SWAMI DAYANANDA
COLLEGE OF ARTS & SCIENCE,
MANJAKKUDI-612610

DEPERTMENT OF MATHEMATICS

Calculus & Fourier Series(16SACMM1)

Study Material

Class : I-B.Sc Physics

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B.Sc. Physics / Chemistry / Industrial Electronics / Geology - Students

(For the candidates admitted from the academic year 2016-17 onwards)

ALLIED MATHEMATICS

ALLIED COURSE I

CALCULUS AND FOURIER SERIES

Objects :

1. To learn the basic need for their major concepts
2. To train the students in the basic Integrations

UNIT I

Successive Differentiation – n^{th} derivative of standard functions (Derivation not needed) simple problems only-Leibnitz Theorem (proof not needed) and its applications- Curvature and radius of curvature in Cartesian only (proof not needed)–Total differential coefficients (proof not needed) - Jacobians of two & three variables –Simple problems in all these.

UNIT II

Evaluation of integrals of types

1] $\int \frac{px+q}{ax^2+bx+c}dx$ 2] $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ 3] $\int \frac{dx}{(x+p)\sqrt{ax^2+bx+c}}$

4] $\int \frac{dx}{a+b\cos x}$ 5] $\int \frac{dx}{a+b\sin x}$ 6] $\int \frac{(a\cos x+b\sin x+c)}{(p\cos x+q\sin x+r)}dx$

Integration by trigonometric substitution and by parts of the integrals

1] $\int \sqrt{a^2-x^2}\,dx$ 2] $\int \sqrt{a^2+x^2}\,dx$ 3] $\int \sqrt{x^2-a^2}\,dx$

UNIT III

General properties of definite integrals – Evaluation of definite integrals of types

1] $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$ 2] $\int_a^b \sqrt{(x-a)(b-x)}\,dx$ 3] $\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx$

Reduction formula (When n is a positive integer) for

1] $\int_a^b e^{ax}x^n\,dx$ 2] $\int_a^b \sin^n x\,dx$ 3] $\int_a^b \cos^n x\,dx$

4] $\int_0^x e^{ax} x^n dx$ 5] $\int_0^{\frac{\pi}{2}} \sin^n x dx$

6] Without proof $\int_0^{\frac{\pi}{2}} \sin^n x \cos^m x dx$ - and illustrations

UNIT IV

Evaluation of Double and Triple integrals in simple cases –Changing the order and evaluating of the double integration. (Cartesian only)

UNIT V

Definition of Fourier Series – Finding Fourier Coefficients for a given periodic function with period 2π and with period 2ℓ - Use of Odd & Even functions in evaluating Fourier Coefficients - Half range sine & cosine series.

TEXT BOOK(S)

1. S. Narayanan, T.K. Manichavasagam Pillai, Calculus, Vol. I, S. Viswanathan Pvt Limited, 2003
2. S. Arumugam, Isaac and Somasundaram, Trigonometry & Fourier Series, New Gamma Publishers, Hosur, 1999.

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5 SUCCESSIVE DIFFERENTIATION

❑ SUCCESSIVE DIFFERENTIATION

Let y be a function of x, its differential coefficient $\frac{dy}{dx}$ will be in general a function of x which can be differentiated once again. The differential coefficient of $\frac{dy}{dx}$ is called the second differential coefficient of y. Similarly the differential coefficient of second differential coefficient is called the third differential coefficient, and so on. It is denoted by

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^2}, \ldots\ldots$

The n^{th} differential coefficient of y is denoted by $\frac{d^ny}{dx^n}$. The n^{th} differential coefficient can also be denoted in the following ways :

$$D^ny,\ y_n,\ \frac{d^ny}{dx^n},\ y^{(n)},\ \left(\frac{d}{dx}\right)^n y.$$

Example 1 *If* $y = \frac{x+1}{x+2}$ *find* $\frac{d^2y}{dx^2}$.

Solution

Given
$$y = \frac{x+1}{x+2}$$
$$\therefore \frac{dy}{dx} = \frac{(x+2)-(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{(x+2)^2}\right] = \frac{0-2\,(x+2)}{(x+2)^4}$$
$$= \frac{-2}{(x+2)^3}$$

Example 2 *If* $x = a\,(\cos\theta + \theta\sin\theta)$ *and* $y = a\,(\sin\theta - \theta\cos\theta)$ *find* $\frac{d^2y}{dx^2}$.

Solution

Given
$$x = a\,(\cos\theta + \theta\sin\theta)$$
$$\frac{dx}{d\theta} = a\,[-\sin\theta + \theta\cos\theta + \sin\theta]$$
$$= a\,\theta\cos\theta$$

$$\frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta - \cos\theta)$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta}$$

$$= \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}[\tan\theta] = \frac{d}{d\theta}(\tan\theta)\cdot\frac{d\theta}{dx}$$

$$= \sec^2\theta\,\frac{1}{a\theta\cos\theta}$$

$$= \frac{\sec^3\theta}{a\theta}$$

Example 3

If $y = a\cos 2x + b\sin 3x$, find $\frac{d^2y}{dx^2}$.

Solution

$$\frac{dy}{dx} = -2a\sin 2x + 3b\cos 3x$$

$$\frac{d^2y}{dx^2} = -4a\cos 2x - 9b\sin 3x$$

Example 4

If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Solution

$$y = a\cos(\log x) + b\sin(\log x)$$

$$\frac{dy}{dx} = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

i.e.,

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Differentiating w.r.t. 'x', we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

i.e., $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = -[a\cos(\log x) + b\sin(\log x)]$

$= -y$

$\therefore x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

Example 5 *If $y = a\,e^{mx} + b\,e^{-mx}$ show that $\dfrac{d^2y}{dx^2} - m^2y = 0.$*

Solution

Given $y = a\,e^{mx} + b\,e^{-mx}$

$$\frac{dy}{dx} = m\,a\,e^{mx} - m\,b\,e^{-mx}$$

$$\frac{d^2y}{dx^2} = m^2\,a\,e^{mx} + m^2\,b\,e^{-mx}$$

$$= m^2(a\,e^{mx} + b\,e^{-mx})$$

$$\frac{d^2y}{dx^2} = m^2 y$$

i.e., $\dfrac{d^2y}{dx^2} - m^2y = 0$

Exercises

1. If $y = e^x \sin x$, prove that $y_2 - 2x\,y_1 + 2y = 0$.
2. If $y = \cos^{-1} x$, show that $(1 - x^2)\,y_2 - x\,y_1 = 0$.
3. If $y = e^{m\cos^{-1}x}$, prove that $(1 - x^2)\,y_2 - x\,y_1 - m^2\,y = 0$.
4. If $y = x\sin(\log x) + x\log x$, show that $x^2\,y_2 - x\,y_1 + 2y = x\log x$.
5. If $y = (\log x)^2$, show that $x^2\,y_2 + x\,y_1 = 2$.
6. If $y = a\,e^{2x} + B\,x\,e^{2x}$, show that $y_2 - 4y_1 + 4\,y = 0$.
7. If $y = a\,x^2 + bx$, show that $x^2\,y_2 - 2\,x\,y_1 + 2y = 0$.
8. If $y = \dfrac{x^2+1}{x}$, show that $y_2 - x^2\,y = 0$.
9. If $xy = ax^2 + \dfrac{b}{x}$, show that $x^2\,y_2 + 2\,(xy_1 - y) = 0$.
10. If $y = \log[\,x + \sqrt{x^2 + a^2}\,]$ show that $(a^2 + x^2)\,y_2 + xy_1 = 0$.

IMPORTANT RESULTS ON n^{TH} DERIVATIVES

1. *To find the n^{th} derivative of e^{ax}.*

Let $y = e^{ax}$

$$y_1 = \frac{dy}{dx} = a\,e^{ax}$$

$$y_2 = \frac{d^2y}{dx^2} = a^2\,e^{ax}$$

$$y_3 = \frac{d^3y}{dx^2} = a^3\,e^{ax}$$

... .

... .

$$y_n = \frac{d^ny}{dx^n} = a^n\,e^{ax}$$

$$\therefore \boxed{D^n\,(e^{ax}) = a^n\,e^{ax}}$$

2. *To find the n^{th} derivative of $\frac{1}{ax+b}$*

Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$

$$\frac{dy}{dx} = y_1 = -1\,(ax+b)^{-2}\,.\,a$$

$$\frac{d^2y}{dx^2} = y_2 = -1\cdot(-2)\,(ax+b)^{-3}\,.\,a^2$$

$$\frac{d^3y}{dx^2} = y_3 = (-1)\,(-2)\,(-3)\,(ax+b)^{-4}\,.\,a^3$$

....

....

$$\frac{d^ny}{dx^n} = y_n = (-1)\,(-2)\,(-3)\,......\,(-n)\,(ax+b)^{-(n+1)}\,.\,a^n$$

i.e., $$\frac{d^ny}{dx^n} = (-1)^n\,1\cdot 2\cdot 3\,......\,n\,(ax+b)^{-(n+1)}\,.\,a^n$$

i.e., $$\boxed{D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n\,n!\,a^n}{(ax+b)^{n+1}}}$$

3. *To find the n^{th} derivative of $\frac{1}{(ax+b)^2}$*

Let $$y = \frac{1}{(ax+b)^2}$$

$$i.e., \quad y = (ax+b)^{-2}$$

$$\frac{dy}{dx} = y_1 = (-2)(ax+b)^{-3}\cdot a$$

$$\frac{d^2y}{dx^2} = y_2 = (-2)(-3)(ax+b)^{-4}\cdot a^2$$

....

....

$$\frac{d^ny}{dx^n} = y_n = (-2)(-3)\,\ldots\ldots\,[-(n+1)]\,(ax+b)^{-(n+2)}\,.\,a^n$$

$$\boxed{D^n\left[\frac{1}{(ax+b)^2}\right] = \frac{(-1)^n\,(n+1)!\,a^n}{(ax+b)^{n+2}}}$$

4. *To find the n^{th} derivative of log (ax + b)*

Let $\quad y = \log(ax+b)$

$$\frac{dy}{dx} = y_1 = \frac{1}{ax+b}\cdot a$$

$$y_n = D^n[\log(ax+b)]$$

$$= D^{n-1}\left[\frac{1}{ax+b}\cdot a\right]$$

$$= a\,D^{n-1}\left[\frac{1}{ax+b}\right]$$

$$= a\,\frac{(-1)^{n-1}\,(n-1)!\,a^{n-1}}{(ax+b)^n}$$

$$\therefore \quad \boxed{Dn\,[\log(ax+b)] = \frac{(-1)^{n-1}\,(n-1)!\,a^n}{(ax+b)^n}}$$

5. *To find the n^{th} derivative of sin (ax + b)*

Let $\quad y = \sin(ax+b)$

$$\frac{dy}{dx} = y_1 = a\cos(ax+b)$$

$$= a\sin\left(ax+b+\frac{\pi}{2}\right)\cdot a$$

$$\frac{d^2y}{dx^2} = y_2 = a\cos\left(ax+b+\frac{\pi}{2}\right)\cdot a$$

$$= a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^2 \sin\left(ax + b + \frac{2\pi}{2}\right)$$

....

....

$$\boxed{\frac{d^n y}{dx^n} = y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)}$$

6. To find the n^{th} derivative of cos (ax + b)

Let $y = \cos(ax + b)$

$$\frac{dy}{dx} = y_1 = -a \cdot \sin(ax + b)$$

$$= a \cos\left(ax + b + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} = y_2 = -a^2 \sin\left(ax + b + \frac{\pi}{2}\right)$$

$$= a^2 \cos\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^2 \cos\left(ax + b + \frac{2\pi}{2}\right)$$

....

....

$$\boxed{\frac{d^n y}{dx^n} = y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)}$$

7. To find the n^{th} derivative of e^{ax} sin (bx + c)

Let $y = e^{ax} \sin(bx + c)$

$$\frac{dy}{dx} = y_1 = e^{ax} \cos(bx + c) \cdot b + e^{ax} \cdot a \sin(bx + c) \quad \text{... (A)}$$

Put $a = r \cos\theta$ ... (1)

$b = r \sin\theta$ ... (2)

where r and θ are two new constants.

From (1) and (2), we get

$$r^2 = a^2 + b^2 \Rightarrow r = \sqrt{a^2 + b^2} \quad \text{... (3)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{r \sin\theta}{r \cos\theta} = \frac{b}{a}$$

i.e., $\tan\theta = \dfrac{b}{a}$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad \ldots (4)$$

$\therefore$ From *(A)*, we get

$$y_1 = r\sin\theta\, e^{ax}\cos(bx+c) + r\cos\theta\, e^{ax}\sin(bx+c)$$

$$= r\,e^{ax}\,[\sin\theta\cos(bx+c)+\cos\theta\sin(bx+c)]$$

$$= r\,e^{ax}\sin[bx+c+\theta]$$

Now $y_2 = r\;[e^{ax}\cos(bx+c+\theta)\cdot b + \sin(bx+c+\theta)\,a\,e^{ax}]$

$$= r^2 e^{ax}\,[\sin\theta\cos(bx+c+\theta)+\cos\theta\sin(bx+c+\theta)]$$

$$y_2 = r^2 e^{ax}\sin(bx+c+2\theta)$$

In general,

$$y_n = r^n e^{ax}\sin(bx+c+n\theta)$$

$$= \left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left[bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$= (a^2+b^2)^{n/2}\, e^{ax}\sin\left[bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$\therefore \boxed{\begin{aligned}&\mathbf{D}^n\,[e^{ax}\sin(bx+c)]\\ &= (a^2+b^2)^{n/2}\, e^{ax}\sin\left[bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right]\end{aligned}}$$

Note : $D^n\,[\,e^{ax}\cos(bx+c)\,]$

$$= (a^2+b^2)^{n/2}\, e^{ax}\cos\left[bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right]$$

TABLE

S.No.	$f(x)$	$D^n\,[\,f(x)\,]$
1	e^{ax}	$a^n\cdot e^{ax}$
2	$\frac{1}{ax+b}$	$\frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$
3	$\frac{1}{(ax+b)^2}$	$\frac{(-1)^n\,(n+1)!\, a^n}{(ax+b)^{n+2}}$
4	$\log(ax+b)$	$\frac{(-1)^{n-1}\,(n-1)!\, a^n}{(ax+b)^n}$
5	$\sin(ax+b)$	$a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$
6	$\cos(ax+b)$	$a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$

4. $\frac{1}{16}\left\{2\sin\left(x+\frac{n\pi}{2}\right)+3^n\sin\left(3x+\frac{n\pi}{2}\right)-5^n\sin\left(5x+\frac{n\pi}{2}\right)\right\}$

5. $\frac{1}{4}(a^2+9b^2)^{n/2}\, e^{ax}\cos\left\{3bx+n\tan^{-1}\left(\frac{3b}{a}\right)\right\}$

$$+\frac{3}{4}(a^2+b^2)^{n/2}\, e^{ax}\cos\left\{bx+n\tan^{-1}\left(\frac{b}{a}\right)\right\}$$

6. $\frac{1}{2}e^x\left\{2^{n/2}\cos\left(x+\frac{n\pi}{4}\right)-10^{n/2}\cos(3x+n\tan^{-1}3)\right\}$

7. $\frac{(-1)^{n-1}c^{n-1}(ad-bc)\,n!}{(cx+d)^{n+1}}$

8. $\frac{4(-1)^n n!}{(x+2)^{n+1}}-\frac{3(-1)^n n!}{(x+1)^{n+1}}+\frac{(1)^n (n+1)!}{(x+1)^{n+2}}$

9. $\frac{(-1)^n n!}{2a}\left\{\frac{1}{(x-a)^{n+1}}-\frac{1}{(x+a)^{n+1}}\right\}$

10. $\frac{(-1)^{n-1}(n-1)!}{(x+2)^n}-\frac{(n-1)!}{(2-x)^n}$.

