SWAMI DAYANANDA
COLLEGE OF ARTS & SCIENCE,
MANJAKKUDI-612610

DEPERTMENT OF MATHEMATICS

Operations Research(16SMBEMM1:1)

Study Material

Class : III-B.Sc Mathematics

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MAJOR BASED ELECTIVE I (A)

OPERATIONS RESEARCH

Objectives:

1. To introduce the various techniques of Operations Research.
2. To make the students solve real life problems in Business and Management

UNIT I

- Linear programming problem - Mathematical formulation – Illustrations on Mathematical formulation on Linear Programming Problems – Graphical solution method - some exceptional cases - Canonical and standard forms of Linear Programming Problem - Simplex method.

UNIT II

- Use of Artificial Variables (Big M method - Two phase method) – Duality in Linear Programming - General primal-dual pair - Formulating a Dual problem - Primal-dual pair in matrix form -Dual simplex method.

UNIT III

- Transportation problem - LP formulation of the TP - Solution of a TP - Finding an initial basic feasible solution (NWCM - LCM -VAM) – Degeneracy in TP - Transportation Algorithm (MODI Method) - Assignment problem - Solution methods of assignment problem – special cases in assignment problem.

UNIT IV

- Queuing theory - Queuing system - Classification of Queuing models - Poisson Queuing systems Model I (M/M/1)(∞/FIFO) only - Games and Strategies – Two person zero sum - Some basic terms - the maximin-minimax principle -Games without saddle points- Mixed strategies - graphic solution 2xn and mx2 games.

UNIT V

- PERT and CPM – Basic components – logical sequencing - Rules of network construction- Critical path analysis - Probability considerations in PERT.

Book for Study:

- Kanti Swarup, P.K. Gupta and ManMohan, Operations Research, 13th edition, Sultan Chand and Sons, 2007.

Unit 1: Chapter 2 Sec 2.1 to 2.4, Chapter 3 Sec 3.1 to 3.5, Chapter 4 Sec 4.1 , 4.3

Unit 2: Chapter 4 Sec 4.4, Chapter 5 Sec 5.1 to 5.4, 5.9

Unit 3: Chapter 10 Sec 10.1, 10.2, 10.8, 10.9, 10.12, 10.13, Chapter 11 Sec 11.1 to 11.4

Unit 4: Chapter 21 Sec 21.1, 21.2, 21.7 to 21.9, Chapter 17 Sec 17.1 to 17.6

Unit 5: Chapter 25 Sec 25.1 to 25.4, 25.6, 25.7

Book for Reference:

- 1. Sundaresan.V, Ganapathy Subramanian. K.S. and Ganesan.K, Resource Management Techniques, A.R. Publications, 2002.
- 2. Taha H.A., Operations Research: An introduction, 7th edition, Pearson Prentice Hall, 2002.

UNIT-1

Origin and Development of O.R

During World war II, military management called on scientists from various disciplines and organised strategic and tactical problems. (i.e) to discuss, evolve and suggest ways and means to improve the execution of various military projects. By their joint efforts, experience and deliberations. They suggested certain approaches that systematic and scientific study of the operations of the system was called the Operations Research or Operational Research (abbreviated as O.R).

Definition of O.R

O.R is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem.

Significant features of O.R

a) Decision - making
b) Scientific Approach
c) Objective
d) Inter - disciplinary Team Approach
e) Digital computer.

a) Decision - making

Primarily, O.R is addressed to managerial decision-making or problem - solving. A major premise of O.R is that decision-making, irrespective of the situation involved, can be considered as a general systematic process.

b) Scientific Approach

O.R employs scientific methods for the purpose of solving problems. It is a formalised process of reasoning.

c) Objective

O.R attempts to locate the best or optimal solution to the problem under consideration. For this purpose, it is necessary that a measure of effectiveness is defined which is based on the goals of the organisation. This measure is then used as the basis to compare the alternative courses of action.

d) Inter-disciplinary Team Approach

OR is inter-disciplinary in nature and requires a team approach to a solution of the problem. Managerial Problems have economic, physical, psychological, biological, sociological and engineering aspects. This requires a blend of people with expertive in the areas of mathematics, statics, engineering, economics, managerent, computer science and so on.

19.06.19 e) Digital Computer

Use of a digital computer has become an integral part of the O.R approach to decision making. The computer may be required due to the complexity of the model, volume of data required and the computations to be made.

Modelling in Operations Research

A model in O.R is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered.

Types of model are

i) Account model

ii) Mathematical model

iii) Quantitative model

iv) Physical model.

~~The main characteristic~~

The main characteristics of good model in O.R

1. A good model should be capable of taking into account new formulations without having any significant change in its frame.

2. Assumptions made in the model should be as small as possible.

3. It should be simple and coherent. Number of variables used should be less.

4. It should be open to parametric type of treatment.

5. It should not take much time in its construction for any problem.

Limitations of model

1. Models are only an attempt in understanding operations and should never be considered as absolute in any sense.

2. Validity of any model with regard to corresponding operation only be verified by carrying the experiment and relevant data characteristics.

Linear programming problem

Linear programming is a technique for determining an optimum schedule of interdependent activities in view of the available resources.

Programming is just another word for planning and refers to the process of determining a particular plan of action from amongest several alternatives.

The word linear stands for indicating that all relationships involved in a particular problem are linear.

Mathematical Formulation of the problem.

Step-1:

Study the given situation to find the key decision to be made.

Step-2:

Identify the variables involved and designate them by symbols x_j $(j=1,2,3,\ldots)$

Step-3

State the feasible alternatives which generally are: $x_j \geq 0$, for all j.

Step-4

Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variable.

Step-5

Identify the objective function and express it as a linear function of the decision variables.

21.06.19 Problem

1, A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of clothes namely suitings, shirtings and wollens yielding profit of Rs 2, Rs 4 and Rs 2 per metre respectively. One metre of suiting requires 2 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly, one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of wollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively.

Formulate the linear programming problem to find the product mix to maximize the profit.

Soln:-

x_1 - suiting (in metres)

x_2 - Shirting (in metres)

x_3 - Wollen (in metres)

$\max z = 2x_1 + 4x_2 + 3x_3$

Subject to the constraints

$3x_1 + 4x_2 + 3x_3 \leq 60$ (in hrs)

$2x_1 + x_2 + 3x_3 \leq 40$ (in hrs)

$x_1 + 3x_2 + 3x_3 \leq 80$ (in hrs)

$\max z = 2x_1 + 4x_2 + 3x_3$

Subject to the constraints

$3x_1 + 4x_2 + 3x_3 \leq 3600$ (in min)

$2x_1 + x_2 + 3x_3 \leq 2400$ (in min)

$x_1 + 3x_2 + 3x_3 \leq 4800$ (in min)

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

2, A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. The size B contains

1 grain of aspirin, 8 grains of bicarbonate and 6 grains of Codeine. It is found by users that it requires atleast 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP.

Soln:-

x_1 - Size A

x_2 - Size B

$\min z = x_1 + x_2$

Subject to the constraints

$$2x_1 + x_2 \geq 12$$

$$5x_1 + 8x_2 \geq 74$$

$$x_1 + 6x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

24.6.19

3, An animal feed company must produce 200 lbs of a mixture containing the ingrediants x_1 and x_2. x_1 costs Rs 3 per lb and x_2 costs Rs. 8 per lb. Not more than 80 lbs of x_1 can be used and minimum quantity to be used for x_2 is 60 lbs. Find how much of each ingredient should be used if the company wants to minimise the cost. Formulate.

Soln:-

x_1 - ingredient x_1

x_2 - ingredient x_2

min $Z = 3x_1 + 8x_2$

Subject to the constraints

$x_1 + x_2 = 200$

$x_1 \leq 80$

$x_2 \geq 60$

$x_1 \geq 0, x_2 \geq 0.$

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4, A factory engaged in the manufacturing of pistons, rings and valves for which the profit per unit are Rs 10, 6 and 4 respectively wants to decide the most profitable mix. It takes one hour of preparatory work, ten hours of machining and two hours of packing and allied formulaties for a piston. Corresponding time requirements for rings and valves are 1, 4 and 2 and 1, 5 and 6 hours respectively. The total number of hours available for preparatory work, machining and packing and allied formalities are 100, 600 and 300 respectively. Determine the most profitable mix, assuming that what all produced can be sold. Formulate the LPP.

Soln:-

x_1 - Pistons

x_2 - Rings

x_3 - Valves

max $Z = 10x_1 + 6x_2 + 4x_3$

Subject to the constraints,

$x_1 + x_2 + x_3 \leq 100$

$10x_1 + 4x_2 + 5x_3 \leq 600$

$2x_1 + 2x_2 + 6x_3 \leq 300$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

5, The manager of an oil refinary must decide on the optimum mix of two possible blending processes of which the input and output production runs are as follows

Process	Input		Output	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	6	4	6	9
2	5	6	5	5

The maximum amounts available of crude A and crude B are 250 units and 200 units respectively. Market demand shows that atleast 150 units of Gasoline X and 130 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs 4 and Rs 5 respectively. Formulate the problem for maximising the profit.

Soln:-

x_1 - Process 1

x_2 - Process 2

Crude A and crude B

$6x_1 + 5x_2 \leq 250$

$4x_1 + 6x_2 \leq 200$

Gasoline X and Gasoline Y

$6x_1 + 5x_2 \geq 150$

$9x_1 + 5x_2 \geq 130$

$\max z = 4x_1 + 5x_2$

Subject to the constraints

$6x_1 + 5x_2 \leq 250$

$4x_1 + 6x_2 \leq 200$

$6x_1 + 5x_2 \geq 150$

$9x_1 + 5x_2 \geq 130$

$x_1 \geq 0, x_2 \geq 0.$

.6.19 Graphical Solution method

Step-1:

Identify the problem - the decision variables, the objective and the restriction.

Step-2:

Set up the mathematical formulation of the problem.

Step-3

Plot a graph representing all the constraints of the problem and identify the feasible region (Solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step-4

The feasible region obtained in step 3 may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step-5

Find out the value of the objective function at each corner (solution) points determined in Step-4.

Step-6

Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

26.06.19 Solve by Graphical method

$\min z = 5x_1 + 2x_2$

st

$$x_1 + 2x_2 \leq 4$$

$$2x_1 + x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

Soln:-

Given $\min z = 5x_1 + 2x_2$

st

$$x_1 + 2x_2 \leq 4$$

$$2x_1 + x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

Consider,

$$x_1 + 2x_2 = 4 \quad \text{—} \; ①$$

$$2x_1 + x_2 = 10 \quad \text{—} \; ②$$

For ①,

Put $x_1 = 0$

$$2x_2 = 4$$

$$\therefore x_2 = 2.$$

$\therefore$ Point is $(0, 2)$

Put $x_2 = 0$

$$x_1 = 4$$

$\therefore$ Point is $(4, 0)$

For ②,

Put $x_1 = 0$

$$x_2 = 10$$

Point is $(0, 10)$

Put $x_2 = 0$

$$2x_1 = 10$$

$$x_1 = 5$$

Point is $(5, 0)$

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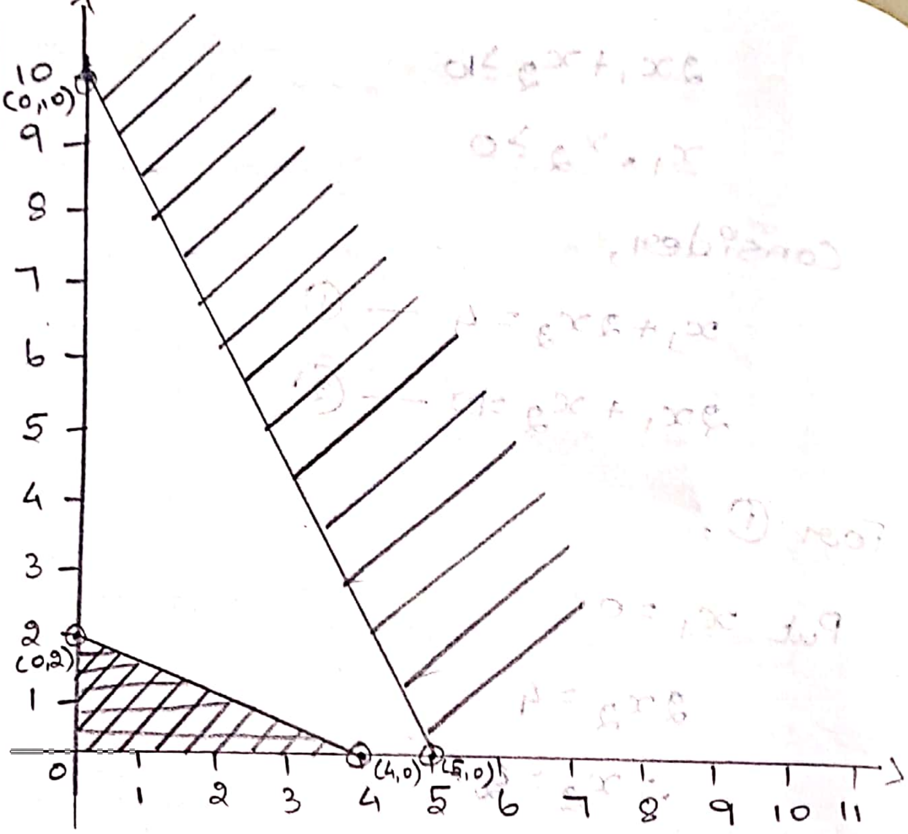


Intersection area not exist.

$\therefore$ Given LPP has no solution

2, Solve $\max z = 2x_1 + 3x_2$

St. $x_1 + 3x_2 \leq 3$

$x_1 + 2x_2 \geq 6$

$x_1 \geq 5$

$x_1, x_2 \geq 0$

Soln:-

Given $\max z = 2x_1 + 3x_2$

St. $x_1 + 3x_2 \leq 3$

$x_1 + 2x_2 \geq 6$

$x_1 \geq 5$

$x_1, x_2 \geq 0$

Consider

$x_1 + 3x_2 = 3$ — (1)

$x_1 + 2x_2 = 6$ — (2)

$x_1 = 5$ — (3)

For ①,

Put $x_1 = 0$

$3x_2 = 3$

$x_2 = 1$

$\therefore$ Point is $(0,1)$

Put $x_2 = 0$

$x_1 + 0 = 3$

$x_1 = 3$

$\therefore$ Point is $(3,0)$

For ②

Put $x_1 = 0$

$2x_2 = 6$

$\boxed{x_2 = 3}$

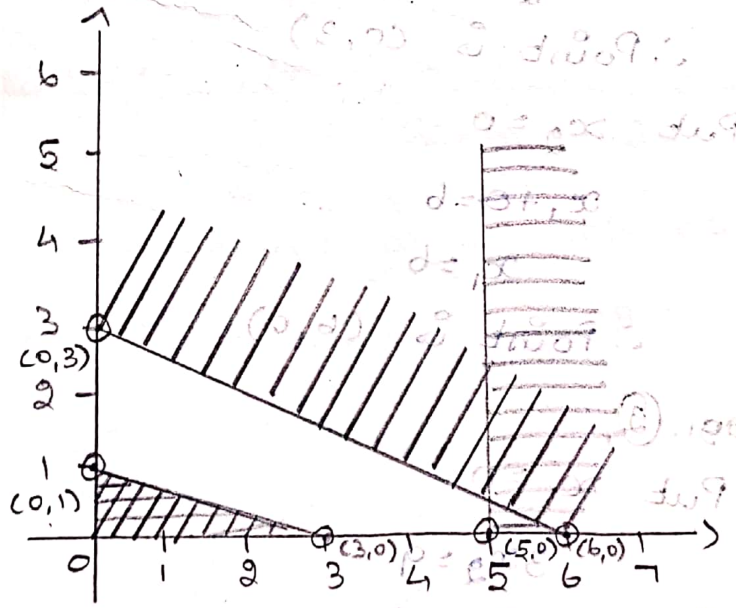
$\therefore$ Point is $(0,3)$

Put $x_2 = 0$

$x_1 + 0 = 6$

$x_1 = 6$

$\therefore$ Point is $(6,0)$



Intersection area not exist

$\therefore$ Given LPP has no solution.

2, Solve max $z = 2x_1 + 3x_2$

St. $x_1 + 2x_2 \leq 6$

$x_1 + 3x_2 \leq 9$

$2x_1 + 3x_2 \leq 3$

$x_1, x_2 \geq 0$

Soln:-

Given max $z = 2x_1 + 3x_2$

St $x_1 + 2x_2 \leq 6$

$x_1 + 3x_2 \leq 9$

$2x_1 + 3x_2 \leq 3$

$x_1, x_2 \geq 0$

Consider $x_1 + 2x_2 = 6$ — (1)

$x_1 + 3x_2 = 9$ — (2)

$2x_1 + 3x_2 = 3$ — (3)

For (1),

Put $x_1 = 0$

$2x_2 = 6$

$x_2 = 3$

$\therefore$ Point is $(0, 3)$

Put $x_2 = 0$

$x_1 + 0 = 6$

$x_1 = 6$

$\therefore$ Point is $(6, 0)$

For (2),

Put $x_1 = 0$

$3x_2 = 9$

$x_2 = 3$

Point is $(0, 3)$

Put $x_2 = 0$

$x_1 + 0 = 9$

$x_1 = 9$

$\therefore$ Point is $(9, 0)$

For ③

Put $x_1 = 0$

$3x_2 = 3$

$x_2 = 1$

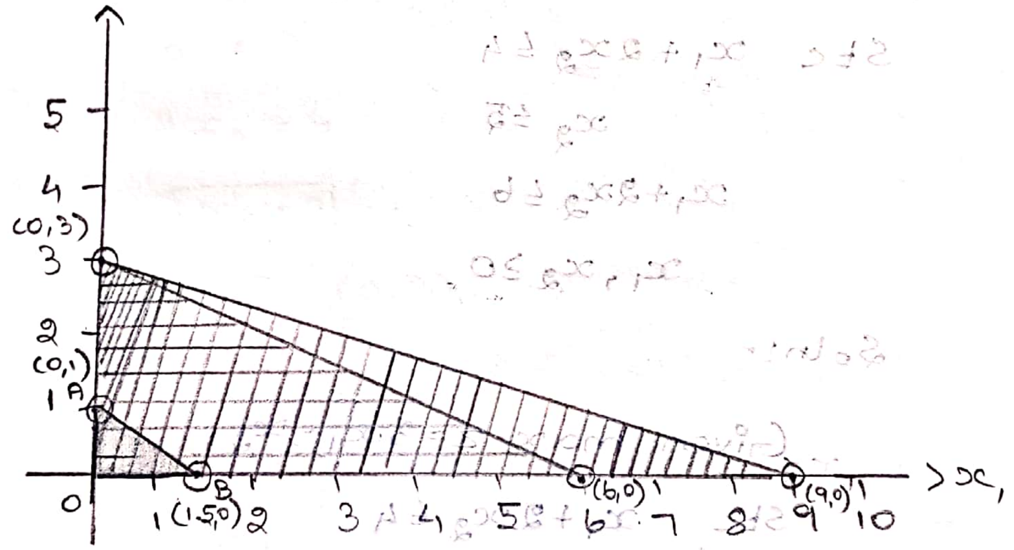
$\therefore$ Point is $(0, 1)$

Put $x_2 = 0$

$2x_1 = \cancel{6}\ 3$

$x_1 = 3/2$

Point is $(3/2, 0)$



$\therefore$ Intersection area AOB exist

$\therefore$ Feasible region is AOB

Points	$z=2x_1+3x_2$
A(0,1)	3
O(0,0)	0
B(1.5,0)	3

$\therefore$ Max $z=3$

Point is (0,1) or (1.5,0)

Solution is $x_1=0, x_2=1$
(or) $x_1=1.5, x_2=0$

28.06.19

4, Solve max $z=3x_1-x_2$

Stc $x_1+2x_2\le 4$

$x_2\le 5$

$x_1+2x_2\le 6$

$x_1, x_2\ge 0$

Soln:-

Given max $z=3x_1-x_2$

Stc $x_1+2x_2\le 4$

$x_2\le 5$

$x_1+2x_2\le 6$

$x_1, x_2\ge 0$

Consider, $x_1+2x_2=4$ —(1)

$x_2=5$ —(2)

$x_1+2x_2=6$ —(3)

For ①

Put $x_1 = 0$

$2x_2 = 4$

$\boxed{x_2 = 2}$

Point is $(0, 2)$

Put $x_2 = 0$

$x_1 + 0 = 4$

$x_1 = 4$

Point is $(4, 0)$

For ②

$x_2 = 5$

For ③

Put $x_1 = 0$

$2x_2 = 6$

$\boxed{x_2 = 3}$

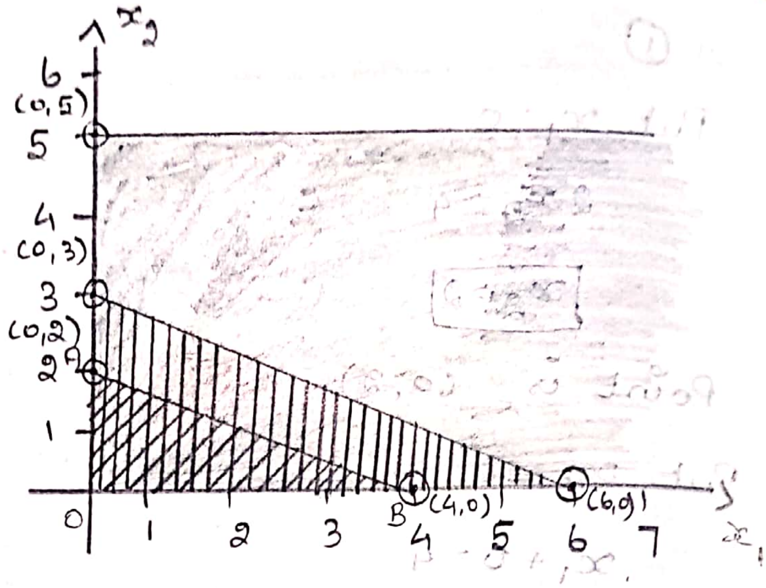
Point is $(0, 3)$

Put $x_2 = 0$

$x_1 + 0 = 6$

$x_1 = 6$

Point is $(6, 0)$



∴ Intersection area AOB exist

∴ Feasible region is AOB

Points	$z = 3x_1 - x_2$
A (0, 2)	-2
O (0, 0)	0
B (4, 0)	12

Max $z = 12$

Point is $(4, 0)$

∴ Solution is $x_1 = 4, x_2 = 0$

∴ Max $z = 12$.

5, Solve $\min z = x_1 + 4x_2$

Stc $x_1 + 2x_2 \geq 4$

$x_2 \leq 7$

$x_1 + x_2 \leq 8$

$x_1, x_2 \geq 0$

Soln:-

Given $\min z = x_1 + 4x_2$

Stc $x_1 + 2x_2 \geq 4$

$x_2 \leq 7$

$x_1 + x_2 \leq 8$

$x_1, x_2 \geq 0$

Consider,

$x_1 + 2x_2 = 4$ — (1)

$x_2 = 7$ — (2)

$x_1 + x_2 = 8$ — (3)

For (1)

Put $x_1 = 0$

$2x_2 = 4$

$x_2 = 2$

Point is $(0, 2)$

Put $x_2 = 0$

$x_1 + 0 = 4$

$x_1 = 4$

Point is $(4, 0)$

For (2)

$x_2 = 7$

For ③

Put $x_1 = 0$

$0 + x_2 = 8$

$x_2 = 8$

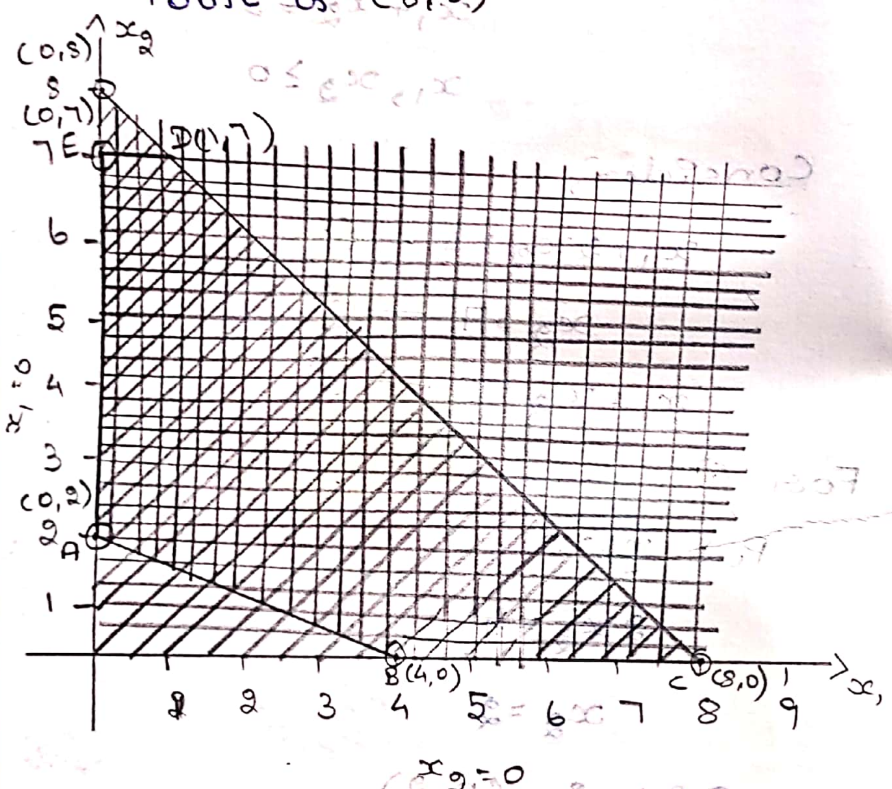
Point is $(0, 8)$

Put $x_2 = 0$

$x_1 + 0 = 8$

$x_1 = 8$

Point is $(8, 0)$



$x_2 = 0$

∴ Intersection area ABCDE exist

∴ Feasible region is ABCDE

To find Point D

Solve ② & ③

Put $x_2 = 7$ in ③

$x_1 + 7 = 8$

$x_1 = 1$

∴ Point D is $(1, 7)$

Point	$z=x_1+4x_2$
A(0,2)	8
B(4,0)	4
C(8,0)	8
D(1,7)	29
E(0,7)	28

min $z=4$

Point is $(4,0)$

$\therefore$ Solution is $x_1=4$, $x_2=0$

$\therefore$ min $z=4$.

6, Solve max $z=x_1+3x_2$

stc $x_1+2x_2 \geq 4$

$2x_1+x_2 \leq 8$

$x_2 \geq 1$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Soln:-

Given max $z=x_1+3x_2$

stc $x_1+2x_2 \geq 4$

$2x_1+x_2 \leq 8$

$x_2 \geq 1$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

$2x_1+1=8$
$2x_1=7$
$x_1=7/2$
$x_1=3.5$

Consider, $x_1+2x_2=4$ — (1)

$2x_1+x_2=8$ — (2)

$x_2=1$ — (3)

$x_1=4$ — (4)

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For ①

Put $x_1 = 0$

$2x_2 = 4$

$x_2 = 2$

Point is $(0, 2)$

Put $x_2 = 0$

$x_1 + 0 = 4$

$x_1 = 4$

Point is $(4, 0)$

For ②

Put $x_1 = 0$

$0 + x_2 = 8$

$x_2 = 8$

Point is $(0, 8)$

Put $x_2 = 0$

$2x_1 = 8$

$x_1 = 4$

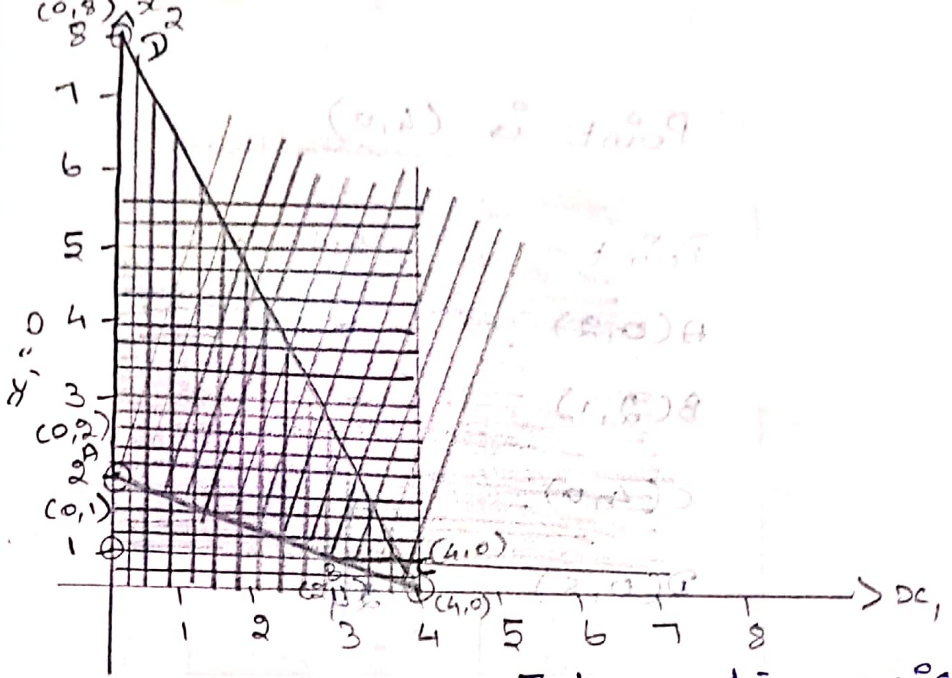
Point is $(4, 0)$

For ③

$x_2 = 1$

For ④

$x_1 = 4$



Intersection exist at ABCD

To find the point B and C

Solve ① & ③

Put $x_2 = 1$ in ①

$$x_1 + 2x_2 = 4$$

$$x_1 + 2 = 4$$

$$x_1 = 2$$

$$2 + 2x_2 = 4$$

$$2x_2 = 2$$

$$x_2 = 1$$

Point is $(2,1)$

Solve ② & ④

Put $x_1 = 4$ in ②

$$8 + x_2 = 8$$

$$x_2 = 0$$

Point is

$$2x_1 + 0 = 8$$

$$2x_1 = 8$$

$$\boxed{x_1 = 4}$$

Point is (4,0)

Point	$z = x_1 + 3x_2$
A(0,2)	6
B(2,1)	5
C(4,0)	4
D(0,8)	24

Max $z = 24$

$\therefore$ Point is (0,8)

Solution is $x_1 = 0, x_2 = 8$

max $z = 24$.

