$.

Let $f(x) = \log(1 + x)$, whenever $-1 < x \leq 1$.

Then $f^n(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$, whenever $x > -1$.

We shall consider the following cases :

(*i*) Let $0 \leq x \leq 1$. Writing Lagrange's remainder after n terms, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x),$$

$$= \frac{x^n}{n!} (-1)^{n-1} \frac{(n-1)!}{(1+\theta x)^n},$$

$$= \frac{(-1)^{n-1}}{n} \cdot \left(\frac{x}{1+\theta x}\right)^n$$

Since $0 \leq x \leq 1, 0 < \theta < 1$, therefore,

$$0 < \frac{x}{1+\theta x} < 1.$$

$\therefore \quad |R_n| < \frac{1}{n}$, and $\frac{1}{n} \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} R_n = 0.$$

(*ii*) Let $-1 < x < 0$. Since in this case $\left|\frac{x}{1+\theta x}\right|$ need not be less than unity, therefore, we may not be able to show easily that $R_n \to 0$ as $n \to \infty$ by considering Lagrange's remainder. Writing Cauchy's remainder, we have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x),$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{1+\theta x},$$

Since $|x| < 1$, therefore,

$$\left|\frac{1-\theta}{1+\theta x}\right| < 1, \text{ so that } \left|\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\right| < 1,$$

and

$$\left|\frac{1}{1+\theta x}\right| < \frac{1}{1-|x|}.$$

Consequently,

$$|R_n| < \frac{|x|^n}{1-|x|},$$

and this implies that $\lim_{n\to\infty} R_n = 0$, since $|x| < 1$.

From (*i*) and (*ii*) above we find that if $-1 < x \le 1$, then $\lim_{n\to\infty} R_n = 0$.

Hence

$$f(x) = f(0) + x f'(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \ldots,$$

whenever $-1 < x \le 1$.

Substituting the values of $f(x), f(0), f'(0), \ldots, f^{n-1}(0), \ldots$, we have

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^2}{3} \ldots, \text{ whenever } -1 < x \leq 1.}$$

The following example shows that Taylor series corresponding to a function f may converge and yet it may not be possible to have a series expansion for the function.

Example 7. *The function f defined on* **R** *by*

$$f(x) = e^{-1/x^2},\ x \neq 0,$$

$$f(0) = 0,$$

possesses continuous derivatives of all orders for all $x \in$ **R** *but cannot be expanded as a Maclaurin's series.*

Proof. *Step* 1. We shall first show that for each positive integer n, $f^n(x)$ is defined whenever $x \neq 0$ and is of the form $e^{-1/x^2}\ p(1/x)$, where $p(1/x)$ is a polynomial in $1/x$. We shall prove this by induction on n.

First, $$f'(x) = \frac{2}{x^3}\, e^{-1/x^2}, \text{ if } x \neq 0,$$

so that $f'(x)$ is of the form $e^{-1/x^2}\ p\left(\frac{1}{x}\right)$.

Now let us assume that for some positive integer k, $f^k(x)$ is of the form $e^{-1/x^2}\ p(1/x)$.

$$\text{Then } f^{k+1}(x) = \frac{d}{dx}\left\{e^{-1/x^2}\ p\left(\frac{1}{x}\right)\right\},$$

$$= \frac{2}{x^3}\, p\left(\frac{1}{x}\right) e^{-1/x^2} + e^{-1/x^2}\, p'\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right),$$

$$= e^{-1/x^2}\left\{\frac{2}{x^3}\, p\left(\frac{1}{x}\right) - \frac{1}{x^2}\, p'\left(\frac{1}{x}\right)\right\},$$

where $p'(1/x)$ is the derivative of $p(1/x)$ with respect to $1/x$, and is therefore, a polynomial in $1/x$.

We thus find that $f^{k+1}(x)$ is the product of e^{-1/x^2} and some polynomial in $1/x$.

By the principle of finite induction, it follows that for each positive integer n, $f^n(x)$ is defined whenever $x \neq 0$ and is of the form $e^{-1/x^2}\ p(1/x)$.

Step 2. We shall show that for each positive integer n, $f^n(0)$ is defined, and that $f^n(0) = 0$. We shall prove this also by induction on n.

First, $$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}, \text{ provided the limit exists,}$$

$$= \lim_{x\to 0} \left\{ \frac{e^{-1/x^2}}{x} \right\},$$

$$= 0 \text{ (see step 3).}$$

Now let us assume that for some positive integer k, $f^k(0)$ is defined, and that $f^k(0) = 0$.

$$\text{Then,} \quad f^{k+1}(0) = \lim_{x\to 0} \frac{f^k(x) - f^k(0)}{x}$$

$$= \lim_{x\to 0} \left\{ e^{-1/x^2} \times \text{polynomial in} \frac{1}{x} \right\},$$

$$= 0 \text{ (see step 3).}$$

Thus $f^{k+1}(0)$ is defined and $f^{k+1}(0) = 0$.

By the principle of finite induction it follows that for each positive integer n, $f^n(0)$ is defined, and that $f^n(0) = 0$.

Thus, by steps 1 and 2, for each positive integer n, $f^n(x)$ is defined for all $x \in \mathbf{R}$.

Step 3. We shall show that

$$\lim_{x\to 0} \left\{ e^{-1/x^2} \times \text{a polynomial in } \frac{1}{x} \right\} = 0.$$

For this, it is enough to show that for each positive integer n, $\lim_{x\to 0} (x^{-n} e^{-1/x^2}) = 0$.

Let k be *any* fixed positive integer. Then

$$(i)\ e^{1/x^2} > \frac{1}{(k+1)!} \left(\frac{1}{x^2} \right)^{k+1},$$

or $$0 < x^{-2k} e^{-1/x^2} < (k+1)!\, x^2,$$

or $$| x^{-2k} e^{-1/x^2} | < (k+1)!\, | x |^2,$$

showing that $x^{-2k} e^{-1/x^2} \to 0$ as $x \to 0$.

(*ii*) Also, $(\lim_{x\to 0} (x^{-(2k-1)} e^{-1/x^2}) = (\lim_{x\to 0} x) (\lim_{x\to 0} x^{-2k} e^{-1/x^2}),$

$$= 0, \text{ by } (i).$$

From (*i*) and (*ii*), we find that $\lim_{x\to 0} (x^{-n} e^{-1/x^2}) = 0$, for each positive integer n, and consequently,

$$\lim_{x\to 0}\left\{e^{-1/x^2}\times \text{a polynomial in } \frac{1}{x}\right\} = 0.$$

Step 4. From steps 1–3, we find that for each positive integer *n*.

$$\lim_{x\to 0} f^{n}(x) = \lim_{x\to 0}\left\{e^{-1/x^2}\times \text{a polynomial in } \frac{1}{x}\right\},$$

$$= 0,$$

$$= f^{n}(0),$$

so that f^{n} is continuous at $x = 0$.

Step 5. Maclaurin's series for *f* is

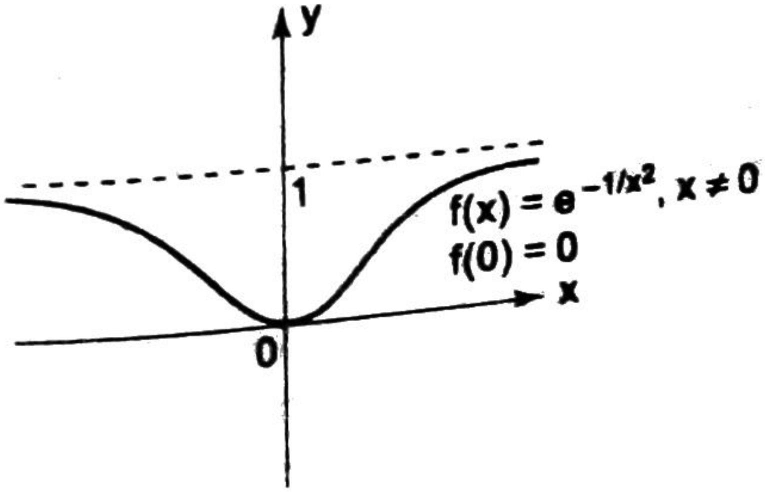
$$f(0) + xf'(0) + \ldots + \frac{x^n}{n!} f^{n}(0) + \ldots$$

$$= 0 + x.0 + \ldots + \frac{x^n}{n!}\, 0 + \ldots,$$

which is not equal to e^{-1/x^2} except for $x = 0$.

Thus *f* cannot be expanded as Maclaurin's series.

Remark. The graph of the function *f* discussed in the above example is as shown below :



6. If D_1, D_2 be two divisions such that $D_2 \supset D_1$, we shall say the division D_2 is *finer than* the division D_1.

7. Norm. The length of the greatest of all the sub-intervals $[x_{r-1}, x_r]$ of a division D will be called the **norm** of D and denoted by $| D |$.

8. The sums S, s corresponding to a division D will be denoted by the symbols $S(D)$ and $s(D)$ respectively. Clearly,

$$S(D) \geq s(D),$$

for all divisions D of $[a, b]$.

9. Oscillatory Sum. We have

$$S(D) - s(D) = \Sigma M_r \delta_r - \Sigma m_r \delta_r = \Sigma (M_r - m_r)\, \delta_r = \Sigma O_r \delta_r$$

where O_r denotes the oscillation of the function in the sub-interval δ_r. The sum $\Sigma O_r \delta_r$ which is called the *oscillatory* sum is denoted by $w(D)$.

As the oscillation O_r cannot be negative, it follows that each oscillatory sum consists of the sum of a finite number of non-negative terms.

EXAMPLES

1. *If f is defined on $[0, 1]$ by $f(x) = x \ \forall \ x \in [0, 1]$, then prove that $f \in \mathbf{R}\,[0, 1]$, and*

$$\int_0^1 f(x)\, dx = \frac{1}{2}. \quad \textit{(Poorvanchal 91; Garhwal 97)}$$

Sol. Let any partition of [0, 1] be $D = \left\{0 = \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$.