LEIBNITZ'S THEOREM

If u and v be any two functions of x, then

$$D^n(uv) = D^n(u)\,v + nC_1\, D^{n-1}(u)\cdot D(v) + nC_2\, D^{n-2}(u)\cdot D^2(v) + \ldots\ldots + u\cdot D^n(v)$$

Note : **This theorem is useful for finding the n^{th} differential coefficient of a product.**

Example 1

Find the n^{th} differential coefficient of $x^3 e^{5x}$.

Solution

For convenience, choose $v = x^3$, $u = e^{5x}$.

$$\therefore D^n(e^{5x}\cdot x^3) = D^n(e^{5x})\cdot x^3 + nC_1\, D^{n-1}(e^{5x}).D(x^3) + nC_2\, D^{n-2}(e^{5x})\,D^2(x^3) + nC_3\, D^{n-3}(e^{5x})\,D^3(x^3)$$

$$= 5^n\cdot e^{5x}\cdot x^3 + nC_1\,(5)^{n-1}\cdot e^{5x}\cdot 3x^2 + nC_2\,(5)^{n-2}\cdot e^{5x}\cdot 6x + nC_3\,(5)^{n-3}\cdot e^{5x}\cdot 6$$

Example 2

Find the n^{th} differential coefficient of $x^2 \cos 2x$.

Solution

Take $u = \cos 2x,\quad v = x^2$

$$D^n [x^2 \cos 2x] = D^n (\cos 2x)\, x^2 + nC_1\, D^{n-1} (\cos 2x)\; 2x + nC_2\, D^{n-2} (\cos 2x) \cdot 2$$

$$= 2^n \cos\left(2x + \frac{n\pi}{2}\right) \cdot x^2 + n\, 2^{n-1} \cos\left[2x + \frac{(n-1)\pi}{2}\right] \cdot 2x + \frac{n(n-1)}{2} \cdot 2^{n-2} \cos\left[2x + \frac{(n-2)\pi}{2}\right] \cdot 2.$$

Example 3

If $y = a \cos(\log x) + b \sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$ and $x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$

Solution

Given $\quad y = a \cos(\log x) + b \sin(\log x)$

Differentiating w.r.t. x, we get

$$y_1 = -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x}$$

i.e., $\quad xy_1 = -a \sin(\log x) + b \cos(\log x)$

Differentiating $\quad xy_2 + y_1 = -a \cos(\log x) \cdot \frac{1}{x} - b \sin(\log x) \cdot \frac{1}{x}$

$$x^2 y_2 + xy_1 = -[a \cos(\log x) + b \sin(\log x)] = -y$$

i.e., $\quad x^2 y_2 + xy_1 + y = 0 \quad \ldots (1)$

Differentiating (1) w.r.t. x, n times, we get

$$D^n (x^2 y_2) + D^n (xy_1) + D^n (y) = 0$$

$$\left.\begin{array}{r} D^n (y_2) \cdot x^2 + nC_1\, D^{n-1} (y_2) \cdot 2x + \\ nC_2\, D^{n-2} (y_2) \cdot 2 + D^n (y_1) \cdot x + \\ nC_1\, D^{n-1} (y_1) \cdot 1 + D^n (y) \end{array}\right\} = 0$$

$$\left.\begin{array}{r} y_{n+2}\, x^2 + 2nx\, y_{n+1} + \frac{n(n-1)}{2}\, y_n \cdot 2 + \\ x\, y_{n+1} + n\, y_n + y_n \end{array}\right\} = 0$$

$$x^2 y_{n+2} + (2n+1)\, x\, y_{n+1} + n^2 y_n - n\, y_n + n\, y_n + y_n = 0$$

$$x^2 y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2+1)\, y_n = 0$$

Example 4

If $y = \sin^{-1} x$, prove that $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0.$

Solution

Given $\quad y = \sin^{-1} x$

Diff. w.r.to x, $\quad y_1 = \dfrac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\; y_1 = 1$

Squaring both sides $(1-x^2)\, y_1^{\,2} = 1$

Diff. again w.r.to x,

$$(1-x^2)\, 2\, y_1\, y_2 - 2x\, y_1^{\,2} = 0$$

Divide both sides by $2\, y_1$, we get

$$(1-x^2)\, y_2 - xy_1 = 0$$

Differentiating it n times by Leibnitz's rule,

$$\left[(1-x^2)\, y_{n+2} + n\,(-2x)\, y_{n+1} + \frac{n\,(n-1)}{2!}(-2)\, y_n \right] - [x\, y_{n+1} + n\,(1)\, y_n] = 0$$

$$(1-x^2)\, y_{n+2} - 2n\, xy_{n+1} - (n^2-n)\, y_n - xy_{n+1} - n\, y_n = 0$$

$$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2-n+n)\, y_n = 0$$

i.e., $$(1-x^2)\; y_{n+2} - (2n+1)\, xy_{n+1} - n^2\, y_n = 0$$

Example 5

If sin (m sin^{-1} x), show that
$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} + (m^2-n^2)\, y_n = 0.$

Solution

Given $\quad y = \sin(m \sin^{-1} x)$

Diff. w.r. to x, $\quad y_1 = \cos(m \sin^{-1} x)\, \dfrac{m}{\sqrt{1-x^2}}$

Cross multiply, $\sqrt{(1-x^2)}\; y_1 = m \cos(m \sin^{-1} x)$

Squaring both sides

$$(1-x^2)\, y_1^{\,2} = m^2 \cos^2(m \sin^{-1} x)$$

$$= m\, [1 - \sin^2(m \sin^{-1} x)]$$

$$(1-x^2)\, y_1^{\,2} = m^2\, (1-y^2)$$

Diff. again w.r. to x

$$(1-x^2)\, 2y_1\, y_2 - 2x\, y_1^{\,2} = m^2\, (-2yy_1)$$

Divide throughout by $2y_1$, we have

$$(1-x^2)\, y_2 - xy_1 = -m^2\, y$$

i.e., $$(1-x^2)\, y_2 - xy_1 + m^2\, y = 0 \qquad \ldots (2)$$

Differentiating (2) n times by Leibnitz's theorem, we have

$$[(1-x^2)\,y_{n+2} + nC_1\,(-2x)\,y_{n+1} + nC_2\,(-2)\,y_n] - [\,xy_{n+1} + nC_1\,(1)\,y_n\,] + m^2 y_n = 0$$

$$\therefore [(1-x^2)\,y_{n+2} - 2n\,xy_{n+1} - \frac{2n\,(n-1)}{2}\,y_n\,] - [xy_{n+1} + n\,y_n] + m^2 y_n = 0$$

$$(1-x^2)\,y_{n+2} - 2n\,xy_{n+1} - (n^2-n)\,y_n - xy_{n+1} - n\,y_n + m^2 y_n = 0$$

i.e.,

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-n+n-m^2)\,y_n = 0$$

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-m^2)\,y_n = 0$$

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2-n^2)\,y_n = 0$$

Example 6 *If $y = \cos\,(m\cos^{-1} x)$, show that $(1-x^2)\,y_{n-2} - (2n+1)\,xy_{n+1} + (m^2-n^2)\,y_n = 0$.*

Solution

Given $\qquad y = \cos\,(m\cos^{-1} x)$

Diff. w.r.to 'x' $\qquad \therefore y_1 = -\sin\,(m\cos^{-1} x)\,\dfrac{-m}{\sqrt{1-x^2}}$

$$\sqrt{1-x^2}\cdot y_1 = m\sin\,(m\cos^{-1} x)$$

$$(1-x^2)\cdot y_1{}^2 = m\sin\,(m\cos^{-1} x)$$

$$(1-x^2)\cdot y_1{}^2 = m^2\sin^2(m\cos^{-1} x)$$

$$= m^2\,[1-\cos^2\,(m\cos^{-1} x)]$$

$$(1-x^2)\cdot y_1{}^2 = m^2\,(1-y^2)$$

Diff. again w.r. to x

$$(1-x^2)\,2\,y_1\,y_2 - 2\,xy_1{}^2 = -m^2\,2\,yy_1$$

Dividing both sides by $2\,y_1$

$$(1-x^2)\,y_2 - xy_1 + m^2 y = 0$$

Differentiating it n times by Leibnitz rule

$$[(1-x^2)\,y_{n+2} + n\,(-2x)\,y_{n+1} + \frac{n\,(n-1)}{2}\,(-2)\,y_n\,] - (xy_{n+1} + n\cdot 1\cdot y_n) + m^2 y_n = 0$$

i.e., $\quad (1-x^2)\,y_{n+2} - 2n\,xy_{n+1} + (n^2-n)\,y_n - xy_{n+1} - n\,y_n + m^2 y_n = 0$

i.e., $\quad (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-n+n-m^2)\,y_n = 0$

i.e., $\quad (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-m^2)\,y_n = 0$

i.e., $\quad (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-m^2)\,y_n = 0$

Example 7 *If $y = \sin h\,(m \sinh^{-1} x)$ prove that $(1+x^2)\, y_{n+2} + (2n-1)\, xy_{n+1} - (n^2 - m^2)\, y_n = 0.$*

Solution

Given $y = \sin h\,(m \sin h^{-1} x)$

Diff. w.r.to x, $y_1 = \cos h\,(m \sin h^{-1} x)\, \dfrac{m}{\sqrt{1+x^2}}$

$$\therefore \sqrt{1+x^2}\cdot y_1 = m \cos h\,(m \sin h^{-1} x)$$

$$(1+x^2)\, {y_1}^2 = m^2 \cos h^2\,(m \sin h^{-1} x)$$

$$= m^2\,[1 + \sin h^2\,(m \sin h^{-1} x]$$

$$(1+x^2)\, {y_1}^2 = m^2\,(1+y^2)$$

Diff. again w.r.to x,

$$(1+x^2)\, 2y_1\, y_2 + 2x\, {y_1}^2 = m^2\, 2y\, y_1$$

Divide both sides by $2y_1$

$$(1+x^2)\, y_2 + xy_1 = m^2 y$$

$$(1+x^2)\, y_2 + xy_1 - m^2 y = 0 \qquad \ldots (1)$$

Diff. (1) n times using Leibnitz's theorem,

$$\left[(1+x^2)\, y_{n+2} + n\cdot 2x\, y_{n+1} + \frac{n\,(n-1)}{2!}\, 2\, y_n\right] + (xy_{n+1} + n\cdot 1\, y_n) - m^2\, y_n = 0$$

$$(1+x^2)\, y_{n+2} + 2n\, xy_{n+1} + (n^2 - n)\, y_n + xy_{n+1} + n\, y_n - m^2\, y_n = 0$$

$$(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - n + n - m^2)\, y_n = 0$$

$$(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$$

Example 8 *If $y = (x + \sqrt{1+x^2}\,)^m$ show that $(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$*

Solution

Given $y = (x + \sqrt{1+x^2}\,)^m$

Diff. w.r.to 'x' $\therefore y_1 = m\,(x + \sqrt{1+x^2}\,)^{m-1}\cdot\left(1 + \dfrac{1}{2\sqrt{1+x^2}}\cdot 2x\right)$

$$= m\,(x + \sqrt{1+x^2}\,)^{m-1}\cdot\left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$= m\,(x + \sqrt{1+x^2}\,)^{m-1}\cdot\left(\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \sqrt{1+x^2}\; y_1 = m\,(x + \sqrt{1+x^2}\,)^m = my$$

Squaring both sides $(1+x^2)\,y_1^2 = m^2 y^2$

Diff. again w.r. to x

$$(1+x^2)\,2\,y_1 y_2 + 2\,x y_1^2 = 2\,m^2 y\,y_1$$

Dividing through out by $2\,y_1$

$$(1+x^2)\,y_2 + x y_1 - m^2 y = 0 \quad \ldots (1)$$

Diff. (1) n times using Leibnitz rule, we have

$$\left[(1+x^2)\,y_{n+2} + n\,(2x)\,y_{n+1} + \frac{n\,(n-1)}{2!}\,(2)\,y_n\right]$$

$$+\,(x y_{n+1} + n\cdot 1\cdot y_n) - m^2 y_n = 0$$

i.e., $(1+x^2)\,y_{n+2} + 2n\,x y_{n+1} + (n^2-n)\,y_n + x y_{n+1} + n\,y_n - m^2 y_n = 0$

i.e., $(1+x^2)\,y_{n+2} + (2n+1)\,x y_{n+1} + (n^2 - n + n - m^2)\,y_n = 0$

i.e., $(1+x^2)\,y_{n+2} + (2n+1)\,x y_{n+1} + (n^2 - m^2)\,y_n = 0$

which is the required result.