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7, Solve $\max z = x_1 + 3x_2$

Sb $x_1 + 2x_2 \geq 4$

$2x_1 + x_2 \geq 8$

$x_2 \geq 1$

$x_1 \geq 4$

$x_1, x_2 \geq 0$

Soln:-

Given $\max z = x_1 + 3x_2$

St $x_1 + 2x_2 \geq 4$

$2x_1 + x_2 \geq 8$

$x_2 \geq 1$

$x_1 \geq 4$

$x_1, x_2 \geq 0$

Consider,

$$x_1 + 2x_2 = 4 \text{ --- } (1)$$

$$2x_1 + x_2 = 8 \text{ --- } (2)$$

$$x_2 = 1 \text{ --- } (3)$$

$$x_1 = 4 \text{ --- } (4)$$

For (1)

Put $x_1 = 0$

$2x_2 = 4$

$x_2 = 2$

Point is $(0, 2)$

Put $x_2 = 0$

$x_1 + 0 = 4$

$x_1 = 4$

Point is $(4, 0)$

For (2)

Put $x_1 = 0$

$0 + x_2 = 8$

$x_2 = 8$

Point is $(0, 8)$

Put $x_2 = 0$

$2x_1 = 8$

$x_1 = 4$

Point is $(4, 0)$

For (3)

$x_2 = 1$

For (4)

$x_1 = 4$

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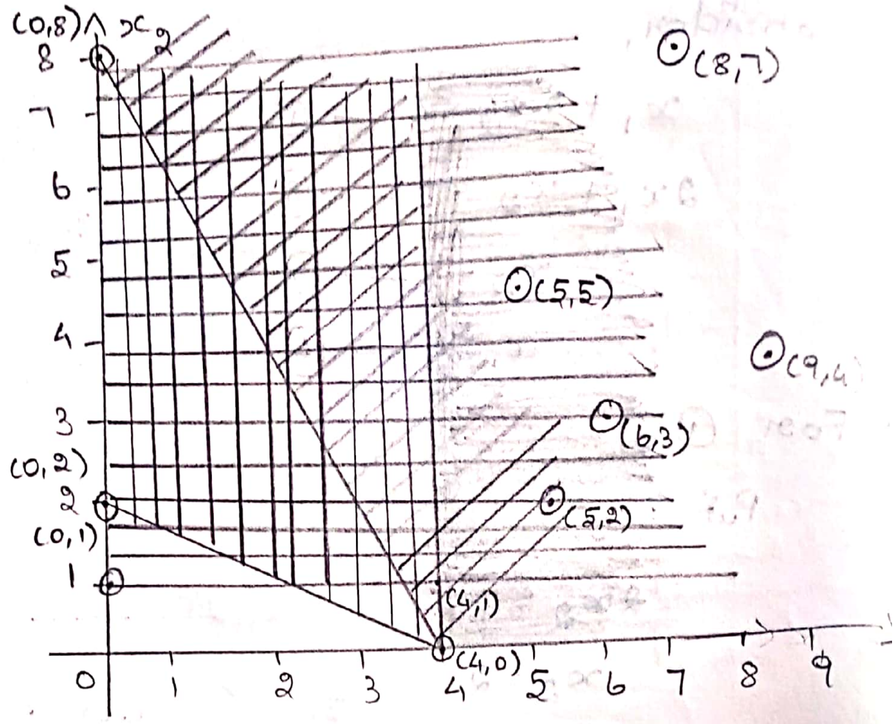


Intersection area exist and feasible region is unbounded.

Points	$z = x_1 + 3x_2$
(4,1)	7
(5,2)	11
(5,5)	20
(6,3)	15
(8,7)	29
(9,4)	21

$\therefore$ Given LPP has unbounded solution.

8, Solve min $z = 2x_1 + 3x_2$

st $x_1 + x_2 \geq 6$

$x_1 \geq 2$

$2x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Soln:-

Given $\min z = 2x_1 + 3x_2$

st $x_1 + x_2 \geq 6$

$x_1 \geq 2$

$2x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Consider,

$x_1 + x_2 = 6$ — ①

$x_1 = 2$ — ②

$2x_1 + x_2 = 4$ — ③

For ①

Put $x_1 = 0$

$0 + x_2 = 6$

$x_2 = 6$

$\therefore$ Point is $(0,6)$

Put $x_2 = 0$

$x_1 + 0 = 6$

$x_1 = 6$

$\therefore$ Point is $(6,0)$

For ②

$x_1 = 2$.

For ③

Put $x_1 = 0$

$0 + x_2 = 4$

$x_2 = 4$

Point is $(0,4)$

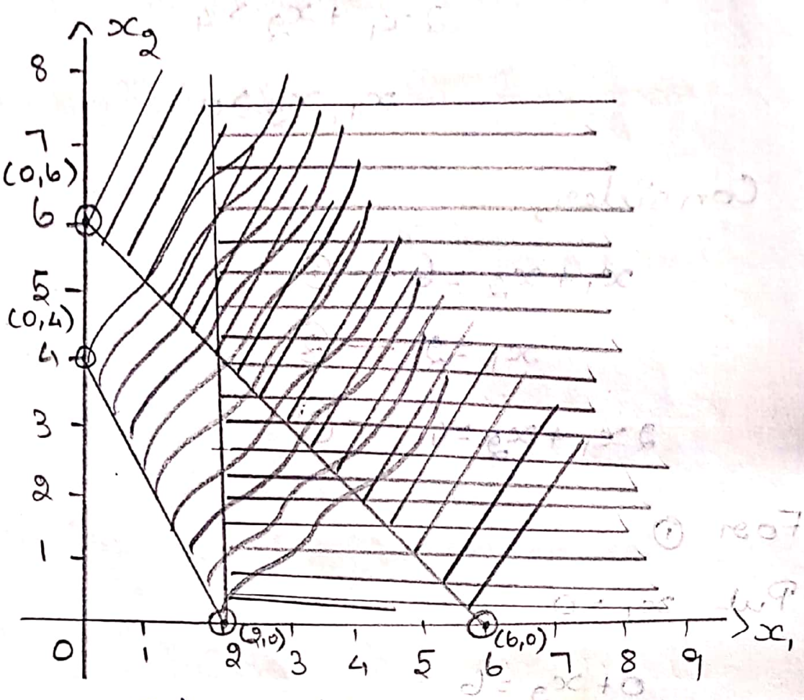
Scanned with CamScanner

Put $x_2 = 0$

$2x_1 = 4$

$x_1 = 2$

Point is $(2,0)$



Intersection area exist and feasible region is unbounded

Points	$z = 2x_1 + 3x_2$
(6,0)	12
(2,4)	16

$\therefore$ min $z = 12$

Point is $(6,0)$

$\therefore$ Solution is $x_1 = 6,\ x_2 = 0$

$\therefore$ min $z = 12$

3.7.19 Infeasible Solution

When the constraints are not satisfied simultaneously, the linear programming problem has no feasible solution. This situation can never occur if all the conditions are of the "$\leq$" type.

Definition (General Linear Programming Problem)

Let z be a linear function on R^n defined by

(a) $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

where c_j's are constants.

Let (a_{ij}) be an $m \times n$ real matrix and let $\{b_1, b_2, \dots, b_m\}$ be a set of constants such that

(b) $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq \text{or} \leq \text{or} = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq \text{or} \leq \text{or} = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq \text{or} \leq \text{or} = b_m$

and finally let

(c) $x_j \geq 0$, $j = 1, 2, 3, \dots, n$

The problem of determining an n-tuple $(x_1, x_2, \ldots, x_n)$ which makes z a minimum (or maximum) and which satisfies (b) and (c) is called the general linear programming problem.

Definition: (Objective function)

The linear function

$$z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

which is to be minimized (or maximized) is called the objective function of the general LPP.

Definition: (constraints)

In LPP

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq \text{or} \leq \text{or} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq \text{or} \leq \text{or} = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq \text{or} \leq \text{or} = b_m$$

are called constants.

Non-negative restrictions

The set of inequations

$$x_j \geq 0, \; j = 1, 2, \ldots, n \quad (c)$$

is usually known as the set of non-negative restrictions of the general LPP.

Definition (Solution)

An n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers which satisfies the constraints of a general LPP is called a solution to the General LPP.

Definition (Feasible Solution)

Any solution to a general LPP which also satisfies the non-negative restriction of the problem, is called a feasible solution to the General LPP.

Definition (Optimum Solution)

Any feasible solution which optimizes (minimizes or maximizes) the objective function of a General LPP is called an optimum solution to the General LPP.

Definition (Slack variables)

Let the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \ , \ i = 1, 2, \ldots k$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \ , \ i = 1, 2, \ldots, k$$

are called slack variables.

Definition: (Surplus variables)

Let the constraints of a General LPP ~~are~~ be

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i \ , \ i = 1, 2, \ldots, k$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1} a_{ij} x_j - x_{n+i} = b_i \ , \ i = 1, 2, \ldots, k$$

are called surplus variables.

4.7.19 Canonical form

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to the constraints

$a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + \ldots + a_{in} x_n \le b_i$

$i = 1, 2, 3, \ldots m$

$x_1, x_2, x_3, \ldots, x_n \ge 0.$

by making use of some elementary transformation. This form of LPP is called the Canonical form of LPP.

2m/5m/10m Characteristics of the Canonical form

i) The objectives function is of the maximization type.

The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function, $-f(x)$.

(i.e) minimize $f(x) = -\text{maximize}\{-f(x)\}$

For example, the linear objective function

minimize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$.

is equivalent to

maximize $h = -z = -c_1x_1 - c_2x_2 - \dots - c_nx_n$

with $z = -h$.

ii) All the constraints are of the "$\leq$" type, except for the non-negative restrictions.

An inequality of $\geq$ type can be changed to an inequality of the "$\leq$" type by multiplying both sides of inequality by -1

For eg, the linear constraints

$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i$

is equivalent to

$-a_{i1}x_1 - a_{i2}x_2 - \dots - a_{in}x_n \leq -b_i$

An equation may be replaced by two weak inequalities in opposite directions.

For example,

$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$

is equivalent to

$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i$

and $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i$

iii) All the variables are non-negative.

A variable which is unrestricted in sign (i.e., positive, negative (or) zero) is equivalent to the difference between two non-negative variables. Thus, if x_j is unrestricted in sign, it can be replaced by $(x_j' - x_j'')$. where x_j' and x_j'' are both non-negative.

(i.e) $x_j = x_j' - x_j''$ where, $x_j' \geq 0$ and $x_j'' \geq 0$

The Standard form

The General LPP in the form

maximize (or) minimize

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to the constraints:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i,$$

$$x_1, x_2, \dots, x_n \geq 0$$

is known as in Standard form.

Characteristics of Standard form

i) All the constraints are expressed in the form of equations, except for the non-negative restrictions.

ii) The right hand side of each constraint equation is non-negative.

The inequality constraint can be changed into equation by introducing a non-negative variable on the left hand side of such constrait. It is to be added (slack variable) if the constraint is of "$\leq$" type and subtracted, (surplus variables) if the constraint is of $\geq$ type.

1, Give Canonical form of

$$\max z = 2x_1 + 3x_2$$

Stc $2x_1 - 4x_2 \leq 4$

$x_1 + x_2 \geq 2$

$x_1 + 7x_2 \leq 7$

$x_1, x_2 \geq 0$,

Soln:-

The canonical form is

$$\max z = 2x_1 + 3x_2$$

Stc. $2x_1 - 4x_2 \leq 4$

$-x_1 - x_2 \leq -2$.

$x_1 - 7x_2 \leq -7$

$x_1, x_2 \geq 0$.

2, Find Canonical form for LPP

$$\min z = 2x_1 + 3x_2$$

Stc $2x_1 - 4x_2 \leq 4$

$x_1 + x_2 \geq 3$

$x_1 \geq 0$

x_2 unrestricted.

Soln:

Given

$$\min z = 2x_1 + 3x_2$$

stc

$$2x_1 - 4x_2 \le 4$$

$$x_1 + x_2 \ge 3$$

$$x_1 \ge 0$$

x_2 unrestricted.

x_2 is unrestricted.

$\therefore$ We put $x_2 = x_2' - x_2''$

$\therefore$ The canonical form is

$$\max z^* = -2x_1 - 3x_2$$

$$\max z^* = -2x_1 - 3(x_2' - x_2'')$$

stc.

$$2x_1 - 4(x_2' - x_2'') \le 4$$

$$-x_1 - (x_2' - x_2'') \le -3$$

$$x_1 \ge 0,\ x_2' \ge 0,\ x_2'' \ge 0.$$

$$\therefore \max z^* = -2x_1 - 3x_2' + 3x_2''$$

stc

$$2x_1 - 4x_2' + 4x_2'' \le 4$$

$$-x_1 - x_2' + x_2'' \le -3.$$

$$x_1, x_2', x_2'' \ge 0.$$

2, Find the canonical & standard form for $\min z = 2x_1 + 3x_2$

stc $2x_1 - 4x_2 \leq 4$

$x_1 + x_2 \geq 2$

$x_1 \geq 0, x_2 \geq 0$

Soln:-

Given $\min z = 2x_1 + 3x_2$

stc $2x_1 - 4x_2 \leq 4$

$x_1 + x_2 \geq 2$

$x_1 \geq 0, x_2 \geq 0$

$\therefore$ Standard form is

S_1 - slack variable S_2 - surplus variable

$\min z = 2x_1 + 3x_2 + \underline{0S_1 + 0S_2}$

no change

stc

$2x_1 - 4x_2 + S_1 + 0S_2 = 4$

$x_1 + x_2 + 0S_2 - S_2 = 2$

$x_1, x_2, S_1, S_2 \geq 0$

Definition (Basic Solution)

Given a system of m simultaneous linear equations is a unknowns (m<n)

$Ax = b, x^T \in R^n$

where A is a $m \times n$ matrix of rank m. Let B be any $m \times n$ submatrix formed by m linearly independent columns of A. Then a solution obtained by setting $n-m$ variables not associated with the columns of B equal to zero, and solving the resulting system, is called a "basic solution to the given system of equation"

The m variables which may be all different from zero, are called basic variables. The $m \times m$ non singular submatrix B is called a Basic matrix with the columns of B as basic vectors.

Definition (Degenerate solution)

A basic solution to the system is called degenerate if one or more of the basic variable vanish.

5.7.19 Definition (Basic feasible Solution)

A feasible solution to an LPP, which is also a basic Solution to the problem is called a basic feasible solution to the LPP.

Definition (Optimum Basic feasible Solution)

Maximize $z = cx$.

Subject to: $Ax = b$ and $x \geq 0$

is called an optimum basic feasible solution if $z_0 = c_B x_B \geq z^*$ where z^* is the value of objective function for any feasible solution

2m Fundamental theorem of Linear Programming

If the feasible region of an LPP is a convex polyhedron, then there exists an optimal solution to the LPP and atleast one basic feasible solution must be optimal.

Conditions of Optimality

A sufficient condition for a feasible solution to an LPP to be an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which the column vector $a_j \in A$ is not in the basis B.

The Simplex Algorithm:

Step-1:

Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximizing it by using the result

Minimum $Z = -$ Maximum $(-Z)$.

Step-2

Check whether all $b_i (i = 1, 2, \ldots, m)$ are non-negative. If any one of b_i is negative, then multiply the corresponding in equation of the constraints by -1, so as to get all $b_i (i = 1, 2, \ldots, m)$ non-negative.

Step-3

Convert all the inequations of the constraints into equations by introducing slack and / or surplus variables in the constraints. Put the costs of these variables equal to zero.

Step-4

Obtain an initial basic feasible solution to the problem in the form.

$$X_B = B^{-1} b$$

and put it in the first column of the simple table.

Step-5

Compute the net evaluations $z_j - c_j (j=1,2,\ldots,n)$ by using the relation $z_j - c_j = c_B y_j - c_j$.

Examine the sign $z_j - c_j$

i) If all $(z_j - c_j) \geq 0$ then the initial basic feasible solution X_B is an optimum basic feasible solution.

ii) If at least one $(z_j - c_j) < 0$, proceed on the next step.

Step-6:

If there are more than one negative $z_j - c_j$, then choose the most negative of them. Let it be $z_r - c_r$ for some $j = r$.

i) If all $y_{ir} \leq 0$ $(i = 1, 2, \ldots, m)$, then there is an unbounded solution to the given problem.

ii) If atleast one $y_{ir} > 0$ $(i = 1, 2, \ldots, m)$ then the corresponding vector y_r enters the basis y_B.

2m Step-7 (Leading elemen or pivotal element)

Compute the ratios

$$\left\{ \frac{x_{Bi}}{y_{ir}},\ y_{ir} > 0,\ i = 1, 2, \ldots, m \right\}$$

and choose the minimum of them.

Let the minimum of these ratios be $\frac{x_{Bk}}{y_{kr}}$. Then the vector y_k will level the basis y_B. The common element y_{kr}, which is in the k^{th} row and the rth column is known as the leading element (or pivotal element) of the table.

Step-8

Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeros by making use of the relations:

$$\hat{y}_{ij} = y_{ij} - \frac{y_{kj}}{y_{kr}} y_{ir}$$

$i = 1, 2, \ldots, m+1, i \neq k$

and $\hat{y}_{kj} = \frac{y_{kj}}{y_{kr}}, \; j = 0, 1, 2, \ldots, n$

Step-9

Go to step 5 and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

.7.19 What are the special cases of graphical method?

1. Alternative Optima
2. Unbounded Solution
3. Infeasible solution or No solution.

When the graphical method will fail?

When the LPP have more than two variables. In this case, the graphical method is failed.

When we use the simplex method?

When the LPP have more than two variables, we use the simplex method.

(i.e) When the graphical method is failed in such a case we use the simplex method.

1, Solve the LPP

$$\max z = 2x_1 + 3x_2$$

st $x_1 + 2x_2 \leq 4$

$x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

Soln:-

Given, $\max z = 2x_1 + 3x_2$

st $x_1 + 2x_2 \leq 4$

$x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

Given LPP becomes

$\max z = 2x_1 + 3x_2 + 0S_1 + 0S_2.$

st $x_1 + 2x_2 + S_1 + 0S_2 = 4$

$x_1 + x_2 + 0S_1 + S_2 = 2$

$x_1, x_2, S_1, S_2 \geq 0.$

C_j	2	3	0	0				
x_j	x_1	x_2	S_1	S_2	RHS	B	C_B	Ratio
	1	2	1	0	4	S_1	0	$4/2 = 2$
	1	1	0	1	2	S_2	0	$2/1 = 2$
	0	0	0	0				
	0	0	0	0				
z_j	0	0	0	0				
$z_j - c_j$	-2	-3↑	0	0				
	-1	0	1	-2	0	S_1	0	
	1	1	0	1	2	x_2	3	
z_j	3	3	0	3				
	1	0	0	3				

All $z_j - c_j \geq 0$

We reach optimum stage

Solution :

$x_1 = 0, x_2 = 0$

$\max z = 2(0) + 3(2)$

$= 6$

8.07.19

2, Solve using simplex method

$$\max z = 2x_1 + 3x_2$$

stc

$$x_1 + x_2 \le 4$$

$$-x_1 + x_2 \le 1$$

$$x_1 + 2x_2 \le 5$$

$$x_1, x_2 \ge 0$$

Soln:-

Given,

$$\max z = 2x_1 + 3x_2$$

stc $x_1 + x_2 \le 4$

$$-x_1 + x_2 \le 1$$

$$x_1 + 2x_2 \le 5$$

$$x_1, x_2 \ge 0$$

Given LPP becomes

$$\max z = 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

stc $x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 4$

$$-x_1 + x_2 + 0s_1 + s_2 + 0s_3 = 1$$

$$x_1 + 2x_2 + 0s_1 + 0s_2 + s_3 = 5$$

$$x_1, x_2, s_1, s_2, s_3 \ge 0$$

C_j	2	3	0	0	0				
x_j	x_1	x_2	S_1	S_2	S_3	RHS	B	C_B	Ratio
	1	1	1	0	0	4	S_1	0	$4/1=4$
	-1	1	0	1	0	1	S_2	0	$1/1=1$ ←
	1	2	0	0	1	5	S_3	0	$5/2=2.5$
Z_j	0	0	0	0	0				
Z_j-C_j	-2	-3 ↑	0	0	0				
	2	0	1	-1	0	3	S_1	0	$3/2=1.5$
	-1	1	0	1	0	1	x_2	3	-
	3	0	0	-2	1	3	S_3	0	$3/3=1$ →
Z_j	-3	3	0	3	0				
Z_j-C_j	-5 ↑	0	0	3	0				
	1	0	1	$1/3$	$-2/3$	1	S_1	0	$1/1/3=3$ →
	0	1	0	$1/3$	$1/3$	2	x_2	3	$2/1/3=6$
	1	0	0	$-2/3$	$1/3$	1	x_1	2	-
Z_j	2	3	0	$-1/3$	$5/3$				
Z_j-C_j	0	0	0	$-1/3$ ↑	$5/3$				
	3	0	3	1	-2	3	S_2	0	
	-1	1	-1	0	1	1	x_2	3	
	3	0	2	0	-1	-1	x_1	2	
Z_j	3	3	1	0	1				
Z_j-C_j	1	0	1	0	1				

Scanned with CamScanner

All $z_j - c_j \geq 0$

We reach optimum stage

Solution is $x_1 = 3, x_2 = 1$

$\max z = 2(3) + 3(1)$

$z = 9$.

---

v.i.f. 3, Solve by simplex method

$\leq$ - simplex method (+)

objective fun

$\max z = 107x_1 + x_2 + 2x_3$

$=$ - slack variable

stc $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + x_2 - 6x_3 \leq 5$

non-negative restriction

$3x_1 - x_2 - x_3 \leq 0$

$\geq$ - surplus variable (-).

$x_1, x_2, x_3, x_4 \geq 0$

Soln:-

$\max z = 107x_1 + x_2 + 2x_3$

stc $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + x_2 - 6x_3 \leq 5$

$3x_1 - x_2 - x_3 \leq 0$

$x_1, x_2, x_3, x_4 \geq 0$

Given LPP becomes standard form.

$\max z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0S_1 + 0S_2$.

$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0S_1 + 0S_2 = 7/3$

$16x_1 + x_2 - 6x_3 + 0x_4 + S_1 + 0S_2 = \cancel{0}\ 5$

$3x_1 - x_2 - x_4 + 0x_4 + 0S_1 + S_2 = 0$.

$x_1, x_2, x_3, x_4, S_1, S_2 \geq 0$.

b - RHS

B - Identity matrix

IBFS - Initial Basic Feasible Solution

Basic Soln - $C_B = B^{-1} X_B$

$\frac{-14}{9}+2 \quad \frac{-14+18}{9} \quad \frac{-14}{9}-\frac{1}{3} \quad \frac{-14-3}{9}$

C_j	107	1	2	0	0	0				
x_j	x_1	x_2	x_3	x_4	S_1	S_2	RHS	B	C_B	Ratio
	$\frac{14}{3}$	$\frac{1}{3}$	-2	1	0	0	$\frac{7}{3}$	x_4	0	$\frac{7/3}{14/3}=\frac{1}{2}$
	16	1	-6	0	1	0	5	S_1	0	$\frac{5}{16}=0.3$
	3	-1	-1	0	0	1	0	S_2	0	$\frac{0}{3}=0$
Z_j	0	0	0	0	0	0				
Z_j-C_j	-107	-1	-2	0	0	0				
	0	$\frac{17}{9}$	$-\frac{4}{9}$	$+1$	0	$\frac{-14}{9}$	$\frac{7}{3}$	x_4	0	-
	0	$19/3$	$-2/3$	0	1	$\frac{-16}{3}$	5	S_1	0	-
	1	$-1/3$	$-1/3$	0	0	$1/3$	0	x_1	107	-
Z_j	107	$\frac{-107}{3}$	$\frac{-107}{3}$	0	0	$\frac{107}{3}$				
Z_j-C_j	0	$\frac{-110}{3}$	$\frac{-113}{3}$	0	0	$\frac{107}{3}$				

All entries of the column ($x_3 \le 0$) have negative values. So, we cannot find the ratio of LPP.

$\therefore$ The given LPP have unbounded solution.

UNIT - II

Artificial Variable

In order to obtain an initial basic feasible solution, we first put the given LPP into its standard form and then a non-negative variable is added to the left side of each of equation that lacks the much needed starting basic variables. The so-added variable is called an artificial variable.

It plays a same role as a slack variable. It is denoted by A.

Two - Phase Method

Step-1:

Write the given LPP into its standard form and check whether there exists a starting basic feasible solution.

a) If there is a ready starting basic feasible solution, go to phase 2.

b) If there does not exist a ready starting basic feasible solution, go on to the next step.

Phase-1

Step-2

Add the artificial variable to the left side of the each equation that lacks the needed starting basic variables. Construct an auxilliary objective function aimed at minimising the sum of all artificial variables.

Thus, the new objective is to

$$\text{Minimize } z = A_1 + A_2 + \dots + A_n$$

(i.e) $\text{Maximize } z^* = -A_1 - A_2 - \dots - A_n$

where A_i $(i = 1, 2, \dots, n)$ are the non-negative artificial variables.

Step-3

Apply simplex algorithm to the specially constructed LPP.

The following three cases may arise at the least interaction.

a) max $z^* < 0$ and atleast one artificial variable is present in the basis with positive value. In such a case, the original LPP does not posses any feasible solution.

b). max $z^* = 0$ and atleast one artificial variable is present in the basis at zero value. In such a case, the original LPP posses the feasible solution. In order to get basic feasible solution we may proceed directly to phase 2 or else eliminate the artificial basic variable and then proceed to phase 2.

c) max $z^* = 0$ and no artificial variable present in the basis. In such a case, a basic feasible solution to the original LPP has been found. Go to phase 2.

Phase-2

Step-4

Consider the optimum basic feasible solution of phase 1 as a starting basic feasible solution for the original LPP. Assing actual coefficients to the variable in the objective function and a value zero to the artificial variables that appear at zero value in the final simplex table of phase-1.

Apply usual simplex algorithm to modified simplex table to get the optimum solution of the original problem.

11.4.19

1. Use two-phase method to solve

min $z = -x_1 - 2x_2 - x_3$

st $x_1 + 2x_2 \leq 2$

$x_1 + 2x_2 + x_3 \geq 6$

$x_1 + x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

Soln:

Given

$$\min z = -x_1 - 2x_2 - x_3$$

$$\text{stc } \quad x_1 + 2x_2 \leq 2$$

$$x_1 + 2x_2 + x_3 \geq 6$$

$$x_1 + x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

LPP becomes

$$\max z^* = x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + 0s_3 - A$$

$$\text{st } \quad x_1 + 2x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 + 0A = 2.$$

$$x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 - s_3 + A = 6$$

$$x_1 + 0x_2 + x_3 + 0s_1 + s_2 + 0s_3 + 0A = 4$$

$$x_1, x_2, x_3, s_1, s_2, s_3, A \geq 0$$

Phase - I

C_j	0	0	0	0	0	0	-1				
x_j	x_1	x_2	x_3	S_1	S_2	S_3	A	RHS	B	C_B	Ratio
	1	2	0	1	0	0	0	2	S_1	0	$\frac{2}{2}=1$
	1	2	1	0	0	-1	1	6	A	-1	$\frac{6}{2}=3$
	1	0	1	0	1	0	0	4	S_2	0	-
z_j	-1	-2	-1	0	0	1	-1				
z_j-c_j	-1	-2↑	-1	0	0	1	0				
	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	0	1	x_2	0	-
	0	0	1	-1	0	-1	1	4	A	-1	$\frac{4}{1}=4$
	1	0	1	0	1	0	0	4	S_2	0	$\frac{4}{1}=4$
z_j	0	0	-1	1	0	1	-1				
z_j-c_j	0	0	-1↑	1	0	1	0				
	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	0	1	x_2	0	
	-1	0	0	-1	-1	-1	1	0	A	-1	
	1	0	1	0	1	0	0	4	x_3	0	
z_j	1	0	0	1	1	1	-1				
z_j-c_j	1	0	0	1	1	1	0				

All $z_j-c_j \geq 0$

$\therefore$ We reach the optimum stage and artificial variable present in the table at zero level.

∴ We go to phase - II.

Phase II

C_j	1	2	1	0	0	0	-1				
x_j	x_1	x_2	x_3	s_1	s_2	s_3	R	RHS	B	C_B	Ratio
	1/2	1	0	$\frac{1}{2}$	0	0	0	1	x_2	2	
	-1	0	0	-1	-1	-1	1	0	R	-1	
	1	0	1	0	1	0	0	4	x_3	1	
Z_j	3	2	1	2	2	1	-1				
$Z_j - C_j$	2	0	0	2	2	1	0				

All $Z_j - C_j \geq 0$

∴ We reach the optimum stage.

∴ Solution is $x_1 = 0, x_2 = 1, x_3 = 4$

$\max z^* = 1(0) + 2(1) + 1(4)$

$= 6$,

$\min z = -6$,

---

2. Solve $\max z = 5x_1 + 3x_2$

st $2x_1 + x_2 \leq 1$

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$

Soln:-

Given, $\max z = 5x_1 + 3x_2$

st $2x_1 + x_2 \leq 1$

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$

LPP becomes

$$\max z = 5x_1 + 3x_2 + 0S_1 + 0S_2 - A$$

$$\text{St} \quad 2x_1 + x_2 + S_1 + 0S_2 + 0A = 1$$

$$x_1 + 4x_2 + 0S_1 - S_2 + A = 6$$

$$x_1, x_2, S_1, S_2, A \geq 0$$

Phase-I

c_j	0	0	0	0	-1				
x_j	x_1	x_2	S_1	S_2	A	RHS	B	C_B	Rat
	2	1	1	0	0	1	S_1	0	$\frac{1}{1}$ =
	1	4	0	-1	1	6	A	-1	$\frac{6}{4}$ =
z_j	-1	-4	0	1	-1				
$z_j - c_j$	-1	-4↑	0	1	0				
	2	1	1	0	0	1	x_2	0	
	-7	0	-4	-1	1	2	A	-1	
z_j	7	0	4	1	-1				
$z_j - c_j$	7	0	4	1	0				

All $z_j - c_j \geq 0$

∴ We reach optimum stage

But artificial variable present in the table with positive level.