Let the sub-intervals be $I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right]$, for $r = 1, 2, \ldots, n$. If S_r be the length of this interval I_r, then

$$S_r = \frac{r}{n} - \frac{r-1}{n} = \frac{1}{n}$$

Also, if M_r and m_r be respectively the supremum and infimum of the function f in I_r, then $M_r = \frac{r}{n}$ and $m_r = \frac{r-1}{n}$, as $f(x) = x$.

$$\therefore \quad S(D) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \left(\frac{r}{n} \cdot \frac{1}{n}\right)$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} r = \frac{1}{n^2}\left[\frac{1}{2} n(n+1)\right] = \frac{n+1}{2n}$$

$$= \frac{1}{2}\left(1 + \frac{1}{n}\right) \quad \ldots(i)$$

Also, $$s(D) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \left(\frac{r-1}{n}\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} (r-1)$$

$$= \frac{1}{n^2} \left[\frac{1}{2}(n-1)(n-1+1)\right] = \frac{1}{2} \frac{(n-1)}{n}$$

$$= \frac{1}{2}\left(1 - \frac{1}{n}\right) \qquad \text{...(ii)}$$

Again, $$\overline{\int_0^1} f(x)\, dx = \text{inf.}\left[s(D)\right]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2} \qquad \text{...(iii)}$$

And $$\underline{\int_0^1} f(x)\, dx = \text{sup.}\ [s(D)]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2} \qquad \text{...(iv)}$$

From (*iii*) and (*iv*), we find that

$$\underline{\int_0^1} f\, dx = \overline{\int_0^1} f\, dx = \frac{1}{2}$$

Hence $f \in \mathbf{R}\ [0, 1]$ and $\int_0^1 f(x)\, dx = \frac{1}{2}$.

2. *Show that a constant function is Reimann-integrable.*

(*Poorv. 93*)

Sol. Let a function f be defined on $[a, b]$ by $f(x) = c,\ \forall\, x \in [a, b]$, where c is a constant.

Let any partition of $[a, b]$ be $D = \{a = x_0, x_1, x_2, ..., x_r, ..., x_n = b\}$

Let its sub-intervals be $I_r = [x_{r-1}, x_r]$ for $r = 1, 2, ..., n$.

If δ_r be the length of this interval I_r, then

$$\delta_r = x_r - x_{r-1}.$$

Let M_r and m_r be respectively the supremum and infimum of the function f in I_r, then $M_r = c$, $m_r = c$, as $f(x) = c \ \forall \ x \in (a, b)$

$$\therefore \quad S(D) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} c\left(x_r - x_{r-1}\right)$$

$$= c\,[(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})]$$

$$= c\,(x_n - x_0) = c\,(b - a) = \text{constant.}$$

And
$$s(D) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} c\left(x_r - x_{r-1}\right)$$

$$= c\,(b - a) = \text{constant.}$$

$$\therefore \quad \overline{\int_a^b} f(x)\,dx = \text{infimum } S(D)$$

$$= c\,(b - a)$$

and
$$\underline{\int_a^b} f(x)\,dx = \text{sup.}\left[s(D)\right]$$

Hence
$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = c\,(b-a)$$

and so $f \in$ **R** $[a, b]$, *i.e.*, the function f is **R**-integrable and

$$\int_a^b f(x)\,dx = c\,(b - a).$$

3. *If $f(x)$ be defined on $[0, 1]$ as follows —*

$$f(x) = 1, \quad \text{when } x \text{ is rational}$$
$$= -1, \quad \text{when } x \text{ is irrational}$$

then prove that f is not Riemann integrable over $[0, 1]$.

Sol. Let any partition of [0, 1] be $D = \{0 = x_0, x_1, x_2, \ldots, x_r, \ldots, x_n = 1\}$.

Let its sub-interval be $I_r = [x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

Clearly, $M_r = 1$ and $m_r = -1$.

$$\therefore \quad S(D) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1.\left(x_r - x_{r-1}\right)$$

$$= x_n - x_0 = 1 - 0 = 1.$$

And
$$s(D) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} (-1)\left(x_{r-1} - x_r\right)$$

$$= x_0 - x_n = -1.$$

$$\therefore \quad \overline{\int_0^1} f(x)\,dx = \text{inf. } \{s(D)\} = 1$$

and $$\int_{\underline{0}}^{1} f(x)\,dx = \text{sup. } \{s(D)\} = -1$$

$$\therefore \quad \int_{0}^{\overline{1}} f(x)\,dx \neq \int_{\underline{0}}^{1} f(x)\,dx$$

Hence $f \notin \mathbf{R}\,[0, 1]$, as the necessary and sufficient condition of Riemann-integrability is not satisfied.

EXERCISES

1. If

$$f(x) = \begin{cases} 0, \text{ where } x \text{ is rational,} \\ 1, \text{ where } x \text{ is irrational} \end{cases}$$

show that f is not integrable in any interval. (*Lucknow 95; Garhwal 90*)

2. Show that

$$\int_{a}^{b} k\,dx = \int_{a}^{b} k\,dx = k\,(b-a),$$

where k is a constant.

(This proves that every constant function is integrable.)

3. A function f is bounded in $[a, b]$; show that

$$(i) \quad \int_{a}^{b} kf(x)\,dx = k\int_{a}^{\overline{b}} f(x)\,dx, \quad \int_{\underline{a}}^{b} kf(x)\,dx = k\int_{\underline{a}}^{b} f(x)\,dx,$$

where k is a positive constant.

$$(ii) \quad \int_{a}^{\overline{b}} kf(x)\,dx = k\int_{\underline{a}}^{b} f(x)\,dx, \quad \int_{\underline{a}}^{b} kf(x)\,dx = k\int_{a}^{\overline{b}} f(x)\,dx,$$

where k is a negative constant.

Deduce that if f is bounded and integrable over $[a, b]$ then so is kf, where k is a constant, and that

6. If D_1, D_2 be two divisions such that $D_2 \supset D_1$, we shall say the division D_2 is *finer than* the division D_1.

7. Norm. The length of the greatest of all the sub-intervals $[x_{r-1}, x_r]$ of a division D will be called the **norm** of D and denoted by $| D |$.

8. The sums S, s corresponding to a division D will be denoted by the symbols $S(D)$ and $s(D)$ respectively. Clearly,

$$S(D) \geq s(D),$$

for all divisions D of $[a, b]$.

9. Oscillatory Sum. We have

$$S(D) - s(D) = \Sigma M_r \delta_r - \Sigma m_r \delta_r = \Sigma (M_r - m_r) \delta_r = \Sigma O_r \delta_r$$

where O_r denotes the oscillation of the function in the sub-interval δ_r. The sum $\Sigma O_r \delta_r$ which is called the *oscillatory* sum is denoted by $w(D)$.

As the oscillation O_r cannot be negative, it follows that each oscillatory sum consists of the sum of a finite number of non-negative terms.

EXAMPLES

1. *If f is defined on $[0, 1]$ by $f(x) = x \ \forall \ x \in [0, 1]$, then prove that $f \in \mathbf{R}[0, 1]$, and*

$$\int_0^1 f(x)\, dx = \frac{1}{2}.$$ (*Poorvanchal 91; Garhwal 97*)

Sol. Let any partition of [0, 1] be $D = \left\{0 = \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$.

Let the sub-intervals be $I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right]$, for $r = 1, 2, \ldots, n$. If δ_r be the length of this interval I_r, then

$$\delta_r = \frac{r}{n} - \frac{r-1}{n} = \frac{1}{n}$$

Also, if M_r and m_r be respectively the supremum and infimum of the function f in I_r, then $M_r = \frac{r}{n}$ and $m_r = \frac{r-1}{n}$, as $f(x) = x$.

$$\therefore \qquad S(D) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \left(\frac{r}{n} \cdot \frac{1}{n}\right)$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} r = \frac{1}{n^2}\left[\frac{1}{2} n(n+1)\right] = \frac{n+1}{2n}$$

$$= \frac{1}{2}\left(1 + \frac{1}{n}\right) \qquad \ldots(i)$$

Also, $$s(D) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \left(\frac{r-1}{n}\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} (r-1)$$

$$= \frac{1}{n^2}\left[\frac{1}{2}(n-1)(n-1+1)\right] = \frac{1}{2}\frac{(n-1)}{n}$$

$$= \frac{1}{2}\left(1-\frac{1}{n}\right) \qquad \text{...(ii)}$$

Again, $$\int_0^{\overline{1}} f(x)\,dx = \text{inf.}\,[S(D)]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1+\underset{\to 0}{\frac{1}{n}}\right) = \frac{1}{2} \qquad \text{...(iii)}$$

And $$\int_{\underline{0}}^{1} f(x)\,dx = \text{sup.}\,[s(D)]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1-\underset{\to 0}{\frac{1}{n}}\right) = \frac{1}{2} \qquad \text{...(iv)}$$

From (*iii*) and (*iv*), we find that

$$\int_{\underline{0}}^{1} f\,dx = \int_0^{\overline{1}} f\,dx = \frac{1}{2}$$

Hence $f \in \mathbf{R}\,[0, 1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$.

2. *Show that a constant function is Reimann-integrable.*

(*Poorv. 93*)

Sol. Let a function f be defined on $[a, b]$ by $f(x) = c$, $\forall\, x \in [a, b]$, where c is a constant.

Let any partition of $[a, b]$ be $D = \{a = x_0, x_1, x_2, \ldots, x_r, \ldots, x_n = b\}$

Let its sub-intervals be $I_r = [x_{r-1}, x_r]$ for $r = 1, 2, \ldots, n$.