Exercises

Find the n^{th} differential coefficient of the following :

1. $x^2 e^{ax}$ [*Ans* : $a^{n-2} e^{ax}\,[a^2 x^2 + 2nax + n\,(n-1)\}]$

2. $x^3 \cos x$ [*Ans* : $x\,\{x^2 - 3n\,(n-1)\}\cos\left(x + \frac{n\pi}{2}\right) +$ $n\,\{3x^2 - (n-1)\,(n-2)\}\sin\left(x + \frac{n\pi}{2}\right)$]

3. $x^3 \log x$ [*Ans* : $(-1)^n\,(n-4)\,!\,6\,x^{-n+3}$]

4. If $y = a\cos(\log x) + b\sin(\log x)$ prove that $x^2 y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2+1)\,y_n = 0.$

5. If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - (n^2+a^2)\,y_n = 0.$

6. If $\cos^{-1}\left(\frac{y}{n}\right) = n\log\left(\frac{x}{b}\right)$ prove that $x^2 y_{n+2} + (2n+1)\,x\,y_{n+1} + 2\,n^2 y_n = 0.$

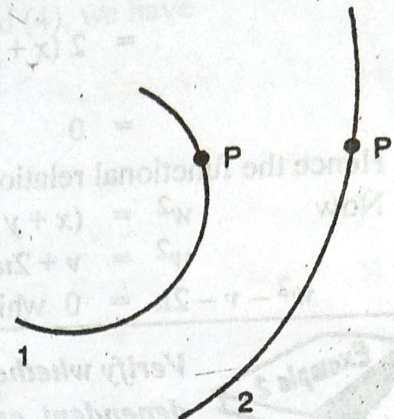
7. If $y = (\sin^{-1} x)^2$ prove that $(1-x^2)\,y_{n+2} - (2n+1)\,x y_{n+1} - n^2 y_n = 0.$

8. If $y^{1/m} + y^{-1/m} = 2x$ show that $(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0.$

9. If $y = (x^2-1)^n$ prove that $(x^2-1)\,y_{n+2} + 2x\,y_{n+1} - n(n+1)\,y_n = 0.$

CURVATURE

In figure, we have two curves. Of these two curves, one bends more sharply than the other. In otherwords, one has a greater **curvature** than the other. Let P be a given point on the curve. The part of the curve in a neighbourhood of P be regarded roughly as an arc of a circle. Here we notice from the figure that the radius of such a circle would be small when the curvature is a great, and vice versa.



Definitions : Let P be a given point on a given curve, and Q any other point on it. Let the normals at P and Q intersect in N. If N tends to a definite position C as Q tends to P, then C is called the **centre of curvature** of the curve at P.

The reciprocal of the distance CP is called the **curvature** of the curve at P.

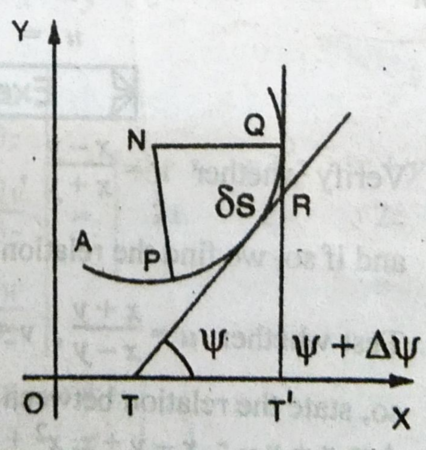
The circle with its centre at C and radius CP is called the **circle of curvature** of the curve at P.

The distance CP is called the **radius of curvature** of the curve at P. The radius of curvature is usually denoted by the Greek letter ρ.

☐ FORMULA FOR THE RADIUS OF CURVATURE

Let $y = f(x)$ be the given curve. Let P be a given point on the curve APQ, and let Q be any other point. Let the length of the curve AP be s and PQ be a small length δs such that the length AQ $= s + \delta s$.

Let the tangent PT at P to the curve make with X-axis the angle ψ, and let the tangent QT′ at Q make with the X-axis the angle $\psi + \delta\psi$. Let these tangents meet at R.



Let the normals at P and Q be PN and QN respectively, N being their point of intersection. Join PQ.

The radius of curvature at P $= \rho = \lim_{\delta s \to 0}$ PN.

From the triangle PNQ,

$$\frac{PN}{\sin \angle NQP} = \frac{\text{chord } PQ}{\sin \angle PNQ} \quad \left(\text{using } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$$

i.e.,
$$\frac{PN}{\text{chord } PQ} = \frac{\sin \angle NQP}{\sin \angle PNQ} = \frac{\sin \angle NQP}{\sin \angle TRT'}$$
$$= \frac{\sin \angle NQP}{\sin \delta\psi}$$

i.e.,
$$PN = \frac{\text{chord } PQ \times \sin \angle NQP}{\sin \delta\psi}$$

As $\delta s \to 0, \delta\psi \to 0$

$$\therefore\ \rho = \lim_{\delta s \to 0} PN = \lim_{\delta\psi \to 0} PN$$
$$= \lim_{\delta\psi \to 0} \frac{\text{chord } PQ \times \sin \angle NQP}{\sin \delta\psi}$$
$$= \lim_{\delta\psi \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \sin \angle NQP$$

As $\delta\psi \to 0, \dfrac{\text{chord } PQ}{\delta s} \to 1, \dfrac{\delta\psi}{\sin \delta\psi} \to 1$ and

$$\angle NQP = \frac{\pi}{2}$$

Since $\angle NQP = \dfrac{\pi}{2} - \angle PQT'$ and $\angle PQT' \to 0$ and $\dfrac{\delta s}{\delta\psi} \to \dfrac{ds}{d\psi}$

Hence
$$\rho = \frac{\delta s}{\delta\psi}$$

Note 1 : The relation between s and ψ for any curve is called the **intrinsic equation** of the curve.

Note 2 : The circle whose centre is at N and radius PN has the same tangent and the same curvature as the curve has at P. Hence this circle is called the **circle of curvature** at P.

$\boxed{\text{Example 1}}$ ***Find ρ for the catenary whose intrinsic equation is $s = a \tan \psi$.***

Solution :
$$\rho = \frac{ds}{d\psi} = a \sec^2 \psi.$$

$\boxed{\text{Example 2}}$ ***Find ρ for the cycloid $s = 4a \sin \psi$.***

Solution :
$$\rho = \frac{ds}{d\psi} = 4a \cos \psi.$$

❑ CARTESIAN FORMULA FOR RADIUS OF CURVATURE

We know that $\frac{dy}{dx} = \tan\psi$

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{ds}(\tan\psi)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds} = \frac{d(\tan\psi)}{d\psi}\cdot\frac{d\psi}{ds}$$

$$\frac{d^2y}{dx^2}\cdot\frac{dx}{ds} = \sec^2\psi\cdot\frac{d\psi}{ds} = \sec^2\psi\cdot\frac{1}{\rho} \qquad (\because \rho = \frac{ds}{d\psi})$$

$$\therefore \rho = \frac{\sec^2\psi}{\frac{d^2y}{dx^2}\cdot\frac{dx}{ds}}$$

$$= \frac{\sec^2\psi}{\frac{d^2y}{dx^2}\cdot\cos\psi} \qquad (\because \cos\psi = \frac{dx}{ds})$$

$$= \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = \frac{(\sec\psi)^3}{\frac{d^2y}{dx^2}} = \frac{\left(\sqrt{1+\tan^2\psi}\right)^3}{\frac{d^2y}{dx^2}}$$

$$= \frac{(1+\tan^2\psi)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Note 1 : We consider the sign of ρ is always positive.

Note 2 : The definition of the radius of curvature shows that its value depends only on the curve and not on the axes. Therefore by interchanging the axes of x and y, we obtain

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

This formula is useful when the tangent is parallel to the Y-axis, *i.e.*, when $\frac{dy}{dx} = \infty$.

Example 1 *Find the radius of curvature for the curve* $\sqrt{x}+\sqrt{y}=1$ *at* $\left(\frac{1}{4},\frac{1}{4}\right)$.

Solution

Given $$\sqrt{x}+\sqrt{y} = 1$$

Differentiating, we get,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} = 0$$

i.e., $$\frac{dy}{dx} = -\sqrt{\frac{y}{x}} \qquad \ldots (1)$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = -\sqrt{\frac{\frac{1}{4}}{\frac{1}{4}}} = -1 \qquad \ldots (2)$$

Differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{x}\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}+\sqrt{y}\frac{1}{2\sqrt{x}}}{x}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = \frac{\frac{\sqrt{\frac{1}{4}}}{2\sqrt{\frac{1}{4}}}-\frac{\sqrt{\frac{1}{4}}}{2\sqrt{\frac{1}{4}}}\left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)}}{\frac{1}{4}}$$

$$= \frac{\frac{1}{2}-\frac{1}{2}(-1)}{\frac{1}{4}} = 4 \qquad \ldots (3)$$

We know that $$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \qquad \ldots (4)$$

Substituting (2) and (3) in (4), we get,

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{4} = \frac{2^{\frac{3}{2}}}{4} = \frac{2^1\cdot2^{\frac{1}{2}}}{4}$$

$$= \frac{2^{\frac{1}{2}}}{2} = \frac{1}{\sqrt{2}}$$

Example 2

Find the radius of curvature at (x, y) for the curve $a^2y = x^3 - a^3$.

Solution

Given $$a^2y = x^3 - a^3$$

Differentiating w.r.t. x, we get

$$a^2\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{a^2}$$

$$\frac{d^2y}{dx^2} = \frac{6x}{a^2}$$

$$\text{Radius of curvature } \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+\frac{9x^4}{a^4}\right\}^{\frac{3}{2}}}{\frac{6x}{a^2}}$$

$$= \frac{(a^4+9x^4)^{\frac{3}{2}}}{a^6} \infty \frac{a^2}{6x}$$

$$= \frac{(a^4+9x^4)^{\frac{3}{2}}}{6a^4x}$$

Example 3

Prove that at the point $x = \frac{\pi}{2}$ of the curve $y = 4\sin x - \sin 2x$, $\rho = \frac{5\sqrt{5}}{4}$.

Solution

Given $$y = 4\sin x - \sin 2x$$

$$\frac{dy}{dx} = 4\cos x - 2\cos 2x$$

$$\frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = 4\cos\frac{\pi}{2} - 2\cos 2\frac{\pi}{2}$$

$$= 0 - 2(-1) = 2 \qquad [\because \cos\pi = -1]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{2}} = -4\sin\frac{\pi}{2} + 4\sin\pi = -4$$

$$\therefore\ \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+4)^{\frac{3}{2}}}{-4}$$

$$= \frac{5^{\frac{3}{2}}}{-4} = \frac{5^1 \cdot 5^{\frac{1}{2}}}{-4}$$

Hence $\rho = \frac{5\sqrt{5}}{4}$ (omitting –ve sign)

Example 4

Find the radius of curvature for the curve $y^2 = x^3 + 8$ at $(-2, 0)$.

Solution

Given $\qquad y^2 = x^3 + 8$

Differentiating we get, $\quad 2y\frac{dy}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{3x^2}{2y} \qquad \text{... (1)}$$

$$\left(\frac{dy}{dx}\right)_{(-2,\,0)} = \frac{3\times 4}{0} = \infty$$

Here we get the value of $\frac{dy}{dx}$ at $(-2, 0)$ is ∞.

Therefore to find ρ we use the formula

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} \qquad \text{... (2)}$$

Form (1) we get, $\quad \frac{dx}{dy} = \left(\frac{2y}{3x^2}\right) \qquad \text{... (3)}$

$$\left(\frac{dx}{dy}\right)_{(-2,\,0)} = 0 \qquad \text{... (4)}$$

Differentiating (3), w.r.t. y, we get,

$$\frac{d^2x}{dy^2} = \frac{3x^2 \cdot 2 - 2y \cdot 6x \cdot \frac{dx}{dy}}{9x^4}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(-2,\,0)} = \frac{24-0}{9\times 16} \qquad \left[\because \left(\frac{dx}{dy}\right)_{(-2,\,0)} = 0\right]$$

$$= \frac{1}{6} \qquad \ldots (5)$$

Substituting (4) and (5) in (2) we get,

$$\rho = \frac{(1+0)^{\frac{3}{2}}}{\frac{1}{6}} = 6$$

Example 5 *Find ρ for the curve $y = c\cos h\frac{x}{c}$ at the point $(0, c)$.* [April '93]

Solution

Given $\qquad y = c\cos h\left(\frac{x}{c}\right)$

$$\frac{dy}{dx} = c\cdot\sin h\left(\frac{x}{c}\right)\cdot\frac{1}{c} = \sin h\left(\frac{x}{c}\right)$$

$$\left(\frac{dy}{dx}\right)_{(0,\,c)} = \sin h\left(\frac{0}{c}\right)$$

$$= \sin h0 = 0 \qquad (\because \sin h0 = 0)$$

$$\frac{d^2y}{dx^2} = \frac{1}{c}\cos h\left(\frac{x}{c}\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{(0,\,c)} = \frac{1}{c}\cos h\,(0) = \frac{1}{c} \qquad (\because \cos h0 = 1)$$

We know that $\qquad \rho = \dfrac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \dfrac{(1+0)^{\frac{3}{2}}}{\frac{1}{c}}$

$$= c$$

Example 6 *Find the radius of curvature of the curve $xy^2 = a^3 - x^3$ at $(a, 0)$.* [April, '93]

Solution

Given $\qquad xy^2 = a^3 - x^3 \qquad \ldots (1)$

Differentiating (1) w.r.t. x, we get,

$$x\cdot 2y\frac{dy}{dx} + y^2 = -3x^2$$

$$\frac{dy}{dx} = \frac{-(3x^2+y^2)}{2xy} \quad \text{... (2)}$$

$$\left(\frac{dy}{dx}\right)_{(a,\,0)} = \frac{-(3a^2+0)}{0} = \infty$$

Here $\frac{dy}{dx}$ at $(a, 0)$ is ∞.