∴ We cannot proceed phase-II.

∴ Given LPP has no solution.

16.8.19

3, Use two-phase simplex method

$\min z = x_1 + x_2$

st $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Soln:

Given $\min z = x_1 + x_2$

st $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

LPP becomes

$\max z^* = -x_1 + x_2 + 0s_1 + 0s_2 - A_1 - A_2$

st $2x_1 + x_2 - s_1 + 0s_2 + A_1 = 4$

$x_1 + 7x_2 + 0s_1 - s_2 + A_2 = 7$

$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

Phase - I

c_j	0	0	0	0	-1	-1				
x_j	x_1	x_2	s_1	s_2	A_1	A_2	RHS	B	C_B	Ratio
	2	1	-1	0	1	0	4	A_1	-1	$\frac{4}{1} = 4$
	1	7	0	-1	0	1	7	A_2	-1	$\frac{7}{7} = 1$ →
z_j	-3	-8	1	1	-1	-1				
$z_j - c_j$	-3	-8 ↑	1	1	0	0				
	+13/7	0	-1	+1/7	+1	-1/7	+3	A_1	-1	$\frac{21}{13} = 1.62$ →
	$\frac{1}{7}$	1	0	$\frac{-1}{7}$	0	1/7	1	x_2	0	$\frac{1}{1/7} = 7$

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z_j	$-13/7$	0	1	$-1/7$	-1	$1/7$			
$z_j - c_j$	$-\frac{13}{7}\uparrow$	0	1	$-1/7$	0	$8/7$			
	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	$\frac{7}{13}$	$-\frac{1}{13}$	$\frac{21}{13}$	x_1	0
	0	$13+13$	$\frac{-2}{\cancel{12}/7}$	$\frac{-13}{-13}$	2	$\frac{-39}{-8/7}$		x_2	0
z_j	0	0	0	0	0	0			
$z_j - c_j$	0	0	0	0	1	1			

All $z_j - c_j \geq 0$

$\therefore$ We reach the optimum stage and artificial variables abses in the table.

$\therefore$ We proceed phase II

c_j	-1	-1	0	0	-1	-1				
x_j	x_1	x_2	S_1	S_2	A_1	A_2	RHS	B	C_B	Ratio
	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	$\frac{7}{13}$	$-\frac{1}{13}$	$\frac{21}{13}$	x_1	-1	
	0	13	$+1$	-2	$\frac{-13}{+1}$	$+2$	$\frac{-39}{-8/7}$	x_2	-1	
z_j	-1	-13	$+\frac{8}{13}$	$\frac{25}{13}$	$\frac{82}{13}$	$+\frac{25}{13}$				
$z_j - c_j$	0	0	$\frac{8}{13}$	$\frac{27}{13}$	$\frac{7}{13}$	3				

All $z_j - c_j \geq 0$,

$\therefore$ We reach the Optimum stage

$\therefore$ Solution is

$$x_1 = \frac{21}{13}, \quad x_2 = \frac{8}{7}$$

$$\max z^* = -\frac{21}{13} + \frac{8}{7}$$

$$= \frac{-147 + 104}{91}$$

$$= \frac{-43}{91}$$

$$\min z = \frac{43}{91}$$

Big-M method (Method of Penalities)

Charne's method

The Big-M method is an alternative method of solving a linear programming problem involving artificial variables. In this method we assign a very high penalty (say M) to the artificial variables in the objective function.

The iteration procedure of the algorithm is given below.

Step-1

Write the given LPP into the standard form and check whether there exists a starting basic feasible solution.

a) If there is a ready starting basic feasible solution, move on to step-4.

b) If there does not exist a ready starting basic feasible solution move on to step-2.

Step-2

Add artificial variables to the left side of each equation that has no obvious starting basic variables. Assign a very high penalty (say M) to these variables in the objective function.

Step-3

Apply simplex method to the modified LPP. following cases may arise of the last iteration.

a) Atleast one artificial variable is present in the basis with zero value. In such a case the current optimum basic feasible solution is degenerate.

b) Atleast one artificial variable is present in the basis with a positive value. In such a case, the given LPP does not posses an optimum basic feasible solution. The given problem is said to have a pseudo-optimum basic feasible solution

1. Solve using Big-M method

$\max z = x_1 + 3x_2$

st $x_1 + 2x_2 + x_3 \geq 4$

$2x_1 + 2x_2 - x_3 = 6$

$x_1, x_2, x_3 \geq 0$

Soln:

Given, $\max z = x_1 + 3x_2$

st $x_1 + 2x_2 + x_3 \geq 4$

$2x_1 + 2x_2 - x_3 = 6$

$x_1, x_2, x_3 \geq 0$

LPP becomes

$\max z = x_1 + 3x_2 + 0x_3 + 0S_1 - MA_1 - MA_2$

st $x_1 + 2x_2 + x_3 - S_1 + A_1 + 0A_2 = 4$

$2x_1 + 2x_2 - x_3 + 0S_1 + 0A_1 + A_2 = 6$

$x_1, x_2, x_3, A_1, A_2 \geq 0$

c_j	1	3	0	0	$-M$	$-M$				
x_j	x_1	x_2	x_3	S_1	A_1	A_2	RHS	B	C_B	Ratio
	1	2	1	-1	1	0	4	A_1	$-M$	$\frac{4}{2} = 2 \rightarrow$
	2	2	-1	0	0	1	6	A_2	$-M$	$\frac{6}{2} = 3$
z_j	$-3M$	$-4M$	0	M	$-M$	$-M$				
$z_j - c_j$	$-3M-1$	$-4M-3 \uparrow$	0	M	0	0				
	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	2	x_2	3	-
	1	0	-2	1	-1	1	2	A_2	$-M$	$\frac{2}{1} = 2 \rightarrow$
z_j	$\frac{3}{2}-M$	3	$\frac{3}{2}+2M$	$-\frac{3}{2}-M$	$\frac{3}{2}+M$	$-M$				
$z_j - c_j$	$\frac{1}{2}-M$	0	$\frac{3}{2}+2M$	$-\frac{3}{2}-M \uparrow$	$\frac{3}{2}+2M$	0				

	x_1	x_2	x_3	S_1	A_1	A_2	RHS	B	C_B	Ratio
	~~2~~	~~$\frac{3}{2}$~~	~~2~~	~~0~~	~~1~~	~~$\frac{1}{2}$~~	$\frac{1}{2}$			
	2	1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	3	x_2	3	
	1	0	-2	1	-1	1	0	S_1	0	
z_j	3	3	$-\frac{3}{2}$	0	0	$3/2$				
$z_j - c_j$	2	0	$-\frac{3}{2}\uparrow$	0	M	$M+\frac{3}{2}$				

Here, all entries of the column x_3 are negative values

$\therefore$ The given LPP has unbounded solution.

22.07.19

2, Solve using penalty method

$\max z = x_1 + 3x_2 + x_3$

st $x_1 + 2x_2 + x_3 \leq 4$

$x_1 - 4x_2 + x_3 \geq 2$

$x_1 + 2x_2 + 2x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

Soln:-

Given LPP becomes

$\max z = x_1 + 3x_2 + x_3 + 0S_1 + 0S_2 + 0S_3 - MA$

st $x_1 + 2x_2 + x_3 + S_1 + 0S_2 + 0S_3 + 0A = 4$

$x_1 - 4x_2 + x_3 + 0S_1 - S_2 + 0S_3 + A = 2$

$x_1 + 2x_2 + 2x_3 + 0S_1 + 0S_2 + S_3 + 0A = 2.$

$x_1, x_2, x_3, S_1, S_2, S_3, A \geq 0.$

Scanned with CamScanner

	x_1	x_2	x_3	s_1	s_2	s_3	A	RHS	B	C_B	Ratio
c_j	1	3	1	0	0	0	-M				
	1	2	1	1	0	0	0	4	s_1	0	$\frac{4}{1}=4$
	1	-4	1	0	-1	0	1	2	A	-M	$\frac{2}{1}=2$
	1	2	2	0	0	1	0	2	s_3	0	$\frac{2}{1}=2$ →
z_j	-M	4M	-M	0	M	0	-M				
z_j-c_j	-M-1	4M-3	-M-1	0	M	0	0				
	0	0	-1	1	0	-1	0	2	s_1	0	
	0	-6	-1	0	-1	-1	1	0	A	-M	
	1	2	2	0	0	1	0	2	x_1	1	
z_j	1	6M+2	M+2	0	M	M+1	-M				
z_j-c_j	0	6M-1	M+1	0	M	M+1	0				

$\therefore$ All $z_j - c_j \geq 0$

$\therefore$ We reach the optimum state

$\therefore$ Solution is $x_1 = 2$, $x_2 = 0$, $x_3 = 0$

$\max z = 2 + 3(0) + 1(0)$

$= 2$

$\max z = 2.$

3, Solve $\max z = 2x_1 + 3x_2$

$$\text{st } x_1 + 2x_2 \le 4$$
$$x_1 + x_2 = 3$$
$$x_1, x_2 \ge 0$$

Soln :-

Given LPP becomes

$$\max z = 2x_1 + 3x_2 + 0S_1 - MA$$
$$\text{St } x_1 + 2x_2 + S_1 + \cancel{0S_2} + 0A = 4$$
$$x_1 + x_2 + 0S_1 + A = 3$$
$$x_1, x_2, S_1, A \ge 0$$

C_j	2	3	0	$-M$				
x_j	x_1	x_2	S_1	A	RHS	B	C_B	Ratio
	1	2	1	0	4	S_1	0	$\frac{4}{2} = 2 \rightarrow$
	1	1	0	1	3	A	$-M$	$\frac{3}{1} = 3$
z_j	$-M$	$-M$	0	$-M$				
$z_j - c_j$	$-M-2$	$-M-3 \uparrow$	0	$-2M$				
	$\frac{1}{2}$	1	$\frac{1}{2}$	0	2	x_2	3	$\frac{2}{1/2} = 4$
	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	1	A	$-M$	$\frac{1}{1/2} = 2 \rightarrow$
z_j	$\frac{3}{2} - \frac{M}{2}$	3	$\frac{3}{2} + \frac{M}{2}$	$-M$				
$z_j - c_j$	$-\frac{1}{2} - \frac{M}{2} \uparrow$	0	$\frac{3}{2} + \frac{M}{2}$	0				
	0	1	1	-1	1	x_2	3	
	1	0	-1	2	2	x_1	2	

z_j	2	3	1	1
$z_j - c_j$	0	0	1	1+M

$\therefore$ All $z_j - c_j \geq 0$

$\therefore$ We reach the optimum stage

$\therefore$ Solution is $x_1 = 2,\ x_2 = 1$

$$\max z = 2(2) + 3(1)$$

$$= 4 + 3$$

$$\max z = 7$$

$$\max z = 7.$$

z_j	2	3	1	1
$z_j - c_j$	0	0	1	1+M

$\therefore$ All $z_j - c_j \geq 0$

$\therefore$ We reach the optimum stage

$\therefore$ Solution is $x_1 = 2$, $x_2 = 1$

$\max z = 2(2) + 3(1)$

$= 4 + 3$

$\max z = 7$

$\max z = 7$.

13.07.19 Duality in Linear Programming

2m
II CIA

Associated with every linear programming problem (maximization or minimization) there always exist another linear programming problem which is based upon the same data and having the same solution. The original problem is called the primal problem while the associated one is called its dual problem.

It is important to note that either of the two linear programming problems can be treated as primal and the other as its dual. The two problems. Thus, constitute a primal-dual pair.

2m I, CIA

General Primal - Dual pair

Definition-1: (Standard Primal Problem)

Maximize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to the constraints

$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i;$

$i = 1, 2, \dots, m$

$x_j \geq 0; \ j = 1, 2, \dots, n.$

Dual Problem

Minimize $z^* = b_1w_1 + b_2w_2 + \dots + b_mw_m$

Subject to the constraints:

$a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_m \geq c_j;$

$j = 1, 2, \dots, n$

$w_i \ (i = 1, 2, \dots, m)$ unrestricted.

Definition-2: (Standard Primal problem)

Minimize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to the constraints

$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i;$

$\leq$ $\quad i = 1, 2, \dots, m$

$x_j \geq 0; \ j = 1, 2, \dots, n$

Dual Problem

Maximize $z^* = b_1 w_1 + b_2 w_2 + \dots + b_m w_m$

Subject to the constraints:

$a_{1j} w_1 + a_{2j} w_2 + \dots + a_{mj} w_m \geq c_j$

$j = 1, 2, \dots, n$

$w_i \, (i = 1, 2, \dots, m)$ unrestricted.

Formulating a Dual Problem

TCIA 5m

Step-1:

Put the given linear programming problem into its standard form. Consider it as the primal problem.

Step-2

Identify the variables to be used in the dual problem. The number of these variables equals the number of constraint equations in the primal.

Step-3:

Write down the objective function of the dual, using the right-hand side constants of the primal constraints.

If the primal problem is of maximization type, the dual will be a minimization problem and vice-versa.

Step-4

Making use of dual variable identified in step 2, write the constraints for the dual problem.

a) If the primal is a maximization problem, the dual constraints must be all of $\geq$ type. If the primal is a minimization problem, the dual constraints must be all of '$\leq$' type.

b) The column coefficients of the primal contraints become the row coefficients of the dual constraints.

c) The coefficient of the primal objective function becomes the right-hand side constants of the dual constraints.

d) The dual variables are defined to be unrestricted in sign.

Step-5

Using step 3 and 4, write down the dual of the given LPP.

4.07.19 Primal - Dual pair in Matrix form:

2m Standard Primal problem:

Definition 1: (Standard Primal Problem)

Find $x^T \in R^n$ so as to

maximize $z = cx$, $c \in R^n$

Subject to the constraints:

$Ax = b$ and $x \geq 0$, $b^T \in R^m$

where A is an $m \times n$ real matrix.

Dual Problem

Find $w^T \in R^m$ so as to

minimize $z^* = b^T w$, $b \in R^m$

Subject to the constraints:

$A^T w \geq c^T$, $c \in R^n$

where A^T is the transpose of an $m \times n$ real matrix A and w is unrestricted in sign.

Definition-2: (Standard Primal Problem)

Find $x^T \in R^n$ so as to minim

minimize $z = cx$, $c \in R^n$

Subject to the constraints:

$A x = b$ and $\cancel{x \geq 0}$ $x \leq 0$, $b^T \in R^m$

where A is an $m \times n$ real matrix.

Dual Problem

Find $w^T \in R^n$ so as to

maximize $z^* = b^T w$, $b \in R^n$.

Subject to the constraints:

$A^T w \geq c^T$, $c \in R^n$.

where A^T is the transpose of an $m \times n$ real matrix A and w is unrestricted in sign.

Dual Simplex method

ICIA 5m

Dual Simplex method is applicable to those linear programming problems that start with infeasible but otherwise optimum solution. The method may be summarized as follows:

Step-1

Write the given linear programming problem in its standard form and obtain a starting basic solution.

Step-2:

a) If the current basic solution is feasible, use simplex method to obtain an optimum solution.

b) If the current basic solution is infeasible (i.e) values of basic variables are ≤ 0, go to the next step.

Step-3

Check whether the solution is optimum.

a) If the solution is not optimum, add an artificial constraint in such a way that the condition of optimality is satisfied.

b) If the solution is optimum, go to next step.

Step-4:

Select the basic variable having most negative value. The basic variable becomes the leaving variable and the row corresponding to its become the key row.

Step-5:

Obtain the ratios of the net evaluations to the corresponding coefficients in the key row. Ignore the ratios associated with positive and zero denominators. The entering vector is the one with the smallest absolute value of the ratios. Column corresponding to the entering vector becomes the key column.

Step-6

Reduce the leading element

into unity and all other entries of the key column, to zero by elementary row operations.

Step-7

Go to step 2 and repeat the procedure until an optimum basic feasible solution is attained.

1, Formulate the dual of the following linear programming problem

maximize $z = 5x_1 + 3x_2$

stc $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0.$

Soln:-

Given LPP is

maximize $z = 5x_1 + 3x_2$

stc $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0$

The standard primal problems is

maximize $z = 5x_1 + 3x_2 + 0s_1 + 0s_2$

stc $3x_1 + 5x_2 + s_1 + 0s_2 = 15$

$5x_1 + 2x_2 + 0s_1 + s_2 = 10$

$x_1, x_2, s_1, s_2 \ge 0.$

Dual

minimize $z^* = 15w_1 + 10w_2$

stc $3w_1 + 5w_2 \geq 5$

$5w_1 + 2w_2 \geq 3$

$w_1 + 0w_2 \geq 0$

$0w_1 + w_2 \geq 0$

w_1, w_2 are unrestricted.

$\therefore$ Dual problem is

minimize $z^* = 15w_1 + 10w_2$

stc $3w_1 + 5w_2 \geq 5$

$5w_1 + 2w_2 \geq 3$

$w_1 \geq 0, w_2 \geq 0.$

2. Write the dual of the LPP

min $z = 4x_1 + 6x_2 + 18x_3$

stc $x_1 + 3x_2 \geq 3$

$x_2 + 2x_3 \geq 5$

and $x_j \geq 0.$

Soln:-

Given,

min $z = 4x_1 + 6x_2 + 18x_3$

stc $x_1 + 3x_2 \geq 3$

$x_2 + 2x_3 \geq 5$

$x_1, x_2, x_3 \geq 0,$

The standard primal problem

$\min z = 4x_1 + 6x_2 + 18x_3 + 0s_1 + 0s_2$

stc

$x_1 + 3x_2 + 0x_3 - s_1 + 0s_2 = 3$

$0x_1 + x_2 + 2x_3 + 0s_1 - s_2 = 5$

$x_1, x_2, x_3, s_1, s_2 \geq 0.$

Dual

$\max z^* = 3w_1 + 5w_2$

stc

$w_1 + 0w_2 \leq 4$

$3w_1 + w_2 \leq 6$

$0w_1 + 2w_2 \leq 18$

$-w_1 + 0w_2 \leq 0$

$0w_1 + w_2 \leq 0$

w_1, w_2 are unrestricted.

∴ Dual problem is

maximize $z^* = 3w_1 + 5w_2$

stc $w_1 \leq 4$

$3w_1 + w_2 \leq 6$

$2w_2 \leq 18$

$w_1 \geq 0, w_2 \geq 0.$

3, Obtain the dual problem of the following problem

$$\min z = x_1 - 3x_2 - 2x_3$$

stc $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 - 4x_2 \ge 12$

$-4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \ge 0$ & x_3 is unrestricted.

Soln:-

Given LPP is

$$\min z = x_1 - 3x_2 - 2x_3$$

stc $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 - 4x_2 \ge 12$ $x_3 = x_3' - x_3''$

$-4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \ge 0$ & x_3 is unrestricted.

Given LPP becomes

$$\min z = x_1 - 3x_2 - 2x_3' + 2x_3''$$

stc $3x_1 - x_2 + 2x_3' - 2x_3'' \le 7$

$2x_1 - 4x_2 + 0x_3' - 0x_3'' \ge 12$

$-4x_1 + 3x_2 + 8x_3' - 8x_3'' = 10$

$x_1, x_2, x_3', x_3'' \ge 0.$

The standard primal problem is

min z

~~max z*~~ $= x_1 - 3x_2 - 2x_3' + 2x_3'' + 0s_1 + 0s_2$

stc $3x_1 - x_2 + 2x_3' - 2x_3'' + s_1 + 0s_2 = 7$

$$2x_1 - 4x_2 + 0x_3' - 0x_3'' + 0s_1 - s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3' - 8x_3'' + 0s_1 + 0s_2 = 10$$
$$x_1, x_2, x_3', x_3'', s_1, s_2 \geq 0$$

Dual

$$\max z^* = 7w_1 + 12w_2 + 10w_3$$

stc $3w_1 + 2w_2 - 4w_3 \leq 1$

$-w_1 - 4w_2 + 3w_3 \leq -3.$ $<$ - Simplex

$+2w_1 + 0w_2 + 8w_3 \leq -2$ min - 2 - Phase

$-2w_1 - 0w_2 - 8w_3 \leq 2$ max - Big - M

$w_1 + 0w_2 + 0w_3 \leq 0$

$0w_1 + 0w_2 + 0w_3 \leq 0$

w_1, w_2, w_3 are unrestricted.

$\therefore$ The dual problem is

$$\max z^* = 7w_1 + 12w_2 + 10w_3$$

stc $3w_1 + 2w_2 - 4w_3 \leq 1$

$-w_1 - 4w_2 + 3w_3 \leq -3$

$2w_1 + 8w_3 \leq -2$

$-2w_1 - 8w_3 \leq 2$

$w_1 \leq 0$, $w_2 \geq 0$, & w_3 is unrestricted

$\therefore$ The dual problem is

$$\max z^* = 7w_1 + 12w_2 + 10w_3$$

stc $3w_1 + 2w_2 - 4w_3 \leq 1$

$w_1 + 4w_2 - 3w_3 \geq 3$

$2w_1 + 8w_3 = 2$

$w_1 \leq 0, w_2 \geq 0$ & w_3 is unrestricted.

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27.07.19

4, Use duality to solve the following LPP maximum $z = 2x_1 + x_2$

stc $x_1 + 2x_2 \le 10$

$x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_1 - 2x_2 \le 1$

$x_1, x_2 \ge 0$

Soln:-

Given LPP is

$\max z = 2x_1 + x_2$

stc $x_1 + 2x_2 \le 10$

$x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_1 - 2x_2 \le 1$

$x_1, x_2 \ge 0$

Given LPP becomes

$\max z = 2x_1 + x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

stc $x_1 + 2x_2 + S_1 + 0S_2 + 0S_3 + 0S_4 = 10$

$x_1 + x_2 + 0S_1 + S_2 + 0S_3 + 0S_4 = 6$

$x_1 - x_2 + 0S_1 + 0S_2 + S_3 + 0S_4 = 2$

$x_1 - 2x_2 + 0S_1 + 0S_2 + 0S_3 + S_4 = 1$

$x_1, x_2, S_1, S_2, S_3, S_4 \ge 0.$

The standard primal problem is

$\min z^* = 2x_1 + x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

stc

Dual

$\min z^* = 10w_1 + 6w_2 + 2w_3 + w_4$

$w_1 + w_2 + w_3 + w_4 \geq 2$

$2w_1 + w_2 - w_3 - 2w_4 \geq 1$

$w_1 + 0w_2 + 0w_3 + 0w_4 \geq 0$

$0w_1 + w_2 + 0w_3 + 0w_4 \geq 0$

$0w_1 + 0w_2 + w_3 + 0w_4 \geq 0$

$0w_1 + 0w_2 + 0w_3 + w_4 \geq 0$

$\therefore$ The Dual problem is

$\min z^* = 10w_1 + 6w_2 + 2w_3 + w_4$

$w_1 + w_2 + w_3 + w_4 \geq 2$

$2w_1 + w_2 - w_3 - 2w_4 \geq 1$

$w_1, w_2, w_3, w_4 \geq 0.$

To solve the dual problem:

The Dual problem becomes

$\max z_1 = -10w_1 - 6w_2 - 2w_3 - w_4 + 0S_1 + 0S_2 - A_1 - A_2$

$2w_1 + w_2 + w_3 + w_4 - S_1 + 0S_2 + A_1 + 0A_2 = 2.$

$2w_1 + w_2 - w_3 - 2w_4 + 0S_1 - S_2 + 0A_1 + A_2 = 1$

$w_1, w_2, w_3, w_4, S_1, S_2, A_1, A_2 \geq 0.$

Phase-I

C_j	0	0	0	0	0	0	-1	-1				
x_j	w_1	w_2	w_3	w_4	S_1	S_2	A_1	A_2	RHS	B	C_B	Ratio
	1	1	1	1	-1	0	1	0	2	A_1	-1	$\frac{2}{1}=2$
	2	1	-1	-2	0	-1	0	1	1	A_2	-1	$\frac{1}{2}=0.5$ →
Z_j	-3	-2	0	1	1	1	-1	-1				
Z_j-C_j	-3↑	-2	0	1	1	1	0	0				
	0	1	3	4	-2	1	2	-1	3	A_1	-1	$\frac{3}{4}=0.75$ →
	1	1/2	-1/2	-1	0	-1/2	0	1/2	1/2	w_1	0	-
Z_j	0	-1	-3	-4	2	-1	-2	1				
Z_j-C_j	0	-1	-3	-4↑	2	-1	-1	2				
	0	1/4	3/4	1	-1/2	1/4	1/2	-1/4	3/4	w_4	0	
	4	3	1	0	-2	-1	2	1	5	w_1	0	
Z_j	0	0	0	0	0	0	0	0				
Z_j-C_j	0	0	0	0	0	0	1	1				

All $Z_j - C_j \geq 0$

$\therefore$ We reach the optimum stage and artificial variable absent in the table.

$\therefore$ We proceed phase-II.

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Phase - II

c_j	-10	-6	-2	-1	0	0	-1	-1				
x_j	w_1	w_2	w_3	w_4	S_1	S_2	A_1	A_2	RHS	B	C_B	Ratio
	0	1/4	3/4	1	-1/2	1/4	1/2	-1/4	3/4	w_4	0	
	4	3	1	0	-2	-1	2	1	5	w_1	0	
z_j	0	0	0	0	0	0	0	0				
z_j-c_j	10	6	2	1	0	0	1	1				

All $z_j - c_j \geq 0$

$\therefore$ We reach the optimum stage.

$\therefore$ Solution is $w_1 = 5, w_2 = 0, w_3 = 0, w_4 = \frac{3}{4}$

$$\max z_1 = -10(5) - \frac{3}{4}$$

$$= -50 - \frac{3}{4}$$

$$= \frac{-200-3}{4}$$

$$= \frac{-203}{4}$$

$$\min z^* = \frac{203}{4}$$

26.08.19

UNIT - III

Transportation Problem (275 Books)

The Transportation problem is one of the subclasses of LPPs in which the objective is to transport various quantities of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is minimum.

General Transportation Problem:

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \dots, n$$

and $x_{ij} \geq 0$ for all i and j

where

a_i = quantity of commodity available at origin i,

b_j = quantity of commodity needed at destination j

c_{ij} = Cost of transporting one unit of commodity from origin i to destination j.

and, x_{ij} = quantity transported from origin i to destination j.

The Transportation Table

	Destination 1	2	....	n	Supply
1	x_{11} c_{11}	x_{12} c_{12}	.....	x_{1n} c_{1n}	a_1
2	x_{21} c_{21}	x_{22} c_{22}	.....	x_{2n} c_{2n}	a_2
Origin ⋮	....	.....	.....	......	⋮
m	x_{m1} c_{m1}	x_{m2} c_{m2}		x_{mn} c_{mn}	a_m
Demand	b_1	b_2	...	b_n	

Types of Transportation Problem (T.P)

2m

In a T.P if the total capacity (supply) from all the origin equals the total requirements (Demand) in all the destination. Then it is said to be balanced T.P

(i.e) $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Otherwise it is called unbalanced T.P.

(i.e) $\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j$

Note:

Suppose $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$ then the destination is considered with requirement $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$

The unit cost of transportation to the destination from all the m origins may be taken as zeros.

Similar procedure can be taken for the case $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

Different type of Solution

Feasible Solution

A Feasible solution to T.P is a set of non-negative values x_{ij}, x_{ij}, $i=1$ to m, $j=1$ to n that satisfies the constraints.

Basic Feasible Solution

A Feasible solution is called the basic feasible solution.

If it contain no more than m+n-1 non-negative allocations, m is number of rows and n is the number of column in a transportation table.

Optimum Solution

Optimum Solution is a feasible solution not necessarily basic which minimize the total transportation cost.

Non-degenerate basic feasible solution

If a basic feasible solution to a TP contains exactly m+n-1 allocations is independent position. Then it is called a non degenerate basic feasible solution.

Degenerate basic feasible solution

If a basic feasible solution contain less than m+n-1 non-negative allocations then it is said to be degenerate.

27.08.19 Occupied Cells

The allocate cells in the transportation table is called occupied cells.

Unoccupied Cells

The empty (non allocated) cells in the transportation table is called Unoccupied Cells.

Finding an ~~it~~ initial basic feasible solution

There are several methods available to obtain an initial basic feasible solution. However, we shall discuss here the following three methods

i) North-West Corner Method,

ii) Least-Cost Method, and

iii) Vogel's approximation method.

1, North-West Corner Method

Step-1

Select the North-West (Upper Left hand) corner cell of the transportation table and allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied.

(i.e) $x_{11} = \min(a_1, b_1)$

Step-2

If $b_1 > a_1$, we move down vertically to the second row and make the second allocation of magnitude.

$x_{21} = \min(a_2, b_1 - x_{11})$ in the cell $(2,1)$

If $b_1 < a_1$, we move right horizontally to the second column and make the second allocation of magnitude.

$x_{21} = \min(a_1 - x_{11}, b_2)$ in the cell $(1,2)$

If $b_1 = a_1$, there is a tie for the second allocation one can make the second allocations of magnitude

$x_{12} = \min(a_1 - a_1, b_2) = 0$ in the cell $(1,2)$

$x_{21} = \min(a_2, b_1 - b_1) = 0$ in the cell $(2,1)$

Step-3

Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

2. Least-Cost Method (or) Matrix Minima Method

Step-1

Determine the smallest cost in the cost matrix of the transportation table. Let it be c_{ij}. Allocate

$x_{ij} = \min(a_i, b_j)$ in the cell (i,j)

Step-2

If $x_{ij} = a_i$ cross of the i^{th} row of the transportation table and decrease b_j by a_i. Go to Step-3

If $x_{ij} = b_j$ cross of the j^{th} column of the transportation table and decrease a_i by b_j. Go to Step-3.

If $x_{ij} = a_i = b_j$ cross off either the i^{th} row or j^{th} column but not the both.

Step-3

Repeat steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied whenever the minimum cost is not unique make an arbitary choice among the minima.

28.08.19

1, Find the IBFS to the following TP by North-West Corner Method.

Warehouse

	1	2	3	4	5	Factory Available
I	50	80	60	60	30	800
II	40	70	70	60	50	600
III	80	40	60	60	40	1100
Warehouse Requirements	400	400	500	400	800	2500

(Factories: I, II, III)

Soln:-

$$\sum a_i = 800 + 600 + 1100 = 2500$$

$$\sum b_j = 400 + 400 + 500 + 400$$
$$= 2500$$

$$\therefore \sum a_i = \sum b_j$$

$\therefore$ Given transportation problem is balanced.