If δ_r be the length of this interval I_r, then

$$\delta_r = x_r - x_{r-1}.$$

Let M_r and m_r be respectively the supremum and infimum of the function f in I_r, then $M_r = c$, $m_r = c$, as $f(x) = c \ \forall \ x \in (a, b)$

$$\therefore \qquad S(D) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} c\left(x_r - x_{r-1}\right)$$

$$= c\,[(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})]$$

$$= c\,(x_n - x_0) = c\,(b - a) = \text{constant.}$$

And
$$s(D) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} c\left(x_r - x_{r-1}\right)$$

$$= c\,(b - a) = \text{constant.}$$

$$\therefore \qquad \int_a^{\bar{b}} f(x)\,dx = \text{infimum } S(D)$$

$$= c\,(b - a)$$

and
$$\int_{\underline{a}}^{b} f(x)\,dx = \text{sup.}\left[s(D)\right]$$

$$\text{Hence } \int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx = c\,(b - a)$$

and so $f \in \mathbf{R}\,[a, b]$, *i.e.*, the function f is **R**-integrable and

$$\int_a^{b} f(x)\,dx = c\,(b - a).$$

3. *If $f(x)$ be defined on $[0, 1]$ as follows —*

$$f(x) = 1, \qquad \textit{when } x \textit{ is rational}$$

$$= -1, \qquad \textit{when } x \textit{ is irrational}$$

then prove that f is not Riemann integrable over $[0, 1]$.

Sol. Let any partition of $[0, 1]$ be $D = \{0 = x_0, x_1, x_2, \ldots, x_r, \ldots, x_n = 1\}$.

Let its sub-interval be $I_r = [x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

Clearly, $M_r = 1$ and $m_r = -1$.

$$\therefore \qquad S(D) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1.\left(x_r - x_{r-1}\right)$$

$$= x_n - x_0 = 1 - 0 = 1.$$

And
$$s(D) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} (-1)\left(x_{r-1} - x_r\right)$$

$$= x_0 - x_n = -1.$$

$$\therefore \qquad \int_0^{\bar{1}} f(x)\,dx = \text{inf. } \{S(D)\} = 1$$

and $$\int_{\underline{0}}^{1} f(x)\,dx = \text{sup. } \{s(D)\} = -1$$

$$\therefore \quad \int_{0}^{\bar{1}} f(x)\,dx \neq \int_{\underline{0}}^{1} f(x)\,dx$$

Hence $f \notin \mathbf{R}\,[0, 1]$, as the necessary and sufficient condition of Riemann-integrability is not satisfied.

EXERCISES

H.W

1. If

$$f(x) = \begin{cases} 0, \text{ where } x \text{ is rational,} \\ 1, \text{ where } x \text{ is irrational} \end{cases}$$

show that f is not integrable in any interval. *(Lucknow 95; Garhwal 90)*

2. Show that

$$\int_{a}^{\bar{b}} k\,dx = \int_{\underline{a}}^{b} k\,dx = k(b-a),$$

where k is a constant.

(This proves that every constant function is integrable.)

3. A function f is bounded in $[a, b]$; show that

$$(i) \quad \int_{a}^{b} kf(x)\,dx = k\int_{a}^{\bar{b}} f(x)\,dx, \quad \int_{\underline{a}}^{b} kf(x)\,dx = k\int_{\underline{a}}^{b} f(x)\,dx,$$

where k is a positive constant.

$$(ii) \quad \int_{a}^{\bar{b}} kf(x)\,dx = k\int_{\underline{a}}^{b} f(x)\,dx, \quad \int_{\underline{a}}^{b} kf(x)\,dx = k\int_{a}^{\bar{b}} f(x)\,dx,$$

where k is a negative constant.

Deduce that if f is bounded and integrable over $[a, b]$ *then so is kf, where k is a constant, and that*

$$\int_{a}^{b} kf(x)\,dx = k\int_{a}^{b} f(x)\,dx.$$

[If M_r, m_r be the bounds of f in δ_r then kM_r, km_r, (km_r, kM_r) are the bounds of kf in δ_r, where k is positive, (k is negative).

4. A bounded function f is integrable over $[a, b]$ and M, m are the bounds of f, show that

$$m(b-a) \leq \int_{a}^{b} f(x)\,dx \leq M(b-a).$$

6.3. DARBOUX'S THEOREM

To every $\varepsilon > 0$*, there corresponds* $\delta > 0$ *such that*

$$S(D) < \overline{\int_a^b} f(x)\,dx + \varepsilon \qquad \textit{(Poorv. 91, 93)}$$

$\forall$ *D with* $|D| \leq \delta$.

Lemma. Let $\left|f(x)\right| \leq k \; \forall \; x \in [a, b]$.

Let δ be a positive number and D_1 a division of $[a, b]$ such that

$$\left|D_1\right| \leq \delta.$$

Let D_2 be a division of $[a, b]$ consisting of all the points of D_1 and at the most some p more.

Then we shall show that

$$S(D_1) - 2pk\delta \leq S(D_2) \leq S(D_1).$$

In particular, it will follow that

$$D_2 \supset D_1 \Rightarrow S(D_2) \leq S(D_1).$$

Firstly suppose that $p = 1$ so that only one interval, say δ_r, of D_1 is divided into two intervals, say δ'_r, and δ''_r. Let M_r, M'_r, M''_r be the suprema of f in δ_r, δ'_r, δ''_r respectively.

We have

$$\begin{aligned} S(D_1) - S(D_2) &= M_r\delta_r - (M'_r\delta'_r + M_r''\delta_r'') \\ &= (M_r - M'_r)\,\delta_r + (M_r - M''_r)\,\delta''_r, \end{aligned}$$

for $\delta_r = \delta'_r + \delta''_r$.

Now $\qquad \left|f(x)\right| \leq k, \forall \; x \in [a, b]$

$\Rightarrow \qquad -k \leq M'_r \leq M_r \leq k,$

$\Rightarrow \qquad 0 \leq M_r - M'_r \leq 2k.$

Similarly we have

$$0 \leq M_r - M''_r \leq 2k.$$

It follows that

$$0 \leq S(D_1) - S(D_2) \leq 2k(\delta'_r + \delta''_r) = 2k\delta_r \leq 2k\delta.$$

Now supposing that each additional point is introduced one by one, we obtain the result.

We now prove the *main theorem.*

As f is bounded, there exists $k > 0$, such that

$$\left| f(x) \right| \leq k \; \forall \; x \in [a, b].$$

Since $$\int_a^{\bar{b}} f(x)\, dx$$

is the infimum of the set of upper sums S, there exists a division

$$D_1 \left\{ a = x_0, x_1, x_2, \ldots, x_{p-1}, x_p = b \right\}$$

such that $$S(D_1) < \int_a^b f(x)\, dx + \frac{\varepsilon}{2}.$$

The points of D_1 are $(p + 1)$ in number.

Let δ be the positive number such that

$$2k\,(p - 1)\,\delta = \frac{1}{2}\varepsilon.$$

Let D be any division with norm less than or equal to δ.

Let D_2 be the division consisting of the points of D_1 as well as those of D. Applying the lemma to the divisions D and D_2, we have

$$S(D) - 2(p-1)k\delta \leq S(D_2) \leq S(D).$$

Also $$D_2 \supset D_1 \Rightarrow S(D_2) \leq S(D_1).$$

Thus we obtain

$$S(D) - 2(p-1)\,k\delta \leq S(D_1)$$

$$S(D) \leq 2(p-1)\,k\delta + S(D_1)$$

$$< \frac{\varepsilon}{2} + \int_a^{\bar{b}} f(x)\, dx + \frac{\varepsilon}{2} = \int_a^{\bar{b}} f(x)\, dx + \varepsilon.$$

Hence the result.

6.3. DARBOUX'S THEOREM

To every $\varepsilon > 0$, *there corresponds* $\delta > 0$ *such that*

$$S(D) < \int_a^{\bar{b}} f(x)\,dx + \varepsilon \qquad \text{(Poorv. 91, 93)}$$

$\forall$ *D with* $|D| \le \delta$.

Lemma. Let $|f(x)| \le k \;\forall\; x \in [a, b]$.

Let δ be a positive number and D_1 a division of $[a, b]$ such that

$$|D_1| \le \delta.$$

Let D_2 be a division of $[a, b]$ consisting of all the points of D_1 and at the most some p more.

Then we shall show that

$$S(D_1) - 2pk\delta \le S(D_2) \le S(D_1).$$

In particular, it will follow that

$$D_2 \supset D_1 \Rightarrow S(D_2) \le S(D_1).$$

Firstly suppose that $p = 1$ so that only one interval, say δ_r, of D_1 is divided into two intervals, say δ'_r, and δ''_r. Let M_r, M'_r, M''_r be the suprema of f in δ_r, δ'_r, δ''_r respectively.

We have

$$\begin{aligned} S(D_1) - S(D_2) &= M_r\delta_r - (M'_r\delta'_r + M_r''\delta_r'') \\ &= (M_r - M'_r)\,\delta_r + (M_r - M''_r)\,\delta''_r, \end{aligned}$$

for $\delta_r = \delta'_r + \delta''_r$.

$$
\begin{aligned}
S(D_1) - S(D_2) &= M_\gamma \delta_\gamma - (M_\gamma{}' \delta_\gamma{}' + M_\gamma{}'' \delta_\gamma{}'') \\
&= M_\gamma \delta_\gamma - \left(M_\gamma{}' (\delta_\gamma - \delta_\gamma{}'') + M_\gamma{}'' \delta_\gamma{}''\right) \\
&= M_\gamma \delta_\gamma - M_\gamma{}' \delta_\gamma + M_\gamma{}' \delta_\gamma{}'' - M_\gamma{}'' \delta_\gamma{}'' \\
&= \delta_\gamma (M_\gamma - M_\gamma{}') + \delta_\gamma{}'' (M_\gamma{}' - M_\gamma{}'')
\end{aligned}
$$

Now $$|f(x)| \leq k, \forall\ x \in [a, b]$$

$\Rightarrow$ $$-k \leq M'_r \leq M_r \leq k,$$

$\Rightarrow$ $$0 \leq M_r - M'_r \leq 2k.$$

Similarly we have

$$0 \leq M_r - M''_r \leq 2k.$$

$-k < f(x) < k$

$-k \leq M_r' \leq M_r \leq k$

$0 \leq M_r' + k \leq M_r + k \leq 2k$

$0 \leq M_r - M_r' \leq 2k$

Similarly we have $0 \leq M_r - M_r'' \leq 2k$

$$0 \leq S(D_1) - S(D_2) \leq 2k(\delta'_r + \delta''_r) = 2k\delta_r \leq 2k\delta.$$

Now supposing that each additional point is introduced one by one we obtain the result.