Therefore we use the formula $\rho = \dfrac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$ ...(3)

From (2) we get , $\left(\frac{dx}{dy}\right)_{(a,\,0)} = 0$ ... (4)

$$\frac{dx}{dy} = \frac{-2xy}{3x^2+y^2}$$

$$\left(\frac{d^2x}{dy^2}\right) = \frac{-2(3x^2+y^2)\left(x+y\frac{dx}{dy}\right)+2xy\left(6x\frac{dx}{dy}+2y\right)}{(3x^2+y^2)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,\,0)} = \frac{-2(3a^2)(a)+0}{(3a^2+0)^2}$$

$$= \frac{-6a^3}{9a^4} = \frac{-2}{3a} \quad \text{... (5)}$$

Substituting (4) and (5) in (3) we get

$$\rho = \frac{(1+0)^{\frac{3}{2}}}{\frac{-2}{3a}} = \frac{-3a}{2}$$

Hence $\rho = \frac{3a}{2}$ (omitting negative sign)

Example 7 *Find ρ for the curve $x^3+y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.*

Solution

Given $\quad x^3+y^3 = 3axy$

Differentiating w.r.t. x, we get

$$3x^2+3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx}+y\right)$$

$$\frac{dy}{dx}(3y^2-3ax) = 3ay-3x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \quad \ldots (1)$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{a \cdot \frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \cdot \frac{3a}{2}} = \frac{-3a^2}{3a^2} = -1 \quad \ldots (2)$$

Differentiating (1) w.r.t. x, we get,

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left(a\frac{dy}{dx} - 2x\right) - (ay-x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2-ax)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} =$$

$$\frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\left[a(-1) - \frac{6a}{2}\right] - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)\left[\frac{6a}{2}(-1) - a\right]}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\left(\frac{3a^2}{4}\right)\left(\frac{-8a}{2}\right) - \left(\frac{-3a^2}{4}\right)\left(\frac{-8a}{2}\right)}{\frac{9a^4}{16}}$$

$$= (-3a^3 - 3a^3) \times \frac{16}{9a^4}$$

$$= \frac{-32}{3a} \quad \ldots (3)$$

We know that $$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= \frac{[1 + (-1)^2]^{\frac{3}{2}}}{\frac{-32}{3a}} = \frac{-2^{\frac{3}{2}} \times 3a}{32}$$

$$= \frac{-2^1 \cdot 2^{\frac{1}{2}} \cdot 3a}{32} = \frac{3a\sqrt{2}}{16}$$

Hence $$\rho = \frac{3a\sqrt{2}}{16}$$ (Omitting negative sign)

Exercises

1. Find the radius of curvature of the following curves.

(a) $x^4 + y^4 = 2$ at (1, 1). [Ans. $\frac{\sqrt{2}}{3}$]

(b) $y = e^x$ at the point where it crosses the Y axis. [Ans. $2\sqrt{2}$]

(c) $xy = 30$ at (3, 10). [Ans. $\frac{(109)^{\frac{3}{2}}}{60}$]

(d) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at any point (x, y). [Ans. $3a^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}$]

(e) $x^2y = a(x^2 + y^2)$ at $(-2a, 2a)$. [Ans. $-2a$]

(f) $y^2 = \frac{x(x-2)}{x-5}$ at $x = 2$. [Ans. $\frac{1}{3}$]

(g) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the vertex. [Ans. $\frac{b^2}{a}$]

(h) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$. [Ans. $\rho \propto a$]

(i) $xy^3 = a^4$ at (a, a). [Ans. $\frac{10^{\frac{3}{2}} a}{12}$]

(j) $y^2 = \frac{x^2(x+4)}{x+1}$ at $y = 0$. [Ans. $\frac{8}{3}$]

2. Show that the radius of curvature at any point of the curve $x^m + y^m = 1$ is $\frac{\{x^{2(m-1)} + y^{2(m-1)}\}^{\frac{3}{2}}}{(m-1)\,x^{m-2} \cdot y^{m-2}}$.

3. Show that the curves $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$ and $y = \frac{a}{2}\left(2 + \frac{x^2}{a^2}\right)$ have the same curvature at the point $(0, a)$.

4. Show that at any point on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, the radius of curvature is twice the normal at that point.

5. Show that at any point P on the rectangular hyperbola $xy = c^2$, $\rho = \frac{r^3}{2c^2}$ where r is the distance of P from the origin.

❑ TOTAL DIFFERENTIAL COEFFICIENT

If $u = f(x, y)$ is a function of x and y where $x = f(t)$ and $y = g(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \qquad \dots (1)$$

In the differential form, (1) can be written as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

du is called the total differential of u.

Note: The above result can be extended to functions of several variables. Suppose $u = f(x_1, x_2, \dots\dots, x_n)$ where $x_1, x_2, \dots\dots, x_n$ are all functions of 't' then

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dt} + \dots\dots + \frac{\partial u}{\partial x_n} \frac{dx_n}{dt}$$

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \dots\dots + \frac{\partial u}{\partial x_n} dx_n$$

Note: If $u = f(x_1, x_2, \dots\dots, x_n)$ where $x_1, x_2, \dots\dots, x_n$ are all functions of several independent variables $t_1, t_2, \dots\dots\dots, t_n$ then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt_1} + \frac{\partial u}{\partial x_2} \cdot \frac{dx_2}{dt_1} + \dots\dots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_2} + \dots\dots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_2}$$

Example 1 Find $\frac{du}{dt}$ if $u = x^3 y^4$ where $x = t^3$ and $y = t^2$.

Solution

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

$$= 3x^2 y^4 \cdot 3t^2 + 4x^3 y^3 \cdot 2t$$

$$= 3t^6 t^8 3t^2 + 4t^9 t^6 2t$$

$$= 9t^{16} + 8t^{16} = 17t^{16}$$

Example 2 If $u = x^2 y^3$, $x = \log t$, $y = e^t$ find $\frac{du}{dt}$.

Solution

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

$$= 2xy^3 \cdot \frac{1}{t} + 3x^2 y^2 \cdot e^t$$

$$= 2 \log t \cdot e^{3t} \cdot \frac{1}{t} + 3(\log t)^2 e^{2t} \cdot e^t$$

$$= \frac{2 \log t \, e^{3t}}{t} + 3(\log t)^2 e^{3t}$$

Example 3 If $u = xy + yz + zx$ where $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$, find $\frac{du}{dt}$.

Solution

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

$$= -\frac{(y+z)}{t^2} + (x+z)e^t - (x+y)e^{-t}$$

Example 4 Find the total differential of the function $u = \tan(3x - y) + 6^{y+z}$ of three variable x, y, z.

Solution

Given $u = \tan(3x - y) + 6^{y+z}$

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z}\, dz$$

$$= 3\sec^2(3x-y)\,dx$$

$$+ [-\sec^2(3x-y) + 6^{y+z}(\log 6)]\,dy + 6^{y+z}\log 6\,dz$$

Example 5

If $u = e^x \sin y$ where $x = st^2$ and $y = s^2t$ find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

Solution

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial s}$$

$$= e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial t}$$

$$= e^x \sin y \cdot 2st + e^x \cos y\, s^2$$

DIFFERENTIATION FROM IMPLICIT FUNCTION

Let $u = f(x, y) = c$ be a given implicit function of x and y, we know that

$$du = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$$

Since $u = c$ is a constant $du = 0$.

$$\therefore \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

i.e.,

$$\frac{dy}{dx} = -\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)$$

Example 1

Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3axy$.

Solution

Given $x^3 + y^3 - 3axy = 0$

$$\text{Let } f(x, y) = x^3 + y^3 - 3axy$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\therefore \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{(3x^2 - 3ay)}{(3y^2 - 3ax)}$$

$$= \frac{ay - x^2}{y^2 - ax}$$

Example 2

Find $\frac{dy}{dx}$ if $3x^2 + xy - y^2 + 4x - 2y + 1 = 0$.

Solution

Let

$$f(x, y) = 3x^2 + xy - y^2 + 4x - 2y + 1$$

$$\frac{\partial f}{\partial x} = 6x + y + 4$$

$$\frac{\partial f}{\partial y} = x - 2y - 2$$

$$\therefore \frac{dy}{dx} = -\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right) = -\left(\frac{6x + y + 4}{x - 2y - 2}\right)$$

Example 3

Find $\frac{du}{dx}$ if $u = \cos(x^2 + y^2)$ where $a^2 x^2 + b^2 y^2 = c^2$.

Solution

We know that $\quad du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$

i.e., $\quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad$ ... (1)

Now $\quad \frac{\partial u}{\partial x} = -\sin(x^2 + y^2) \cdot 2x \quad$ ... (2)

$\quad \frac{\partial u}{\partial y} = -\sin(x^2 + y^2)\, 2y \quad$ ... (3)

Given $\quad a^2 x^2 + b^2 y^2 = c^2$

Differentiating w.r.t. 'x', we get

$$2a^2x + 2b^2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-a^2x}{b^2y} \quad \text{... (4)}$$

Substituting (2), (3) and (4) in (1), we get

$$\frac{du}{dx} = -\sin(x^2+y^2)\cdot 2x - \sin(x^2+y^2)\, 2y\left(\frac{-a^2x}{b^2y}\right)$$

$$= 2\sin(x^2+y^2)\left[\frac{a^2x}{b^2}\sin(x^2+y^2) - x\right]$$

Example 4 *Find* $\frac{du}{dx}$ *if* $u = \tan^{-1}\left(\frac{x}{y}\right)$ *where* $x^2+y^2 = a^2$.

Solution

We know that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}\cdot dy$

$$\text{i.e., } \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \qquad \ldots (1)$$

Given $u = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}$$

$$\frac{\partial u}{\partial x} = \frac{y}{x^2+y^2} \qquad \ldots (2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{-x}{x^2+y^2} \qquad \ldots (3)$$

Given $x^2+y^2 = a^2$

Differentiating w.r.t. 'x', we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y} \qquad \ldots (4)$$

Substituting (2), (3) and (4), we get

$$\frac{du}{dx} = \frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\left(\frac{-x}{y}\right)$$

$$= \frac{y}{x^2+y^2} + \frac{x^2}{y(x^2+y^2)} = \frac{(y^2+x^2)}{(x^2+y^2)\,y}$$

$$\therefore \frac{du}{dx} = \frac{1}{y}$$

Example 5

If $u = f(x-y, y-z, z-x)$ show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Solution

Let
$$\left.\begin{aligned} x-y &= x_1 \\ y-z &= x_2 \\ z-x &= x_3 \end{aligned}\right\} \qquad \ldots (A)$$

$$\begin{aligned} \therefore\ u &= f(x-y, y-z, z-x) \\ &= f(x_1, x_2, x_3) \end{aligned}$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial x}$$

Now from *(A)*, we get

$$\frac{\partial x_1}{\partial x} = 1, \quad \frac{\partial x_2}{\partial x} = 0 \text{ and } \frac{\partial x_3}{\partial x} = -1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_3} \qquad \ldots (1)$$

Similarly
$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial y} \\ &= \frac{\partial u}{\partial x_1}(-1) + \frac{\partial u}{\partial x_2}(1) + \frac{\partial u}{\partial x_3}(0) \end{aligned}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \qquad \ldots (2)$$

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial z} \\ &= \frac{\partial u}{\partial x_1}(0) + \frac{\partial u}{\partial x_2}(-1) + \frac{\partial u}{\partial x_3}(1) \\ &= -\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} \end{aligned} \qquad \ldots (3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

If $z = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$, show that
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\sin\theta \quad \dots (1)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\therefore \frac{1}{r}\cdot\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta \quad \dots (2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Example 7

If $z = f(x, y)$ where $x = e^u + e^{-u}$, $y = e^{-u} - e^v$, prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Solution

Given

$$z = f(x, y) \text{ and } x = e^u + e^{-u},\ y = e^{-u} - e^v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u} \quad \dots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= -\frac{\partial z}{\partial x}(e^{-v}) - \frac{\partial z}{\partial y}(e^v) \quad \dots (2)$$

$$(1) - (2) \Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v)$$

$$= x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y}$$

Exercises

1. Find $\frac{du}{dt}$ from the following

(a) $u = x^3y^2 + x^2y^3$ where $x = at^2, y = 2at$. *Ans* : $8a^5t^6(4t+7)$

(b) $u \sin(xy^2)$ where $x = \log t, y = e^t$ *Ans* : $y^2\left(\frac{1}{t} + 2x\right)\cos(xy^2)$

(c) $u = \log(x + y^2), x = \sqrt{1+t}, y = 1 - \sqrt{t}$

2. Find $\frac{dy}{dx}$ if

(a) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *Ans* : $-\left(\frac{ax + hy + g}{hx + by + f}\right)$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *Ans* : $\frac{b^2x}{a^2y}$

(c) $x^y + y^x = c$ *Ans* : $-\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$

3. Find $\frac{du}{dx}$ if

(i) $u = x\log(xy)$, where $x^3 + y^3 + 3xy = 1$

Ans : $1 + \log(xy) - \frac{x(x^2+y)}{y(x+y^2)}$

(ii) $u = x^2y$ where $x^2 + xy + y^2 = 1$ *Ans* : $\frac{x(xy + 4y^2 - 2x^2)}{x + 2y}$

4. If $z = f(u, v)$ where $u = x^2 - 2xy - y^2$, $v = y$ show that

$$(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$$

5. If $V = f(x, y)$ and $x = u^2 + v^2$, $y = 2uv$, show that

$$u\frac{\partial V}{\partial u} - v\frac{\partial V}{\partial v} = 2(x^2 - y^2)^{1/2}\,\frac{\partial V}{\partial x}$$

JACOBIANS

If u and v are functions of the two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the ***Jacobian*** of u, v w.r.t. x, y. It is denoted by

$$\frac{\partial (u, v)}{\partial (x, y)} \text{ [or] } J\left[\frac{uA\ v}{xA\ y}\right]$$

Note: The Jacobian of u, v, w w.r.t. x, y, z is

$$\frac{\partial (uA\ vA\ w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Example 1

If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial (x, y)}{\partial (r, \theta)}$.