	1	2	3	4	5	a_i
I	$\boxed{400}$ 50	$\boxed{400}$ 80	$\boxed{0}$ 60	60	30	800, 400, 0
II	40	70	$\boxed{500}$ 70	$\boxed{100}$ 60	50	600, 100
III	80	40	60	$\boxed{300}$ 60	$\boxed{800}$ 40	1100, 800
b_j	400	400	500	400, 300	800	

$\therefore$ IBFS is

$x_{11} = 400, x_{12} = 400, x_{13} = 0, x_{23} = 500, x_{24} = 100,$

$x_{34} = 300, x_{35} = 800$

Total cost $= (400 \times 50) + (400 \times 80) + (0 \times 60)$
$+ (500 \times 70) + (100 \times 60) +$
$(300 \times 60) + (800 \times 40)$
$= 20000 + 32000 + 0 + 35000 +$
$6000 + 18000 + 32000$
$= 1,43,000$

2, Find the IBFS by using NWC method to

From \ To	A	B	C	Availability
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirements	4	2	2	

Soln:.

$\sum a_i = 1 + 3 + 4 = 8$

$\sum b_j = 4 + 2 + 2 = 8$

$\therefore \sum a_i = \sum b_j$

$\therefore$ Given Transportation Problem is balanced.

	A	B	C	a_i
I	$\boxed{1}$ 50	30	220	$\cancel{1}$
II	$\boxed{3}$ 90	$\boxed{0}$ 45	170	$\cancel{3}$ $\cancel{0}$
III	250	$\boxed{2}$ 200	$\boxed{2}$ 50	$\cancel{4}$ $\cancel{2}$
b_j	$\cancel{4}$ $\cancel{3}$	$\cancel{2}$	$\cancel{2}$	

$\therefore$ IBFS is

$x_{11} = 1, x_{21} = 3, x_{22} = 0, x_{32} = 2, x_{33} = 2.$

Total cost $= 1(50) + 3(90) + 0(45) + 2(200) + 2(50)$

$= 50 + 270 + 400 + 100$

Total Cost $= 820$

2, Find the Solution by Least-Cost Method

From \ To	A	B	C	Availability
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirements	4	2	2	

Soln:-

$\Sigma a_i = 1 + 3 + 4 = 8$

$\Sigma b_j = 4 + 2 + 2 = 8$

$\Sigma a_i = \Sigma b_j$

$\therefore$ Given Transportation Problem is balanced.

	A	B	C	a_i
I	50	$\boxed{1}$ 30	220	$\cancel{1}$
II	$\boxed{2}$ 90	$\boxed{1}$ 45	170	$\cancel{3}$ 2
III	$\boxed{2}$ 250	200	$\boxed{2}$ 50	$\cancel{4}$ 2
b_j	$\cancel{4}$ 2	$\cancel{2}$ $\cancel{1}$	$\cancel{2}$	

$\therefore$ IBFS is

$x_{12} = 1,\ x_{21} = 2,\ x_{22} = 1,\ x_{33} = 2,\ x_{31} = 2.$

$$\text{Total cost} = 1(30) + 2(90) + 1(45) + 2(50) + 2(250)$$

$$= 30 + 180 + 45 + 100 + 500$$

$$= 855$$

Total Cost = 855

4, Find the solution by Least - Cost Method

	Warehouse 1	2	3	4	5	Factorial available
Factories I	50	80	60	60	30	800
II	40	70	70	60	50	600
III	80	40	60	60	40	1100
Warehouse Requirements	400	400	500	400	800	2500

Soln:.

$\Sigma a_i = 800 + 600 + 1100$

$\Sigma a_i = 2500$

$\Sigma b_j = 400 + 400 + 500 + 400 + 800$

$\Sigma b_j = 2500$

$\therefore$ Given transportation problem is balanced.

	1	2	3	4	5	a_i
I	50	80	$\boxed{0}$ 60	60	$\boxed{800}$ 30	~~800~~ ~~0~~
II	$\boxed{400}$ 40	70	70	$\boxed{200}$ 60	50	~~600~~ ~~200~~
III	80	$\boxed{400}$ 40	$\boxed{500}$ 60	$\boxed{200}$ 60	40	~~1100~~ ~~700~~ ~~200~~
b_j	~~400~~	~~400~~	~~500~~	~~400~~ ~~200~~	~~800~~	

$\therefore$ IBFS is

$x_{13} = 0,\ x_{15} = 800,\ x_{21} = 400,\ x_{24} = 200,$

$x_{32} = 400,\ x_{33} = 500,\ x_{34} = 200$

Total Cost $= 0(60) + 800(30) + (400)(40)$
$+ (200)(60) + (400)40 +$
$500(60) + 200(60)$

$= 0 + 24000 + 16000 + 12000 +$

$16000 + 30000 + 12000$

$= 1,10,000$

Total Cost = 1,10,000

4.19 Vogel's Approximation method (VAM)

Step-1

For each row of the transportation table identify the smallest and the next-to-smallest cost. Determine the difference between them for each row. Display them alongside the transportation table by enclosing them in parenthesis against the respective rows, similarly, compute the diffences for each column.

Step-2

Identify the row or column with the largest difference among all the rows and columns. If a tie occurs, use any arbitary tie-breaking choice. Let the greatest difference correspond to i^{th} row and let c_{ij} be the smallest cost in the i^{th} row.

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Allocate the maximum feasible amount $x_{ij} = \min(a_i, b_j)$ in the $(i,j)^{th}$ cell and cross off the i^{th} row or the j^{th} column in the usual manner.

Step-3

Recompute the column and row differences for the reduced transportation table and go to step 2. Repeat the procedure until all the rim requirements are satisfied.

1, Use Vogel's Approximation method to obtain an initial basic feasible solution of the transportation problem.

	D	E	F	G	Available
A	11	12	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250.	

Soln:

$\Sigma a_i = 250 + 300 + 400 = 950$

$\Sigma b_j = 200 + 225 + 275 + 250 = 950$

$\therefore \Sigma a_i = \Sigma b_j$

$\therefore$ Given Transportation problem is balanced.

	D	E	F	G	a_i
A	$\boxed{200}$ 11	$\boxed{50}$ 13	17	14	~~250~~ ~~50~~ (2) (1) - -
B	16	$\boxed{175}$ 18	14	$\boxed{125}$ 10	~~300~~ ~~125~~ (4) (4) (4) (4) ←
C	21	24	$\boxed{275}$ 13	$\boxed{125}$ 10	~~400~~ ~~275~~ (3) (3) (3) (3)
b_j	200	~~225~~ 175	~~275~~	~~250~~ ~~125~~	
	(5)↑	(5)↑	(1)	(0)	
	-	(5)↑	(1)	(0)	
	-	(6)↑	(1)	(0)	
	-	-	(1)	(0)	

$$\begin{array}{r} 175 \\ \times 18 \\ \hline 1400 \\ 175+ \\ \hline 3150 \end{array} \qquad \begin{array}{r} 3150 \\ 1300 \\ \hline 4450 \end{array}$$

$\therefore$ IBFS is

$x_{11} = 200,\ x_{12} = 50,\ x_{22} = 175,\ x_{24} = 125,$

$x_{33} = 275,\ x_{34} = 125$

Total Cost $= 200(11) + 13(50) + 18(175) + 10(125) + 13(275) + 10(125)$

$= 2200 + 650 + 3150 + 1250 + 3575$ ~~4450~~ $+ 1250$

$= 12075$

Total Cost $= 12075$/-

2. Find the Solution

	D_1	D_2	D_3	D_4	
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
	5	8	7	15	

Soln:-

$\Sigma a_i = 7 + 10 + 18 = 35$

$\Sigma b_j = 5 + 8 + 7 + 15 = 35$

$\therefore \Sigma a_i = \Sigma b_j$

$\therefore$ Given Transportation problem is balanced.

	D_1	D_2	D_3	D_4	a_i	
P_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 12	~~7~~ ~~2~~	(7)(18)(38) (38)
P_2	70	30	$\boxed{7}$ 40	$\boxed{3}$ 60	~~10~~ 3	(10)(10)(20) (20)
P_3	40	$\boxed{8}$ 10	60	$\boxed{10}$ 20	~~18~~ ~~10~~	(10)(10)(40)
b_j	~~5~~	~~8~~	~~7~~	~~15~~ ~~7~~ 3		
	(21)	(20)	(10)	(8)		
	-	(20)	(10)	(8)		
	-	-	(10)	(8)		
	-	-	(10)	(48)		

$\therefore$ IBFS is

$x_{11} = 5, \; x_{14} = 2, \; x_{23} = 7, \; x_{24} = 3, \; x_{32} = 8,$

$x_{34} = 10$

$$\text{Total Cost} = 5(19) + 2(12) + 7(40) + 3(60) + 8(10) + 10(20)$$

$$= 95 + 24 + 280 + 180 + 80 + 200$$

$$= 859$$

Total Cost = 859.

2, Use Vogel's Approximation method to obtain IBFS of the TP

	Warehouse 1	2	3	4	5	Factorial available
I	50	80	60	60	30	800
II	40	70	70	60	50	600
III	80	40	60	60	40	1100
Warehouse requirements	400	400	500	400	800	2500

Soln:

$$\Sigma a_i = 800 + 600 + 1100 = 2500$$

$$\Sigma b_j = 400 + 400 + 500 + 400 + 800 = 2500$$

$$\Sigma a_i = \Sigma b_j$$

$\therefore$ Given Transportation table is balanced.

	1	2	3	4	5	a_i
I	50	$\boxed{400}$ 80	60	60	800 30	~~800~~ (20)(20) 400
II	40	70	70	60	50	600 (10)(10)
III	80	40	60	60	40	1100 (0)(20)
b_j	400	~~400~~	500	400	~~800~~ 0	
	(10)	(30)↑	(0)	(0)	(10)	
	(10)	~~(30)~~	(00)	(0)	(10)	

	1	2	3	4	5	a_i
I	50	80	60	60	$\boxed{800}$ 30	~~800~~ (20)(20) ←
II	$\boxed{400}$ 40	70	70	$\boxed{200}$ 60	50	~~600~~ ~~200~~ (10)(10)(10)(10) (10)
III	80	$\boxed{400}$ 40	$\boxed{500}$ 60	$\boxed{200}$ 60	$\boxed{0}$ 40	~~1100~~ ~~700~~ ~~200~~ (0)(20)(20) (20) ←
b_j	~~400~~	~~400~~	~~500~~	~~400~~ 200	~~800~~ 0	
	(10)	(30)↑	(0)	(0)	(10)	
	(10)	-	(0)	(0)	(10)	
	(40)↑	-	(10)	(0)	(10)	
	-	-	(10)	(0)	(10)	
	-	-	(10)↑	(0)		

∴ IBFS is

$x_{15} = 800, x_{21} = 400, x_{24} = 200, x_{32} = 400,$

$x_{33} = 500, x_{34} = 200, x_{35} = 0.$

Total Cost = 30(800) + 40(400) + 60(200)

+ 40(400) + 60(500) + 60(200)

+ 0(40)

= 24000 + 16000 + 12000 + 16000 +

30000 + 12000 + 0

= 1,10,000.

Total Cost = 1,10,000.

8.19

4, Solve the transportation problem

	B_1	B_2	B_3	
A_1	12	14	16	150
A_2	21	31	72	100
A_3	18	8	20	150
	100	150	250	

Soln:

$\Sigma a_i = 150 + 100 + 150 = 400$

$\Sigma b_j = 100 + 150 + 250 = 500$

$\Sigma a_i \neq \Sigma b_j$

∴ Given TP is unbalanced

$\therefore$ We add dummy row A_4 with

$a_4 = 100$

$\therefore$ The balanced TP is

	B_1	B_2	B_3	a_i
A_1	12	14	16	150
A_2	21	31	72	100
A_3	18	8	20	150
A_4	0	0	0	100
b_j	100	150	250	

To solve by NWC method.

				a_i
	$\boxed{100}$ 12	$\boxed{50}$ 14	16	~~150~~ 50
	21	$\boxed{100}$ 31	72	~~100~~
	18	$\boxed{0}$ 8	$\boxed{150}$ 20	~~150~~
	0	0	$\boxed{100}$ 0	~~100~~
b_j	~~100~~	~~150~~	~~250~~	
		~~100~~	~~100~~	
		ø		

Solution is

$a_{11} = 100, a_{12} = 50, a_{22} = 100, a_{32} = 0,$

$a_{33} = 150, a_{43} = 100$

$\text{Total} = (100 \times 12) + 50(14) + 100(31)$

$+ (0 \times 8) + 150(20) + 100(0)$

$= 1200 + 700 + 3100 + 0 + 3000 + 0$

$\text{Total} = 8000.$

5. Find IBFS using matrix minima method

	O	P	Q	
A	100	102	204	1
B	18	117	103	5
C	96	132	72	7
	2	3	5	

Soln:

$\Sigma a_i = 1 + 5 + 7 = 13$

$\Sigma b_j = 2 + 3 + 5 = 10$

$\therefore \Sigma a_i \neq \Sigma b_j$

$\therefore$ Given TP is unbalanced

We add dummy column R

with $b_4 = 3$.

$\therefore$ Balanced TP is

	O	P	Q	R	a_i
A	100	102	204	0	1
B	18	117	103	0	5
C	96	132	72	0	7
b_j	2	3	5	3	

To solve by Matrix minima method

	O	P	Q	R	a_i
A	100	102	204	$\boxed{1}$ 0	1
B	$\boxed{2}$ 18	117	103	$\boxed{2}$ 0	5
C	96	132	72	0	7
b_j	2	3	5	3	

~~Total~~

IBFS is $x_{14} = 1$; $x_{21} = 2$; $x_{22} = 1$; $x_{24} = 2$

$x_{32} = 2$; $x_{33} = 72$

Total $= 1(0) + 2(18) + 1(117) + 2(0) +$

$2(132) + 5(72)$

$= 0 + 36 + 117 + 0 + 264 + 360$

Total $= 777$.

Degeneracy in Transportation Problem

A basic feasible solution for the general transportation problem must consist of $(m+n-1)$ occupied cells. The basic solution will be called degenerate when the number of occupied cells in less then the required number, $m+n-1$. Degenerate can occur in initial solution or it may arise in some subsequent iterations. We now discuss procedure to deal with the problem of degeneracy.

Case (1): (Degeneracy at the initial solution)

To resolve degeneracy at the initial solution, a very small quantity $\varepsilon (>0)$ is allocated in an unoccupied cell so as to get $m+n-1$ number of unoccupied cells. In a minimum transportation problem, it is better to allocate to unoccupied cells that have lowest transportation costs. In some cases, ε must be added in one of those unoccupied cells which make possible the determination of u_i and v_j uniquely.

The quantity ε is considered to be so small that if it is transferred to an occupied cell it does not change the quantity of allocation.

That is, $x_{ij} + \varepsilon = x_{ij} - \varepsilon = x_{ij}$ but $\varepsilon - \varepsilon = 0$. Also ε does not effect the total transportation cost of allocation. Hence the quantity ε is used to evaluate unoccupied cells and once the purpose is over, ε must be removed from the scene.

Case 2: (Degeneracy at subsequent Iterations)

The resolve degeneracy which occur during optimality test, the quantity ε may be allocated to one or more cells which have become unoccupied recently to have m+n-1 number of occupied cells in the new solution. It may be removed once the purpose is over.

Transportation Algorithm (MODI method)

Step-1

Find the initial basic feasible solution by using any of the three methods discussed above.

Step-2

Check the number of occupied cells. If there are less than m+n-1, there exists degeneracy and we introduce a very small positive assignment of $\varepsilon (\approx 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to m+n-1.

Step-3

For each occupied cell in the current solution, solve the system of equations $u_i + v_j = c_{ij}$. Starting initially with some $u_i = 0$ (or) $v_j = 0$ and entering successively, the values of u_i and v_j in the transportation table margins.

Step-4

Compute the net evaluations $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$ for all unoccupied basic cells and enter them in the lower left corners of the corresponding cells.

Step-5

Examine the sign of each $z_{ij}-c_{ij}$. If $z_{ij}-c_{ij} \leq 0$, then the current basic feasible solution is all optimum one. If atleast one $z_{ij}-c_{ij} > 0$, select the unoccupied cell, having the largest positive net evaluations to enter the basis.

Step-6

Let the unoccupied cell (r,s) enter the basis. Allocate an unknown quantity say θ to the cell (r,s). Identify a loop that starts and ends at the cell (r,s) and connects some of the basic cells. Add and subract interchangeably, θ to and from the transition cells of the loop in such a way that the rim requirement remain satisfied.

Step-7

Assign a maximum value to θ in such a way that the value of one basic variable becomes zero and the other basic variables remain non-negative. The basis cell whose allocation has been reduced to zero, leaves the basis.

Step-8

Return to step 3 and repeat the process until an optimum basic feasible solution has been obtained.

Definition : (Loop)

In a transportation table, an ordered set of four or more cells is said to form a LOOP if

i) any two adjacent cells in the ordered set lie either in the same row or in the same column and

ii) any three or more adjacent cells in the ordered set do not lie in the same ~~or~~ row or the same column. The first cell of the set is considered to follow the last in the set.

(i.e) each cell (except the last) must appear only one in the ordered set.

1, Find Optimum solution for

	D_1	D_2	D_3	D_4	Supply
S_1	3	7	6	4	5
S_2	2	4	3	2	2
S_3	4	3	8	5	3
Demand	3	3	2	2	

Soln:-

$\Sigma a_i = 5+2+3 = 10$

$\Sigma b_j = 3+3+2+2 = 10$

$\therefore \Sigma a_i = \Sigma b_j$

$\therefore$ Given TP is balanced

To find IBFS using VAM:

$\boxed{3}$ 3	7	6	$\boxed{2}$ 4	$\cancel{5}$ $\cancel{2}$	(1) (1) (1) (1)
2	4	$\boxed{2}$ 3	$\boxed{0}$ 2	$\cancel{2}$ 0	(0) (0) - -
4	$\boxed{3}$ 3	8	$\boxed{0}$ 5	$\cancel{3}$ 0	(1) (1) (1) (1)
$\cancel{3}$	$\cancel{3}$	$\cancel{2}$	$\cancel{2}$ 0		
(1)	(1)	(3)↑	(2)		
(1)	(1)	-	(2)↑		
(1)	(4)↑	-	(1)		
(1)	-	-	(1)		

$\therefore$ IBFS is

$x_{11} = 3,\ x_{14} = 2,\ x_{23} = 2,\ x_{24} = 0,\ x_{32} = 3,$

$x_{34} = 0.$

$$\text{Total} = 3(3) + 2(4) + 2(3) + 0(2) + 3(3) + 0(5)$$

$$= 9 + 8 + 6 + 9$$

$$= 32.$$

To find Optimum Solution using MODI method (u-v method)

				u_i
$\boxed{3}$ 3	$\boxed{-5}$ 7	$\boxed{-1}$ 6	$\boxed{2}$ 4	4
$\boxed{-1}$ 2	$\boxed{-4}$ 4	$\boxed{2}$ 3	$\boxed{0}$ 2	2
$\boxed{0}$ 4	$\boxed{3}$ 3	$\boxed{-2}$ 8	$\boxed{0}$ 5	5

v_j -2 0

All $u_i + v_j - c_{ij} \leq 0$

$\therefore$ We attain optimum stage

$\therefore$ Optimum solution is

$x_{11} = 3,\ x_{14} = 2,\ x_{23} = 2,\ x_{24} = 0,\ x_{32} = 3,\ x_{34} = 0$

$$\text{Total} = 3(3) + 2(4) + 2(3) + (0 \times 2) + 3(3) + 0(5)$$

$$= 9 + 8 + 6 + 0 + 9 + 0$$

$$= 32.$$

2, Find Optimum Solution for the transportation problem

				Available units
6	1	9	3	70
11	5	2	8	55
10	12	4	7	90
Required Units: 85	35	50	45	

Soln:

$\sum a_i = 70 + 55 + 90$

$= 215$

$\sum b_j = 85 + 35 + 50 + 45$

$= 215$

$\therefore \sum a_i = \sum b_j$

$\therefore$ Given Transportation problem is balanced.

To find IBFS using VAM.

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				a_i				
70 / 6	1	9	3	~~70~~	(2)	-	-	
11	35 / 5	20 / 2	8	~~55~~ ~~20~~	(3)	(3)	(6) ←	
15 / 10	12	30 / 4	45 / 7	~~90~~ ~~60~~ ~~15~~	(3)	(3)	(3)	~~(3)~~
b_j								
~~85~~ ~~15~~	~~35~~	~~50~~ 30	45					
~~(4)~~ ↑	(4)	(2)	(4)					
(1)	(7) ↑	(2)	(1)					
(1)	-	(2)	(1)					

∴ IBFS is

$x_{11} = 70, x_{22} = 35, x_{23} = 20, x_{31} = 15, x_{33} = 30,$

$x_{34} = 45.$

Total $= 6(70) + 5(35) + 2(20) + 15(10) + 30(4) + 7(45)$

$= 420 + 175 + 40 + 150 + 120 + 315$

Total $= 1220$

To find Optimum Solution using MODI method (u-v method)

$m = 3$

$n = 4$

$3 + 4 - 1 = 6$

				u_i
70 $-\theta$ / 6	θ (2) / 1	(-9) / 9	(0) / 3	-4
(-3) / 11	35 $-\theta$ / 5	20 θ / 2	(-3) / 8	-2
15 θ / 10	(-5) / 12	30 $-\theta$ / 4	45 / 7	0
10	7	4	7	v_j

$\theta = \min\{70, 30, 35\}$

$\theta = 30$

				u_i
40 / 6	30 / 1	(-11) 9	(0) 3	0
(-1) 11	5 / 5	50 / 2	(-1) 8	4
45 / 10	(-7) 12	(-2) 4	45 / 7	4
v_j: 6	1	-2	3	

All $u_i + v_j - c_{ij} \le 0$

∴ We reach the Optimum Stage

~~∴ IBFS~~ is Optimum Solution is

$x_{11} = 40$, $x_{12} = 30$, $x_{22} = 5$, $x_{23} = 50$,

$x_{31} = 45$, $x_{34} = 45$

$\text{Total} = 40(6) + 30(1) + 5(5) + 50(2)$

$+ 45(10) + 45(7)$

$= 240 + 30 + 25 + 100 + 450 + 315$

$= 1160$

2, Find IBFS by North-West Corner Method and also find Optimum Solution for

				Available units
6	1	9	3	70
11	5	2	8	55
10	12	4	7	90
Required units: 85	35	50	45	

Soln:-

$\Sigma a_i = 70 + 55 + 90 = 215$

$\Sigma b_j = 85 + 35 + 50 + 45 = 215$

$\Sigma a_i = \Sigma b_j$

$\therefore$ Given TP is balanced.

				a_i
$\boxed{70}$ 6	1	9	3	$\cancel{70}$
$\boxed{15}$ 11	$\boxed{35}$ 5	$\boxed{5}$ 2	8	$\cancel{55}$ $\cancel{40}$ $\cancel{5}$
10	12	$\boxed{45}$ 4	$\boxed{45}$ 7	$\cancel{90}$ $\cancel{45}$
b_j: $\cancel{85}$ $\cancel{15}$	$\cancel{35}$	$\cancel{50}$ $\cancel{45}$	$\cancel{45}$	

$\therefore$ IBFS is

$x_{11} = 70$, $x_{21} = 15$, $x_{22} = 35$, $x_{23} = 5$, $x_{33} = 45$, $x_{34} = 45$

Total = 6(70) + 15(11) + 5(35) + 5(2)
+ 45(4) + 45(7)

= 420 + 165 + 175 + 10 + 180 + 315

Total = 1265

To find Optimum Solution using MODI method (u-v method)

				u_i
$\boxed{70}$ 6	(-1) 1	(-12) 9	(-3) 3	-5
$\boxed{15}$ $-\theta$ 11	$\boxed{35}$ 5	$\boxed{5}$ θ 2	(-3) 8	0
θ (3) 10	(-5) 12	$\boxed{45}$ $-\theta$ 4	$\boxed{45}$ 7	2
v_j: 11	5	2	5	

$\theta = \min\{45, 15\}$

$\theta = 15$

				u_i
$\boxed{70}$ $-\theta$ 6	θ (2) 1	(-9) 9	(0) 3	-4
(-3) 11	$\boxed{35}$ $-\theta$ 5	$\boxed{20}$ θ 2	(-3) 8	-2
$\boxed{15}$ θ 10	(-5) 12	$\boxed{30}$ $-\theta$ 4	$\boxed{45}$ 7	0
v_j: 10	7	4	7	

$\theta = \min\{70, 30, 35\}$

$\theta = 30$

				u_i
$\boxed{40}$ 6	$\boxed{30}$ 1	-11 9	0 3	0
-1 11	$\boxed{5}$ 5	$\boxed{50}$ 2	-1 8	4
$\boxed{45}$ 10	-7 12	-2 4	$\boxed{45}$ 7	4
v_j 6	1	-2	3	

All $u_i + v_j - c_{ij} \le 0$

$\therefore$ We reach the Optimum stage

Optimum Solution is

$x_{11} = 40, x_{12} = 30, x_{22} = 5, x_{23} = 50, x_{31} = 45,$

$x_{34} = 45$

$\text{Total} = 40(6) + 30(1) + 5(5) + 50(2) + 45(10) + 45(7)$

$= 240 + 30 + 25 + 100 + 450 + 315$

$\text{Total} = 1160$

06.09.19 Assignment Problem

II - CIA 2m

The assignment problem is special case of transportation problem in which the objective is to assign a number of origins to the equal number of destinations at a minimum cost (or maximum profit).

Assignment problem is a completely degenerate form of a transportation problem. The units available at each origin and units demanted at each destination are all equal to one. That means exactly one occupied cell in each row and each column of the transportation table.

(i.e) only n occupied cells in place of the required $n+n-1$ $(=2n-1)$.

II - CIA 2m

Mathematical Formulation of the problem

The general Assignment problem is

Activity

Resource	A_1	A_2	$\cdots$	A_n	Available
R_1	C_{11}	C_{12}	$\cdots$	C_{1n}	1
R_2	C_{21}	C_{22}	$\cdots$	C_{2n}	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_n	C_{n1}	C_{n2}	$\cdots$	C_{nn}	1
Required	1	1	$\cdots$	1	

Let x_{ij} denote the assignment of ith resource to jth activity, such that

$$x_{ij} = \begin{cases} 1, & \text{if resource } i \text{ is assigned to activity } j. \\ 0, & \text{otherwise} \end{cases}$$

Then, the mathematical formulation of the assignment problem is

$$\text{minimize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ and } \sum_{j=1}^{n} x_{ij} = 1 ; x_{ij} = 0 \text{ or } 1,$$

for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

The Assignment Problem

Step-1:-

Determine the cost table from the given problem.

(i) If the number of sources is equal to the number of destinations, go to step-3.

(ii) If the number of sources is not equal to the number of destinations, go to step-2.

Step-2:-

Add a dummy source or dummy destination, so that the cost table becomes a square matrix. The cost entries of dummy source / destinations are always zero.

Step-3

Locate the smallest element in each row of the given cost matrix and then subract the same from each element of that row.

Step-4

In the reduced matrix obtained in step-3, locate the smallest element of each column and then subract the same from each element of that column. Each column and row now have atleast one zero.

Step-5

In the modified matrix obtained in step 4, search for an optimal assignment as follows:

a) Examine the rows successively until a row with a single zero is found. Enrectangle this zero (□) and cross off (X) all other zeros in its column. Continue in this manner until all the rows have been taken care of.

9.9.19 b) Repeat procedure for each column of the reduced matrix.

c) If a row and / or column has two or more zeros and one cannot be choose by inspection. Then assign arbitary any one of these zeros and cross off all other zeros of that row/column.

d) Repeat (a) throug(c), above successively until the chain of assigning (□) or cross (x) ends.

Step-6

If the number of assigments (□) is equal to n (the order of the cost matrix), an optimum solution is reached

If the number of assignments is less than n (the order of the cost matrix), go to the next step

Step-7

Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of the reduced matrix. This can be conveniently done by using a simple procedure.

a, Mark (✓) rows that do not have any assigned zero.

b) Mark (✓) columns that have zero in the marked row.

c) Mark (✓) rows that have assigned zeros in the marked column.

d) Repeat (b) and (c) above until the chain of marking is completed.

e) Draw lines through all the unmarked rows and marked columns. This gives us the desired minimum number of lines.

Step-8

Develop the new revised cost matrix as follows:

a) Find the smallest element of the reduced matrix not covered by any of the lines.

b) Subract this element from all the uncovered elements and add the same to all the elements lying at the intersection of any two lines.

Step-9

Go to Step-6 and repeat the procedure until an optimum solution is obtained.

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1, Solve the assignment Problem

	E	F	G	H
A	12	9	12	21
B	11	25	27	29
C	15	16	17	12
D	23	15	12	11

Soln:

Number of rows = 4
Number of column = 4

∴ Given assignment problem is balanced.

Step-1 (for row) Select minimum element from each row and it is subracted from each element of the corresponding rows.

3	0	4	12
0	14	16	18
8	9	0	5
12	4	1	0

Step-2 (for column) Select minimum element from each column and it is subracted from each element of the corresponding column.

3	$\boxed{0}$	4	12
$\boxed{0}$	14	16	18
8	9	$\boxed{0}$	5
12	4	1	$\boxed{0}$

Step-3

Starting with row 1, mark assignment enrectangle (□) and for all rows and columns.

Now, since each row and each column has one and only one assignment,

∴ We reach the optimum stage.

∴ Optimum Solution is

A -> F, B -> E, C -> G, D -> H

Total = 9+11+7+11

= 38

10.9.19

2, A department head has four subordinates and four tasks to be performed. The subordinates differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate of the time each man would take to perform each task, is given in the matrix below:

Tasks	Men			
	E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man-hours?

Soln:-

Number of rows = 4

Number of columns = 4

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4 = 4

∴ Given Assignment problem is balanced.