We now prove the *main theorem.*

As f is bounded, there exists $k > 0$, such that

$$|f(x)| \le k \; \forall \; x \in [a, b].$$

Since

$$\int_a^{\bar{b}} f(x)\,dx$$

is the infimum of the set of upper sums S, there exists a division

$$D_1 \; \{a = x_0, x_1, x_2, \ldots, x_{p-1}, x_p = b\}$$

such that

$$S(D_1) < \int_a^{b} f(x)\,dx + \frac{\varepsilon}{2}.$$

The points of D_1 are $(p + 1)$ in number.

Let δ be the positive number such that

$$2k\,(p - 1)\,\delta = \frac{1}{2}\varepsilon.$$

Let D be any division with norm less than or equal to δ.

Let D_2 be the division consisting of the points of D_1 as well as those of D. Applying the lemma to the divisions D and D_2, we have

$$S(D) - 2(p-1)k\delta \le S(D_2) \le S(D).$$

Also

$$D_2 \supset D_1 \Rightarrow S(D_2) \le S(D_1).$$

Thus we obtain

$$S(D) - 2(p-1)\,k\delta \le S(D_1)$$

$$S(D) \le 2(p-1)\,k\delta + S(D_1)$$

$$< \frac{\varepsilon}{2} + \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} = \int_a^{\bar{b}} f(x)\,dx + \varepsilon.$$

Hence the result.

Darboux's theorem II. *To every $\varepsilon > 0$ there corresponds $\delta > 0$, such that*

$$s(D) > \int_a^{b} f(x)\,dx - \varepsilon$$

for every division D, with

$$|D| \le \delta.$$

This proof is similar to that of the corresponding result on that upper integral proved above.

Note. Darboux's theorem may be symbolically exhibited as follows :

$$\lim s(D) = \int_{\underline{a}}^{b} f(x)\,dx, \quad \lim S(D) = \int_{\underline{a}}^{b} f(x)\,dx,$$

when the norm $| D |$ tends to zero.

Cor. I. *For every bounded function f*

$$\int_{a}^{b} f(x)\,dx \geq \int_{\underline{a}}^{b} f(x)\,dx,$$

so that the upper integral ≥ the lower integral.

If possible, let

$$\int_{a}^{\bar{b}} f(x)\,dx < \int_{\underline{a}}^{b} f(x)\,dx.$$

Let, k be any number lying between the upper and lower integrals.

Now there exists by Darboux's theorem, a positive number δ_1 such that for every division whose norm is $\leq \delta_1$,

$$S < k.$$

Also, there exists a positive number δ_2 such that for every division whose norm is $\leq \delta_2$

$$s > k.$$

If, δ, be any positive number smaller than δ_1 as well as δ_2, then for every division whose norm is $\leq \delta$, we have

$$S < k < s \Rightarrow S < s,$$

which is not true.

Hence the result.

Cor. II.

$$S(D_1) \leq s(D_2)$$

even when D_1, D_2 are two *different* divisions.

This at once follows from the cor. 1 above.

6.5. CONDITIONS FOR INTEGRABILITY

6.5.1. First Form

A necessary and sufficient condition for the integrability of a bounded function f is, that to every $\varepsilon > 0$, there corresponds a $\delta > 0$ such that for every division D, whose norm is $\leq \delta$, the oscillatory sum $w(D)$ is $< \varepsilon$.

(Garhwal 92, 93, 95, Poorv. 91; Rohilkhand 90, 94)

The condition is necessary. The bounded function f being integrable, we have

$$\int_a^{\overline{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Let ε be any positive number. By Darboux's theorem, there exists $\delta > 0$ such that for every division D whose norm is $\leq \delta$,

$$\left\{\begin{array}{l} S(D) < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \frac{\varepsilon}{2}, \\ s(D) > \int_{\underline{a}}^b f(x)\,dx - \frac{\varepsilon}{2} = \int_a^b f(x)\,dx - \frac{\varepsilon}{2}. \end{array}\right.$$

$$\Rightarrow \qquad \int_a^b f(x)\,dx - \frac{\varepsilon}{2} < s(D) \leq S(D) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

$$\Rightarrow \qquad w(D) = S(D) - s(D) < \varepsilon$$

for every division D whose norm is $\leq \delta$.

The condition is sufficient. Let ε be any positive number. There exists a division D such that

$$S(D) - s(D) = \left[S(D) - \int_a^{\bar{b}} f(x)\,dx\right] + \left[\int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^b f(x)\,dx\right] + \left[\int_{\underline{a}}^b f(x)\,dx - s(D)\right] < \varepsilon.$$

Since each one of the three numbers

$$S(D) - \int_a^{\bar{b}} f(x)\,dx, \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^b f(x)\,dx, \int_a^b f(x)\,dx - s(D)$$

is non-negative, we see that

$$0 \leq \int_a^{\bar{b}} f(x)\,dx - \int_a^b f(x)\,dx < \varepsilon.$$

As ε is an arbitrary positive number, we see that the non-negative number

$$\int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^b f(x)\,dx,$$

is less than every positive number, and hence

$$\int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^b f(x)\,dx = 0 \Rightarrow \int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^b f(x)\,dx,$$

so that f is integrable.

6.7.2. Integrability of the Sum, Difference, Product and Quotient of Integrable Functions

Before taking up the main question, we state and prove a simple lemma.

Lemma. *The oscillation of a bounded function f in an interval $[a, b]$ is the supremum of the set of numbers*

$$\{|f(\alpha) - f(\beta)| : \alpha, \beta \in [a, b]\}.$$

Let m, M be the bounds of f in $[a, b]$. We have

$$m \le f(\alpha), f(\beta) \le M; \alpha, \beta \in [a, b]$$

$$\Rightarrow \qquad |f(\alpha) - f(\beta)| \le M - m; \qquad \text{...(1)}$$

and as such $M - m$ is an upper bound of the set in question.

Let $\varepsilon > 0$ be given.

Since M is the supremum of f, there exists a_1 $[a, b]$ such that

$$M > f(\alpha_1) > M - \frac{1}{2}\varepsilon. \qquad \text{...(2)}$$

Since m is the infimum of f, there exists $\beta_1 \in [a, b]$ such that

$$m < f(\beta_1) < m + \frac{1}{2}\varepsilon. \qquad \text{...(3)}$$

From (2) and (3), we have

$$f(\alpha_1) - f(\beta_1) > M - m - \varepsilon.$$

$$\Rightarrow \qquad |f(\alpha_1) - f(\beta_1)| \ge f(\alpha_1) - f(\beta_1) > M - m - \varepsilon.$$

There exist therefore a pair of numbers α_1, β_1 such that

$$\left|f(\alpha_1)-f(\beta_1)\right| > M-m-\varepsilon_1 \qquad \text{...(4)}$$

where $\varepsilon > 0$ is arbitrary, so that no number less than $M - m$ is an upper bound of the set in question.

From (1) and (4), it follows that $M - m$ is the supremum of the set of numbers

$$\left\{\left|f(\alpha)-f(\beta)\right|;\ \alpha,\beta \in [a,b]\right\}$$

6.7.3. Integrability of the Sum and Difference

If f and g are two functions both bounded and integrable in [a, b] then f ± g are also bounded and integrable in [a, b], and

$$\int_a^b \left[f(x) \pm g(x)\right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

(*Garhwal 90, 91, 93; Poorv. 91*)

Let

$$D\left\{a = x_0, x_1, \ldots, x_{r-1}, x_r, \ldots, x_n = b\right\}$$

by *any* division of $[a, b]$.

Let

$$M_r', m_r';\ M_r'', m_r'';\ M_r, m_r$$

be the bounds of f, g and $f + g$ in $\delta_r = [x_{r-1}, x_r]$. If α_1, α_2 be any two points of δ_r, we have

$$\left|\left[f(\alpha_2)+g(\alpha_2)\right]-\left[f(\alpha_1)+g(\alpha_1)\right]\right| \le \left|f(\alpha_2)-f(\alpha_1)\right| + \left|g(\alpha_2)-g(\alpha_1)\right|$$

$$\le (M_r' - m_r') + (M_r'' - m_r'')$$

$$\Rightarrow \qquad M_r - m_r \le (M_r' - m_r') + (M_r'' - m_r''). \qquad \text{...(1)}$$

Let $\varepsilon > 0$ be a given number.

Since f, g are integrable, there exists $\delta > 0$ such that for every division of norm $\le \delta$, the oscillatory sums of f and g are both less than $\frac{1}{2}\varepsilon$.

We now suppose that D is a division with norm $\le \delta$, so that for D, we have from (1),

$$\Sigma (M_r - m_r)\,\delta_r \le \Sigma (M_r' - m_r')\,\delta_r + \Sigma (M_r'' - m_r'')\,\delta_r < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

i.e., the oscillatory sum $\Sigma(M_r - m_r)\delta_r$ of $f+g$ for the division D is less than ε. Thus $f+g$ is integrable in $[a, b]$. Now to prove that

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Let ε be a positive number.

Since f, g are integrable, there exists $\delta > 0$ such that for every division of norm $\leq \delta$ and for every $\xi_r \in \delta_r$,

$$\left|\Sigma f(\xi_r)\delta_r - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2}, \left|\Sigma g(\xi_r)\delta_r - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2}.$$

It follows that

$$\left|\Sigma[f(\xi_r)+g(\xi_r)]\delta_r - \left[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right]\right| < \varepsilon.$$

Thus we obtain

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \; (\S\, 6.4)$$

The case of difference may be similarly discussed.

6.7.4. Integrability of Product

If f, g are two functions, both bounded and integrable in $[a, b]$, *then their product fg is also bounded and integrable in* $[a, b]$.

(Garhwal 90, Poorv. 92)

Since f, g are bounded, there exists k, such that

$$|f(x)| \leq k, |g(x)| \leq k, \forall\, x \in [a, b],$$

$$\Rightarrow |f(x)\,g(x)| \leq k^2, \forall\, x \in [a, b]$$

Thus fg is bounded.