Solution

Given

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{dy}{d\theta} = r\cos\theta$$

$$\therefore \frac{\partial (x, y)}{\partial (r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Example 2 *If $x = \cosh\phi \cos\theta$, $y = a \sinh\phi \sin\theta$, show that*

$$\frac{\partial(x, y)}{\partial(\phi, \theta)} = \frac{a^2}{2}\,[\cosh 2\phi - \cos 2\theta]$$

Solution

$$x = a \cos h\,\phi \cos\theta \qquad y = a \sin\ h\,\phi \sin\theta$$

$$\frac{\partial x}{\partial \phi} = a \sin h\,\phi \cos\theta \qquad \frac{\partial y}{\partial \phi} = a \cos h\,\phi \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -a \cos h\,\phi \sin\theta \qquad \frac{\partial y}{\partial \theta} = a \sin h\,\phi \cos\theta$$

$$\frac{\partial(x, y)}{\partial(\phi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a \sin h\,\phi \cos\theta & -a \cos h\,\phi \sin\theta \\ a \cos h\,\phi \sin\theta & a \sin h\,\phi \cos\theta \end{vmatrix}$$

$$= a^2 \sin h^2\,\phi \cos^2\theta + a^2 \cos h^2\,\phi \sin^2\theta$$

$$= a^2\,[\sin h^2\,\phi\,(1 - \sin^2\theta) + (1 + \sin h^2\,\phi) \sin^2\theta]$$

$$= a^2\,[\sin h^2\,\phi - \sin h^2\,\phi \sin^2\theta + \sin^2\theta \pm \sin h^2\,\phi \sin^2\theta]$$

$$= a^2\,[\sin h^2\,\phi + \sin^2\theta]$$

$$= a^2\left[\frac{\cosh 2\phi - 1}{2} + \frac{1 - \text{cso } 2\theta}{2}\right]$$

$$= \frac{a^2}{2}\,[\cosh 2\phi - \cos 2\theta]$$

Example 3 *If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$. Show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.*

Solution

Given: $\quad y_1 = \frac{x_2 x_3}{x_1}, \quad y_2 = \frac{x_3 x_1}{x_2}, \quad y_3 = \frac{x_1 x_2}{x_3}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_3 x_1 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1)$$

$$= 0 + 2 + 2 = 4$$

Example 4

If $x = r \sin\theta \cos\phi$; $y = r \sin\theta \sin\phi$;

$z = r \cos\theta$

show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$.

Solution

Given :

$x = r \sin\theta \cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi,\quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi,\quad \frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\quad \frac{\partial z}{\partial \theta} = -r\sin\theta$

$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,\quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,\quad \frac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2[\sin\theta\cos\phi\,(0+\sin^2\theta\cos\phi) - \cos\theta\cos\phi\,(0-\sin\theta\cos\theta\cos\phi) - \sin\theta\sin\phi\,(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)]$$

$$= r^2[\sin^3\theta\cos^2\phi + \sin\theta\cos^2\theta\cos^2\phi + \sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta\sin^2\phi]$$

$$= r^2[\sin^3\theta(\cos^2\phi+\sin^2\phi) + \sin\theta\cos^2\theta(\cos^2\phi+\sin^2\phi)]$$

$$= r^2[\sin^3\theta + \sin\theta\cos^2\theta] = r^2\sin\theta(\sin^2\theta+\cos^2\theta)$$

$$= r^2\sin\theta$$

❑ PROPERTIES OF JACOBIANS

1. First Property

If u and v are the functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Proof: Let $u = f(x,y)$

$$v = \phi(x,y)$$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

On interchanging the rows and columns of second determinant

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix} \quad \ldots (3)$$

7. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}$.

8. If $u = x + y + z$, $u^2 v = y + z$, $u^3 w = z$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = u^5$.

9. If $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$. [Ans : u]

10. Find the value of $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$, if $y_1 = (1 - x_1)$, $y_2 = x_1(1 - x_2)$, $y_3 = x_1 x_2 (1 - x_3)$.

11. Fill in the blanks:

(i) If $x = r\cos\theta$, $y = r\sin\theta$, then the value of the Jacobian $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ is .... [Ans : r]

(ii) $\dfrac{\partial(u, v)}{\partial(r, s)} \times \dfrac{\partial(r, s)}{\partial(x, y)} =$ .... [Ans : $\dfrac{\partial(u, v)}{\partial(x, y)}$]

(iii) If $u = x(1 - y)$, $v = xy$, then the value of the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)} =$ .... [Ans : x]

(iv) $\dfrac{\partial(x, y)}{\partial(r, \theta)} \cdot \dfrac{\partial(r, \theta)}{\partial(x, y)} =$ .... [Ans : 1]

(v) If $u = x^2$, $v = y^2$, then the value of the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)} =$ .... [Ans : $4xy$]

3. Third Property

If functions u, v, w of three independent variables x, y, z are not independent, then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

Proof: As u, v, w are not independent, then $f(u, v, w) = 0$. ... (1)

Differentiating (1) w.r.t. x, y, z, we get

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x} = 0 \quad \text{... (2)}$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y} = 0 \quad \text{... (3)}$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z} = 0 \quad \text{... (4)}$$

Eliminating $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ from (2), (3) and (4), we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

On interchanging rows and columns, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

or $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = 0.$$

Example 1 *$u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$, determine whether there is a functional relationship between u, v, w and if so, find it.*

Solution

$$u = xy + yz + zx, \quad v = x^2 + y^2 + z^2$$
$$w = x + y + z$$

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad R_1 + R_2$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0 \qquad [R_1 = R_3]$$

Hence the functional relationship exists between, u, v and w.

Now
$$w^2 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$$
$$w^2 = v + 2u$$
$$w^2 - v - 2u = 0 \text{ which is the required relationship.}$$

Example 2 ***Verify whether the following functions are functionally dependent, and if so, find the relation between them.***

Solution

$$u = \frac{x+y}{1-xy}, \quad v = \tan^{-1} x + \tan^{-1} y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

Hence u, v are functionally related.

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

or
$$v = \tan^{-1} u$$
$$u = \tan v$$

Exercises

1. Verify whether $u = \dfrac{x-y}{x+y}$, $v = \dfrac{x+y}{x}$ are functionally dependent, and if so, we find the relation between them. *[Ans : $u = \dfrac{2-v}{v}$]*
2. Test whether $u = \dfrac{x+y}{x-y}$, $v = \dfrac{xy}{(x-y)^2}$ are functionally dependent, if so, state the relation between them. *[Ans : $u^2 - 4v = 1$]*
3. Are $x+y-z$, $x-y+z$, $x^2+y^2+z^2-2yz$ functionally dependent? If so, find a relation between them. *[Ans : $u^2+v^2 = 2w$]*
4. If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = x^3+y^3+z^3-3xyz$, prove that u, v, w are not independent and find the relation between them. *[Ans : $2w = u(3v-u^2)$]*

Integration formulae

1. $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$

2. $\int \frac{1}{x^n} \cdot dx = -\frac{1}{(n-1)x^{n-1}} + c$

3. $\int \frac{1}{x} \cdot dx = \log x + c$

4. $\int e^x \cdot dx = e^x + c$

5. $\int a^x \cdot dx = \frac{a^x}{\log a} + c$

6. $\int \sin x \, dx = -\cos x + c$

7. $\int \cos x \cdot dx = \sin x + c$

8. $\int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

9. $\int \sec^2 x \cdot dx = \tan x + c$

10. $\int \sec x \cdot \tan x \cdot dx = \sec x + c$

11. $\int \operatorname{cosec} x \cot x \cdot dx = -\operatorname{cosec} x + c$

12. $\int \frac{1}{1+x^2} \cdot dx = \tan^{-1} x + c$ (or) $\cot^{-1} x + c$

13. $\int \frac{1}{\sqrt{1-x^2}} \cdot dx = \sin^{-1} x + c$ (or) $\cos^{-1} x + c$

$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x$ $\qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} x/a$

14. $\int (ax+b)^n \cdot dx = \frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right] + c \quad (n \neq -1)$

15. $\int \frac{1}{ax+b} \cdot dx = \frac{1}{a} \log(ax+b) + c$

16. $\int e^{ax+b} \cdot dx = \frac{1}{a} e^{ax+b} + c$

17. $\int \sin(ax+b) \cdot dx = -\frac{1}{a} \cos(ax+b) + c$

18. $\int \cos(ax+b)\, dx = \frac{1}{a} \sin(ax+b) + c$

19. $\int \sec^2(ax+b) \cdot dx = \frac{1}{a} \tan(ax+b) + c$

20. $\int \operatorname{cosec}^2(ax+b) \cdot dx = -\frac{1}{a} \overset{\cot}{\cancel{\operatorname{cosec}}}(ax+b) + c$

21. $\int \operatorname{cosec}(ax+b) \cot(ax+b) \cdot dx = -\frac{1}{a} \operatorname{cosec}(ax+b) + c$

22. $\int \frac{1}{1+(ax)^2} \cdot dx = \frac{1}{a} \tan^{-1}(ax) + c$

23. $\int \frac{1}{\sqrt{1-(ax)^2}} \cdot dx = \frac{1}{a} \sin^{-1}(ax) + c$

Hyperbolic function

$\int \sinh x \cdot dx = \cosh x$

$\int \cosh x \cdot dx = \sinh x$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \log(x+\sqrt{a^2+x^2})$

or $\sinh^{-1}(x/a)$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cos^{-1}(x/a)$

$\log(x+\sqrt{x^2-a^2})$

$\int a f(x) \cdot dx = a \int f(x) \cdot dx$

$\int a f(x) \pm b g(x) \cdot dx = a \int f(x) \cdot dx \pm b \int g(x) \cdot dx$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$

UNIT-IV

Integration

$$\int \sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{x^2-a^2}\cdot dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{x^2+a^2}\cdot dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$$

1. Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$

Sol

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}} \qquad \int \frac{1}{\sqrt{a^2-x^2}}\cdot dx = \sin^{-1}\left(\frac{x}{a}\right)+c$$

$$= \sin^{-1}\left(\frac{x}{3}\right)+c$$

2. Find $\int \frac{dx}{\sqrt{x^2+7}}$

Soln

$$I = \int \frac{dx}{\sqrt{x^2+7}}$$

$$= \int \frac{dx}{\sqrt{x^2+(\sqrt{7})^2}}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{x^2+(\sqrt{7})^2}} = \sinh^{-1}\left(\frac{x}{\sqrt{7}}\right)+c$$

3. $\int \frac{dx}{\sqrt{x^2-25}}$

Soln

$$I = \int \frac{dx}{\sqrt{x^2-25}}$$

$$= \int \frac{dx}{\sqrt{x^2-5^2}}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \int \frac{dx}{\sqrt{x^2-5^2}} = \cosh^{-1}\left(\frac{x}{5}\right) + c$$

4. Obtain $\int \frac{dx}{x^2+16}$

Soln

$$I = \int \frac{dx}{x^2+16}$$

$$= \int \frac{dx}{x^2+4^2}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{x^2+4^2} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$$

5. Evaluate $\int \frac{dx}{x^2-17}$

Soln

Let $I = \int \frac{dx}{x^2-17}$

$$= \int \frac{dx}{x^2-(\sqrt{17})^2}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

$$\therefore \int \frac{dx}{x^2-(\sqrt{17})^2} = \frac{1}{2\sqrt{17}}\log\left(\frac{x-\sqrt{17}}{x+\sqrt{17}}\right) + c$$

6. Find $\int \frac{dx}{\sqrt{81-x^2}}$

Soln

Let $I = \int \frac{dx}{81-x^2}$

$= \int \frac{dx}{9^2-x^2}$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$

$\int \frac{dx}{9^2-x^2} = \frac{1}{18} \log\left(\frac{9+x}{9-x}\right)$

7. Evaluate $\int \frac{dx}{\sqrt{x^2+3x+7}}$

Soln

Let $I = \int \frac{dx}{\sqrt{x^2+3x+7}}$

$= \int \frac{dx}{\sqrt{x^2+2\left(\frac{3}{2}\right)x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+7}}$

$= \int \frac{dx}{\sqrt{(x+3/2)^2+19/4}}$

$= \frac{dx}{\sqrt{(x+3/2)^2+\left(\frac{\sqrt{19}}{2}\right)^2}}$

$= \sinh^{-1}\left(\frac{x+3/2}{\sqrt{19}/2}\right) + C$

$= \sinh^{-1}\left(\frac{2x+3}{\sqrt{19}}\right) + C$

8. Evalute $\int \frac{2x+1}{x^2+3x+1} \cdot dx$

Soln

Let $I = \int \frac{2x+1}{x^2+3x+1}$

$$2x+1 = A\frac{d}{dx}(x^2+3x+1)+B$$

$$2x+1 = A(2x+3)+B$$

compare the coefficient on both sides

$$2 = 2A$$

$$A = 1$$

$$1 = 3A+B$$

$$1-3 = B$$

$$B = -2$$

$$\therefore I = \int \frac{(2x+3)-2}{x^2+3x+1}\,dx$$

$$= \int \frac{2x+3}{x^2+3x+1}\,dx + \int \frac{-2}{x^2+3x+1}\,dx$$

$$= \int \frac{d(x^2+3x+1)}{x^2+3x+1} - 2\int \frac{dx}{x^2+2\left(\frac{3}{2}\right)x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+1}$$

$$= \int \frac{d(x^2+3x+1)}{x^2+3x+1} - 2\int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\frac{5}{4}}$$

$$= \int \frac{d(x^2+3x+1)}{x^2+3x+1} - 2\int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \log(x^2+3x+1) - 2\,\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left(\frac{x+\frac{3}{2}-\frac{\sqrt{5}}{2}}{x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right)+c$$

$$= \log(x^2+3x+1) - \frac{2}{\sqrt{5}}\log\left(\frac{2x+3-\sqrt{5}}{2x+3+\sqrt{5}}\right)+c$$