Step-1 (for row)

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Step-2: (for column

7	11	5	$\boxed{0}$
$\boxed{0}$	11	$\cancel{0}$	13
25	$\boxed{0}$	2	$\cancel{0}$
9	12	13	$\cancel{0}$

Step-2

2	6	$\boxed{0}$	$\cancel{0}$
$\boxed{0}$	11	$\cancel{0}$	18
23	$\boxed{0}$	2	5
4	7	8	$\boxed{0}$

∴ We reach the Optimum stage

Optimum Solution is

A -> G, B -> E, C -> F, D -> H

Total = 17 + 12 + 19 + 10

= 59

Balanced and Unbalanced Problems

II CIA 2m

In Assignment problem, if no. of rows = no. of columns, then given Assignment problem is balanced.

Example

2	3	4	6
7	8	9	12
12	14	16	17
21	31	41	76

If no. of rows ≠ no. of columns then given assignment problem is unbalanced.

Example

2	3	4
5	6	7
8	11	12
17	16	17

1	2	3	7
11	13	14	16
21	2	4	6

3. Solve the Assignment Problem

	B_1	B_2	B_3
A_1	2	3	4
A_2	5	6	7
A_3	8	11	13
A_4	17	16	17

Soln:-

No. of rows = 4

No. of columns = 3.

No. of rows $\neq$ No. of columns.

$\therefore$ Given A.P is unbalanced.

$\therefore$ We add dummy column.

$\therefore$ Balanced Assignment problem is

	B_1	B_2	B_3	B_4
A_1	2	3	4	0
A_2	5	6	7	0
A_3	8	11	13	0
A_4	17	16	17	0

Step-1 (for row)

$$\begin{bmatrix} 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 0 \\ 8 & 11 & 13 & 0 \\ 17 & 16 & 17 & 0 \end{bmatrix}$$

Step-2 (for column)

$$\begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 3 & 3 & 3 & \boxed{0} \\ 6 & 8 & 9 & \cancel{0} \\ 15 & 13 & 13 & \cancel{0} \end{bmatrix}$$

Step-3

$$\begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} & 3 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \\ 3 & 5 & 6 & \boxed{0} \\ 12 & 10 & 10 & \cancel{0} \end{bmatrix}$$

Step-4

$$\begin{bmatrix} \cancel{0} & \boxed{0} & \cancel{0} & 6 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 \\ \boxed{0} & 2 & 3 & \cancel{0} \\ 9 & 7 & 7 & \boxed{0} \end{bmatrix}$$

∴ We reach the Optimum Stage

Optimum Solution is

$A_1 \rightarrow B_2$, $A_2 \rightarrow B_3$, $A_3 \rightarrow B_1$, $A_4 \rightarrow B_4$.

∴ Total = 3 + 7 + 8 + 0

= 18

12.09.19

4, Solve the Assignment Problem

	D	E	F	G
A	1	2	3	7
B	11	13	14	16
C	21	2	4	6

Soln:-

Numbers of rows = 3

Number of column = 4.

∴ No. of rows ≠ No. of column.

∴ Given A.P is unbalanced.

∴ Balanced Assignment Problem is

	D	E	F	G
A	1	2	3	7
B	11	13	14	16
C	21	2	4	6
H	0	0	0	0

Step-1 (for row)

$$\begin{bmatrix} 0 & 1 & 2 & 6 \\ 0 & 2 & 3 & 5 \\ 19 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-2 (for column

$$\begin{bmatrix} \boxed{0} & 1 & 2 & 6 \\ \cancel{0} & 2 & 3 & 5 \\ 19 & \boxed{0} & 2 & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & \cancel{0} \end{bmatrix}$$

Step-3

$$\begin{bmatrix} \cancel{0} & \cancel{0} & 1 & 5 \\ \boxed{0} & 1 & 2 & 4 \\ 20 & \boxed{0} & 2 & 4 \\ 1 & \cancel{0} & \boxed{0} & \boxed{0} \end{bmatrix}$$

Step-4

$$\begin{bmatrix} \cancel{0} & \cancel{0} & \boxed{0} & 4 \\ \boxed{0} & 1 & 1 & 3 \\ 20 & \boxed{0} & 1 & 3 \\ 2 & 1 & \cancel{0} & \boxed{0} \end{bmatrix}$$

∴ We reach the optimum stage

Optimum Solution is

A -> F, B -> D, C -> E, H -> G

Total = 3 + 11 + 2 + 0

Total = 16

Special case in Assignment Problems

Maximization case in Assignment Problem

In some cases, the pay off elements of the assignment problem may represent revenues or profits instead of costs so that the objective will be to maximize the total revenues or profits. The Hungarian method explained earlier can also be used for maximization case. The problem of maximization can be converted into a minimization case by selecting the largest element among all elements of the profit matrix, and then subtracting it from all other elements in the matrix. We can then proceed as usual and obtain the optimum solution by adding the original values of these cells to which the assignments have been made.

12.09.19

1, Find the Optimum assignment and the maximum sales for

Zones

Sales Engineer	A	B	C	D
P	140	112	98	154
Q	90	72	63	99
R	110	88	77	121
S	80	64	56	88

Soln:-

We convert maximization problem into minimization problem.

	A	B	C	D
P	14	42	56	0
Q	64	82	91	55
R	44	66	77	33
S	74	90	98	66

No. of rows = 4

No. of column = 4

∴ Given A.P is balanced.

Step-1 (for row)

$$\begin{bmatrix} 14 & 42 & 56 & 0 \\ 9 & 27 & 36 & 0 \\ 11 & 33 & 44 & 0 \\ 8 & 24 & 32 & 0 \end{bmatrix}$$

Step-2 (for Column)

$$\begin{bmatrix} 6 & 18 & 24 & \boxed{0} \\ 1 & 3 & 4 & \cancel{0} \\ 3 & 9 & 12 & \cancel{0} \\ \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Step-3.

$$\begin{bmatrix} 5 & 17 & 23 & \boxed{0} \\ \boxed{0} & 2 & 3 & \cancel{0} \\ 2 & 8 & 11 & \cancel{0} \\ \cancel{0} & \boxed{0} & \cancel{0} & 1 \end{bmatrix}$$

Step-4

$$\begin{bmatrix} 3 & 15 & 21 & \boxed{0} \\ \boxed{0} & 2 & 3 & 2 \\ \cancel{0} & 6 & 9 & \cancel{0} \\ \cancel{0} & \boxed{0} & \cancel{0} & 3 \end{bmatrix}$$

Step-5

3	13	19	$\boxed{0}$
$\cancel{0}$	$\boxed{0}$	1	2
$\boxed{0}$	4	7	$\cancel{0}$
2	$\cancel{0}$	$\boxed{0}$	5

∴ We reach the Optimum state

∴ Optimum Solution is P->D, Q->B, R->A, S->C

~~P->D, B->Q, C->R, D->~~ 56

Total = 154 + 72 + 110 + 56

= 392

Total = 392.

Aliter

1, A company has 5 jobs to be done on five machines. Any job can be done on any machine. The cost of doing the jobs on different machines are given below. Assign the jobs for different machines so as to minimize the total cost

Machines

Jobs	A	B	C	D	E
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

Soln:-

We form the first modified matrix by subtracting the minimum element from all the elements in the respective row, and the same with respective columns.

Step-1:

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

Since each column has the minimum element 0, we have the first modified matrix. Now we draw the minimum number of lines to cover all zeros.

Step-2

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

No. of lines drawn to cover

zero is N=4 < the order of matrix n=5.

We find the second modified matrix by subtracting the smallest uncovered element (3) from all the uncovered elements and adding to the element that is the point of intersection of lines

Step-3

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 5 & 0 & 5 & 10 & 8 \\ 0 & 6 & 12 & 0 & 0 \\ 11 & 8 & 0 & 3 & 0 \\ 0 & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & 0 & 5 \end{bmatrix} \end{array}$$

No. of lines drawn to cover all zeros = 5

which is the order of matrix.

Hence, we can form an assignment

Assignment

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 5 & \boxed{0} & 5 & 10 & 8 \\ \cancel{0} & 6 & 12 & \cancel{0} & \boxed{0} \\ 11 & 8 & \boxed{0} & 3 & \cancel{0} \\ \boxed{0} & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & \boxed{0} & 5 \end{bmatrix} \end{array}$$

21:1. INTRODUCTION

A flow of customers from infinite/finite population towards the service facility forms a *queue* (*waiting line*) on account of lack of capability to serve them all at a time. The queues may be of persons waiting at a doctor's clinic or at railway booking office, these may be of machines waiting to be repaired or of ships in the harbour waiting to be unloaded or of letters arriving at a typist's desk. In the absence of a perfect balance between the service facilities and the customers, waiting is required either of the service facilities or for the customer's arrival.

By the term '*customer*' we mean the arriving unit that requires some service to be performed. The customer may be of persons, machines, vehicles, parts, etc. ***Queues*** (***waiting line***) stands for a number of customers waiting to be serviced. The queue does not include the customer being serviced. The process or system that performs the services to the customer is termed by *service channel* or *service facility*.

The subject of queueing is not directly concerned with optimization (maximisation or minimization). Rather, it attempts to explore, understand, and compare various queueing situations and thus indirectly achieves optimization approximately.

21:2. QUEUEING SYSTEM

The mechanism of a queueing process is very simple. Customers arrive at a service counter and are attended to by one or more of the servers. As soon as a customer is served, it departs from the system. Thus a queueing system can be described as consisting of customers arriving for service, waiting for service if it is not immediate, and leaving the system after being served.

The general framework of a queueing system is shown in *Fig. 21.1* :

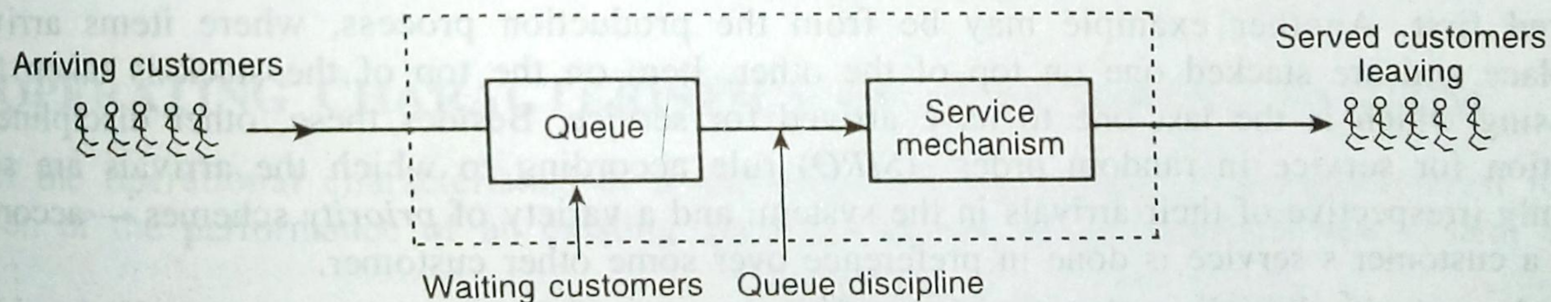


Fig. 21.1. Queueing System

21:3. ELEMENTS OF A QUEUEING SYSTEM

The basic elements of a queueing system are as follows :

1. **Input (or Arrival) Process.** This element of queueing system is concerned with the pattern in which the customers arrive for service. Input source can be described by following three factors :

(*a*) ***Size of the queue.*** If the total number of potential customers requiring service are only few, then size of the input source is said to be *finite*. On the other hand, if potential customers requiring service are sufficiently large in number, then the input source is considered to be *infinite*.

Also, the customers may arrive at the service facility in batches of fixed size or of variable size or one by one. In the case when more than one arrival is allowed to enter the system simultaneously (entering the system does not necessarily mean entering into service), the input is said to occur in *bulk* or in *batches*. Ships discharging cargo at a dock, families visiting restaurants, etc. are the examples of bulk arrivals.

(*b*) ***Pattern of arrivals.*** Customers may arrive in the system at known (regular or otherwise) times, or they may arrive in a random way. In case the arrival times are known with certainty, the queueing problems are categorized as deterministic models. On the other hand, if the time between successive arrivals (inter-arrival times) is uncertain, the arrival pattern is measured by either mean arrival rate or inter-arrival time. These are characterised by the probability distribution associated with this random process. The most common stochastic queueing models assume that arrival rate follow a Poisson distribution and/or the inter-arrival times follow an exponential distribution.

(*c*) ***Customers' behaviour.*** It is also necessary to know the reaction of a customer upon entering the system. A customer may decide to wait no matter how long the queue becomes (patient customer), or if the queue is too long to suit him, may decide not to enter it (impatient customer). Machines arriving at the maintenance shop in a plant are examples of patient customers. For impatient customers,

(*i*) if a customer decides not to enter the queue because of its length, he is said to have *balked*.

(*ii*) if a customer enters the queue, but after some time loses patience and decides to leave, then he is said to have *reneged*.

(*iii*) if a customer moves from one queue to another (providing similar/different services) for his personal economic gains, then he is said to have *jockeyed* for position.

The final factor to be considered regarding the input process is the manner in which the arrival pattern changes with time. The input process which does not change with time is called a *stationary* input process. If it is time dependent then the process is termed as *transient*.

2. **Queue Discipline.** It is a rule according to which customers are selected for service when a queue has been formed. The most common queue discipline is the "first come, first served" (*FCFS*), or the "first in, first out" (*FIFO*) rule under which the customers are serviced in the strict order of their arrivals. Other queue discipline include : "last in, first out" (*LIFO*) rule according to which the last arrival in the system is serviced first.

This discipline is practised in most cargo handling situations where the last item loaded is removed first. Another example may be from the production process, where items arrive at a workplace and are stacked one on top of the other. Item on the top of the stack is taken first for processing which is the last one to have arrived for service. Besides these, other disciplines are : "selection for service in random order" (*SIRO*) rule according to which the arrivals are serviced randomly irrespective of their arrivals in the system; and a variety of *priority* schemes — according to which a customer's service is done in preference over some other customer.

Under *priority* discipline, the service is of two types : (*i*) *Pre-emptive priority*. Under this rule, the customers of high priority are given service over the low priority customers. That is, lower priority customer's service is interrupted (pre-empted) to start service for a priority customer. The interrupted service is resumed again as soon as the highest priority customer has been served.

(*ii*) *Non pre-emptive priority*. In this case the highest priority customer goes ahead in the queue, but his service is started only after the completion of the service of the currently being served customer.

3. **Service Mechanism.** The service mechanism is concerned with service time and service facilities. Service time is the time interval from the commencement of service to the completion of service. If there are infinite number of servers then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite, then the customers are served according to a specific order. Further, the customers may be served in batches of fixed size or of variable size rather than individually by the same server, such as a computer with parallel processing or people boarding a bus. The service system in this case is termed as *bulk service system*.

In the case of parallel channels "fastest server rule" (*FSR*) is adopted. For its discussion we suppose that the customers arrive before parallel service channels. If only one service channel is free, then incoming customer is assigned to free service channel. But it will be more efficient to assume that an incoming customer is to be assigned a server of largest service rate among the free ones.

Service facilities can be of the following types :

(*a*) ***Single queue-one server***, *i.e.*, one queue-one service channel, wherein the customer waits till the service point is ready to take him in for servicing.

(*b*) ***Single queue-several servers*** wherein the customers wait in a single queue until one of the service channels is ready to take them in for servicing.

(*c*) ***Several queues-one server*** wherein there are several queues and the customer may join any one of these but there is only one service channel.

(*d*) ***Several servers.*** When there are several service channels available to provide service, much depends upon their arrangements. They may be arranged in *parallel* or in *series* or a more complex combination of both, depending on the design of the system's service mechanism.

By *parallel channels,* we mean a number of channels providing identical service facilities. Further, customers may wait in a single queue until one of the service channels is ready to serve, as in a barber shop where many chairs are considered as different service channels; or customers may form separate queues in front of each service channel as in the case of super markets.

For *series channels*, a customer must pass through all the service channels in sequence before service is completed. The situations may be seen in public offices where parts of the service are done at different service counters.

4. **Capacity of the System.** The source from which customers are generated may be finite or infinite. A *finite source* limits the customers arriving for service, *i.e.*, there is a finite limit to the maximum queue size. The queue can also be viewed as one with forced balking where a customer is forced to balk if he arrives at a time when queue size is at its limit. Alternatively, an *infinite source* is forever "abundant" as in the case of telephone calls arriving at a telephone exchange.

21:4. OPERATING CHARACTERISTICS OF A QUEUEING SYSTEM

Some of the operational characteristics of a queueing system, that are of general interest for the evaluation of the performance of an existing queueing system and to design a new system are as follows :

1. *Expected number of customers in the system* denoted by $E(n)$ or L is the average number of customers in the system, both waiting and in service. Here, n stands for the number of customers in the queueing system.

2. *Expected number of customers in the queue* denoted by $E(m)$ or L_q is the average number of customers waiting in the queue. Here $m = n - 1$, *i.e.*, excluding the customer being served.

3. E

4. *Expected waiting time in queue* denoted by $E(w)$ or W_q is the average time spent by a customer in the queue before the commencement of his service.

5. *The server utilizaiion factor (or busy period)* denoted by $P (= \lambda/\mu)$ is the proportion of time that a server actually spends with the customers. Here, λ stands for the average number of customers arriving per unit of time and μ stands for the average number of customers completing service per unit of time.

The server utilization factor is also known as *traffic intensity* or the clearing ratio.

21:5. DETERMINISTIC QUEUEING SYSTEM

A queueing system wherein the customers arrive at regular intervals and the service time for each customer is known and constant, is known as a *deterministic* queueing system.

Let the customers come at the teller counter of a bank for withdrawl every 3 minutes. Thus the interval between the arrival of any two successive customers, that is the inter-arrival time, is exactly 3 minutes. Further, suppose that the incharge of that particular teller takes exactly 3 minutes to serve a customer. This implies that the arrival and service rates are both equal to 20 customers per hour. In this situation there shall never be a queue and the incharge of the teller shall always be busy with serving work.

Now suppose instead, that the incharge of the teller can serve 30 customers per hour, *i.e.*, he takes 2 minutes to serve a customer and then has to wait for one minute for the next customer to come for service. Here also, there would be no queue, but the teller is not always busy.

Further, suppose that the incharge of the teller can serve only 15 customers per hour, *i.e.*, he takes 4 minutes to serve a customer. Clearly, in this situation he would be always busy and the queue length will increase continuously without limit with the passage of time. This implies that when the service rate is less than the arrival rate, the service facility cannot cope with all the arrivals and eventually the system leads to an *explosive* situation. In such situations, the problem can be resolved by providing additional service facilities, like opening parallel counters. We can summarize the above as follows :

Let the arrival rate be λ customers per unit time and the service rate be μ customers per unit time. Then,

(*i*) if $\lambda > \mu$, the waiting line (queue) shall be formed and will increase indefinitely; the service facility would always be busy and the service system will eventually fail.

(*ii*) if $\lambda \le \mu$, there shall be no queue and hence no waiting time; the proportion of time the service facility would be idle is $1 - \lambda/\mu$.

However, it is easy to visualize that the condition of uniform arrival and uniform service rates has a very limited practicability. Generally, the arrivals and servicing time are both variable and uncertain. Thus, variable arrival rates and servicing times are the more realistic assumptions. The probabilistic queueing models are based on these assumptions.

21:6. PROBABILITY DISTRIBUTIONS IN QUEUEING SYSTEMS

It is assumed that customers joining the queueing system arrive in a random manner and follow a *Poisson distribution* or equivalently the inter-arrival times obey *exponential distribution*. In most of the cases, service times are also assumed to be exponentially distributed. It implies that the probability of service completion in any short time period is constant and independent of the length of time that the service has been in progress.

In this section, the arrival and service distributions for Poisson queues are derived. The basic assumptions (axioms) governing this type of queues are stated below :

Axiom 1. The number of arrivals in non-overlapping intervals are statistically independent, that is, the process has independent increments.

Axiom 2. The probability of more than one arrival between time t and time $t + \Delta t$ is $o(\Delta t)$; that is, the probability of two or more arrivals during the small time interval Δt is negligible.

Thus
$$P_0(\Delta t) + P_1(\Delta t) + o(\Delta t) = 1.$$

Axiom 3. The probability that an arrival occurs between time t and time $t + \Delta t$ is equal to $\lambda \Delta t + o(\Delta t)$.

Thus
$$P_1(\Delta t) = \lambda \Delta t + o(\Delta t),$$

where λ is a constant and is independent of the total number of arrivals upto time t, Δt is an incremental element, and $o(\Delta t)$ represents the terms such that $\lim\limits_{\Delta t \to 0} \dfrac{o(\Delta t)}{\Delta t} = 0.$

1. Distribution of Arrivals (*Pure Birth Process*)

The model in which only arrivals are counted and no departure takes place are called *pure birth models*. Stated in terms of queueing, birth-death processes usually arise when an additional customer increases the arrival (referred as birth) in the system and decreases by departure (referred as death) of a serviced customer from the system.

Let $P_n(t)$ denote the probability of n arrivals in a time interval of length t (both waiting and in service), where $n \geq 0$ is an integer. Then $P_n(t + \Delta t)$ being the probability of n arrivals in a time interval of length $t + \Delta t$ (making use of *axiom 1*) is as follows :

$$\begin{aligned} P_n(t + \Delta t) = {} & P\{\, n \text{ arrivals in time } t \text{ and one arrival in time } \Delta t \,\} \\ & + P\{\, (n-1) \text{ arrivals in time } t \text{ and one arrival in time } \Delta t \,\} \\ & + P\{\, (n-2) \text{ arrivals in time } t \text{ and two arrivals in time } \Delta t \,\} \\ & + \ldots + P\{\, \text{no arrival in time } t \text{ and } n \text{ arrivals in time } \Delta t \,\}, \quad \text{for } n \geq 1. \end{aligned}$$

Making use of *axiom 2* and *axiom 3*, this difference equation reduces to

$$\begin{aligned} P_n(t + \Delta t) &= P_n(t)\, P_0(\Delta t) + P_{n-1}(t)\, P_1(\Delta t) + o(\Delta t) \\ &= P_n(t)\,[1 - \lambda \Delta t - o(\Delta t)] + P_{n-1}(t)\,\{\lambda \Delta t + o(\Delta t)\} + o(\Delta t) \end{aligned}$$

where the last term, $o(\Delta t)$, represents the terms

$$P\,[(n-k) \text{ arrivals in time } t \text{ and } k \text{ arrivals in time } \Delta t] \qquad 2 \leq k \leq n$$

The above equation can be re-written as

$$P_n(t + \Delta t) - P_n(t) = -\lambda \Delta t \cdot P_n(t) + \lambda \Delta t \cdot P_{n-1}(t) + o(\Delta t)$$

Dividing it by Δt on both sides and then taking the limit as $\Delta t \to 0$, the equation reduces to

$$\frac{d}{dt} P_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \quad n \geq 1. \qquad \text{...(A)}$$

For the case when $n = 0$,

$$P_0(t + \Delta t) = P_0(t)\, P_0(\Delta t) = P_0(t)\,[1 - \lambda \Delta t - o(\Delta t)]$$

Rearranging the terms and then dividing on both sides by Δt, taking the limit as $\Delta t \to 0$, we have

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t) \qquad \text{...(B)}$$

To solve the $n + 1$ differential-difference equations given in (A) and (B), we make use of the generating function

$$\phi(z, t) = \sum_{n=0}^{\infty} P_n(t) \cdot z^n,$$

in the unit circle $|z| \leq 1$.

Now multiplying the differential-difference equations given in (B) and (A) by $z^0, z^1, z^2, \ldots, z^n$ respectively and then taking summation over n from 0 to ∞, we get

$$\sum_{n=0}^{\infty} \frac{d}{dt} P_n(t) z^n = -\lambda \phi(z, t) + \lambda z \phi(z, t).$$

This can also be written as

$$\frac{d}{dt} \phi(z, t) = \lambda (z-1) \phi(z, t).$$

An obvious solution of this differential equation is

$$\phi(z, t) = C e^{\lambda (z-1) t},$$

where C is an arbitrary constant.

To determine the value of C, we use the initial condition that there is no arrival by time $t = 0$ and this gives

$$\phi(z, 0) = P_0(0) + \sum_{n=1}^{\infty} P_n(0) z^n = 1$$

Now, $P_n(0) = 0$ for $n \geq 1$. Therefore, $C = 1$.

Hence,

$$\phi(z, t) = e^{\lambda (z-1) t}$$

Now,

$$\frac{d}{dz} \phi(z, t) \Big|_{z=0} = P_1(t), \quad \frac{d^2}{dz^2} \phi(z, t) \Big|_{z=0} = 2!\, P_2(t), \ldots, \quad \ldots (C)$$

$$\frac{d^n}{dz^n} \phi(z, t) \Big|_{z=0} = n!\, P_n(t)$$

Using the value of $\phi(z, t)$ as given in equation (C), we get

$$P_0(t) = e^{-\lambda t}, \qquad P_1(t) = (\lambda t) e^{-\lambda t}$$

$$P_2(t) = \frac{1}{2!} (\lambda t) e^{-\lambda t}, \qquad P_n(t) = \frac{1}{n!} (\lambda t)^n e^{-\lambda t}$$

The general formula, therefore, is

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad \text{for } n \geq 0$$

which is the well-known *Poisson probability law* with mean λt. Thus, the random variable defined as the number of arrivals to a system in time t, has the *Poisson distribution* with a mean of λt arrivals, or a mean arrival rate of λ.

Example. For a Poisson arrival process with a mean 2 per unit time, if an arrival occurred at time $t = 30$, what is the probability that an arrival will occur by $t = 40$?

Solution. Exercise to the reader.

2. Distribution of Inter-arrival Times (*Exponential Process*)

Inter-arrival times are defined as the time intervals between two successive arrivals. Here, we shall show that if the arrival process follows the Poisson distribution, an associated random variable defined as the time between successive arrivals (inter-arrival time) follows the exponential distribution $f(t) = \lambda e^{-\lambda t}$ and *vice-versa*.

Let the random variable T be the time between successive arrivals; then

$$P(T > t) = P(\text{no arrival in time } t) = P_0(t) = e^{-\lambda t}.$$

The cumulative distribution function of T denoted by $F(t)$ is given by

$$F(t) = P(T \leq t) = 1 - P(T > t)$$
$$= 1 - P_0(t) = 1 - e^{-\lambda t}, \quad t > 0$$

The density function $f(t)$ for inter-arrival times, therefore, is

$$f(t) = \frac{d}{dt} F(t) = \lambda e^{-\lambda t}, \quad t > 0$$

The *expected* (or mean) inter-arrival time is given by

$$E(t) = \int_0^\infty t \cdot f(t)\, dt = \int_0^\infty \lambda t e^{-\lambda t}\, dt = 1/\lambda,$$

where λ is the *mean arrival rate.*

Thus, T has the *exponential distribution* with mean $1/\lambda$. We would intuitively expect that, if the mean arrival rate is λ, then the mean time between arrivals is $1/\lambda$. Conversely, we can also show that if the inter-arrival times are independent and have the same exponential distribution then the arrival rate follows the Poisson distribution.

3. Distribution of Departures (*Pure Death Process*)

The model in which only departures are counted and no other arrivals allowed are called *pure death models.* The queueing system starts with N customers at time $t = 0$, where $N \geq 1$. Departures occur at the rate of μ customers per unit time. To develop the differential-difference equations for the probability of n customers remaining after 't' time units, $P_n(t)$, we make use of similar assumptions as was done for arrivals. Let the three axioms, given at the beginning of this section, be changed by using the word service instead of arrival and condition the probability statements by requiring the system to be non-empty. Let us define

$\mu \Delta t$ = probability that a customer in service at time t will complete service during time Δt.

For small time interval $\Delta t > 0$, $\mu \Delta t$ gives probability of one departure during Δt. Using the same arguments as in pure birth process case, the differential-difference equations for this can easily be obtained :

$$P_n(t + \Delta t) = P_n(t)\{1 - \mu \Delta t + o(\Delta t)\} + P_{n+1}(t) \cdot \{\mu \Delta t + o(\Delta t)\}, \quad 1 \leq n \leq N-1$$
$$P_0(t + \Delta t) = P_0(t) + P_1(t)\{\mu \Delta t + o(\Delta t)\}, \quad n = 0$$
$$P_N(t + \Delta t) = P_N(t) \cdot \{1 - \mu \Delta t + o(\Delta t)\}, \quad n = N$$

Re-arranging the above equations, dividing them by Δt on both sides and then taking the limits as $\Delta t \to 0$, we get

$$\frac{d}{dt} P_n(t) = -\mu P_n(t) + \mu P_{n+1}(t); \quad 0 \leq n \leq N-1,\ t > 0$$
$$\frac{d}{dt} P_0(t) = \mu P_1(t); \quad n = 0,\ t \geq 0$$
$$\frac{d}{dt} P_N(t) = -\mu P_N(t); \quad n = N,\ t \geq 0$$

The solution of these equations with initial conditions :

$$P_n(0) = \begin{cases} 1; & n = N \neq 0 \\ 0; & n \neq N \end{cases}$$

can easily be obtained as earlier. The general solution to the above equation so obtained is

$$P_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}; \quad 1 \leq n \leq N \quad \text{and} \quad P_0(t) = 1 - \sum_{n=1}^{N} P_n(t)$$

which is known as a *truncated Poisson law.*

4. Distribution of Service Times

Making similar assumption as done above, for arrivals, one could utilize the same type of process to describe the service pattern. Let the three axioms be changed by using the word *service* instead of *arrival* and condition the probability statements by requiring the system to be *non-empty.* Then we can easily show that, the time t to complete the service on a customer follows the exponential distribution :

$$s(t) = \begin{cases} \mu e^{-\mu\tau} & ; \quad t > 0 \\ 0 & ; \quad t < 0 \end{cases}$$

where μ is the mean service rate for a particular service channel. This shows that service times follows exponential distribution with mean $1/\mu$. The number, n, of potential services in time T will follow the *Poisson distribution* given by

$$\phi(n) = P\,[n \text{ service in time } T, \text{ if servicing is going on throughout } T] = \frac{(\mu T)^N}{n\,!}\,e^{-\mu T}$$

Consequently, we can also show that

$$P\,[\text{no service in } \Delta t] = 1 - \mu\Delta t + o(\Delta t) \quad \text{and} \quad P\,[\text{one service in } \Delta t] = \mu\Delta t + o(\Delta t).$$

21:7. CLASSIFICATION OF QUEUEING MODELS

Generally queueing model may be completely specified in the following symbolic form :

$$(a/b/c) : (d/e).$$

The first and second symbols denote the type of distributions of inter-arrival times and of inter-service times, respectively. Third symbol specifies the number of servers, whereas fourth symbol stands for the capacity of the system and the last symbol denotes the queue discipline.