Let

$$D\{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\}$$

be any division of $[a, b]$. Let

$$M_r', m_r'; M_r'', m_r''; M_r, m_r$$

be the bounds of f, g and fg in $\delta_r = [x_{r-1}, x_r]$. We have $\forall\, \alpha_1, \alpha_2 \in \delta_r$

$$f(\alpha_2)g(\alpha_2) - f(\alpha_1)g(\alpha_1) = g(\alpha_2)[f(\alpha_2) - f(\alpha_1)] + f(\alpha_1)[g(\alpha_2) - g(\alpha_1)]$$

$$\left| f(\alpha_2)\, g(\alpha_2) - f(\alpha_1)\, g(\alpha_1) \right| \le \left| g(\alpha_2) - f(\alpha_1) \right|$$

$$+ f \left| (\alpha_1) \right| \left| g(\alpha_2) - g(\alpha_1) \right|$$

$$\le k\,(M'_r - m'_r) + k\,(M''_r - m''_r).$$

$$\Rightarrow \qquad (M_r - m_r) \le k\,(M_r' - m_r') + k\,(M_r'' - m_r''). \qquad ...(1)$$

Now let ε be any positive number.

Since f, g are integrable, there exists $\delta > 0$, such that for every division of norm $\le \delta$, the oscillatory sums of f and g are both $< \varepsilon/2k$. We now suppose that D is a division of norm $\le \delta$, so that for D, we have, from (1).

$$\Sigma\,(M_r - m_r)\delta_r < k\,\Sigma\,(M_r' - m_r')\,\delta_r + k\Sigma\,(M_r'' - m_r'')\,\delta_r$$

$$< k\,(\varepsilon / 2k) + k\,(\varepsilon / 2k) = \varepsilon,$$

so that the oscillatory sum $\Sigma\,(M_r - m_r)\,\delta_r < \varepsilon$.

Hence fg is integrable in $[a, b]$.

Ex. Show by means of an example that the product of two non-integrable functions may be integrable.

6.7.5. Integrability of Quotient

If f, g are two functions, both bounded and integrable in $[a, b]$ and there exists a number, $t > 0$, such that $|\, g\,(x)\, | \ge t$ $\forall\, x \in [a, b]$, then f/g is bounded and integrable in $[a, b]$.

There exist positive numbers k and t such that $\forall\, x \in [a, b]$

$$\left| f(x) \right| \le k,\ \left| g(x) \right| \le k, \left| g(x) \right| \ge t.$$

Thus $\forall\ x \in [a, b]$

$$\left| f(x) / g(x) \right| \le k / t \Rightarrow f / g \text{ is bounded.}$$

Let

$$D \left\{ a = x_0, x_1, ..., x_{r-1}, x_r, ..., x_n = b \right\}$$

be a division of $[a, b]$ and let $M_r', m_r'; M_r'', m_r''; M_r, m_r$ be the bounds of $f, g, f/g$ in $\delta_r = \left[x_{r-1}, x_r\right]$. Now $\forall\, \alpha_1, \alpha_2 \in \delta_r$, we have

$$\left|\frac{f(\alpha_2)}{g(\alpha_2)} - \frac{f(\alpha_1)}{g(\alpha_1)}\right| = \left|\frac{g(\alpha_1)\left[f(\alpha_2) - f(\alpha_1)\right] - f(\alpha_1)\left[g(\alpha_2) - g(\alpha_1)\right]}{g(\alpha_1)\, g(\alpha_2)}\right|$$

$$\leq \left(k/t^2\right)\left|f(\alpha_2) - f(\alpha_1)\right| + \left(k/t^2\right)\left|g(\alpha_2) - g(\alpha_1)\right|$$

$$\leq \left(k/t^2\right)\left(M_r' - m_r'\right) + \left(k/t^2\right)\left(M_r'' - m_r''\right)$$

$$\Rightarrow \qquad \left(M_r - m_r\right) \leq \left(k/t^2\right)\left(M_r' - m_r'\right) + \left(k/t^2\right)\left(M_r'' - m_r''\right). \ldots(1)$$

Let, now ε be any positive number.

Since f, g are integrable, there exists a number $\delta > 0$ such that for every division D of norm $\leq \delta$, the oscillatory sums for f, g are both less than $t^2\varepsilon/2k$. Thus for division D of norm $\leq \delta$, we have from (1)

$$\Sigma\left(M_r - m_r\right)\delta_r \leq \left(k/t^2\right)\Sigma\left(M_r' - m_r'\right)\delta_r + \left(k/t^2\right)\Sigma\left(M_r'' - m_r''\right)\delta_r$$

$$< \left(k/t^2\right)\left(t^2\varepsilon/2k\right) + \left(k/t^2\right)\left(t^2\varepsilon/2k\right) = \varepsilon.$$

Hence f/g is bounded and integrable in $[a, b]$.

6.7.6. Integrability of the Modulus of an Integrable Function

If f is bounded and integrable in $[a, b]$, then $|f|$ is also bounded and integrable in $[a, b]$. (*Poorv. 90*)

Since there exists a positive number k such that $\forall\, x \in [a, b] \left|f(x)\right| \leq k$, the function $|f|$ is bounded.

Let ε be any positive number.

Since f is integrable, there exists a division

$$D\left\{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\right\}$$

such that the corresponding oscillatory sum for f is $< \varepsilon$.

Let $M_r', m_r'; M_r, m_r$ be respectively the bounds of f and $|f|$ in $\delta_r = [x_{r-1}, x_r]$.

Now $\forall\, \alpha_1, \alpha_2 \in \delta_r$, we have

$$\left|\left[\left|f(\alpha_2)\right| - \left|f(\alpha_1)\right|\right]\right| \leq \left|f(\alpha_2) - f(\alpha_1)\right|$$

$$\leq M_r' - m_r'$$

$$\Rightarrow \qquad M_r - m_r \leq M_r' - m_r'$$

This gives

$$\Sigma(M_r - m_r)\delta_r \le \Sigma(M_r' - m_r')\delta_r < \varepsilon,$$

$\Rightarrow$ $$\Sigma(M_r - m_r)\delta_r < \varepsilon.$$

Hence $|f|$ is integrable in $[a, b]$.

Remarks. The converse of this result is not true. If we take

$$f(x) = \begin{cases} 1 \text{ when } x \text{ is rational,} \\ -1, \text{ when } x \text{ is irrational,} \end{cases}$$

then $$\overline{\int_a^b} f(x)\,dx = (b-a), \underline{\int_a^b} f(x)\,dx = -(b-a)$$

so that f is not integrable.

But since $|f(x)| = 1 \ \forall \ x$, therefore

$$\int_a^b |f(x)|\,dx \text{ exists and is equal to } (b-a).$$

6.7.7. Definition

The meaning of

$$\int_a^b f(x)\,dx,$$

where $b \le a$.

If f be bounded and integrable in $[b, a]$ where $a > b$, then, by def.,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Also by def., $$\int_a^a f(x)\,dx = 0.$$

It is easy to show that the results about integrals obtained in §§ 6.6, 6.7 hold true when the upper limit is less than or equal to the lower limit.

Note. The reader may carefully note that the statement :

$$\int_a^b f(x)\,dx \text{ exists}$$

means that f is bounded and integrable in $[a, b]$.

6.7.5. Integrability of Quotient

If f, g are two functions, both bounded and integrable in $[a, b]$ *and there exists a number,* $t > 0$*, such that* $| g(x) | \geq t \ \forall x \in [a, b]$*, then* f/g *is bounded and integrable in* $[a, b]$.

There exist positive numbers k and t such that $\forall x \in [a, b]$

$$\left|f(x)\right| \leq k, \left|g(x)\right| \leq k, \left|g(x)\right| \geq t.$$

Thus $\forall \ x \in [a, b]$

$$\left|f(x)/g(x)\right| \leq k/t \Rightarrow f/g \text{ is bounded.}$$

Let

$$D\left\{a = x_0, x_1, \ldots, x_{r-1}, x_r, \ldots, x_n = b\right\}$$

be a division of $[a, b]$ and let $M_r', m_r'; M_r'', m_r''; M_r, m_r$ be the bounds of f, g, f/g in $\delta_r = \left[x_{r-1}, x_r\right]$. Now $\forall\, \alpha_1, \alpha_2 \in \delta_r$, we have

$$\left|\frac{f(\alpha_2)}{g(\alpha_2)} - \frac{f(\alpha_1)}{g(\alpha_1)}\right| = \left|\frac{g(\alpha_1)\left[f(\alpha_2) - f(\alpha_1)\right] - f(\alpha_1)\left[g(\alpha_2) - g(\alpha_1)\right]}{g(\alpha_1)\, g(\alpha_2)}\right|$$

$$\le \left(k/t^2\right)\left|f(\alpha_2) - f(\alpha_1)\right| + \left(k/t^2\right)\left|g(\alpha_2) - g(\alpha_1)\right|$$

$$\le \left(k/t^2\right)\left(M_r' - m_r'\right) + \left(k/t^2\right)\left(M_r'' - m_r''\right)$$

$$\Rightarrow \qquad \left(M_r - m_r\right) \le \left(k/t^2\right)\left(M_r' - m_r'\right) + \left(k/t^2\right)\left(M_r'' - m_r''\right) \ldots (1)$$

Let, now ε be any positive number.

Since f, g are integrable, there exists a number $\delta > 0$ such that for every division D of norm $\le \delta$, the oscillatory sums for f, g are both less than $t^2 \varepsilon / 2k$. Thus for division D of norm $\le \delta$, we have from (1)

$$\Sigma\left(M_r - m_r\right)\delta_r \le \left(k/t^2\right)\Sigma\left(M_r' - m_r'\right)\delta_r + \left(k/t^2\right)\Sigma\left(M_r'' - m_r''\right)\delta_r$$

$$< \left(k/t^2\right)\left(t^2 \varepsilon / 2k\right) + \left(k/t^2\right)\left(t^2 \varepsilon / 2k\right) = \varepsilon.$$

Hence f/g is bounded and integrable in $[a, b]$.