7.10.2019

8. Evalute $\int \frac{3x+1}{2x^2-x+5} \cdot dx$

Soln

Given $\int \frac{3x+1}{2x^2-x+5} \cdot dx$

Let

$3x+1 = A \frac{d}{dx}(2x^2-x+5) + B$

$3x+1 = A(4x-1)+B$

equate co.efficient on both side

$3 = 4A$

$\boxed{A = 3/4}$

Eqn constant on both side

$1 = -A+B$

$B = 1+A = 1+3/4$

$\boxed{B = 7/4}$

$3x+1 = 3/4(4x-1)+7/4$

$\int \frac{3x+1}{2x^2-x+5} \cdot dx = \int \frac{3/4(4x-1)+7/5}{2x^2-x+5} \cdot dx$

$= \frac{3}{4}\int \frac{4x-1}{2x^2-x+5} \cdot dx + \int \frac{7/5}{2x^2-x+5} \cdot dx$

$= 3/4 \log(2x^2-x+5) + 7/4 \int \frac{dx}{2(x^2-x/2+5/2)}$

$= 3/4 \log(2x^2-x+5) + 7/8 \int \frac{dx}{(x-1/4)^2+39/16}$ $\quad (x-\frac{1}{4})^2-1/16+5/2$

$= 3/4 \log(2x^2-x+5) + 7/8 \int \frac{dx}{(x-1/4)^2+\left(\frac{\sqrt{39}}{4}\right)^2}$

$= 3/4 \log(2x^2-x+5) + 7/8 \left(\frac{1}{2\times\sqrt{39}/4}\right) \tan^{-1}\left(\frac{x-1/4}{\sqrt{39}/4}\right)$

$= 3/4 \log(2x^2-x+5) + \frac{7}{4\sqrt{39}} \tan^{-1}\left(\frac{4x-1}{\frac{\sqrt{39}}{4}}\right)$

9. Evaluate $\int \frac{5x+7}{6x^2+4x-1} \cdot dx$

Soln

Let $5x+7 = A(12x+4)+B$

Equate 'x'

$5 = 12A$

$A = 5/12$

Equate constant coefficient

$7 = 4A+B, \quad B = 7-4A$

$= 7-4\left(5/12\right) = 7 - 5/3$

$B = 16/3$

$5x+1 = \frac{5}{12}(12x+4) + 16/3$

$$\int \frac{5x+1}{6x^2+4x-1} = \int \frac{5/12\,(12x+4)+16/3}{6x^2+4x-1} \cdot dx$$

$$= \frac{5}{12} \int \frac{12x+4}{6x^2+4x-1} \cdot dx + 16/3 \int \frac{dx}{6x^2+4x-1}$$

$$= \frac{5}{12} \log(6x^2+4x-1) + 16/3 \int \frac{dx}{6(x^2+\frac{4x}{6}-1/6)}$$

$$= \frac{5}{12} \log(6x^2+4x-1) + \frac{16}{18} \int \frac{dx}{x^2+\frac{2x}{3}-1/6}$$

$$= \frac{5}{12} \log(6x^2+4x-1) + 8/9 \int \frac{dx}{(x+1/3)^2 - 5/18}$$

$$= \frac{5}{12} \log(6x^2+4x-1) + 8/9 \int \frac{dx}{(x+1/3)^2 - \left(\frac{\sqrt{5}}{18}\right)^2}$$

$$= \frac{5}{12} \log(6x^2+4x-1) + 8/9 \, \frac{1}{\frac{2\times\sqrt{5}}{18}} \log\left[\frac{x+1/3-\frac{\sqrt{5}}{\sqrt{18}}}{x+1/3+\frac{\sqrt{5}}{\sqrt{18}}}\right]$$

10. Evaluate $\int \frac{3x-4}{\sqrt{3x^2+4x+7}} \cdot dx$

Soln

Let $3x-4 = A(6x+4)+B$

Equ co.eff of 'x'

$6A = 3$

$\boxed{A = \frac{1}{2}}$

Eqn constant coeff

$-4 = 4A+B$

$B = -4-4A$

$\boxed{B = -6}$

$3x-4 = \frac{1}{2}(6x+4)-6$

$\therefore \int \frac{3x-4}{\sqrt{3x^2+4x+7}} \cdot dx = \int \frac{\frac{1}{2}(6x+4)-6}{\sqrt{3x^2+4x+7}} \cdot dx$

$= \frac{1}{2}\int \frac{6x+4}{\sqrt{3x^2+4x+7}} \cdot dx - 6\int \frac{dx}{\sqrt{3x^2+4x+7}}$

$= \frac{1}{2} \cdot 2\sqrt{3x^2+4x+7} - 6\int \frac{dx}{\sqrt{3}\sqrt{x^2+\frac{4}{3}x+\frac{7}{3}}}$

$= \sqrt{3x^2+4x+7} - \frac{6}{\sqrt{3}}\int \frac{dx}{\sqrt{(x+\frac{2}{3})^2+\frac{17}{9}}}$

$= \sqrt{3x^2+4x+7} - \frac{6}{\sqrt{3}} \sinh^{-1}\left(\frac{x+\frac{2}{3}}{\sqrt{17}/3}\right)$

$= \sqrt{3x^2+4x+7} - \frac{6}{\sqrt{3}} \sinh^{-1}\left(\frac{3x+2}{\sqrt{17}}\right)$

10.10.19

11 Evaluate $\int \frac{2x+1}{\sqrt{3+4x-x^2}} \cdot dx$

Given

$2x+1 = A \cdot \frac{d}{dx}(-2x+4)+B$

Let

$$2x+1 = A(-2x+4)+B$$

Equate coefficient of x

$$2\cancel{x} = -2A$$

$$\boxed{A=-1}$$

Equ coeff. of constantant

$$1 = 4A+B$$

$$1+4 = B$$

$$\boxed{B=5}$$

$$\therefore 2x+1 = -1(-2x+4)+5$$

$$= 2x-4+5$$

$$\int \frac{2x+1}{\sqrt{3+4x-x^2}}\,dx = \int \frac{-1(-2x+4)+5}{\sqrt{3+4x-x^2}}\,dx$$

$$= -1\int \frac{-2x+4}{\sqrt{3+4x-x^2}} + 5\int \frac{dx}{\sqrt{3+4x-x^2}}$$

$$= -1 \cdot 2\sqrt{3+4x-x^2} + 5\int \frac{dx}{\sqrt{-(x^2-4x-3)}}$$

$$= -2\sqrt{3+4x-x^2} + 5\int \frac{dx}{\sqrt{-(x^2-2(2x)+4-4-3}}$$

$$= -2\sqrt{3+4x-x^2} + 5\int \frac{dx}{\sqrt{-(x-2)^2-(\sqrt{7})^2}}$$

$$= -2\sqrt{3+4x-x^2} + 5\int \frac{dx}{\sqrt{(x-2)^2+(\sqrt{7})^2}}$$

$$= -2\sqrt{3+4x-x^2} + 5\sin h^{-1}\left(\frac{x-2}{\sqrt{7}}\right)$$

$= -2\sqrt{3+4x-x^2} + 5\sin^{-1}\left(\frac{x-2}{\sqrt{7}}\right)$

Evalute $\int \frac{\sqrt{x-1}}{\sqrt{2x+3}} \cdot dx$

Soln

$\int \frac{\sqrt{x-1}}{\sqrt{2x+3}} = \int \frac{x-1}{\sqrt{2x+3} \cdot \sqrt{x-1}} \cdot dx$

$= \int \frac{x-1}{\sqrt{2x+3 \cdot (x-1)}} \cdot dx$

$= \int \frac{x-1}{\sqrt{2x^2+x-3}} \cdot dx$

Put $x-1 = A(4x+1) \cdot dx$

Equate co.eff of x

$1 = 4A$

$\boxed{A = 1/4}$

equate co.eff of const

$-1 = A+B$

$\boxed{B = -5/4}$

$\therefore x-1 = 1/4 (4x+1) - 5/4$

$\int \frac{x-1}{\sqrt{2x^2+x-3}} \cdot dx = \int \frac{1/4 (4x+1) - 5/4}{\sqrt{2x^2+x-3}}$

$= 1/4 \int \frac{4x+1}{\sqrt{2x^2+x-3}} \cdot dx - 5/4 \int \frac{dx}{\sqrt{2x^2+x-3}}$

$= 1/4 \; 2 \cdot \sqrt{2x^2+x-3} - 5/4 \int \frac{dx}{\sqrt{2}\sqrt{x^2+x/2-3/2}}$

$= 1/2 \sqrt{2x^2+x-3} - \frac{5}{4\sqrt{2}} \int \frac{dx}{(x+\frac{1}{4})^2 - \frac{1}{16} - 3/2}$

$\frac{-1-24}{16}$

-25

$= 1/2 \sqrt{2x^2+x-3} - \frac{5}{4\sqrt{2}} \int \frac{dx}{(x+1/4)^2 - (5/4)^2}$

$$= \frac{1}{2}\sqrt{2x^2-x-3} - \frac{5}{4\sqrt{2}} \cosh^{-1}\left(\frac{x+1/4}{5/4}\right)$$

$$= \frac{1}{2}\sqrt{2x^2-x-3} - \frac{5}{4\sqrt{2}} \cosh^{-1}\left(\frac{4x+1}{5}\right)$$

11.10.19

Integrals of types

i) $\int \frac{dx}{a+b\cos x}$

ii) $\int \frac{dx}{a+b\sin x}$

iii) $\int \frac{dx}{a\cos x + b\sin x}$

To evalute the above type of integral

Put $t = \tan\frac{x}{2}$

$$dt = \sec^2\frac{x}{2} \cdot \frac{1}{2} \cdot dx$$

$$dx = \frac{2dt}{\sec^2 x/2}$$

$$= \frac{2dt}{1+\tan^2 x/2}$$

$$dx = \frac{2dt}{1+t^2}$$

w.k.T,

$$\sin x = \frac{2\tan^2\frac{x}{2}}{1+\tan^2 x/2}$$

$$\sin x = \frac{2t^2}{1+t^2}$$

$$\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

$$\therefore \cos x = \frac{1-t^2}{1+t^2}.$$

1. Evalute $\int \frac{dx}{4+5\cos x}$

Soln

Let $I = \int \frac{dx}{4+5\cos x}$

Put $t = \tan x/2$

$$dx = \frac{2dt}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{\frac{2dt}{1+t^2}}{4+5\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{4(1+t^2)+5(1-t^2)}{1+t^2}}$$

$$= \int \frac{2dt}{4+4t^2+5-5t^2}$$

$$= \int \frac{2dt}{9-t^2}$$

$$= \int \frac{2dt}{3^2-t^2}$$

$$= \frac{2}{2\times 3} \log\left(\frac{3+t}{3-t}\right)+c$$

$$I = \frac{1}{3} \log \left(\frac{3+\tan x/2}{3-\tan x/2}\right)+c$$

2. Evaluate $\int \frac{dx}{4+5\sin x}$.

Soln

$$I = \int \frac{dx}{4+5\sin x}.$$

$t = \tan \frac{x}{2}$

$dx = \frac{2dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$$I = \int \frac{\frac{2dt}{1+t^2}}{4+5\left(\frac{2t}{1+t^2}\right)}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{4(1+t^2)+5(2t)}{1+t^2}}$$

$$= \int \frac{2dt}{4+4t^2+10t}$$

$$= \int \frac{2\ \ dt}{4t^2+10t+4}$$

$$= \frac{2}{4} \int \frac{1}{\left(t^2+\frac{10t}{4\times 2}+\frac{4}{4}\right)} \cdot dt$$

$$= \frac{1}{2} \int \frac{1}{\left(t+\frac{5}{4}\right)^2-\frac{25}{16}+1} \cdot dt$$

$$= \frac{1}{2} \int \frac{1}{\left(t+\frac{5}{4}\right)^2-\left(\sqrt{\frac{9}{16}}\right)^2} \cdot dt \quad \frac{9}{16}$$

$$= \frac{1}{2} \int \frac{1}{\left(t+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2} \cdot dt$$

$$= \frac{1}{2} \cdot \frac{1}{2\times\frac{3}{4}} \log\left(\frac{t+\frac{5}{4}-\frac{3}{4}}{t+\frac{5}{4}+\frac{3}{4}}\right)+C$$

$$I = \frac{1}{3} \log\left(\frac{4t+2}{4t+8}\right)+C$$

$$I = \frac{1}{3} \log\left(\frac{4\tan x/2 + 2}{4\tan x/2 + 8}\right) + c$$

3. Evaluate $\int \frac{dx}{3\sin x + 4\cos x}$.

Soln

Given $\int \frac{dx}{3\sin x + 4\cos x}$

$$t = \tan x/2$$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$I = \int \frac{\frac{2dt}{1+t^2}}{3\left(\frac{2dt}{1+t}\right.}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

$$= \int \frac{\frac{2dt}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$$

$$= \int \frac{2dt}{6t+4-4t^2}$$

$$= 2\int \frac{dt}{-4t^2+6t+4}$$

$$= \frac{2}{-4}\int \frac{dt}{t^2 - \frac{6t}{4} - \frac{4}{4}}$$

$$= -\frac{1}{2}\int \frac{dt}{\left(t-\frac{3}{4}\right)^2 - \frac{9}{16} - 1}$$

$$= -\frac{1}{2}\int \frac{dt}{\left(t-3/4\right)^2 - \frac{25}{16}}$$

$$= -\frac{1}{2}\int \frac{dt}{(t-3/4)^2 - \left(\frac{5}{4}\right)^2}$$

$$= -\frac{1}{2}\ \frac{1}{2(5/4)} \log\left(\frac{t-3/4-5/4}{t-3/4+5/4}\right) + c$$