If we specify the following letters as :

$M \equiv$ Poisson arrival or departure distribution,

$E_k \equiv$ Erlangian or Gamma inter-arrival for service time distribution,

$GI \equiv$ General input distribution,

$G \equiv$ General service time distribution,

then $(M/E_k/1) : (\infty/FIFO)$ defines a queueing system in which arrivals follow Poisson distribution, service times are Erlangian, single server, infinite capacity and "first in, first out" queue discipline.

21:8. DEFINITION OF TRANSIENT AND STEADY STATES

A queueing system is said to be in *transient state* when its operating characteristic (like input, output, mean queue length, etc.) are dependent upon time.

If the characteristic of the queueing system becomes independent of time, then the *steady-state* condition is said to prevail.

If $P_n(t)$ denotes the probability that there are n customers in the system at time t, then in the steady-state case, we have

$$\lim_{t \to \infty} P_n(t) = P_n \text{ (independent of } t).$$

Due to practical viewpoint of the steady-state behaviour of the systems, the present chapter is amply focused on studying queueing systems under the existence of steady-state conditions. However, the differential-difference equations which can be used for deriving transient solutions will be presented.

21:9. POISSON QUEUEING SYSTEMS

Queues that follow the Poisson arrivals (exponential inter-arrival time) and Poisson service (exponential service time) are called *Poisson queues*. In this section, we shall study a number of Poisson queues with different characteristics.

Model I {(M/M/1) : (∞/FIFO)}. This model deals with a queueing system having single service channel. Poisson input, Exponential service and there is no limit on the system capacity while the customers are served on a "first in, first out" basis.

The solution procedure of this queueing model can be summarized in the following three steps :

Step 1. Construction of Differential-Difference Equations. Let $P_n(t)$ be the probability that there are n customers in the system at time t. The probability that the system has n customers at time $(t+\Delta t)$ can be expressed as the sum of the joint probabilities of the four mutually exclusive and collectively exhaustive events as follows :

$$\begin{aligned} P_n(t+\Delta t) = {} & P_n(t) \cdot P\,[\text{no arrival in } \Delta t] \cdot P\,[\text{no service completion in } \Delta t] \\ & + P_n(t) \cdot P\,[\text{one arrival in } \Delta t] \cdot P\,[\text{one service completed in } \Delta t] \\ & + P_{n+1}(t) \cdot P\,[\text{no arrival in } \Delta t] \cdot P\,[\text{one service completed in } \Delta t] \\ & + P_{n-1}(t) \cdot P\,[\text{one arrival in } \Delta t] \cdot P\,[\text{no service completion in } \Delta t] \end{aligned}$$

This is re-written as :

$$\begin{aligned} P_n(t+\Delta t) = {} & P_n(t)\,[1 - \lambda\,\Delta t + o(\Delta t)]\,[1 - \mu\,\Delta t + o(\Delta t)] + P_n(t)\,[\lambda\,\Delta t]\,[\mu\,\Delta t] \\ & + P_{n+1}(t)\,[1 - \lambda\,\Delta t + o(\Delta t)]\,[\mu\,\Delta t + o(\Delta t)] + P_{n-1}(t)\,[\lambda\,\Delta t + o(\Delta t)]\,[1 - \mu\,\Delta t + o(\Delta t)] \end{aligned}$$

or $$P_n(t+\Delta t) - P_n(t) = -(\lambda + \mu)\,\Delta t\,P_n(t) + \mu\,\Delta t\,P_{n+1}(t) + \lambda\,\Delta t\,P_{n-1}(t) + o(\Delta t)$$

Since Δt is very small, terms involving $(\Delta t)^2$ can be neglected. Dividing the above equation by Δt on both sides and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt} P_n(t) = -(\lambda + \mu)\,P_n(t) + \mu\,P_{n+1}(t) + \lambda\,P_{n-1}(t)\,; \quad n \geq 1.$$

Similarly, if there is no customer in the system at time $(t + \Delta t)$, there will be no service completion during Δt. Thus for $n = 0$ and $t \geq 0$, we have only two probabilities instead of four. The resulting equation is

$$P_0(t+\Delta t) = P_0(t)\,\{1 - \lambda\,\Delta t + o(\Delta t)\} + P_1(t)\,\{\mu\,\Delta t + o(\Delta t)\}\,\{1 - \lambda\,\Delta t + o(\Delta t)\}$$

or $$P_0(t+\Delta t) - P_0(t) = -\lambda\,\Delta t\,P_0(t) + \mu\,\Delta t\,P_1(t) + o(\Delta t).$$

Dividing both sides of this equation by Δt and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt} P_0(t) = -\lambda\,P_0(t) + \mu\,P_1(t)\,; \quad n = 0.$$

Step 2. Deriving the Steady-State Difference Equations. In the steady-state, $P_n(t)$ is independent of time t and $\lambda < \mu$ when $t \to \infty$. Thus $P_n(t) \to P_n$ and

$$\frac{d}{dt} P_n(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Consequently the differential-difference equations obtained in *Step 1* reduce to

$$0 = -(\lambda + \mu)\,P_n + \mu\,P_{n+1} + \lambda\,P_{n-1}\,; \quad n \geq 1$$

and

$$0 = -\lambda\,P_n + \mu\,P_1\,; \quad n = 0.$$

These constitute the steady-state difference equations.

Step 3. Solution of the Steady-State Difference Equations. For the solution of the above difference equations there exist three methods, namely, the iterative method, use of generating functions and the use of linear operators. Out of these three the first one is the most straightforward and therefore the solution of the above equations will be obtained here by using the iterative method.

Using iteratively, the difference-equations yield

$$P_1 = \frac{\lambda}{\mu} P_0, \quad P_2 = \frac{\lambda + \mu}{\mu} P_1 - \frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda + \mu}{\mu} P_2 - \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^3 P_o, \text{ and in general } P_n = \left(\frac{\lambda}{\mu}\right)^n P_o.$$

Now, $$P_{n+1} = \frac{\lambda + \mu}{\mu} P_n - \frac{\lambda}{\mu} P_{n-1}, \quad n \geq 1.$$

Substituting the values of P_n and P_{n-1}, the equation yields

$$P_{n+1} = \frac{\lambda + \mu}{\mu}\left(\frac{\lambda}{\mu}\right)^n P_o - \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^{n-1} P_o = \left(\frac{\lambda}{\mu}\right)^{n+1} P_o.$$

Thus, by the principle of mathematical induction, the general formulae for P_n, is valid for $n \geq 0$.

To obtain the value of P_o, we make use of the boundary condition $\sum_{n=0}^{\infty} P_n = 1$.

$$\therefore \quad 1 = \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n P_o = P_o \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n; \quad \text{since } P_n = \left(\frac{\lambda}{\mu}\right)^n P_o$$

$$= P_o \frac{1}{1 - \lambda/\mu}, \quad \text{since } \left(\frac{\lambda}{\mu}\right) < 1.$$

This gives $$P_o = 1 - \left(\frac{\lambda}{\mu}\right).$$

Hence, the steady-state solution is

$$P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1 - \frac{\lambda}{\mu}\right) = \rho^n (1 - \rho); \quad \rho = \left(\frac{\lambda}{\mu}\right) < 1, \quad \text{and} \quad n \geq 0.$$

This expression gives us the probability distribution of queue length.

Characteristics of Model I

(*i*) Probability of queue size being greater than or equal to k, the number of customers is given by

$$P(n \geq k) = \sum_{k=n}^{\infty} P_k = \sum_{k=n}^{\infty} (1-\rho)\rho^k = (1-\rho)\rho^n \sum_{k=n}^{\infty} \rho^{k-n} = (1-\rho)\rho^n \sum_{k-n=0}^{\infty} \rho^{k-n}$$

$$= (1-\rho)\rho^n \sum_{x=0}^{\infty} \rho^x = \frac{(1-\rho)\rho^n}{1-\rho} = \rho^n.$$

(*ii*) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1}$$

$$= \rho(1-\rho)\sum_{n=0}^{\infty}\frac{d}{d\rho}\rho^n = \rho(1-\rho)\frac{d}{d\rho}\sum_{n=0}^{\infty}\rho^n, \quad \text{since } \rho < 1$$

$$= \rho(1-\rho)\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}.$$

(*iii*) Average queue length is given by

$$E(m) = \sum_{m=0}^{\infty} mP_n, \quad \text{where } m = n - 1$$

being the number of customers in the queue, excluding the customer which is in service.

$$\therefore \quad E(m) = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n = \sum_{n=0}^{\infty} nP_n - \left[\sum_{n=0}^{\infty} P_n - P_o\right]$$

$$= \frac{\rho}{1-\rho} - [1 - (1-\rho)] = \frac{\rho}{1-\rho} - \rho$$

$$= \rho^2/(1-\rho) = \lambda^2/\mu(\mu - \lambda).$$

(iv) Average length of non-empty queue is given by

$$E(m \mid m > 0) = \frac{E(m)}{P(m > 0)} = \frac{\lambda^2}{\mu(\mu - \lambda)} \times \frac{1}{(\lambda/\mu)^2} = \frac{\mu}{\mu - \lambda},$$

since

$$P(m > 0) = P(n > 1) = \sum_{n=0}^{\infty} P_n - P_0 - P_1 = \left(\frac{\lambda}{\mu}\right)^2$$

(v) The fluctuation (variance) of queue length is given by

$$V(n) = \sum_{n=0}^{\infty} [n - E(n)]^2 P_n = \sum_{n=0}^{\infty} n^2 P_n - [E(n)]^2.$$

Using some algebraic transformations and the value of P_n, the result reduces to

$$V(n) = (1-\rho)\frac{\rho + \rho^2}{(1-\rho)^3} - \left[\frac{\rho}{1-\rho}\right]^2 = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}.$$

Waiting Time Distribution for Model I. The waiting time of a customer in the system is, for the most part, a continuous random variable except that there is a non-zero probability that the delay will be zero, that is a customer entering service immediately upon arrival. Therefore, if we denote the time spent in the queue by w and if $\Psi_w(t)$ denotes its cumulative probability distribution then from the complete randomness of the Poisson distribution, we have

$$\Psi_w(0) = P(w = 0) \quad \text{(No customer in the system upon arrival)}$$
$$= P_0 = (1 - \rho).$$

It is now required to find $\Psi_w(t)$ for $t > 0$.

Let there be n customers in the system upon arrival, then in order for a customer to go into service at a time between 0 and t, all the n customers must have been served by time t. Let $s_1, s_2, \ldots, s_n$ denote service times of n customers respectively. Then

$$w = \sum_{i=1}^{n} s_i, \ (n \geq 1) \quad \text{and} \quad w = 0 \quad (n = 0).$$

The distribution function of waiting time, w, for a customer who has to wait is given by

$$P(w \leq t) = P\left[\sum_{i=1}^{n} s_i \leq t\right]; \quad n \geq 1 \quad \text{and} \quad t > 0.$$

Since, the service time for each customer is independent and identically distributed, therefore its probability density function is given by $\mu e^{-\mu t}$ $(t > 0)$, where μ is the mean service rate. Thus

$$\Psi_n(t) = \sum_{n=1}^{\infty} P_n \times P\,(n-1 \text{ customers are served at time } t) \times P\,(1 \text{ customer is served in time } \Delta t)$$

$$= \sum_{n=1}^{\infty} \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \frac{(\mu t)^{n-1} e^{-\mu t}}{(n-1)!} \cdot \mu \, \Delta t.$$

The expression for $\Psi_w(t)$, therefore, can be written as

$$\Psi_w(t) = P(w \leq t) = \sum_{n=1}^{\infty} P_n \int_0^t \Psi_n(t)\, dt$$

$$= \sum_{n=1}^{\infty} (1-\rho)\rho^n \int_0^t \frac{(\mu t)^{n-1}}{(n-1)!} e^{-\mu t} \cdot \mu \, dt = (1-\rho)\rho \int_0^t \mu e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\mu t \rho)^{n-1}}{(n-1)!} \cdot dt$$

$$= (1-\rho)\rho \int_0^t \mu e^{-\mu t(1-\rho)}\, dt.$$

Hence, the waiting time of a customer who has to wait is given by

$$\Psi(w) = \frac{d}{dt}[\Psi_w(t)] = \rho(1-\rho)\cdot\mu e^{-\mu t(1-\rho)} = \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}.$$

Characteristic of Waiting Time Distribution

(*i*) Average waiting time of a customer (in the queue) is given by

$$E(w) = \int_0^\infty t\cdot\Psi(w)\,dt = \int_0^\infty t\,.\,\rho\mu(1-\rho)\,e^{-\mu t(1-\rho)}\,dt$$

$$= \rho\int_0^\infty \frac{xe^{-x}}{\mu(1-\rho)}\,dx, \quad \text{for } \mu(1-\rho)\,t = x$$

$$= \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

(*ii*) Average waiting time of an arrival who has to wait is given by

$$E(w \mid w>0) = \frac{E(w)}{P(w>0)} = \left\{\frac{\lambda}{\mu(\mu-\lambda)}\right\}\Big/\left(\frac{\lambda}{\mu}\right) = \frac{1}{\mu-\lambda}$$

[Here $P(w>0) = 1 - P(w=0) = 1-(1-\rho) = \rho$.]

(*iii*) For the busy period distribution, let the random variable v denote the total time that a customer has to spend in the system including service. Then the probability density of its cumulative density function is given by

$$\Psi(w \mid w>0) = \frac{\Psi(w)}{P(w>0)} = \left[\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}\right]\Big/\left(\frac{\lambda}{\mu}\right)$$

$$= (\mu-\lambda)\,e^{-(\mu-\lambda)t}, \quad t>0.$$

(*iv*) Average waiting time that a customer spends in the system including service is given by

$$E(v) = \int_0^\infty t\,.\,\Psi(w \mid w>0)\,dt = \int_0^\infty t\,.\,(\mu-\lambda)\,e^{-(\mu-\lambda)t}\,dt$$

$$= \frac{1}{\mu-\lambda}\int_0^\infty xe^{-x}\,dx, \quad \text{for } (\mu-\lambda)\,t = x$$

$$= \frac{1}{\mu-\lambda}.$$

Relationships among Operating Characteristics

We have derived the following important characteristics of an $M/M/1$ queueing system :

$$E(n) = \frac{\lambda}{\mu-\lambda}, \quad E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)}, \quad E(w) = \frac{\lambda}{\mu(\mu-\lambda)} \quad \text{and} \quad E(v) = \frac{1}{\mu-\lambda}.$$

Using these expressions, we observe some general relationships between the average system characteristics as follows :

(*i*) Expected number of customers in the system is equal to the expected number of customers in the queue plus a customer currently in service, *i.e.*,

$$E(n) = E(m) + \frac{\lambda}{\mu}$$

(*ii*) Expected waiting time of a customer in the system is equal to the expected waiting time in the queue plus the expected service time of a customer in service, *i.e.*,

$$E(v) = E(w) + \frac{1}{\mu}.$$

(iii) Expected number of customers in the system is equal to the average number of arrivals per unit of time multiplied by the average time spent by the customer in the system, *i.e.*,

$$E(n) = \lambda E(v)$$

(iv) Expected number of customers in the queue is equal to the average number of arrivals per unit of time multiplied by the average time spent by a customer in the queue, *i.e.*,

$$E(m) = \lambda E(w).$$

Note. Relations between Average Queue Length and Average Waiting Time are known as *Little's Formulae.*

SAMPLE PROBLEMS

2101. *A TV repairman finds that the time spent on his jobs has an Exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought-in?*

[Kerala M.Sc. (Math.) 2001; Delhi M.B.A. (PT) 2008; Madurai M.B.A. 2009]

Solution. We are given,

$$\lambda = 10 \text{ sets per day, and } \mu = 16 \text{ sets per day.}$$

$$\therefore \quad \rho = \lambda/\mu = 10/16 = 0.625$$

The probability for the repairman to be idle is

$$P_0 = 1 - \rho = 1 - 0.625 = 0.375$$

(i) Expected idle time per day $= 8 \times 0.375 = 3$ hours.

(ii) Expected (or average) number of T.V. sets in the system

$$E(n) = \frac{\rho}{1-\rho} = \frac{0.625}{1 - 0.625} = \frac{5}{3} = 2 \text{ (approx.) T.V. sets.}$$

2102. *In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes. Calculate the following :*

(i) the mean queue size (line length), and

(ii) the probability that the queue size exceeds 10.

If the input of trains increases to an average 33 per day, what will be the change in (i) and (ii)?

[Meerut M.Sc. (Math.) 2000; Madras M.B.A. 2006; IGNOU M.B.A. (Dec.) 2006; Lucknow B.M.S. 2008]

Solution. Here, we have

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ and } \mu = \frac{1}{36} \text{ trains per minute.}$$

$$\therefore \quad \rho = \lambda/\mu = 36/48 = 0.75$$

(i)
$$E(m) = \frac{\rho}{1-\rho} = \frac{0.75}{1 - 0.75} = 3 \text{ trains.}$$

(ii)
$$P(\geq 10) = \rho^{10} = (0.75)^{10} = 0.06.$$

When the input increases to 33 trains per day, we have

$$\lambda = \frac{33}{60 \times 24} = \frac{11}{480} \text{ and } \mu = \frac{1}{36} \text{ trains per minute.}$$

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{11}{480} \times 36 = 0.83$$

Then, we get

(i)
$$E(n) = \frac{\rho}{1-\rho} = \frac{0.83}{1 - 0.83} = 4.9 \text{ or } 5 \text{ trains (approx.)}$$

(ii)
$$P(\geq 10) = \rho^{10} = (0.83)^{10} = 0.2 \text{ (approx.)}$$

2103. *The rate of arrival of customers at a public telephone booth follows Poisson distribution with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution, with mean time of 3 minutes.*

(i) What is the probability that a person arriving at the booth will have to wait?

(ii) What is the average length of the non-empty queues that form from time to time?

(iii) The Mahanagar Telephone Nigam Ltd. will install a second booth, when it is convinced that the customers would expect waiting for at least 3 minutes for their turn to make a call. By how much time should the flow of customers increase in order to justify a second booth?

[Delhi B.Sc. (Stat.) 1996; Visvesvaraya M.B.A. (June) 2011]

(iv) Estimate the fraction of a day that the phone will be in use.

[Delhi PG Dip. in Glob. Bus. Oper. 2010; Kerala M.B.A. 2010]

(v) What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call?

[Madras M.B.A. (Nov.) 2006]

Solution. Here, we are given :

$$\lambda = \frac{1}{10}\times 60 \text{ or } 6 \text{ per hour and } \mu = \frac{1}{3}\times 60 \text{ or } 20 \text{ per hour.}$$

(*i*) Probability that a person arriving at the booth will have to wait

$$P(w > 0) = 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{6}{20} \text{ or } 0.3.$$

(*ii*) Average length of non-empty queues

$$E(m \mid m > 0) = \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 6} = 1.43.$$

(*iii*) The installation of a second booth will be justified, if the arrival rate is greater than the waiting time. Now, if λ' denotes the increased arrival rate, expected waiting time is :

$$E(w) = \frac{\lambda'}{\mu(\mu - \lambda')} \Rightarrow \frac{3}{60} = \frac{\lambda'}{20(20 - \lambda')} \text{ or } \lambda' = 10.$$

Hence, the arrival rate should become 10 customers per hour to justify the second booth.

(*iv*) The fraction of a day that the phone will be busy = traffic intensity $\rho = \lambda/\mu = 0.3$.

(*v*) $$P(w \geq 0) = \int_{10}^{\infty} \lambda\left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu - \lambda)t}\, dt = \int_{10}^{\infty} (0.30)(0.23)\, e^{-0.23t}\, dt,$$

where $\lambda = 0.10$ per minute, and $\mu = 0.33$ per minute.

$$\therefore \quad P(w \geq 10) = (0.069)\left.\frac{e^{-0.23t}}{(-0.23)}\right|_{10}^{\infty} = 0.03.$$

This shows that 3 per cent of the arrivals on an average will have to wait for 10 minutes or more before they can use the phone.

2104. *On an average 96 patients per 24-hour day require the service of an emergency clinic. Also on an average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ a patient.*

[Delhi M.B.A. 2008]

Solution. Here,

$$\lambda = \frac{96}{24\times 60} = \frac{1}{15} \text{ and } \mu = \frac{1}{10} \text{ patients per minute}$$

$$\therefore \quad \rho = \lambda/\mu = 2/3.$$

Average number of patients in the queue are given by,

$$E(m) = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1-2/3} = \frac{4}{3}$$

Fraction of the time for which there are no patients is given by,

$$P_0 = 1 - \rho = 1 - 2/3 = 1/3.$$

Now, when the average queue size is decreased from 4/3 patients to 1/2 patient, we are to determine the value of μ. So, we have

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} \Rightarrow \frac{1}{2} = \frac{(1/15)^2}{\mu(\mu - 1/15)^2}$$

$$\mu = 2/15 \text{ patients per minute.}$$

i.e., $\therefore$ Average rate of treatment required $= 1/\mu = 15/2 = 7.5$ minutes.

i.e., a decrease in the average rate of treatment is $(10 - 7.5)$ minutes or 2.5 minutes.

$$\text{Budget per patient} = \text{Rs.}\ (100 + 2.5 \times 10) = \text{Rs. } 125.$$

Hence, in order to get the required size of the queue, the budget should be increased from Rs. 100 per patient to Rs. 125 per patient.

2105. *A road transport company has one reservation clerk on duty at a time. He handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can service 12 customers on an average per hour. After stating your assumptions, answer the following :*

(i) What is the average number of customers waiting for the service of the clerk?

(ii) What is the average time a customer has to wait before getting service?

(iii) The management is contemplating to install a computer system to handle the information and reservations. This is expected to reduce the service time from 5 to 3 minutes. The additional cost of having the new system works out to Rs. 50 per day. If the cost of goodwill of having to wait is estimated to be 12 paise per minute spent waiting before being served. Should the company install the computer system? Assume 8 hours working day. **[Madras M.B.A. 1997; Madurai M.B.A. 2010]**

Solution. We are given

$$\lambda = 8 \text{ customers per hour and } \mu = 12 \text{ customers per hour.}$$

(i) Average number of customers waiting for the service of the clerk (in the system) :

$$E(n) = \frac{\lambda}{\mu - \lambda} = \frac{8}{12-8} = 2 \text{ customers.}$$

The average number of customers waiting for the service of the clerk (in the queue) :

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{8 \times 8}{12(12-8)} \text{ or } 1.33 \text{ customers.}$$

(ii) The average waiting time of a customer (in the system) before getting service :

$$E(v) = \frac{1}{\mu-\lambda} = \frac{1}{12-8} \text{ hour or } 15 \text{ minutes.}$$

The average waiting time of a customer (in the queue) before getting service :

$$E(w) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{12(12-8)} = \frac{1}{6} \text{ hour or } 10 \text{ minutes.}$$

(iii) We now calculate the difference between the goodwill cost of customers with one system and the goodwill cost of customers with an additional computer system. This difference will be compared with the additional cost (of Rs. 50 per day) of installing another computer system.

An arrival waits for $E(w)$ hours before being served and there are λ arrivals per hour. Thus, expected waiting time for all customers in an 8-hour day with one system

$$= 8\lambda \times E(w) = 8 \times 8 \times \tfrac{1}{6} \text{ hrs. or } \tfrac{64}{6} \times 60 \text{ minutes, } i.e., \text{ 640 minutes.}$$

The goodwill cost per day with one system = $640 \times$ Re. 0.12 = Rs. 76.80.

The expected waiting time of a customer before getting service when there is an additional computer system is :

$$E(w^*) = \frac{8}{20\,(20-8)} = \frac{8}{20 \times 12} \text{ or } \frac{1}{30} \text{ hr.}$$

Thus, expected waiting time of customers in an 8-hour day with an additional computer system is :

$$8\lambda \times E(w^*) = 8 \times 8 \times \frac{1}{30} \text{ hr.} = 128 \text{ minutes.}$$

The total goodwill cost with an additional computer system = $128 \times$ Re. 0.12 = Rs. 15.36.

Hence, reduction in goodwill cost with the installation of a computer system

= Rs. 76.80 – Rs. 15.36 = Rs. 61.44.

Whereas the additional cost of a computer system is Rs. 50 per day, Rs. 61.44 is the reduction in goodwill cost when additional computer system is installed, hence there will be net saving of Rs. 11.44 per day. It is, therefore, worthwhile to instal a computer.

2106. *In the production shop of a company the breakdown of the machines is found to be Poisson with an average rate of 3 machines per hour. Breakdown time at one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairmen. One of the repairmen is slow but cheap, the other fast but expensive. The slow-cheap repairman demands Rs. 20 per hour and will repair the broken down machines exponentially at the rate of 4 per hour. The fast-expensive repairman demands Rs. 30 per hour and will repair machines exponentially at an average rate of 6 per hour. Which repairman should be hired?* ***[Delhi M.Com. 2006]***

Solution. In this problem, we compare the total expected daily cost for both the repairmen. This would equal the total wages paid plus the downtime cost.

Case 1. Slow-cheap repairman

λ = 3 machines per hour and μ = 4 machines per hour.

$\therefore$ Average downtime of a machine = $\dfrac{1}{\mu - \lambda} = \dfrac{1}{4-3}$ = 1 hour.

$\therefore$ The downtime of 3 machines that arrive in an hour = 1×3 = 3 hours.

Downtime cost = Rs. 40×3 = Rs. 120, charges paid to the repairman = Rs. 20×3 = Rs. 60

Total cost = Rs. 120 + Rs. 60 = Rs. 180.

Case 2. Fast-expensive repairman

λ = 3 machines per hour and μ = 6 machines per hour.

$\therefore$ Average downtime of machine = $\dfrac{1}{\mu - \lambda} = \frac{1}{3}$ hours

$\therefore$ The downtime of 3 machines that arrive in an hour = $\frac{1}{3} \times 3$ = 1 hour.

Downtime cost = Rs. 40×1 = Rs. 40, charges paid to the repairman = Rs. 30×1 = Rs. 30

Total cost = Rs. 40 + Rs. 30 = Rs. 70.

From the above two cases, the decision of the company should be to engage the fast-expensive repairman.

17:1. INTRODUCTION

Many practical problems require decision-making in a *competitive situation* where there are two or more opposing parties with conflicting interests and where the action of one depends upon the one taken by the opponent. For example, candidates for an election, advertising and marketing campaigns by competing business firms, countries involved in military battles, etc. have their conflicting interests. In a competitive situation the courses of action (alternatives) for each competitor may be either finite or infinite. A competitive situation will be called a 'Game', if it has the following properties :

(*i*) There are a finite number of competitors (participants) called *players*.

(*ii*) Each player has a finite number of strategies (alternatives) available to him.

(*iii*) A play of the game takes place when each player employs his strategy.

(*iv*) Every game results in an outcome, *e.g.*, loss or gain or a draw, usually called *payoff*, to some player.

17:2. TWO-PERSON ZERO-SUM GAMES

When there are two competitors playing a game, it is called a 'two-person game'. In case the number of competitors exceeds two, say *n*, then the game is termed as a '*n*-person game'.

Games having the 'zero-sum' character that the algebraic sum of gains and losses of all the players is zero are called *zero-sum games*. The play does not add a single paisa to the total wealth of all the players; it merely results in a new distribution of initial money among them. Zero-sum games with *two* players are called *two-person zero-sum* games. In this case the loss (gain) of one player is exactly equal to the gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of non-zero sum character or simply a *non-zero sum game*.

17:3. SOME BASIC TERMS

1. *Player.* The competitors in the game are known as players. A player may be individual or group of individuals, or an organisation.

2. *Strategy.* A strategy for a player is defined as a set of rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt. A strategy may be of two types :

(a) *Pure strategy.* If the players select the same strategy each time, then it is referred to as pure-strategy. In this case each player knows exactly what the other player is going to do, the objective of the players is to maximize gains or to minimize losses.

(b) *Mixed strategy.* When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus, there is a probabilistic situation and objective of the player is to maximize expected gains or to minimize expected losses.

3. *Optimum strategy.* A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy.

4. *Value of the game.* It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair, if it is non-zero.

5. *Payoff matrix.* When the players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a matrix called the *payoff matrix*. Since the game is zero-sum therefore gain of one player is equal to the loss of other and vice-versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus, it is sufficient to construct payoff only for one of the players.

Let player A have m strategies $A_1, A_2, \ldots, A_m$ and player B have n strategies $B_1, B_2, \ldots, B_n$. Here, it is assumed that each player has his choices from amongst the pure strategies. Also it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let a_{ij} be the payoff which player A gains from player B if player A chooses strategy A_i and player B chooses strategy B_j. Then the payoff matrix to player A is :

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

The payoff matrix to player B is $(-a_{ij})$.

Example. Consider a two-person coin tossing game. Each player tosses an unbiased coin simultaneously. Player B pays Rs. 7 to A, if $\{H, H\}$ occurs and Rs. 4, if $\{T, T\}$ occurs; otherwise player A pays Rs. 3 to B. This two-person game is a zero-sum game, since the winnings of one player are the losses for the other. Each player has his choices from amongst two *pure* strategies H and T. If we agree conventionally to express the outcome of the game in terms of the payoffs to one player only, say A, then the above information yields the following payoff matrix in terms of the payoffs to the player A. Clearly, the entries in B's payoff matrix will be just the negative of the corresponding entries in A's payoff matrix so that the sum of payoff matrices for player A and player B is ultimately a null matrix. We generally display the payoff matrix of that player who is indicated on the left side of the matrix. For example, A's payoff matrix may be displayed as below :

$$\text{Player } A \quad \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cc} H & T \end{array} \\ \begin{pmatrix} 7 & -3 \\ -3 & 4 \end{pmatrix} \end{array}$$

17:4. THE MAXIMIN-MINIMAX PRINCIPLE

We shall now explain the so-called *Maximin-Minimax Principle* for the selection of the optimal strategies by the two players.

For player A, minimum value in each row represents the least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains. This choice of player A is called the *maximin principle*, and the corresponding gain is called the *maximin value of the game*.

For player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum losses. This choice of player B is called the *minimax principle*, and the corresponding loss is the *minimax value of the game*.