6.7.6. Integrability of the Modulus of an Integrable Function

If f is bounded and integrable in $[a, b]$, then $|f|$ is also bounded and integrable in $[a, b]$. *(Poorv. 90)*

Since there exists a positive number k such that $\forall\, x \in [a, b]\, |f(x)| \le k$, the function $|f|$ is bounded.

Let ε be any positive number.

Since f is integrable, there exists a division

$$D\left\{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\right\}$$

such that the corresponding oscillatory sum for f is $< \varepsilon$.

Let $M_r', m_r'; M_r, m_r$ be respectively the bounds of f and $|f|$ in $\delta_r = [x_{r-1}, x_r]$.

Now $\forall\, \alpha_1, \alpha_2 \in \delta_r$, we have

$$\left|\left[\left|f(\alpha_2)\right| - \left|f(\alpha_1)\right|\right]\right| \le \left|f(\alpha_2) - f(\alpha_1)\right|$$

$$\le M_r' - m_r'$$

$$\Rightarrow \qquad M_r - m_r \le M_r' - m_r'$$

This gives

$$\Sigma(M_r - m_r)\delta_r \leq \Sigma(M_r' - m_r')\delta_r < \varepsilon,$$

$$\Rightarrow \qquad \Sigma(M_r - m_r)\delta_r < \varepsilon.$$

Hence $|f|$ is integrable in $[a, b]$.

Remarks. The converse of this result is not true. If we take

$$f(x) = \begin{bmatrix} 1 \text{ when } x \text{ is rational,} \\ -1, \text{ when } x \text{ is irrational,} \end{bmatrix}$$

then $$\int_a^{\bar{b}} f(x)\,dx = (b-a), \int_{\underline{a}}^{b} f(x)\,dx = -(b-a)$$

so that f is not integrable.

But since $|f(x)| = 1 \ \forall \ x$, therefore

$$\int_a^b |f(x)|\,dx \text{ exists and is equal to } (b - a$$

6.7.7. Definition

The meaning of

$$\int_a^b f(x)\,dx,$$

where $b \leq a$.

If f be bounded and integrable in $[b, a]$ where $a > b$, then, by

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Also by def., $$\int_a^a f(x)\,dx = 0.$$

It is easy to show that the results about integrals obtained in § 6.7 hold true when the upper limit is less than or equal to the lower

Note. The reader may carefully note that the statement :

$$\int_a^b f(x)\,dx \text{ exists}$$

means that f is bounded and integrable in $[a, b]$.

6.7.8. Inequalities for an Integral

Theorem. If f is bounded and integrable in $[a, b]$, and M, m are the bounds of f in $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \text{ if } b \geq a.$$

$$m(b-a) \geq \int_a^b f(x)\,dx \geq M(b-a) \text{ if } b \leq a. \quad (\textit{Allahabad 99})$$

For $a = b$, the result is trivial.

If $b > a$, then for any division D, we have

$$m(b-a) \leq \int_a^b f(x)\,dx \leq S(D) \leq M(b-a) \qquad (\S\ 6.2)$$

$$\Rightarrow \qquad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

If $b < a$, *i.e.*, $a > b$, then, as proved above,

$$m(a-b) \leq S(D) \leq \int_b^a f(x)\,dx \leq M(a-b)$$

$$\Rightarrow \qquad -m(a-b) \geq -\int_b^a f(x)\,dx \geq -M(a-b)$$

$$\Rightarrow \qquad m(b-a) \geq \int_a^b f(x)\,dx \geq M(b-a).$$

Hence the results.

Cor. 1. *If f is bounded and integrable in $[a, b]$, then there exists a number, μ, lying between the bounds of f such that*

$$\int_a^b f(x)\,dx = \mu(b-a). \qquad (\textit{Garhwal 93, Ajmer 99})$$

Cor. 2. *If f is continuous in $[a, b]$, then there exists a number, c, lying between a and b such that*

$$\int_a^b f(x)\,dx = (b-a)\,f(c).$$

(*Allahabad 99; Lucknow 92; Garhwal 94, 97*)

Cor. 3. *If f is bounded and integrable in $[a, b]$, and, k is a number such that $\forall\, x \in [a, b], |f(x)| \leq k$,* (*Garhwal 98*)

then

$$\left|\int_a^b f(x)\,dx\right| \leq k\,|b-a|.$$

For $a = b$, the result is trivial.

We have $\forall\, x \in [a, b]$,

$$-k \le f(x) \le k,$$

so that if M, m be the bounds of f in $[a, b]$,

$$-k \le m \le f(x) \le M \le k,\ \forall\, x \in [a, b]. \qquad \text{...(1)}$$

Let $b > a$. Therefore, from 1,

$$-k(b-a) \le m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \le k(b-a).$$

$$\Rightarrow \qquad \left|\int_a^b f(x)\,dx\right| \le k\,|b-a|.$$

Let $b < a$. We have, from above

$$\left|\int_a^b f(x)\,dx\right| \le k\,|a-b| \Rightarrow \left|\int_a^b f(x)\,dx\right| \le k\,|b-a|.$$

Cor. 4. *If f is bounded and integrable in $[a, b]$ and*

$$\forall\, x \in [a, b],\ f(x) \ge 0,$$

then
$$\int_a^b f(x)\,dx \begin{cases} \ge 0, & \text{when } b \ge a, \\ \le 0, & \text{when } b \le a. \end{cases}$$

For $b = a$, the result is trivial.

Now $f(x) \ge 0\ \forall\, x \in [a, b] \Rightarrow m \ge 0$.

Let $b > a$. We have

$$\int_a^b f(x)\,dx \ge m(b-a) \ge 0. \qquad \because (b-a) \ge 0$$

Let $b < a$. We have, as proved above,

$$\int_b^a f(x)\,dx \ge 0.$$

$$\Rightarrow \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \le 0.$$

Cor. 5. *If*

$$\int_a^b f(x)\,dx,\ \int_a^b g(x)\,dx$$

both exist, then

$$f \geq g \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \text{ when } b \geq a,$$

$$f \geq g \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \text{ when } b \leq a.$$

Under the given condition $[f(x) - g(x)]$ is integrable and ≥ 0, $\forall\, x \in [a, b]$.

Therefore

$$\int_a^b [f(x) - g(x)]\,dx \geq 0 \text{ or } \leq 0,$$

according as $b \geq a$ or $b \leq a$

$$\left[\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right] \geq 0 \text{ or } \leq 0,$$

according as $b \geq a$ or $b \leq a$.

Hence the result.

Cor. 6. *If*

$$\int_a^b |f(x)|\,dx$$

exists, then

$$\left|\int_a^b f(x)\,dx\right| \leq \left|\int_a^b |f(x)|\,dx\right|$$

It has been shown in § 6.7.6, page 217 that

$$\int_a^b |f(x)|\,dx$$

exists. We have $\forall\, x \in [a, b]$,

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

If $b \geq a$, we have

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx, \qquad \textit{(cor. 5)}$$

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx = \left|\int_a^b |f(x)|\,dx\right|.$$

If $b \leq a$, we have, as proved above,

$$\left|\int_a^b f(x)\,dx\right| \leq \left|\int_b^a |f(x)|\,dx\right|,$$

$$\Rightarrow \quad \left|\int_a^b f(x)\,dx\right| \leq \left|\int_a^b |f(x)|\,dx\right|.$$

.7. PROPERTIES OF INTEGRABLE FUNCTIONS

6.7.1. *If a bounded function f is integrable in $[a, b]$, then it is also integrable in $[a, c]$ and $[c, b]$ where c is a point of $[a, b]$.*

Conversely, if f is bounded and integrable in [a, c], [c, b], then it is also integrable in [a, b].

Also in either case

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, a < c < b.$$

(*Garhwal 98; Kumaon 97, 98; Allahabad 98*)

Suppose that f is bounded and integrable in $[a, b]$.

Let ε be a positive number.

There exists $\delta > 0$ such that for each division of $[a, b]$ whose norm is $\leq \delta$, the oscillatory sum is $< \varepsilon$. Let D be a division of $[a, b]$ such that $c \in D$ and $| D | \leq \delta$.

The oscillatory sum for the division D breaks itself into two parts, respectively consisting of the terms which arise from the sub-intervals $[a, c]$ and $[c, b]$. Since the terms of an oscillatory sum are all positive, each part must itself be $< \varepsilon$. Hence f is integrable both in $[a, c]$ and $[c, b]$.

Let now, f be obtained and integrable in $[a, c]$ and $[c, b]$.

Let ε be a positive number. There exist divisions of $[a, c]$ and $[c, b]$ such that the corresponding oscillatory sums are $< \varepsilon/2$. The divisions of $[a, c]$ and $[c, b]$ give rise to a division of $[a, b]$ for which the oscillatory sum is $< (\varepsilon/2 + \varepsilon/2) = \varepsilon$. Hence f is integrable in $[a, b]$.

The relationship of equality is to be proved now.

Let ε be a positive number.

As f is simultaneously integrable in $[a, c]$, $[c, b]$ and $[a, b]$, there exists $\delta > 0$ such that for divisions of norm $\leq \delta$, and of which c, is a point, we have

$$\left| \sum_{(a,c)} f(\xi_r)\,\delta_r - \int_a^c f(x)\,dx \right| < \frac{\varepsilon}{3}, \left| \sum_{(c,b)} f(\xi_r)\,\delta_r - \int_c^b f(x)\,dx \right| < \frac{\varepsilon}{3}$$

$$\left| \sum_{(a,b)} f(\xi_r)\,\delta_r - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{3};$$

where the meanings of the symbols $\sum_{(a,c)} f(\xi_r)\,\delta_r$, etc. are obvious.

Since $$\sum_{(a,c)} f(\xi_r)\,\delta_r + \sum_{(c,b)} f(\xi_r)\,\delta_r = \sum_{(a,b)} f(\xi_r)\,\delta_r,$$

we deduce that

$$\left| \int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx \right| < \varepsilon,$$

$$\Rightarrow \qquad \int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx = 0;$$

ε being an positive number.