$$= -\frac{1}{5} \log\left(\frac{4t-8}{4t+2}\right) + c$$

$$I = -\frac{1}{5} \log\left(\frac{4\tan x/2 - 8}{4\tan x/2 + 2}\right) + c$$

Integration by the method of Partial fraction

1. Evalute $\int \frac{3x+4}{(x-7)(2x+3)} \cdot dx$ — Denomenator degree is less than numeral

Soln

$$I = \int \frac{3x+4}{(x-7)(2x+3)}$$

$$\frac{3x+4}{(x-7)(2x+3)} = \frac{A}{(x-7)} + \frac{B}{(2x+3)}$$

$$\frac{3x+4}{(x-7)(2x+3)} = \frac{A(2x+3) + B(x-7)}{(x-7)(2x+3)}$$

Put $x = 7$

$$3x+4 = A(2x+3) + B(x-7)$$

$$3x+4 = A(2x+3) + 0$$

$$21+4 = A(14+3)$$

$$25 = 17A$$

$$\boxed{A = \frac{25}{17}}$$

Put $x = -\frac{3}{2}$

$$-\frac{9}{2}+4 = A(0) + B\left(\frac{-3}{2}-7\right)$$

$$-\frac{1}{2} = -\frac{17}{2}B$$

$$\boxed{B = \frac{1}{17}}$$

$$I = \int \frac{\frac{25}{17}}{x-7}\cdot dx + \int \frac{\frac{1}{17}\cdot dx}{2x+3}$$

$$\frac{25}{17}\int \frac{dx}{x-7} + \frac{1}{17}\int \frac{dx}{2x+3}$$

$$I = \frac{25}{17}\log(x-7) + \frac{1}{34}\log(2x+3) + C$$

2. Evalute $\int \frac{x^2+1}{(x^2-1)(2x+1)}\cdot dx$

Soln

$$I = \int \frac{x^2+1}{(x^2-1)(2x+1)}\cdot dx$$

$$\frac{x^2+1}{(x+1)(x-1)(2x+1)} = \frac{A}{(x+1)} + \frac{B}{x-1} + \frac{C}{2x+1}$$

$$x^2+1 = A(x-1)(2x+1) + B(x+1)(2x+1) + C(x+1)(x-1)$$

Put $x = -1$

$$2 = 2A$$

$$\boxed{A = 1}$$

Put $x = 1$

$2 = 6B$

$\boxed{B = \frac{1}{3}}$

Put $x = -\frac{1}{2}$

$\frac{5}{4} = \frac{-3C}{4}$

$\boxed{C = -\frac{5}{3}}$

$$\int \frac{x^2+1}{(x+1)(x-1)(2x+1)} = \int \frac{1}{x+1} \cdot dx + \int \frac{\frac{1}{3}}{x-1} \cdot dx + \int \frac{-\frac{5}{3} dx}{2x+1}$$

$$= \log(x+1) + \frac{1}{3} \log(x-1) - \frac{5}{3} \frac{\log(2x+1)}{2}$$

$$I = \log(x+1) + \frac{1}{3} \log(x-1) - \frac{5}{6} \log(2x+1)$$

3. Evalute $\int \frac{3x-1}{(x-1)^2(1+x)} \cdot dx$

Soln

Let $I = \int \frac{3x-1}{(x-1)^2(1+x)} \cdot dx$

$$\int \frac{3x-1}{(x-1)^2(1+x)} \cdot dx = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(1+x)}$$

$3x-1 = A(x-1) + B(1+x) + C(x-1)^2$

Put $x = 1$

$3-1 = A(0) + B(2) + C(0)$

$2 = 2B$

$\boxed{B = 1}$

Put $x = -1$

$-4 = A(0) + B(0) + 4C$

$-4 = 4C$

$\boxed{C = -1}$

Put $x = 0$

$-1 = +A + B(1) - 1$

$\boxed{A = -1}$

$I = -1\int \frac{dx}{1-x} + \int \frac{dx}{(1-x)^2} + -1\int \frac{dx}{1+x}$

$I = -\frac{\log(1-x)}{-1} + \frac{(1-x)^{-1}}{-1(-1)} + -1 \log(1+x) + C$

$I = \log(1-x) + \frac{1}{(1-x)} - \log(1+x) + C$

4. Evaluate $\int \frac{x}{(1+x)(x^2+1)} \cdot dx$

Soln

Let $I = \int \frac{x}{(1+x)(x^2+1)} \cdot dx$

$\int \frac{x}{(1+x)(x^2+1)} \cdot dx = \frac{A}{1+x} + \frac{Bx+C}{x^2+1}$

$x = A(x^2+1) + Bx+C(1+x)$

Put $x = -1$

$-1 = A(2) + Bx + C = 0$

$\boxed{A = -\frac{1}{2}}$

Put $x = 0$

$0 = A + C$

$0 = -\frac{1}{2} + C$ $\boxed{C = \frac{1}{2}}$

Equating co-efficient of x^2 on both sides

$$0 = A + B$$

$$0 = -\frac{1}{2} + B$$

$$\boxed{B = \frac{1}{2}}$$

$$I = \int \frac{-\frac{1}{2}}{1+x} \cdot dx + \int \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)} \cdot dx$$

$$= -\frac{1}{2}\int \frac{dx}{1+x} + \frac{1}{4}\int \frac{2x}{x^2+1} \cdot dx + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$I = -\frac{1}{2}\log(1+x) + \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + c$$

$$= -\frac{1}{2}\int \frac{dx}{1+x} + \frac{1}{4}\int \frac{d(x^2+1)}{x^2+1} + \frac{1}{2}\int \frac{dx}{(x^2+1)}$$

Definite integral

The integration in the interval (a, b) is given by

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where $\int f(x) \cdot dx = F(x) + c$ is called definite integral of $f(x)$

Properties of definite integrals.

1. $\int_a^b f(x) \cdot dx = -\int_b^a f(x) \cdot dx$

2. $\int_a^b f(x) \cdot dx = \int_a^b f(y)\, dy$

3. $\int_a^b f(x)\, dx = \int_a^c f(x) \cdot dx + \int_c^b f(x)\, dx$

Type 8 : Integral of the form $\displaystyle\int \frac{1}{(x+k)\sqrt{ax^2+bx+c}}\,dx$

This type of integral can be evaluated by putting the substitution

$x+k=\dfrac{1}{t}$.

Example 1 Evaluate $\int \frac{1}{(x+1)\sqrt{1-x^2}}\,dx$

Solution

Put $x+1 = \frac{1}{t}$; $dx = -\frac{1}{t^2}\,dt$

$$\int \frac{dx}{(x+1)\sqrt{1-x^2}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{1-\left(\frac{1}{t}-1\right)^2}} = -\int \frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{-\frac{1}{t^2}+\frac{2}{t}}}$$

$$= -\int \frac{dt}{\sqrt{2t-1}}$$

$$= -\sqrt{2t-1}, \qquad \text{where } t = \frac{1}{x+1}$$

Example 2 Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+2x-8}}$

Solution

Put $x-1 = \frac{1}{t}$; $dx = -\frac{1}{t^2}\,dt$

$$\int \frac{dx}{(x-1)\sqrt{x^2+2x-8}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2+2\left(\frac{1}{t}+1\right)-8}}$$

$$= -\int \frac{dt}{t\sqrt{\frac{1}{t^2}+1+\frac{2}{t}+\frac{2}{t}+2-8}} = -\int \frac{dt}{\sqrt{1-5t^2+4t}}$$

$$= -\int \frac{dt}{\sqrt{-5t^2+4t+1}} = -\int \frac{dt}{\sqrt{5}\sqrt{-t^2-\frac{4}{5}t+\frac{1}{5}}}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{\frac{9}{25}-\left(t-\frac{2}{5}\right)^2}} = -\frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{t-\frac{2}{5}}{\frac{3}{5}}\right)$$

$$= -\frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{5t-2}{3}\right), \qquad \text{where } t = \frac{1}{x-1}.$$

Example 3

Evaluate $\displaystyle\int \frac{dx}{(2+x)\sqrt{x}}$.

Solution

Put $\quad 2+x = \frac{1}{t}\ ;\ dx = -\frac{1}{t^2}\,dt$

$$\therefore \int \frac{dx}{(2+x)\sqrt{x}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t}-2}} = -\int \frac{dt}{t\sqrt{\frac{1}{t}-2}}$$

$$= -\int \frac{dt}{\sqrt{t-2t^2}} = -\int \frac{dt}{\sqrt{-2t^2+t}} = -\int \frac{dt}{\sqrt{2}\sqrt{-t^2+\frac{t}{2}}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\frac{1}{16}-\left(t-\frac{1}{4}\right)^2}} = -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t-\frac{1}{4}}{\frac{1}{4}}\right)$$

$$= \frac{-1}{\sqrt{2}}\sin^{-1}(4t-1), \quad t = \frac{1}{2+x}$$

Example 4

Evaluate $\displaystyle\int \frac{dx}{(x+2)\sqrt{x+3}}$

Solution

Put $x+2 = \frac{1}{t};\quad dx = -\frac{1}{t^2}\,dt$

$$\therefore \int \frac{dx}{(x+2)\sqrt{x+3}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t}-2+3}}$$

$$= -\int \frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t}+1}} = -\int \frac{dt}{t\sqrt{\frac{1}{t}+1}} = -\int \frac{dt}{\sqrt{\frac{t^2}{t}+t^2}}$$

$$= -\int \frac{dt}{\sqrt{t+t^2}} = -\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2-\frac{1}{4}}} = -\cos h^{-1}\left(\frac{t+\frac{1}{2}}{\frac{1}{2}}\right)$$

$$= -\cos h^{-1}(2t+1), \quad \text{where} \quad t = \frac{1}{x+2}$$

Type 13 : Integrals of the form $\int \frac{l \cos x + m \sin x + n}{l' \cos x + m' \sin x + n'} dx$

Let $l \cos x + m \sin x + n$ = A × (denominator function)
+ B × (derivative of denominator)
+ C

where A, B and C are constants.
By equating like coefficients we get the values for A, B and C.

Example 1

Evaluate $\int \frac{2 \cos x + 3 \sin x}{3 \cos x + 5 \sin x} dx$

Solution

Let $$2 \cos x + 3 \sin x = A(3 \cos x + 5 \sin x) + B(-3 \sin x + 5 \cos x) + C$$

Equating coefficients of $\sin x$ and $\cos x$, we get

$$3A + 5B = 2 \quad \dots (1)$$
$$5A - 3B = 3 \quad \dots (2)$$
$$C = 0 \quad \dots (3)$$

Solving (1) and (2), we get

$$A = \frac{21}{34}, \quad B = \frac{1}{34'} \quad C = 0$$

$$\therefore 2 \cos x + 3 \sin x = \frac{21}{34}(3 \cos x + 5 \sin x) + \frac{1}{34}(-3 \sin x + 5 \cos x)$$

$$\int \frac{2 \cos x + 3 \sin x}{3 \cos x + 5 \sin x} dx = \left[\frac{\frac{21}{34}(3 \cos x + 5 \sin x) + \frac{1}{34}(-3 \sin x + 5 \cos x)}{3 \cos x + 5 \sin x}\right] dx$$

$$= \frac{21}{34} \int \left(\frac{3 \cos x + 5 \sin x}{3 \cos x + 5 \sin x}\right) dx + \frac{1}{34} \int \left(\frac{-3 \cos x + 5 \sin x}{3 \cos x + 5 \operatorname{isn} x}\right) dx$$

$$= \frac{21}{34} \int dx + \frac{1}{34} \log(3 \cos x + 5 \sin x)$$

$$= \frac{21}{34} x + \frac{1}{34} \log(3 \cos x + 5 \sin x)$$

Example 2 Evaluate $\int \frac{dx}{1+\tan x}$

Solution

$$\int \frac{dx}{1+\tan x} = \int \frac{dx}{1+\frac{\sin x}{\cos x}} = \int \left(\frac{\cos x}{\cos x+\sin x}\right) dx$$

Now $\cos x = A(\cos x + \sin x) + B(-\sin x + \cos x) + C$

Equating coefficients of $\sin x$ and $\cos x$, we get

$$A + B = 1 \qquad \ldots (1)$$

$$A - B = 0 \qquad \ldots (2)$$

$$C = 0 \qquad \ldots (3)$$

Solving (1) and (2), we get

$$A = \frac{1}{2},\ B = \frac{1}{2},\ C = 0$$

$$\therefore \cos x = \frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(-\sin x + \cos x)$$

$$\int \frac{\cos x}{\cos x + \sin x}\, d\ x = \left\{\frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x}\right\} dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \left(\frac{-\sin x + \cos x}{\cos x + \sin x}\right) dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log(\cos x + \sin x)$$

Example 3 Evaluate $\int \frac{2\sin x + \cos x + 3}{5 + 8\cos x}\, dx$

Solution

$2\sin x + \cos x + 3 = A(5 + 8\cos x) + B(-8\sin x) + C$

Equating coefficients of sin x and $\cos x$, we get

$$8A = 1 \Rightarrow A = \frac{1}{8}$$

$$-8B = 2 \Rightarrow C = 3 - 5 \times \frac{1}{8}$$

$$\Rightarrow C = \frac{19}{8}$$

$$\therefore 2\sin x + \cos x + 3 = \frac{1}{8}(5 + 8\cos x) - \frac{1}{4}(-8\sin x) + \frac{19}{8}$$

$$\int\left\{\frac{2\sin x + \cos x + 3}{5 + 8\cos x}\right\}dx = \int\left\{\frac{\frac{1}{8}(5 + 8\cos x) - \frac{1}{4}(-8\sin x) + \frac{19}{8}}{5 + 8\cos x}\right\}dx$$

$$= \frac{1}{8}\int dx - \frac{1}{4}\int\frac{-8\sin x}{5 + 8\cos x}dx + \frac{19}{8}\int\frac{dx}{5 + 8\cos x}$$

$$= \frac{1}{8}x - \frac{1}{4}\log(5 + 8\cos x) + \frac{19}{8}\int\frac{dx}{5 + 8\cos x} \quad \dots (A)$$

To evaluate $\int\frac{dx}{5 + 8\cos x}$

Put $t = \tan\frac{x}{2}$; $dx = \frac{2\,dt}{1 + t^2}$

$\cos x = \frac{1 - t^2}{1 + t^2}$

$$\therefore \int\frac{dx}{5 + 8\cos x} = \int\frac{\frac{2\,dt}{1 + t^2}}{5 + 8\left(\frac{1 - t^2}{1 + t^2}\right)} = 2\int\frac{dt}{5(1 + t^2) + 8(1 - t^2)}$$

$$= 2\int\frac{dt}{13 - 3t^2} = 2\int\frac{dt}{3\left(\frac{13}{3} - t^2\right)}$$

$$= \frac{2}{3}\int\frac{dt}{\left(\sqrt{\frac{13}{3}}\right)^2 - t^2}$$

$$\int\frac{dx}{5 + 8\cos x} = \frac{2}{3}\frac{1}{2 \times \sqrt{\frac{13}{3}}}\log\left\{\frac{\sqrt{\frac{13}{3}} + t}{\sqrt{\frac{13}{3}} - t}\right\}, t = \tan\frac{x}{2} \quad \dots (B)$$

Substituting (B) in (A), we get

$$\int \left\{ \frac{2 \sin x + \cos x + 3}{5 + 8 \cos x} \right\} dx = \frac{1}{8} x - \frac{1}{4} \log (5 + 8 \cos x)$$

$$+ \frac{1}{\sqrt{39}} \log \left\{ \frac{\sqrt{\frac{13}{3}} + t}{\sqrt{\frac{13}{3}} - t} \right\}, \quad t = \tan \frac{x}{2}$$

Exercises

Integrate the following w.r.t. x.