If the maximin value equals the minimax value, then the game is said to have a *saddle (equilibrium) point* and the corresponding strategies are called *optimum strategies*. The amount of payoff at an equilibrium point is known as the *value of the game*.

To illustrate the maximin-minimax principle, let us consider a two-person zero-sum game with the following 3×2 payoff matrix for player A :

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 9 & 2 \\ 8 & \mathbf{6}^{*\dagger} \\ 6 & 4 \end{pmatrix} \end{array}$$

Let the pure strategies of the two players be designated by

$$S_A = \{A_1, A_2, A_3\} \quad \text{and} \quad S_B = \{B_1, B_2\}.$$

Suppose that player A starts the game knowing fully well that whatever strategy he adopts, B will select that particular counter strategy which will minimize the payoff to A. Thus, if A selects the strategy A_1, then B will reply by selecting B_2, as this corresponds to the *minimum* payoff to A in the first row corresponding to A_1. Similarly, if A chooses the strategy A_2, he may gain 8 or 6 depending upon the strategy chosen by B. However, A can guarantee a gain of at least min. $\{8, 6\} = 6$ regardless of the strategy chosen by B. In other words, whatever strategy A may adopt he can guarantee only the minimum of the corresponding row payoffs. Naturally, A would like to maximise his minimum assured gain. In this example the selection of strategy A_2 gives the maximum of the minimum gains to A. We shall call this gain as the *maximin value* of the game and the corresponding strategy as the *maximin strategy*. The maximin value is indicated in bold type with a star.

On the other hand, player B wishes to minimize his losses. If he plays strategy B_1, his loss is at the most max. $\{9, 8, 6\} = 9$ regardless of what strategy A has selected. He can lose no more than max. $\{2, 6, 4\} = 6$ if he plays B_2. This minimum of the maximum losses will be called the *minimax value* of the game and the corresponding strategy the *minimax strategy*. The minimax value is indicated in bold type marked with [†]. We observe that in the present example the maximum of row minima is equal to the minimum of the column maxima. In symbols,

$$\max_i . \{r_i\} = 6 = \min_j . \{c_j\}$$

or

$$\max_i . [\min_j . \{a_{ij}\}] = 6 = \min_j . [\max_i . \{a_{ij}\}],$$

where $i = 1, 2, 3$ and $j = 1, 2$.

Theorem 17-1. *Let (a_{ij}) be the $m \times n$ payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then $\overline{v} \geq \underline{v}$. That is,*

$$\min_{1 \le j \le n} . \; [\max_{1 \le i \le m} . \; \{a_{ij}\}] \geq \max_{1 \le i \le m} . \; [\min_{1 \le j \le n} . \; \{a_{ij}\}].$$

Proof. We have

$$\max_{1 \le i \le m} \{a_{ij}\} \ge a_{ij} \qquad \text{for all } j = 1, 2, \ldots, n$$

and

$$\min_{1 \le j \le n} \{a_{ij}\} \le a_{ij} \qquad \text{for all } i = 1, 2, \ldots, m$$

Let the above maximum be attained at $i = i'$ and the minimum be attained at $j = j'$, *i.e.*,

$$\max_{1 \le i \le m} \{a_{ij}\} = a_{i'j} \quad \text{and} \quad \min_{1 \le j \le n} \{a_{ij}\} = a_{ij'}$$

Then, we must have

$$a_{i'j} \ge a_{ij} \ge a_{ij'} \qquad \text{for all } i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, n$$

From this, we get

$$\min_{1 \le j \le n} \{a_{i'j}\} \ge a_{ij} \ge \max_{1 \le i \le m} \{a_{ij'}\} \qquad \text{for all } i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, m$$

$$\therefore \quad \min_{1 \le j \le n} \ \max_{1 \le i \le m} \{a_{ij}\}] \ge \max_{1 \le i \le m} \ [\min_{1 \le j \le n} \{a_{ij}\}]$$

or

$$\overline{v} \ge \underline{v}.$$

Remarks 1. A game is said to be fair, if $\underline{v} = 0 = \overline{v}$.

2. A game is said to be strictly determinable, if $\underline{v} = v = \overline{v}$.

Rule for determining a Saddle Point

We may now summarize the procedure of locating the saddle point of a payoff matrix as follows :

Step 1. Select the minimum element of each row of the payoff matrix and mark them [*].

Step 2. Select the greatest element of each column of the payoff matrix and mark them [†].

Step 3. If there appears an element in the payoff matrix marked [*] and [†] both, the position of that element is a saddle point of the payoff matrix.

SAMPLE PROBLEMS

1701. *Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in the case of strictly determinable games :*

(*a*) $\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \end{array}$ (*b*) $\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix} \end{array}$

[Madurai M.Com. 1997]

Solution. (*a*) The payoff matrix for player *A* is

Player A	*Player B* B_1	B_2	*Row minima*
A_1	5†	0*	0
A_2	0*	2†	0
Column maxima	5	2	

The payoffs marked with [*] represent the minimum payoff in each row and those marked with [†] represent the maximum payoff in each column of the payoff matrix. The largest component of row minima represents $\underline{v}$ (maximin value) and the smallest component of column maxima represents $\overline{v}$ (minimax value).

Thus obviously, we have

$$\underline{v} = 0 \text{ and } \overline{v} = 2.$$

Since $\underline{v} \ne \overline{v}$, the game is not strictly determinable.

(b) Here, the payoff matrix for player A is

Player A	Player B B_1	B_2	Row minima
A_1	$0^{*\dagger}$	2	0
A_2	-1^{*}	$4^{\dagger}$	-1
Column maxima	0	4	

Since, the payoffs marked with [*] represent the minimum payoff in each row and those marked with [†] the maximum payoff in each column of the payoff matrix, we have

$\underline{v}$ (maximin value) = 0 and $\overline{v}$ (minimax value) = 0.

As $\underline{v} = \overline{v} = 0$, the game is strictly determinable and fair. Optimum strategies for players A and B are given by

$$S_0 = (A_1, B_1).$$

1702. *Solve the game whose payoff matrix is given by*

$$\begin{array}{cc} & \text{Player } A \\ & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \text{Player } B \begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

[Bharathidasan B.Com. 1999]

Solution. Consider the set of pure strategies

$\alpha = \{A_1, A_2, A_3\}$ for player A and $\beta = \{B_1, B_2, B_3\}$ for player B.

Assume that player B starts the game knowing fully well that whatever strategy he adopts, A will counter with a strategy that will minimize the payoff to B. Thus, if B selects B_1, then A will reply by selecting A_1 or A_2 as this corresponds to the minimum payoff to B in the first row corresponding to B_1. Similarly, if B chooses the strategy B_2, he may loose 4 or 3 or may neither loose nor gain depending upon the strategy chosen by A. However, B is assured of a gain of at least min. $\{0, -4, -3\}$; *i.e.*, -4 regardless of the strategy chosen by A. In other words, whatever strategy B may adopt, he can be assured of only the minimum of the corresponding row payoffs These corresponding to $B_i \in \beta$ are indicated by forming a column vector $r = \{1, -4, -1\}$ of the *row minima*. Naturally, B would like to maximize his minimum gain, which is just the largest component of r. Thus, maximum value of the game is maximum of $\{1, -4, -1\}$, *i.e.*, 1 which corresponds to B_1, the *maximin strategy*.

On the other hand, player A wishes to minimize his losses. If he plays strategy A_1, his loss is at the most maximum of $\{1, 0, 1\}$, *i.e.*, 1 regardless of what strategy B has adopted. He loses no more than max. $\{3, -4, 5\}$, if he plays A_2 and no more then max. $\{1\ -3, -1\}$ if he plays A_3. These maximum losses, corresponding to each $A_i \in \alpha$ are indicated by forming a row vector $c = (1, 5, 1)$ of the column maxima. The smallest component of c represents the minimum possible loss to A whatever strategy B may adopt. Thus, the minimax value of the game is min. $(1, 5, 1)$, *i.e.*, which corresponds to A_1 and A_3, the *minimax strategies*.

The maximin value is generally marked by {*} and the minimax value by {†} as shown below :

$$\begin{array}{cccccc} & A_1 & A_2 & A_3 & & \mathbf{r} \\ B_1 & 1^{*\dagger} & 3 & 1^{*\dagger} & & 1^{*} \\ B_2 & 0 & -4^{*} & -3 & & -4 \\ B_3 & 1^{\dagger} & 5^{\dagger} & -1^{*} & & -1 \\ c: & 1^{\dagger} & 5 & 1^{\dagger} & & \end{array}$$

We observe from the above that there exist two saddle points (having * and † both) at positions (1, 1) and (1, 3). Thus, the solution to the game is given by

(*i*) the optimum strategy for player B is B_1,

(*ii*) the optimum strategies for player A are A_1 and A_3,

(*iii*) the value of game is 1 for B and -1 for A.

Note : Since $v \neq 0$, the game is not fair, although it is strictly determinable.

1703. ***Determine the range of value of p and q that will make the payoff element a_{22}, a saddle point for the game whose payoff matrix (a_{ij}) is given below :***

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} \end{array}$$

Solution. Let us first of all ignore the values of p and q and determine the maximin and minimax values of the payoff matrix. For this, we have

	B_1	B_2	B_3	Row minima
A_1	2	4	5	2
A_2	10	7	q	7
A_3	4	p	8	4
Column maxima	10	7	8	

Obviously, the maximin value ($\underline{v}$) is 7 and the minimax value ($\overline{v}$) is also 7. Thus, there exists a saddle point at position (2, 2).

This imposes the condition on p as $p \leq 7$ and on q as $q \geq 7$.

Hence, the required range of values of p and q is

$$7 \leq q, \quad p \leq 7.$$

PROBLEMS

1704. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games :

(*a*)
$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$

(*b*)
$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 10 & 6 \\ 8 & 2 \end{bmatrix} \end{array}$$

[Madurai M.Com. 1993]

1705. Consider the game G with the following payoff matrix :

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix} \end{array}$$

(*a*) Show that G is strictly determinable whatever μ may be.

(*b*) Determine the value of G.

[Amravathi B.E. (Rul.) 1994]

1706. For the game with payoff matrix :

$$\begin{array}{cc} & \text{Player } A \\ \text{Player } B & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

determine the best strategies for players A and B and also the values of the game for them. Is this game (*i*) fair? (*ii*) strictly determinable?

17:5. GAMES WITHOUT SADDLE POINTS—MIXED STRATEGIES

As determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique payoff position—the saddle point.

In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle (equilibrium) point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called *mixed* strategies because they are probabilistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents which least player A can expect to win and the least which player B can lose. The expected payoff to a player in a game with arbitrary payoff matrix (a_{ij}) of order $m \times n$ is defined as :

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = \mathbf{p}^T \mathbf{A q}$$

where $\mathbf{p}$ and $\mathbf{q}$ denote the mixed strategies for players A and B respectively.

Maximin-Minimax Criterion. Consider an $m \times n$ game (a_{ij}) without any saddle point, *i.e.*, strategies are mixed. Let $p_1, p_2, \ldots, p_m$ be the probabilities with which player A will play his moves $A_1, A_2, \ldots, A_m$ respectively; and let $q_1, q_2, \ldots, q_n$ be the probabilities with which player B will play his moves $B_1, B_2, \ldots, B_n$ respectively. Obviously, $p_i \geq 0$ $(i = 1, 2, \ldots, m)$, $q_j \geq 0$ $(j = 1, 2, \ldots, n)$, and $p_1 + p_2 + \ldots + p_m = 1$; $q_1 + q_2 + \ldots + q_n = 1$.

The expected payoff function for player A, therefore, will be given by

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

Making use of maximin-minimax criterion, we have

For Player A.

$$\underline{v} = \max_{\mathbf{p}}.\ \min_{\mathbf{q}}.\ E(\mathbf{p}, \mathbf{q}) = \max_{\mathbf{p}}. \left[\min_{j}. \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\} \right]$$

$$= \max_{\mathbf{p}}. \left[\min_{j}. \left\{ \sum_{i=1}^{m} p_i a_{i1}, \sum_{i=1}^{m} p_i a_{i2}, \ldots, \sum_{i=1}^{m} p_i a_{in} \right\} \right]$$

Here, $\min. \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\}$ denotes the expected gain to player A, when player B uses his jth pure strategy.

For player B.

$$\overline{v} = \min_{\mathbf{q}}. \left[\max_{i}. \left\{ \sum_{j=1}^{n} q_j a_{1j}, \sum_{j=1}^{n} q_j a_{2j}, \ldots, \sum_{j=1}^{n} q_j a_{mj} \right\} \right].$$

Here $\max. \left\{ \sum_{j=1}^{n} q_j a_{ij} \right\}$ denotes the expected loss to player B when player A uses his ith strategy.

The relationship $\underline{v} \leq \overline{v}$ holds good in general and when p_i and q_j correspond to the optimal strategies the relation holds in 'equality' sense and the expected value for both the players becomes equal to the optimum expected value of the game.

Definition. *A pair of strategies* $(\mathbf{p}, \mathbf{q})$ *for which* $\underline{v} = \overline{v} = v$ *is called a* **saddle point** *of* $E(\mathbf{p}, \mathbf{q})$.

Theorem 17-2. *For any* 2×2 *two-person zero-sum game without any saddle point having the payoff matrix for player A*

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \end{array}$$

the optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad \textit{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix},$$

are determined by

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

where $p_1 + p_2 = 1$ *and* $q_1 + q_2 = 1$. *The value v of the game to A is given by*

$$v = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Proof. Let a mixed strategy for player A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$, where $p_1 + p_2 = 1$. Thus, if player B moves B_1 the net expected gain of A will be

$$E_1(p) = a_{11}\,p_1 + a_{21}\,p_2$$

and if B moves B_2, the net expected gain of A will be

$$E_2(p) = a_{12}\,p_1 + a_{22}\,p_2.$$

Similarly, if B plays his mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$, where $q_1 + q_2 = 1$, then B's net expected loss will be

$$E_1(q) = a_{11}\,q_1 + a_{12}\,q_2$$

if A plays A_1, and

$$E_2(q) = a_{21}\,q_1 + a_{22}\,q_2$$

if A plays A_2.

The expected gain of player A, when B mixes his moves with probabilities q_1 and q_2 is, therefore, given by

$$E(\mathbf{p}, \mathbf{q}) = q_1\,[a_{11}\,p_1 + a_{21}\,p_2] + q_2\,[a_{12}\,p_1 + a_{22}\,p_2].$$

Player A would always try to mix his moves with such probabilities so as to maximize his expected gain.

Now,

$$\begin{aligned} E(\mathbf{p}, \mathbf{q}) &= q_1\,[a_{11}\,p_1 + a_{21}\,(1 - p_1)] + (1 - q_1)\,[a_{12}\,p_1 + a_{22}\,(1 - p_1)] \\ &= [a_{11} + a_{22} - (a_{12} + a_{21})]\,p_1 q_1 + (a_{12} - a_{22})\,p_1 + (a_{21} - a_{22})\,q_1 + a_{22} \\ &= \lambda\left(p_1 - \frac{a_{22} - a_{21}}{\lambda}\right)\left(q_1 - \frac{a_{22} - a_{12}}{\lambda}\right) + \frac{a_{11}\,a_{22} - a_{12}\,a_{21}}{\lambda}, \end{aligned}$$

where $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$.

We see that if A chooses $p_1 = \dfrac{a_{22} - a_{21}}{\lambda}$, he ensures an expected gain of at least $(a_{11}a_{22} - a_{12}a_{21})/\lambda$. Similarly, if B chooses $q_1 = \dfrac{a_{22} - a_{12}}{\lambda}$, then B will limit his expected loss to at most $(a_{11}a_{22} - a_{12}a_{21})/\lambda$. These choices of p_1 and q_1 will thus be optimal to the two players. Thus, we get

$$p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \text{and} \quad p_2 = 1 - p_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \text{and} \quad q_2 = 1 - q_1 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

and

$$v = \frac{a_{11}\,a_{22} - a_{12}\,a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Hence, we have

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}; \quad \text{and} \quad v = \frac{a_{11}\,a_{22} - a_{12}\,a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Note : The above formulae for p_1, p_2, q_1, q_2 and v are valid only for 2×2 games without saddle points.

SAMPLE PROBLEMS

1714. *For the game with the following payoff matrix, determine the optimum strategies and the value of the game :*

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

[ICSI (June) 1996; Madurai M.Com. (Nov.) 2002]

Solution. Clearly, the given matrix is without a saddle point. So, the mixed strategies of P_1 and P_2 are :

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ p_1 & p_2 \end{bmatrix}, \quad S_{P_2} = \begin{bmatrix} 1 & 2 \\ q_1 & q_2 \end{bmatrix}; \quad p_1 + p_2 = 1 \quad \text{and} \quad q_1 + q_2 = 1$$

If $E(p, q)$ denotes the expected payoff function, then

$$E(p, q) = 5p_1q_1 + 3(1 - p_1)\,q_1 + p_1(1 - q_1) + 4(1 - p_1)(1 - q_1)$$
$$= 5p_1q_1 - 3p_1 - q_1 + 4 = 5(p_1 - 1/5)(q_1 - 3/5) + 17/5.$$

If P_1 chooses $p_1 = 1/5$, he ensures that his expectation is at least 17/5. He cannot be sure of more than 17/5, because by choosing $q_1 = 3/5$, P_2 can keep $E(p_1, q_1)$ down to 17/5. So P_1 might as well settle for 17/5 and P_2 reconcile to 17/5. Hence, the optimum strategies for P_1 and P_2 are

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ 1/5 & 4/5 \end{bmatrix}, \quad S_{P_2} = \begin{bmatrix} 1 & 2 \\ 3/5 & 2/5 \end{bmatrix}$$

and the value of the game is $v = 17/5$.

1715. *Consider a "modified" form of "matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and Re. 1.00 if the coins turn both tails. The non-matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?*

[Delhi M.B.A. 1999, 2007]

Solution. The payoff matrix for the matching player is given by

$$\begin{array}{ccc} & & \textit{Non-matching Player} \\ & & \begin{array}{cc} H & T \end{array} \\ \text{Matching Player} & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Clearly, the payoff matrix does not possess any saddle point. The players will use mixed strategies. The optimum mixed strategy for matching player is determined by

$$p_1 = \frac{1 - (-3)}{8 + 1 - (-3 - 3)} = \frac{4}{15}, \quad p_2 = \frac{11}{15}$$

and for the non-matching player, by

$$q_1 = \frac{1 - (-3)}{8 + 1 - (-3 - 3)} = \frac{4}{15}, \quad q_2 = \frac{11}{15}.$$

The expected value of the game (corresponding to the above strategies) is given by

$$v = \frac{8 - 3(-3)(-3)}{8 + 1 - 1(-3 - 3)} = -\frac{1}{15}.$$

Thus, the optimum mixed strategies for matching player and non-matching player are given by

$$S_{match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix} \text{ and } S_{non\text{-}match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix}.$$

Clearly, we would like to be the non-matching player.

PROBLEMS

1716. Solve the following game and determine the value of the game :

(a)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array},$$

(b)
$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

[Madras B.E. (Mech.) 1999] *[Allahabad M.B.A. 1998]*

1717. In a game of matching coins with two players, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A.

[Amravathi B.E. (Rul.) 1994]

1718. Two players A and B match coins. If the coins match, then A wins two units of value, if the coins do not match, then B wins 2 units of value. Determine the optimum strategies for the players and the value of the game. *[Madras M.B.A. (Nov.) 2006; Delhi M.Com. 2008]*

1719. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly then the loser has to pay him as many rupees as the sum of the number held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players. *[Jodhpur M.Sc. (Math.) 1994]*

17:6. GRAPHIC SOLUTION OF $2 \times n$ AND $m \times 2$ GAMES

The procedure described in the last section will generally be applicable for any game with 2×2 payoff matrix unless it possesses a saddle point. Moreover, the procedure can be extended to any square payoff matrix of any order. But it will not work for the game whose payoff matrix happens to be a *rectangular* one, say $m \times n$. In such cases a very simple graphical method is available if either m or n is two. The graphic short-cut enables us to reduce the original $2 \times n$ or $m \times 2$ game to a much simpler 2×2 game. Consider the following $2 \times n$ game :

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \end{array}.$$

It is assumed that the game does not have a saddle point. Let the optimum mixed strategy for A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$. The average (expected) payoff for A when he plays S_A against B's pure moves $B_1, B_2, \ldots, B_n$ is given by

B's pure move	*A's expected payoff E (p)*
B_1	$E_1(p_1) = a_{11}p_1 + a_{21}p_2 = a_{11}p_1 + a_{21}(1 - p_1)$
B_2	$E_2(p_1) = a_{12}p_1 + a_{22}p_2 = a_{12}p_1 + a_{22}(1 - p_1)$
$\vdots$	$\vdots$
B_n	$E_n(p_1) = a_{1n}p_1 + a_{2n}p_2 = a_{1n}p_1 + a_{2n}(1 - p_1).$

According to the maximin criterion for mixed strategy games, player A should select the values of p_1 and p_2 so as to maximize his minimum expected payoffs. This may be done by plotting the expected payoff lines :

$$E_j(p_1) = (a_{1j} - a_{2j})\,p_1 + a_{2j} \quad (j = 1, 2, \ldots, n).$$

The highest point on the *lower envelope* of these lines will give maximum of the minimum (*i.e.*, maximin) expected payoffs to player A as also the maximum value of p_i.

The two lines* passing through the *maximin* point identify the two *critical* moves of B which, combined with two of A, yield the 2×2 matrix that can be used to determine the optimum strategies of the two players, for the original game, using the results of the previous section.

The $(m \times 2)$ games are also treated in the same way where the upper envelope of the straight lines corresponding to B's expected payoffs will give the maximum expected payoff to player B and the lowest point on this then gives the minimum expected payoff (minimax value) and the optimum value of q_1.

SAMPLE PROBLEMS

1720. *Solve the following* 2×2 *game graphically :*

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Math.) 1996]

Solution. Clearly, the problem does not possess a saddle point. Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_2 = 1 - p_1$, against B. Then A's expected payoffs against B's pure moves are given by

B's pure move	*A's expected payoff* $E(p_1)$
B_1	$E_1(p_1) = p_1 + 1$
B_2	$E_2(p_1) = p_1$
B_3	$E_3(p_1) = -3p_1 + 3$
B_4	$E_4(p_1) = -4p_1 + 2$

These expected payoff equations are then plotted as functions of p_1 as shown in Fig. 17.1 which shows the payoffs of each column represented as points on two vertical axis *1* and 2, unit distance apart. Thus line B_1 joins the first payoff element 2 in the first column represented by +2 on axis 2, and the second payoff element 1 in the first column represented by +1 on axis *1*. Similarly, lines B_2, B_3 and B_4 join the corresponding representation of payoff elements in the second, third and fourth columns. Since the player A wishes to maximize his minimum expected payoff we consider the highest point of intersection H on the *lower envelope* of the A's expected payoff equations. This point H represents the *maximin* expected value of the game for A. The lines B_2 and B_4, passing through H, define the two relevant moves B_2 and B_4 that alone B needs to play. The solution to the original 2×4 game, therefore, boils down that of the simpler game with the 2×2 payoff matrix :

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \end{array}$$

*If there are more than two lines passing through the maximin point, there are ties for the optimum mixed strategies for player B. Thus any two such lines with *opposite sign* slopes will define an *alternative optimum* for B.

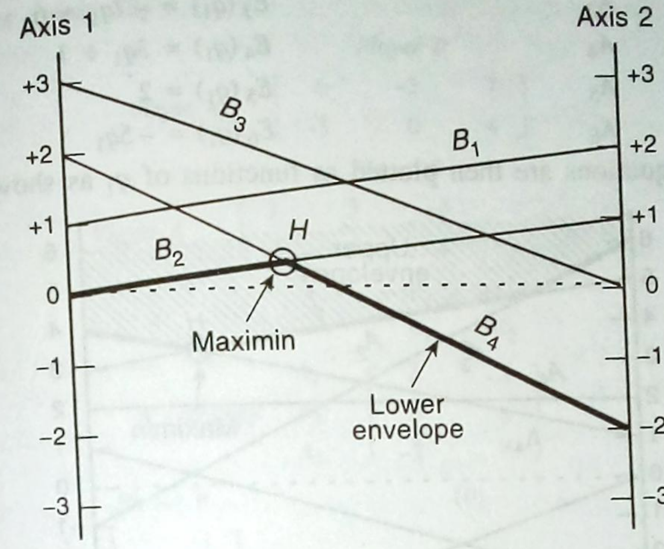


Fig. 17.1. The maximin value

Now if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_2 & B_4 \\ q_2 & q_4 \end{bmatrix}$

be the optimum strategies for A and B, then we have

$$p_1 = \frac{2-0}{1+2-(-2)} = 2/5, \quad p_2 = 1 - p_1 = 3/5,$$

$$q_2 = \frac{2-(-2)}{1+2-(-2)} = 4/5, \quad q_4 = 1 - q_1 = 1/5.$$

Hence, the solution to the game is

(*i*) the optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix}$,

(*ii*) the optimum strategy for B is $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{bmatrix}$

and (*iii*) the expected value of the game is $v = \dfrac{2 \times 1 - 0 \times (-2)}{1 + 2 - (0 - 2)} = \dfrac{2}{5}$.

1721. *Obtain the optimal strategies for both-persons and the value of the game for zero-sum two-person game whose payoff matrix is as follows :*

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

[Guru Nanak Dev Univ. B.Com. 2006]

Solution. Clearly, the given problem does not possess any saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - q_1$ against player A. Then B's expected payoffs against A's pure moves are given by

A's pure move	*B's expected payoff* $E(q_1)$
A_1	$E_1(q_1) = 4q_1 - 3$
A_2	$E_2(q_1) = -2q_1 + 5$

A_3	$E_3(q_1) = -7q_1 + 6$
A_4	$E_4(q_1) = 3q_1 + 1$
A_5	$E_5(q_1) = 2$
A_6	$E_6(q_1) = -5q_1$

The expected payoff equations are then plotted as functions of q_1 as shown in Fig. 17.2 :

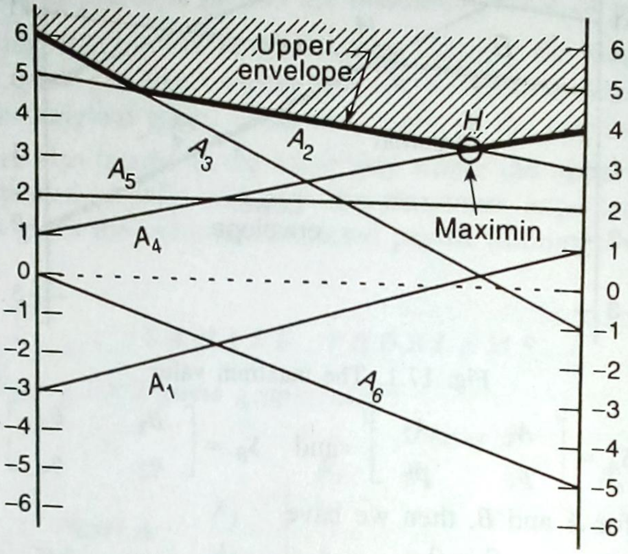


Fig. 17.2. The minimax value

Since, the player B wishes to minimize his maximum expected payoff, we consider the lowest point of intersection H on the *upper envelope* of B's expected payoff equations. This point H represents the minimax expected value of the game for player B. The lines A_2 and A_4 passing through H, define the two relevant moves A_2 and A_4 that alone the player A needs to play. The solution to the original 6×2 game, therefore, reduces to that of the simpler game with 2×2 payoff matrix :

$$\text{Player } A \quad \overset{\text{Player } B}{\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}}$$

If we now, let

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix}, \; p_1 + p_2 = 1 \,; \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, \; q_1 + q_2 = 1$$

then using the usual method of solution for 2×2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix}$$

and the value of the game as $v = 17/5$.

PROBLEMS

1722. Solve the following problem graphically :

$$\text{Player } A \quad \overset{\text{Player } B}{\begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}}$$

[Jodhpur M.Sc. (Math. 1993]

1723. Use graphical method in solving the following game :

$$\text{Player } B \quad \overset{\text{Player } A}{\begin{bmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix}}$$

[Panjab Tech. Univ. M.B.A. (Dec.) 2010]

23.9.19

UNIT - 05

NETWORKS

Network scheduling is a technique used for planning and scheduling in the field of large construction, maintainence, fabrication, purchasing, computer system installation research and development design etc.

Types of Network Scheduling

CPM => Critical Path Method.

PERT => Project Evaluation and Review Technique or (Program).

RAMP => Review Analysis of Multiple Projects.

Network

A network is a graphic representation of a project operators and is composed of activities and events that must be completed to reach the end objective of a project. Showing the planning sequence of their accomplishment their dependence and their inter relationship.

Basic Components of a network

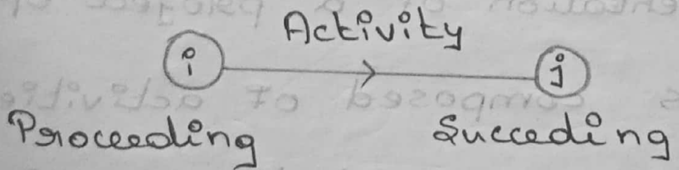
A basic components of a network are

i) Activity

ii) Event.

Activity

An activity is a task (or) item or work to be done, that consumes time, effort, money or other resources. It lies between two events called the preceeding and succeding ones. An activity is represented by an arrow with its head indicating the sequence in which the events are to occur.



Event

An event represent the start of completion of a some activity and as such as if consumed no time. It has no time duration and does not consume any resource. It is knows as a node. An event is not complete until all the activities flowing into it are completed. An event is generally represented on the network by a circle, rectangle, hexagon or some other geometric shape.

Classification Activity

i) Predecessor Activity

An activity which must be completed before one or more other activities start is knows as Predecessor activity.

ii) Successor Activity

An activity which started immediately after one or more of other activities are completed is known as Successor Activity.

iii) Dummy Activity

An activity which does not consume either any resource end time is known as dummy activity. A dummy activity is depicted by dotted line in the network diagram.

25.9.19 When we use dummy activity

The following situation we use dummy activity.