Cor. *If f is bounded and integrable in $[a, b]$, then it is also bounded and integrable in $[\alpha, \beta]$ where $a < \alpha < \beta < b$.*

f is integrable in $[a, b] \Rightarrow f$ is integrable in $[a, \beta]$

$\Rightarrow f$ is integrable in $[\alpha, \beta]$

6.9.1. First Mean Value Theorem

If

$$\int_a^b f(x)\,dx \text{ and } \int_a^b \varphi(x)\,dx,$$

both exist and φ (x) keeps the same sign, positive or negative, throughout the interval of integration, then there exists a number, μ, lying between the bounds of f such that

$$\int_a^b f(x)\,\varphi(x)\,dx = \mu \int_a^b \varphi(x)\,dx. \qquad \text{...(1)}$$

First suppose that $\varphi(x)$ is positive $\forall\, x \in [a, b]$. If M, m be the bounds of f, we have $\forall\, x \in [a, b]$

$$m \le f(x) \le M$$

$$\Rightarrow \qquad m\varphi(x) \le f(x)\,\varphi(x) \le M\,\varphi(x), \text{ for } \varphi(x) \ge 0\ \forall\, x.$$

Thus

$$m\int_a^b \varphi(x)\,dx \leq \int_a^b f(x)\,\varphi(x)\,dx \leq M\int_a^b \varphi(x)\,dx \text{ if } b \geq a$$

$$m\int_a^b \varphi(x)\,dx \geq \int_a^b f(x)\,\varphi(x)\,dx \geq M\int_a^b \varphi(x)\,dx \text{ if } b \leq a$$

In either case we see that there exists a number μ, lying between M and m, such that (1) is true. Hence the result.

The case when φ is negative may be similarly disposed off.

Cor. In addition to the conditions of the theorem, if f is continuous also, then there exists a number, ξ, belonging to the domain of integration such that

$$\int_a^b f(x)\,\phi(x)\,dx = f(\xi)\int_a^b \varphi(x)\,dx.$$

ϕ is negative

$m \leq f(x) \leq M$

$\Rightarrow m\phi(x) \geq f(x)\phi(x) \geq M\phi(x)$ for $\phi(x) \leq 0$

$m\int_a^b \phi(x)dx \geq \int_a^b f(x)\phi(x)dx \geq M\int_a^b \phi(x)dx$ if $b \geq a$

$m\int_a^b \phi(x)dx \leq \int_a^b f(x)\phi(x)dx \leq M\int_a^b \phi(x)dx$ if $b \leq a$

$\exists$ μ between M & m suchthat (1) is true.

Hence the result.

Second MVT (Integration)

If $\int_a^b f(x)dx$ and $f(\xi)\int_a^b \varphi(x)dx$ both exist and φ is monotonic in $[a,b]$

Then There exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)\phi(x)\,dx = \phi(a)\int_a^{\xi} f(x)\,dx + \phi(b)\int_{\xi}^{b} f(x)\,dx$$

Abel's Lemma

(proof of 2nd MVT depends upon this lemma)

Statement

If

(i) $a_1, a_2 \dots a_n$ is a monotonically decreasing set of n positive numbers

(ii) $v_1, v_2 \dots v_n$ is a set of any n numbers

(iii) k, K are two numbers such that

$$k < v_1 + v_2 + \dots + v_p < K \quad \text{for } 1 \le p \le n$$

then

$$a_1 k < a_1 v_1 + a_2 v_2 + \dots + a_n v_n < a_1 K$$

$$a_1 k < \sum_{r=1}^{n} a_r v_r < a_1 K$$

$$S_p = v_1 + v_2 + \dots + v_p$$

$$v_1 = S_1 \qquad v_2 = S_2 - v_1 = S_2 - S_1$$

$$v_3 = S_3 - v_2 - v_1 = S_3 - (S_2 - S_1) - S_1$$
$$= S_3 - S_2$$

$$\vdots$$

$$v_n = S_n - S_{n-1}$$

we have

$$\sum_{r=1}^{n} a_r v_r = a_1 S_1 + a_2 (S_2 - S_1) + \dots + a_r (S_r - S_{r-1}) + \dots + a_n (S_n - S_{n-1})$$

$$= (a_1 - a_2) S_1 + (a_2 - a_3) S_2 + \dots + (a_{n-1} - a_n) S_{n-1} + a_n S_n$$

Since $a_1, a_2 \dots$ are positive & monotonially decreasing

$a_1 > a_2 > a_3 \dots$

$\therefore (a_1 - a_2), (a_2 - a_3) \dots (a_{n-1} - a_n), a_n$

are all positive. and by (iii)

$k < S_p < k \quad \forall \; p \leq n$

Therefore

$$\sum_{r=1}^{n} a_r V_r < (a_1 - a_2) K + (a_2 - a_3) K + \dots + (a_{n-1} - a_n) K + a_n K = a_1 K$$

$$\sum_{r=1}^{n} a_r v_r > (a_1 - a_2)k + (a_2 - a_3)k + \cdots$$

$$+ (a_{n-1} - a_n)k + a_n K$$

$$= a_1 K$$

Hence the

Proof of the theorem. Firstly, we prove the following :

lemma

If

$$\int_a^b f(x)\,dx \text{ and } \int_a^b \psi(x)\,dx$$

both exist. ψ *is monotonically* **decreasing** *and* **positive** *in* $[a, b]$, *then there exists a point,* $\xi \in [a, b]$ *such that*

$$\int_a^b f(x)\,\psi(x)\,dx = \psi(a)\int_a^{\xi} f(x)\,dx.$$

(This result is due to Bonnett).

Let

$$D\left\{a = x_0, x_1, \ldots, x_{r-1}, x_r, \ldots, x_n = b\right\}$$

be any division of $[a, b]$. Let M_r, m_r be the bounds of f in $\delta_r = \left[x_{r-1}, x_r\right]$. Let $\xi_1 = a$ and ξ_r, when $r \neq 1$, be any point of δ_r.

We have,

$$m_r\delta_r \le \int_{x_{r-1}}^{x_r} f(x)\,dx \le M_r\delta_r,\; m_r\delta_r \le f(\xi_r)\,\delta_r \le M_r\delta_r.$$

Putting $r = 1, 2, 3 \ldots P$ $\quad P \leq n$

$$m_1 \delta_1 \leq \int_{a=x_0}^{x_1} f(x)\,dx \leq M_1 \delta_1$$

$$m_2 \delta_2 \leq \int_{x_1}^{x_2} f(x)\,dx \leq M_2 \delta_2$$

$$m_p \delta_p \leq \int_{x_{p-1}}^{x_p} f(x)\,dx \leq M_p \delta_p$$

$$\therefore \sum_{r=1}^{p} m_r \delta_r \leq \int_{a}^{x_p} f(x)\,dx \leq \sum_{r=1}^{p} M_r \delta_r$$

$$0 \le \left| \int_a^{x_p} f(x)\,dx - \sum_{r=1}^{p} m_r \delta_r \right| \le \left| \sum_{r=1}^{p} (M_r - m_r)\delta_r \right|$$

$$0 \le \left| \sum_{r=1}^{p} f(\xi_r)\delta_r - \sum_{r=1}^{p} m_r \delta_r \right| \le \left| \sum_{r=1}^{p} (M_r - m_r)\delta_r \right|$$

thus we have

$$\left| \int_a^{x_p} f(x)\,dx - \sum_{r=1}^{p} f(\xi_r)\delta_r \right| \le \sum_{r=1}^{p} (M_r - m_r)\delta_r$$

$$\le \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

y

$$\left| \int_a^{x_p} f(x)\,dx - \sum_{r=1}^{p} f(\xi_r)\delta_r \right| \le \sum_{r=1}^{n} O_r \delta_r$$

$$\Rightarrow \int_a^{x_p} f(x)\,dx - \sum_{r=1}^{n} O_r \delta_r \le \sum_{r=1}^{p} f(\xi_r)d_r$$

$$\le \int_a^{x_p} f(x)\,dx + \sum_{r=1}^{n} O_r \delta_r$$

where $O_r = (M_r - m_r)$ is the oscillation of f in δ_r.

Now, $\int_a^t f(x)\,dx$, being a continuous function (§ 4.6.1, § 4.6.2) with t as variable, is bounded. Let C, D be its bounds. Therefore we have

$$C - \sum_{r=1}^{r=n} O_r \delta_r \leq \sum_{r=1}^{r=p} f(\xi_r)\,\delta_r \leq D + \sum_{r=1}^{r=n} O_r \delta_r .$$

In the statement of the Abel's lemma, we put, as is justifiable,

$$v_r = f(\xi_r)\,\delta_r, \quad a_r = \psi(\xi_r);$$

$$k = C - \Sigma\, O_r \delta_r, \quad K = D + \Sigma\, O_r \delta_r,$$

$v_1 = f(\xi_1)\delta_1$ $\qquad$ $a_1 = \psi(\xi_1) = \psi(a)$

$r_2 = f(\xi_2)\delta_2$

$\vdots$

by abel's lemma $\quad a_1 K < \sum_{r=1}^{n} a_r v_r < a_1 K$

$$\therefore \psi(a)\left[C - \sum_{r=1}^{n} O_r \delta_r\right] \le \sum_{r=1}^{n} f(\xi_r)\psi(\xi_r)\delta_r$$

$$\le \psi(a)\left[D + \sum_{r=1}^{n} \sigma_r \delta_r\right]$$

Let the norm of division tends to 0 $\Rightarrow \left(\delta_r \to 0\right.$ & $\left.\sum_{r=1} f(\xi_r)\delta_r \to \int_a f(x)dx\right)$

$$\Rightarrow C\psi(a) \le \int_a^b f(x)\psi(x)dx \le D\psi(a)$$

$$\Rightarrow \qquad \int_a^b f(x)\,\psi(x)\,dx = \mu\,\psi(a),$$

where μ is some number between C and D.

The continuous function

$$\int_a^t f(x)\,dx$$

must assume, for some $\xi \in [a, b]$ the value μ which lies between its bounds C, D. (Cor. 2 to § 4.6.4). Thus we obtain

$$\int_a^b f(x)\,\psi(x)\,dx = \psi(a)\int_a^\xi f(x)\,dx.$$

We now turn to the theorem proper.