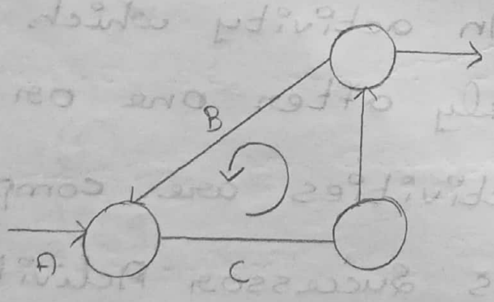
i) Two or more parallel activity joined by a pair of nodes.

ii) Two or more activities have the same immediate prodecessor activities in common.

Types of errors in drawing network diagram

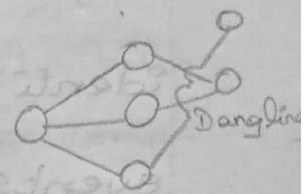
1) Looping

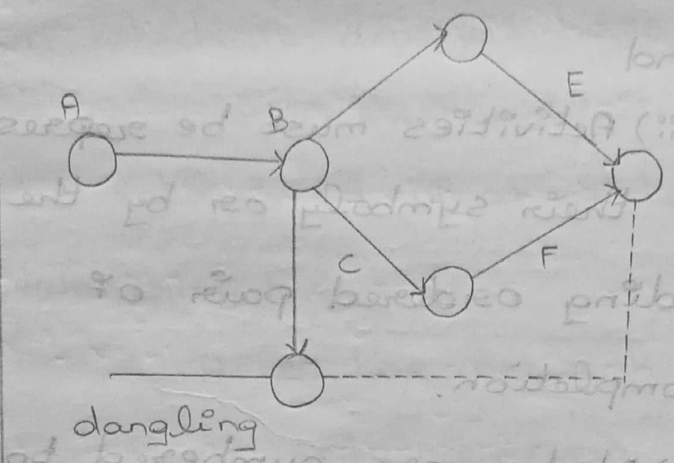
If a activity were represented as going back in time, a closed loop would occur.

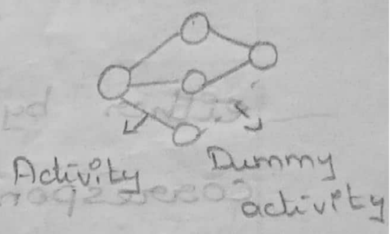


A closed loop would produce an endless cycle in computer programmes without a build in routine for detection (or) identification of cycle. This situation can be avoided checking the procedence relationship of the activity and by numbering them in logical order. Thus one property of a correct constructed network diagram is that it is non-cycle.

Dangling







No activity should end without being joined to the end event. If it is not so, a dummy activity is introduced in order to maintain the continuity of the system. Such and events other than the end of the project as a whole are called dangling. A dammy activity is therefore introduced to avoid this dangling.

Rules of Network construction

1) Each activity is represented by one and only one arrow.

2) Each activity must be identified by its starting and end node which implies that

i) Two activities should not be identified by the same completion events, and

ii) Activities must be represented either by their symbols or by the corresponding ordered pair of starting-completion.

3) Nodes are numbered to identify an activity uniquely. Tail node should be lower than the head node of an activity.

4) Between any pair of the nodes there should be one and only one activity. However, more than one activity may emanate from and terminate to a node.

5) Arrows should be kept straight and not curved or bent.

b) The logical sequence between activities must for the following rules.

i) An event cannot occur until all the incoming activies into it have been completed.

ii) An activity cannot start unless all the proceeding activities on which it depends have been completed.

iii) Dummy activities should only be introduced if absolutely necessary.

27.09.19 Numbering the events

After the network is drawn in a logical sequence, every event is assigned a number. The number sequence must be such so as to refied the flow of the network. In event numbering the following rule should be observed.

a) Event numbers should be unique.

b) Event numbering should be carried out on a sequential basic from left to right.

c) The initial event which has all outgoing arrows with no incoming arrow is numbered 0 to 1.

d) The head of an arrow should always bear a number higher than the one assigned at the tail of the arrow.

e) Gaps should be left in sequence of event numbering to accomodate subsequence inclusion of activities if necessary.

Critical Path

The critical activities of a network that constitute an interupted path which spans the entire network from start to finish is known as Critical Path.

Float

The float of an activity is the amount of time by which it is possible to delay its completion time without affecting the total project complete time.

Event Float

The float of an event is the different between its latest time (L_i) and earliest time (E_i) that is

Event float = $L_i - E_i$

It is a measure of how much later than expected a particular event could without delaying the completion of that project.

Activity Float

i) Total Float

ii) Free Float

iii) Independent Float

Total Float

The total float of an activity represents the amount of time by which an activity can be delayed without delayed in the project completion date.

Total float is the positive difference between the earliest finish time and the latest finish time or the positive difference between the earliest start time and the latest start time of an activity depending which way it is defined.

Free Float

Free Float is that position of total float with in which an activity can be manipulated without affecting the float of subsequence activities.

It is computed for an activity by subtracting the head even slack its difference between the latest and earliest event timings of an activities.

30.09.19 Independent Float

It is the portion of the total float in which an activities can be manipulated without altering the float of subsequence activities

It is computed by subracting the type event slack from the free float of the activity.

Result

The basic difference between slack and float times is that slack is used for events only whereas float is applied for activities.

The differenkce between total float and free float is known as intereference float.

An activity is critical if its total float is zero otherwise it is non-critical.

PERT

PERT network is represented by a probability distribution. This probability distribution of activity time is based upon three are as follows

to - The optimise time is the short possible time to complete the activity of all goes well.

tp - The time is the largest time than an activity could take if ~~every this~~ go everything goes wrong.

tm - The most likely time is the estimate of normal time an activity would take. If only one time were available, this would be Otherwise it is the made of the probability distribution.

Distinguish between PERT and CPM

CPM	PERT
i) CPM is used for repetitive jobs like planning the construction of a house.	i) PERT is used for non-repetive jobs like planning the assembly of the space platform.
ii) It is a model with well known activity time based upon past experience It therefore does not deal with uncertainly in project duration.	ii) It is a probabilistic model with uncertainly in activity duration.

iii) CPM is the activity oriented as the result of calculation are considered in terms of activities or operations of the project.	iii) PERT is said to be even oriented as the result of analysis are expressed interms of events or distinct points in time of programs.
iv) It is employed in construction and business problem.	iv) It incorporates statistical research programs.
v) It does not incorporates statistical analysis in determining time estimates because time is precise and known.	v) It incorporates statistical analysis and threby enables the determination of probability time by which each activity and the entire project would be completed.
vi) It is difficult to use CPM as the controling device for simple seasons. One must repeat the entire	vi) It serves the useful control devices as it assists the management in controlling a project by calling attention through

evaluation of the project. Each time the changes are introduced into the network.	constant review to such delays in activities which might lead to a delay in the project completion date.

Rules of Drawing Network

Write the common errors that occur in the constructing network.

i) Cycle (or) Loop

ii) Dangling

iii) Dummy activity.

What is the expansion of EST and LFT?

EST - Earliest Starting Time

LFT - Latest Finishing Time.

Define Cost Slope

$$\text{cost Slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

What is the usage of CPM?

i) Find the planning of project completed with minimum time.

ii) Find the storage plan to finish the project.

iii) Find the probability of probject completed before or after the expected time.

01.10.19

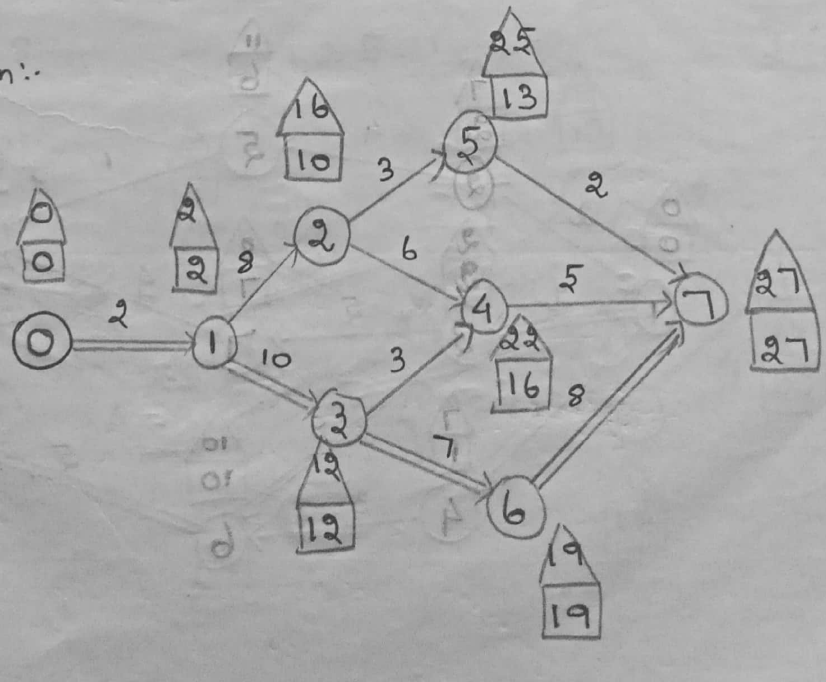
1, Given the following information to draw the graph and find

i) identify the critical method

ii) total project time

Activity	Duration
0 - 1	2
1 - 2	8
1 - 3	10
2 - 4	6
2 - 5	3
3 - 4	3
3 - 6	7
4 - 7	5
5 - 7	2
6 - 7	8

Soln:-



∴ Critical path is

(0) → (1) → (3) → (6) → (7)

Total project time is 27

2, Activity : 1-2 1-3 1-4 2-5 3-6 3-7 4-6 5-8 6-9 7-8 8-9

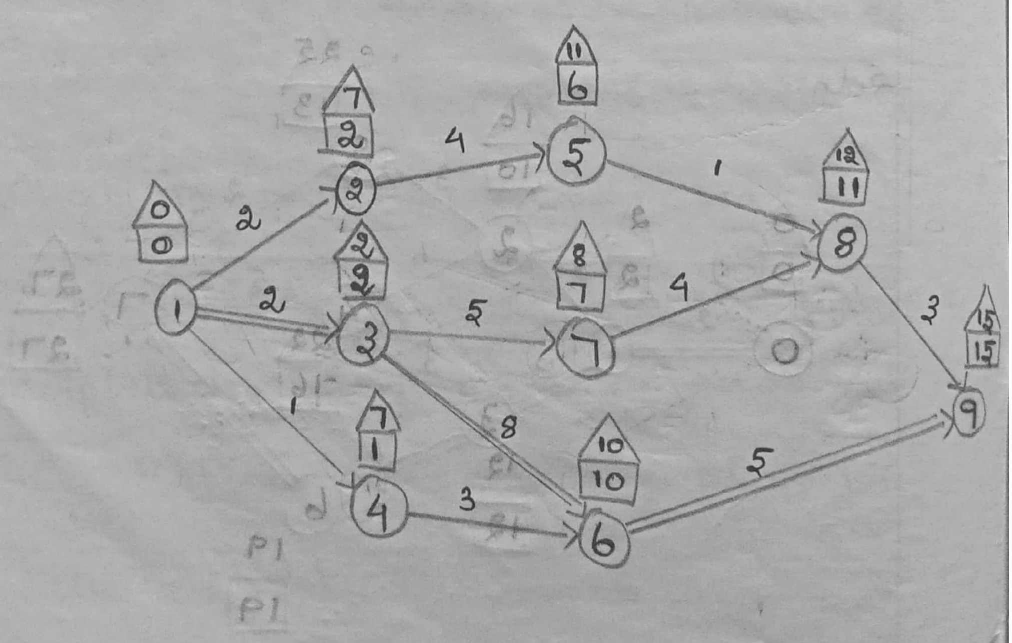
Activity	1-2	1-3	1-4	2-5	3-6	3-7	4-6	5-8	6-9	7-8	8-9
Time in week	2	2	1	4	8	5	3	1	5	4	3

i) Construct the diagram of the given data.

ii) Find the critical path

iii) Total Project duration.

Soln:-



∴ Critical Path is

$① \rightarrow ③ \rightarrow ⑥ \rightarrow ⑨$

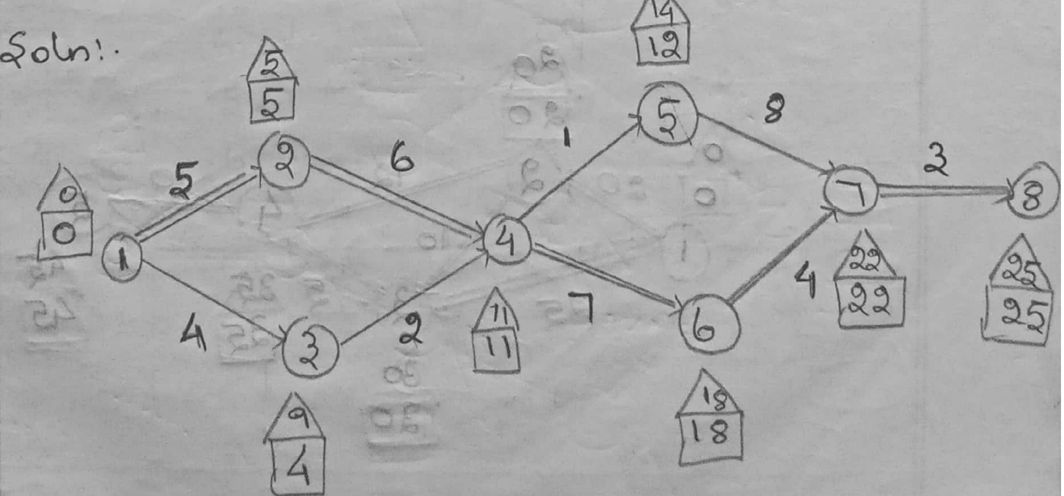
∴ Total project duration is 15

09.10.19

3. Find critical path for the following data:

Activity	Immediate predecessor Activity	Duration
1 - 2	-	5
1 - 3	-	4
2 - 4	1 - 2	6
3 - 4	1 - 3	2
4 - 5	2 - 4	1
4 - 6	2 - 4 & 3 - 4	7
5 - 7	4 - 5	8
6 - 7	4 - 6	4
7 - 8	6 - 7 & 5 - 7	3

Soln:



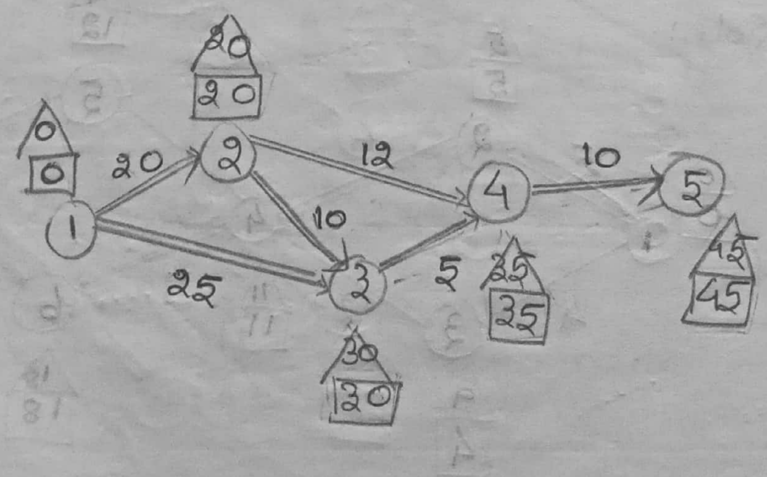
∴ Critical Path is

Total project time is 25

4, Find Critical path and total project time for the following data

Activity	Immediate Precedessor Activity	Duration
1 - 2	–	20
1 - 3	–	25
2 - 3	1 - 2	10
2 - 4	1 - 2	12
3 - 4	1 - 3, 2 - 3	5
4 - 5	2 - 4, 3 - 4	10

Soln :-



The critical Paths are

1 → 2 → 4 → 5

1 → 2 → 3 → 4 → 5

1 → 3 → 4 → 5

Total project time is 45.

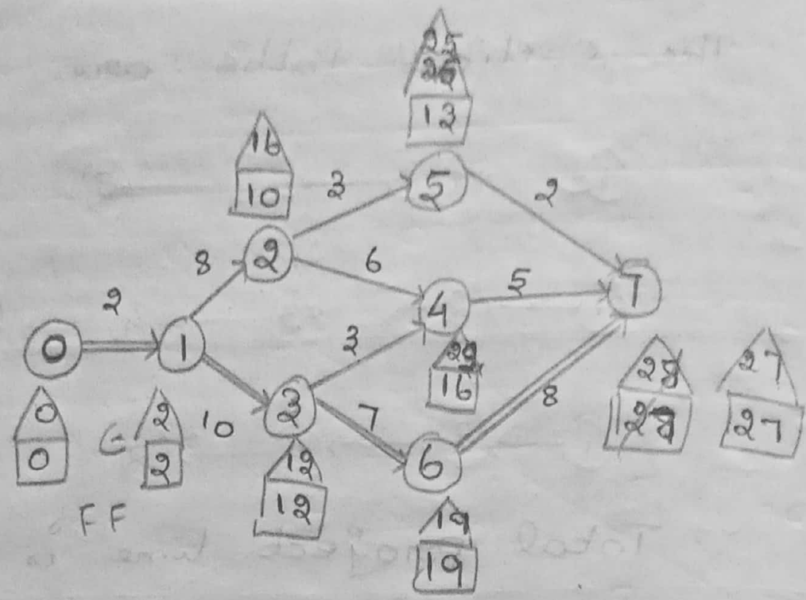
Resource Analysis

10.10.19

5, Find total float, Free float, Independent Float and also find critical path from the following data

Activity	Duration
0 - 1	2
1 - 2	8
1 - 3	10
2 - 4	6
2 - 5	3
3 - 4	3
3 - 6	7
4 - 7	5
5 - 7	2
6 - 7	8

Soln:



Critical Path is

0 → 1 → 3 → 6 → 7

Total Project time is 27

H.D T.D

Activity	Duration	Earliest Time Start (ES)	Earliest Time Finish (EF)	Latest Time Start (LS)	Latest Time Finish (LF)	Total Float TF = ES-ES	Free Float TF-	Independent Float
0-1	2	0 S	2 EF+D	0 EF-D	2 E	0	0-0 =0	0-0 =0
1-2	8	2	10	8	16	6	6-6 =0	0-0 =0
1-3	10	2	12	2	12	0	0-0 =0	0-0 =0
2-4	6	10	16	16	22	6	6-6 =0	0-6 =-6
2-5	3	10	13	22	25	12	12-12 =0	0-6 =-6
3-4	3	12	15	19	22	7	7-6 =1	1-0 =1
3-6	7	12	19	12	19	0	0-0 =0	0-0 =0
4-7	5	16	21	22	27	6	6-0 =6	6-6 =0
5-7	2	13	15	25	27	12	12-0 =12	12-12 =0
6-7	8	19	27	19	27	0	0-0 =0	0-0 =0

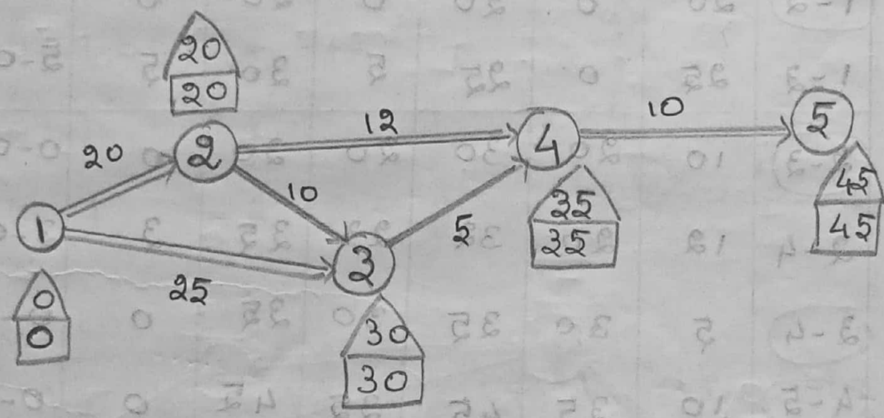
∴ Critical Path is 0 → 1 → 3 → 6 → 7

Total Project time is 27.

6, Resource Analysis in Network scheduling form

Activity	Immediate Predeccesor Activity	Duration
1-2	-	20
1-3	-	25
2-3	1-2	10
2-4	1-2	12
3-4	1-2 & 2-3	5
4-5	2-4 & 3-4	10

Soln:-



Critical path is

1 → 2 → 4 → 5

1 → 2 → 3 → 4 → 5

1 → 2 → 4 → 5

Total Project time is 45

Activity	Duration	Earliest Time Start (ES)	Earliest Time Finish (EF)	Latest Time Start (LS)	Latest Time Finish (LF)	Total Float TF=|ES-LS|	Free Float	Inde. Float
1-2	20	0	20	0	20	0	0-0 =0	0
1-3	25	0	30	0	30	30	0-0 =0	0
2-3	10	20	30	20	30	10	0-0 =0	0
2-4	12	20	35	20	35	15	0-0 =0	0
3-4	05	30	35	30	35	5	0-0 =0	0
4-5	10	35	45	35	45	10	0-0 =0	0

Activity	Duration	Earliest Time Start (ES)	Earliest Time Finish (EF)	Latest Time Start (LS)	Latest Time Finish (LF)	Total float TF=LS-ES	FF	IF
1-2	20	0	20	0	20	0	0-0=0	0-0=0
1-3	25	0	25	5	30	5	5-0=5	5-0=5
2-3	10	20	30	20	30	0	0-0=0	0-0=0
2-4	12	20	32	23	35	3	3-0=3	3-0=3
3-4	5	30	35	30	35	0	0-0=0	0-0=0
4-5	10	35	45	35	45	0	0-0=0	0-0=0

Critical Path is

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$

Total Project Time is 45

15.10.19 Following table shows that jobs at a network along with the time estimate. Draw the projects network and critical path what is the probability of completing the project in 27 days.

Task	t_o	t_p	t_m
1-2	3	15	6
2-3	2	14	5
1-4	6	30	12
2-5	2	8	85
2-6	5	17	11
3-6	3	15	6
4-7	3	27	9
5-7	1	7	4
6-7	2	8	5

Soln:-

The expected activity time te is calculated by using

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

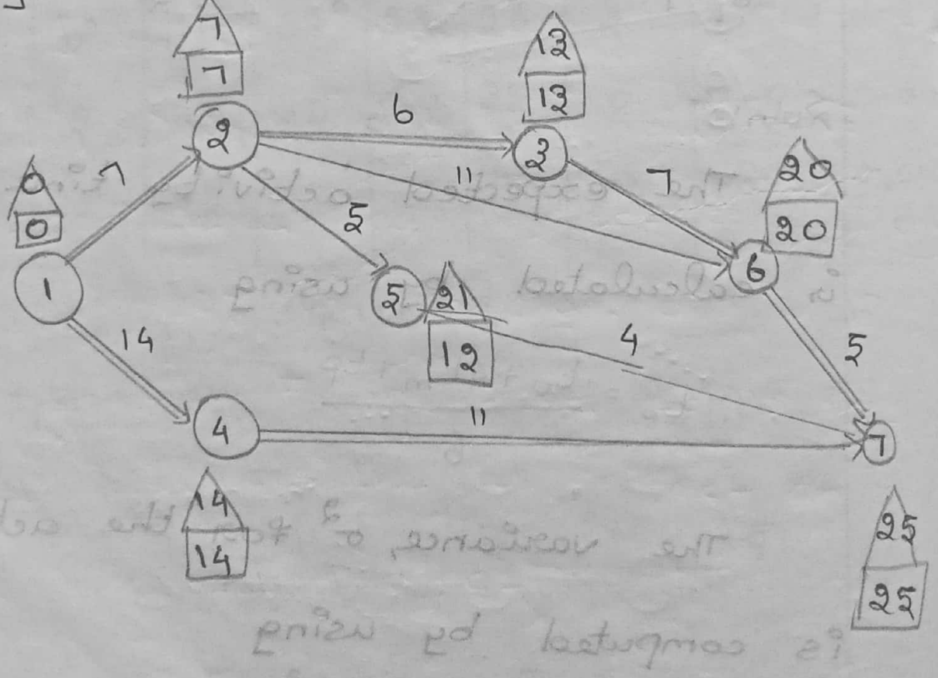
The variance, σ^2 for the activities is computed by using

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

The following table provides us the required information regarding t_e and σ^2.

Activity	t_o	t_p	t_m	t_e	σ^2
1-2	3	15	6	$\frac{3+24+15}{6} = 7$	4
2-3	2	14	5	$\frac{2+20+14}{6} = 6$	4
1-4	6	30	12	$\frac{6+48+30}{6} = 14$	16
2-5	2	8	5	$\frac{2+20+8}{6} = 5$	4
2-6	5	17	11	$\frac{5+44+17}{6} = 11$	4
3-6	3	15	6	$(3+24+15)/6 = 7$	4
4-7	3	27	9	$(3+36+27)/6 = 11$	16
5-7	1	7	4	$(1+16+7)/6 = 4$	1
6-7	2	8	5	$(2+20+8)/6 = 5$	1

Diagram



∴ Critical Path are

$1 \rightarrow 2 \rightarrow 3 \rightarrow 6 \rightarrow 7$

$1 \rightarrow 2 \rightarrow 6 \rightarrow 7$

$1 \rightarrow 4 \rightarrow 7$

Total Project time is 25

$$\text{var}(x) = \left(\frac{t_p - t_o}{6}\right)^2$$

$$= 16 + 16 = 32$$

$$S.D(\sigma) = \sqrt{\text{var}(x)}$$

$$\sigma = \sqrt{32} = 5.656$$

Standard normal variable,

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{\text{Due date} - \text{Expected completion time}}{\sigma}$$

$$z = \frac{27.25}{5.656}$$

$$P(z \leq 0.3536) = P(-\infty \leq z \leq 0) + P(0 \leq z \leq 0.3536)$$

$$= 0.5 + 0.1368$$

$$= 0.6368 \text{ or } 63.68\%.$$

18.10.19

2, A project consists of eight activities with the following relevant information

Activity	Immediate Predecessor	Estimated duration (days)		
		Optimistic	Most likely	Pessimistic
A	-	1	1	7
B	-	1	4	7
C	-	2	2	8
D	A	1	1	1
E	B	2	5	14
F	C	3	5	8
G	D, E	3	6	13
H	F, G	1	2	3

i) Draw the PERT network and find out the expected project completion time.

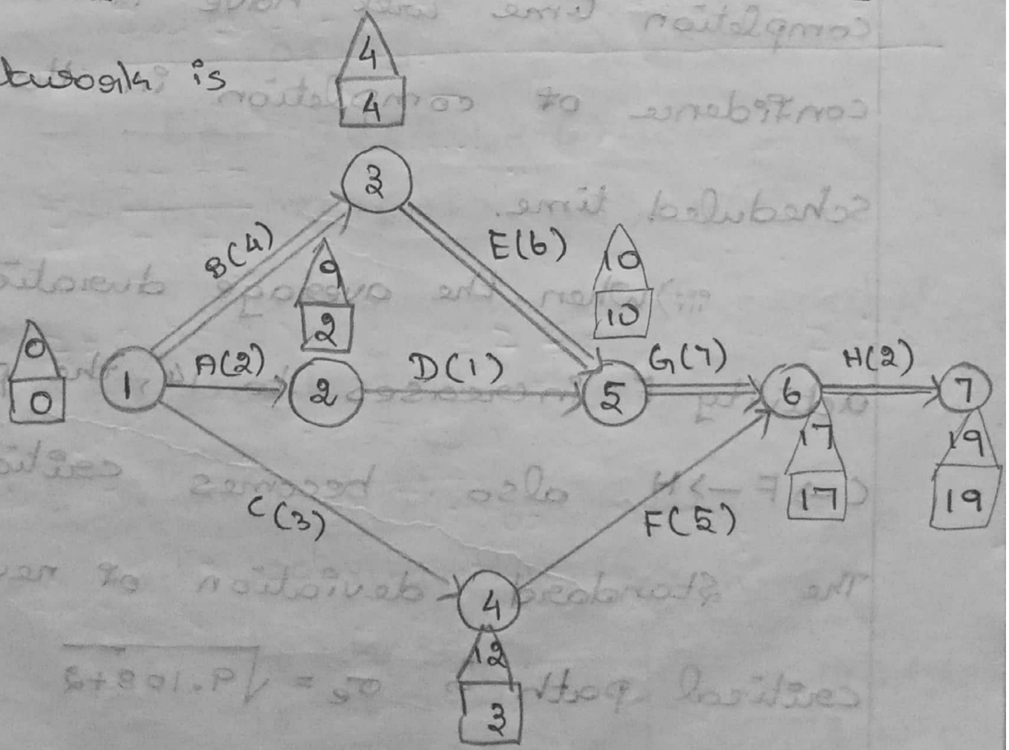
ii) What duration will have 95% confidence for project completion?

iii) If the average duration for activity F increased to 14 days, what will be the its effect on the expected project completion time which will have 95% confidence?

Soln:

Activity	t_o	t_m	t_p	$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\left(\frac{t_p - t_o}{6}\right)^2$
A	1	1	7	2	1
B	1	4	7	4	1
C	2	2	8	3	1
D	1	1	1	1	0
E	2	5	14	6	4
F	2	5	8	5	1
G	3	6	13	7	4
H	1	2	3	2	$(2/6)^2$

i) Using the procedure relationship among the activities, the resulting network is



CPM : 1 -> 3 -> 5 -> 6 -> 7

(i.e) B -> E -> G -> H

Expected duration of the project is 19 days. The variance of the project length is

$$\sigma^2 = 1+4+4+0.108$$
$$= 1+4+4+0.108$$
$$\sigma_e = \sqrt{9.108} = 3.02$$

ii) Since $P(Z \leq 1.645) = 0.5 + 0.45$
$$= 0.95$$

$$\frac{T_S - T_e}{\sigma_e} = 1.645$$

$$\Rightarrow T_S = 19 + 3.02 \times 1.645 = 24 \text{ days}$$

Hence, the 24 days of project completion time will have 95% of confidence of completion in the scheduled time.

iii) When the average duration of activity F increases to 14, the path $C \rightarrow F \rightarrow H$ also becomes critical. The standard deviation of new critical path is $\sigma_e = \sqrt{9.108+2}$

$$= 3.36$$

$P(z < 1.645) = 0.95$ gives us

$$\frac{T_S - 19}{3.36} = 1.645$$

(i.e) $T_S = 19 + 3.36 \times 1.645$

$= 24.52$ days

Hence, the project completion duration of 24.52 days will have 95% confidence.