Let φ be monotonically decreasing so that the function ψ where

$$\psi(x) = \varphi(x) - \varphi(b)$$

is monotonically decreasing and positive.

There exists, therefore, a number, ξ, between a and b, such that

$$\int_a^b f(x)\left[\varphi(x) - \varphi(b)\right]dx = \left[\varphi(a) - \varphi(b)\right]\int_a^\xi f(x)\,dx$$

$$\Rightarrow \qquad \int_a^b f(x)\,\varphi(x)\,dx = \varphi(a)\int_a^\xi f(x)\,dx$$

$$+\,\varphi(b)\left\{\int_a^b f(x)\,dx - \int_a^\xi f(x)\,dx\right\}$$

$$= \varphi(a)\int_a^\xi f(x)\,dx + \varphi(b)\int_\xi^b f(x)\,dx.$$

Let φ be monotonically increasing so that, $-\varphi$, is monotonically decreasing.

There exists, therefore, by the preceding, a number ξ between a and b, such that

$$\int_a^b f(x)\left[-\varphi(x)\right]dx = -\varphi(a)\int_a^\xi f(x)\,dx - \varphi(b)\int_\xi^b f(x)\,dx,$$

$$\Rightarrow \quad \int_a^b f(x)\,\varphi(x)\,dx = \varphi(a)\int_a^\xi f(x)\,dx + \varphi(b)\int_\xi^b f(x)\,dx.$$

Thus we have completely established the second mean value theorem.

Note. The reader may easily show that the theorem holds good even if $a > b$.

6.9.1. First Mean Value Theorem

If

$$\int_a^b f(x)\,dx \text{ and } \int_a^b \varphi(x)\,dx,$$

both exist and $\varphi(x)$ *keeps the same sign, positive or negative, throughout the interval of integration, then there exists a number,* μ, *lying between the bounds of* f *such that*

$$\int_a^b f(x)\,\varphi(x)\,dx = \mu \int_a^b \varphi(x)\,dx. \qquad ...(1)$$

First suppose that $\varphi(x)$ is positive $\forall\ x \in [a, b]$. If M, m be the bounds of f, we have $\forall\ x \in [a, b]$

$$m \leq f(x) \leq M$$

$$\Rightarrow \qquad m\varphi(x) \leq f(x)\,\varphi(x) \leq M\,\varphi(x), \text{ for } \varphi(x) \geq 0\ \forall\ x.$$

Thus

$$m \int_a^b \varphi(x)\, dx \le \int_a^b f(x)\, \varphi(x)\, dx \le M \int_a^b \varphi(x)\, dx \text{ if } b \ge a$$

$$m \int_a^b \varphi(x)\, dx \ge \int_a^b f(x)\, \varphi(x)\, dx \ge M \int_a^b \varphi(x)\, dx \text{ if } b \le a$$

In either case we see that there exists a number μ, lying between M and m, such that (1) is true. Hence the result.

The case when φ is negative may be similarly disposed off.

Cor. In addition to the conditions of the theorem, if f is continuous also, then there exists a number, ξ, belonging to the domain of integration such that

$$\int_a^b f(x)\, \phi(x)\, dx = f(\xi) \int_a^b \varphi(x)\, dx.$$

φ is negative

$$m \leq f(x) \leq M \quad \text{for } \varphi(x) \leq 0$$

$$\Rightarrow m\varphi(x) \geq f(x)\varphi(x) \geq M\varphi(x)$$

Of

$$m\int_a^b \varphi(x)dx \geq \int_a^b f(x)\varphi(x)dx \geq M\int_a^b \varphi(x)dx \quad \text{if } b \geq a$$

$$m\int_a^b \varphi(x)dx \leq \int_a^b f(x)\varphi(x)dx \leq M\int_a^b \varphi(x)dx \quad \text{if } b \leq a$$

$\exists$ μ between M & m such that (1) is true.

Hence the result.

Unit 3

1) S.T $f(x) = |x|$ is not deri at $x=0$

2) S.T Every deriv. is continuous

3) Darboux Thm on derivatives
5m

4) State & prove Inverse function
5m theorem.

5) Chain rule on Diff.
10m

6) If $f(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ find $f'(0)$
2m

7) cf is derivable at x_0.
2m

8) $f(x_0) \neq 0$, Prove that $\frac{1}{f}$ is
5m differentiable at x_0.

9) f cts 1-1 on I & $f'(x_0)$ exists $\neq 0$
10m P.T inv of f is der. at $f(x_0)$ and its der. at $f(x_0)$ is $\frac{1}{f'(x_0)}$

10) $f(x) = x$ find $f'(x)$
2m

11) Give example for cts but not diffble
2m

12) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
5m

13) $f(x) = x^n$ find f'
2m

14) $f(x) = x\sin\frac{1}{x}$
2m
$f(0)=0$
f is cts at 0.

15) $f(x) = |x-1|$
5m
$Rf'(1) = 1$ & $Lf'(1) = -1$

16) Inv. fun thm for derivatives
10m

17) If f+g diffble then f, g diff ?
5m

18) If fg diffble then f, g diff?
5m

Unit 4

1) In $[a,b]$ verify Rolle's thm for
2m

$f(x) = (x-a)(b-x)$

2) Find c of Lagrange's MVT of
2m

$f(x) = x^3$ on $[-2,1]$

3) Rolle's theorem
5m

4) Taylor's development of a function
5m
in a finite form

5) $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} \cdots\cdots$
10m

6) Verify Rolle's thm $f(x) = x^2 : x \in [-1,1]$
2m

7) Writedown Mc. series $\log(1+x)$
2m

8) Cauchy MVT
5m

9) $f(x) = x(x-1)(x-2)$, $a=0$
5m
$b = \frac{1}{2}$ then $\frac{f(b)-f(a)}{b-a} = f'(c)$

10) Taylors thm
10m

11) verify Rolles thm $f(x) = x^2$ in $[-1,1]$
2m

12) State MC Series
2m

13) LMVT $f(x) = x^3$, $a=-2$, $b=1$
5m

14) Gen. MVT
5m

15) Rolles thm $f(x) = \cos x$ in $[-\pi/2, \pi/2]$
2m

Unit 5

2m 1) Let $f(x) = x$ on $[0, 1]$

$P = \{0, \frac{1}{2}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$ Find $L(P, f)$

2m 2) Define Refinement of a partition

5m 3) Prove that $|f| \in R[a,b]$ and $\left| \int_a^b f \right| \leq \int_a^b |f|$

5m 4) Fundamental thm of calculus.

10m 5) $f, g \in R(x)$ on $[a,b]$ P.T $fg \in R(x)$

2m 6) P.T $f(x) = \begin{cases} 0 & \text{if } x \text{ is rat} \\ 1 & \text{if } x \text{ is irrat} \end{cases}$ does not have integral

2m 7) Fundamental thm of calculus.

5m 8) Darboux's thm I

5m 9) f monotonic, P.T intble on $[a, b]$

10m 10) 2nd MVT in integration.

2m 11) Oscillatory sum 12) Condition for intbility

5m 12) $f(x) = x$ P.T $\int_0^1 f(x)\,dx = \frac{1}{2}$

5m 13) P.T cos fun is intble

10m 14) Funda. Int & 1st MVT

Question bank

Unit I

1) 2m - Define abs. value

2) 2m - Define infimum of set

3) 5m. ST there is no rational number whose sequ. square is 5

4) Prove that [0,1] is uncountab

5m

5) Ctble Union of ctb sets is ctble

10m

6) write the order axioms.

2m

7) Define the sup.

2m

8) $|x+y| \leq |x| + |y|$

5m

9) Field axioms

5m

10) $x(-y) = -(xy)$

2m

11) Find sup $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\}$

2m

12) $|x-y| \geq ||x|-|y||$, $|x| = |-x|$

5m

13) NXN is ctble

5m

14) prove that Q is ctble

10m

15) P.T $|x| = \max\{-x, x\}$

2m

~~16) $\lim_{x \to c} f(x) = l$ then $\lim_{x \to c} |f(x)| = |l|$~~

5m

16) $A_1, A_2 \ldots$ Ctble $\bigcup_{n=1}^{\infty}$ An is ctble

5m

17) Trichotomy prop 18) Superset is unctble.

2m 2m

19) Canc laws of multiplication

5m

20) (i) $x(-y) = -xy$ (ii) $(-x)y = -(xy)$

3m

(iii) $(-x)(-y) = xy$

21) Find Sup (i) $\{3,4\}$ (ii) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \ldots$

2m

(iii) $\{x+1, x+\frac{1}{2}, x+\frac{1}{3} + \ldots\}$

22) Q in [0,1] is ctble

10m

Unit 2

1) Define limit
2m

2) Define Discontinuity 1st kind
2m

3) If $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$
5m
Then $\lim_{x\to a} f(x)g(x) = AB$

4) State & prove Intermed. Val. Th
5m

5) $\lim_{x\to 0} (x \sin x)$
10m

6) Define cts function
2m

7) Define remov. disct
2m

8) Explain discts fun of 1st kind & 2nd kind with examples
5m

9) $\lim_{x\to a} g(x) = m$. P.T $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{m}$
10m

10) If ∃ $\delta > 0$ such that $h(x) = 0$
2m
whenever $0 < |x-a| < \delta$ P.T $\lim_{x\to a} h(x) = 0$

11) $f(x) = x \sin\frac{1}{x}$ P.T $\lim_{x\to\infty} f(x) = 1$
5m

12) S.T $f(x) = x^2$ is not Unif. cts
5m

13) Inv. fun. thm cry.
10m

14) $\lim_{x\to} f(x) = l$ then $\lim_{x\to c} |f(x)| = |l|$
5m

15) f cts on closed & bdd on [a,b]
5m
P.T f is unif. cts on I.

16) Draw $f(x) = [x]$ 17) Def. U.C
2m 2m

18) $f(x) = \begin{cases} 1-2x & x<0 \\ 0 & x=0 \\ 1+3x & x>0 \end{cases}$ Find $\lim_{x\to 0} f(x)$
5m

19) f cl.c then pt f is cts.
5m

20) G open, $f^{-1}(x)$ is open (open mapi
10m