1. $\dfrac{5 \cos x}{2 \cos x + 3 \sin x}$ [*Ans* : $2x + \log (2 \cos x + \sin x)$]

2. $\dfrac{2 \sin x + \cos x}{5 \sin x - 3 \cos x}$ [*Ans* : $\dfrac{7}{34} x + \dfrac{11}{34} \log (5 \sin x - 3 \cos x)$]

3. $\dfrac{\sin x + 18 \cos x}{3 \sin x + 4 \cos x}$ [*Ans* : $3x + 2 \log (3 \sin x + 4 \cos x)$]

4. $\dfrac{3 \sin x + 2 \cos x}{3 \cos x + 5 \cos x}$ [*Ans* : $\dfrac{21}{34} x + \dfrac{1}{34} \log (3 \cos x + 5 \sin x)$]

Integration by using trigonometric substitutions

By using trigonometric substitution, we can evaluate the integrals whose integral is of the form $\frac{1}{\sqrt{x^2-a^2}}$, $\frac{1}{\sqrt{a^2-x^2}}$, $\frac{1}{\sqrt{x^2+a^2}}$. The method is explained in the following examples.

Example 1 *Evaluate the integrals of the type*

(a) $\int \frac{dx}{\sqrt{a^2-x^2}}$, (b) $\int \frac{dx}{\sqrt{a^2+x^2}}$, (c) $\int \frac{dx}{\sqrt{x^2-a^2}}$

Solution

(a) Put $x = a\sin\theta$

Differentiating w.r.t. 'θ', we get

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta\, d\theta}{a\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta}{\cos\theta}\, d\theta$$

$$= \int d\theta = \theta = \sin^{-1}\frac{x}{a}$$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}}$$

(b) *To evaluate* $\int \frac{dx}{\sqrt{a^2+x^2}}$

METHOD 1 :

Put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2+a^2\tan^2\theta}}$$

$$= a\int \frac{\sec^2\theta\, d\theta}{a\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \int \sec\theta \, d\theta \quad = \quad \log(\sec\theta + \tan\theta)$$

$$= \log[\sqrt{\tan^2\theta + 1} + \tan\theta)]$$

$$= \log\left[\sqrt{\frac{x^2}{a^2} + 1} + \frac{x}{a}\right]$$

$$= \log\left[\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right]$$

$$= \log\left[\frac{\sqrt{x^2 + a^2} + x}{a}\right]$$

$$= \log(\sqrt{x^2 + a^2} + x) - \log a$$

$$\left[\log\frac{m}{n} = \log m - \log n\right]$$

$$= \log(\sqrt{x^2 + a^2} + x) + c, \quad c = \log a$$

METHOD 2 :

Put $x = a \sin h\,\theta$

$dx = a \cos h\,\theta \, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{a \cos h\,\theta \, d\theta}{\sqrt{a^2 + a^2 \sin h^2\theta}} = \int \frac{a \cos h\,\theta \, d\theta}{a\sqrt{1 + \sin h^2\theta}}$$

$$= \int \frac{\cos h^2\theta}{\sqrt{\cos h^2\theta}} \, d\theta \quad [\because \cos h^2\theta - \sin h^2\theta = 1]$$

$$= \int d\theta = \theta = \sin h^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \int \frac{dx}{\sqrt{a^2 + x^2}} = \sin h^{-1}\left(\frac{x}{a}\right)$$

[OR]

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \log(x + \sqrt{x^2 + a^2})$$

(c) Evaluate $\int \frac{dx}{\sqrt{x^2-a^2}}$

METHOD 1 :

Put $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{a\sqrt{\sec^2\theta-1}}$$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\, d\theta$$

$[\because 1+\tan^2\theta = \sec^2\theta]$

$$= \int \sec\theta = \log(\sec\theta+\tan\theta)$$

$$= \log(\sec\theta + \sqrt{\sec^2\theta-1})$$

$$= \log\left[\frac{x}{a} + \left(\sqrt{\frac{x^2}{a^2}-1}\right)\right]$$

$$= \log\left[\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right]$$

$$= \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)$$

$$= \log(x+\sqrt{x^2-a^2}) - \log a$$

$$= \log(x+\sqrt{x^2-a^2}) + c, \quad c = \log a$$

METHOD 2 :

Put $x = a\cos h\,\theta$

$dx = a\sin h\,\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sin h\,\theta\, d\theta}{\sqrt{a^2\cos h^2\theta-a^2}} = \int \frac{a\sin h\,\theta\, d\theta}{a\sqrt{\cos h^2\theta-1}}$$

$$= \int \frac{\sin h\,\theta}{\sqrt{\sin^2\theta}}\, d\theta \quad [\because \cosh^2\theta - \sinh^2\theta = 1]$$

$$= \int \frac{\sin h\,\theta}{\sin h\,\theta}\, d\theta = \int d\theta = \theta$$

$$= \cos h^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \cos h^{-1}\left(\frac{x}{a}\right)$$

(or)

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right)$$

Example 2 *Evaluate the following integrals*

(a) $\int \frac{dx}{\sqrt{9 - x^2}}$, (b) $\int \frac{dx}{\sqrt{16 - x^2}}$, (c) $\int \frac{dx}{\sqrt{x^2 - 25}}$

Solution

(a) $$\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{dx}{\sqrt{3^2 - x^2}}$$

We know that $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

Here $a = 3$,

$$\therefore \int \frac{dx}{\sqrt{9 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right)$$

(b) $$\int \frac{dx}{\sqrt{16 + x^2}} = \int \frac{dx}{\sqrt{4^2 + x^2}}$$

We know that $$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sin h^{-1}\left(\frac{x}{a}\right)$$

Here $a = 4$, $$\int \frac{dx}{\sqrt{16 + x^2}} = \sin h^{-1}\left(\frac{x}{4}\right)$$

Evaluate the integrals of the type

$$\text{(a)} \int \sqrt{a^2 - x^2}\, dx, \quad \text{(b)} \int \sqrt{x^2 - a^2}\, dx, \quad \text{(c)} \int \sqrt{x^2 + a^2}\, dx$$

Solution

(a) To evaluate $\int \sqrt{a^2 - x^2}\, dx$

Put $x = a \sin\theta$

$dx = a \cos\theta \, d\theta$

$$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2 \sin^2\theta}\; a \cos\theta \, d\theta$$

$$= \int a\sqrt{1-\sin^2\theta}\; a\cos\theta \, d\theta$$

$$= a \int \sqrt{\cos^2\theta}\; a\cos\theta \, d\theta$$

$$= a \int \cos\theta \; a\cos\theta \, d\theta \;=\; a^2 \int \cos^2\theta \, d\theta$$

$$= a^2 \int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$\left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{a^2}{2} \left[\int d\theta + \int \cos 2\theta \, d\theta\right]$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} - \frac{2\sin\theta\cos\theta}{2}\right]$$

$$[\because \sin^2\theta = 2\sin\theta\cos\theta]$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \sin\theta\sqrt{1-\sin^2\theta}\right]$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]$$

$$[\because x = a\sin\theta]$$

$$\therefore \int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a^2}\sqrt{a^2-x^2}\right]$$

$$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

(h) To evaluate $\int\sqrt{x^2-a^2}\,dx$

Put $x = a\cos h\,\theta$

$$dx = a \sin h\,\theta\, d\theta$$

$$\therefore \int \sqrt{x^2 - a^2}\, dx = \int \sqrt{a^2 \cos h^2 \theta - a^2} \cdot a \sin h\,\theta\, d\theta$$

$$= \int a^2 \sqrt{\cos h^2 \theta - 1} \cdot \sin h\,\theta\, d\theta$$

$$[\because \cos h^2 \theta - \sin h^2 \theta = 1]$$

$$= a^2 \int \sqrt{\sin h^2 \theta} \cdot \sin h\,\theta\, d\theta$$

$$= a^2 \int \sin h^2 \theta\, d\theta$$

$$= a^2 \int \left(\frac{\cos h\, 2\theta - 1}{2}\right) d\theta$$

$$[\because \cos h\, 2\theta = 1 + 2 \sin h^2 \theta]$$

$$= \frac{a^2}{2} \left[\int \cos h\, 2\theta\, d\theta - \int d\theta\right]$$

$$= \frac{a^2}{2} \left[\frac{\sin h\, 2\theta}{2} - \theta\right]$$

$$= \frac{a^2}{2} \left[\frac{2 \sin h\,\theta \cos h\,\theta}{2} - \theta\right]$$

$$= \frac{a^2}{2} \left[\cosh \theta \sqrt{\cos h^2 \theta - 1} - \theta\right]$$

$$= \frac{a^2}{2} \left[\frac{x}{a} \sqrt{\frac{x^2}{a^2} - 1} - \cos h^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{a^2}{2} \left[\frac{x}{a^2} \sqrt{x^2 - a^2} - \cos h^{-1}\left(\frac{x}{a}\right)\right]$$

$$\boxed{\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cos h^{-1}\left(\frac{x}{a}\right)}$$

c) To evaluate $\int \sqrt{x^2 + a^2}\, dx$

Put $x = a \sinh\theta$

$dx = a \cosh\theta\, d\theta$

$$\therefore \int \sqrt{x^2+a^2}\, dx = \int \sqrt{a^2 \sinh^2\theta + a^2} \cdot a \cosh\theta\, d\theta$$

$$= \int a^2 \sqrt{\sinh^2\theta + 1} \cdot \cosh\theta\, d\theta$$

$$= a^2 \int \sqrt{\cosh^2\theta} \cdot \cosh\theta\, d\theta \qquad [\because \cosh^2\theta - \sinh^2\theta = 1]$$

$$= a^2 \int \cosh\theta \cosh\theta\, d\theta = a^2 \int \cosh^2\theta\, d\theta$$

$$= a^2 \int \left(\frac{\cosh 2\theta + 1}{2}\right) d\theta$$

$$= \frac{a^2}{2}\left[\int \cosh 2\theta\, d\theta + \int d\theta\right]$$

$$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} + \theta\right] = \frac{a^2}{2}\left[\frac{2\sinh\theta\cosh\theta}{2} + \theta\right]$$

$$[\because \sinh 2\theta = 2\sinh\theta\cosh\theta]$$

$$= \frac{a^2}{2}\left[\sinh\theta\sqrt{1+\sinh^2\theta} + \theta\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}} + \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a^2}\sqrt{a^2+x^2} + \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{x}{a}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$$

$$\boxed{\therefore \int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)}$$

7. $\dfrac{1}{a^2 e^x + b^2 e^{-x}}$ [*Ans* : $\dfrac{1}{ab}\tan^{-1}$ ... b

Type 4 : Integral of the form $\displaystyle\int \frac{px+q}{ax^2+bx^2+c}\,dx$

where a, b, c, p **and** q **are constants.**

In this type of problem, express the numerator $px + q$ as follows :

$px + q$ = **A (derivative of denominator) + B**

By equating like coefficients on both sides we get the values for the two unknowns. The method is explained as below :

Example 1

Evaluate $\displaystyle\int \frac{2x+1}{x^2+3x+1}\,dx$

Solution

$$2x+1 = A\frac{d}{dx}(x^2+3x+1)+B$$

$$2x+1 = A(2x+3)+B$$

Equating coefficients on both sides, we get

$$2A = 2 \Rightarrow A = 1$$

Equating constant coefficients, we get

$$3A + B = 1$$

$$B = 1 - 3A = 1 - 3$$

$$B = -2$$

$$\therefore 2x+1 = 1(2x+3)-2$$

$$\therefore \int \frac{2x+1}{x^2+3x+1}\,dx = \int\left\{\frac{(2x+3)-2}{x^2+3x+1}\right\}dx$$

$$= \int\left(\frac{2x+3}{x^2+3x+1}\right)dx - 2\int\frac{1}{x^2+3x+1}\,dx$$

$$= \log(x^2+3x+1) - 2\int\frac{1}{\left(x+\frac{3}{2}\right)^2-\frac{5}{4}}\,dx$$

$$= \log(x^2+3x+1) - 2\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \log(x^2+3x+1) - 2\frac{1}{2\frac{\sqrt{5}}{2}} \log\left[\frac{x+\frac{3}{2}-\frac{\sqrt{5}}{2}}{x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right]$$

$$= \log(x^2+3x+1) - \frac{2}{\sqrt{5}} \log\frac{2x+3-\sqrt{5}}{2x+3+\sqrt{5}}$$

Example 2

Evaluate $\int \frac{3x+1}{2x^2-x+5}dx$

Solution

Let
$$3x+1 = A\frac{d}{dx}(2x^2-x+5)+B$$
$$3x+1 = A(4x-1)+B$$

Equating coefficient of 'x' on both sides we get

$$3 = 4A \Rightarrow A = \frac{3}{4}$$

Equating constant coefficients we get,

$$1 = -A+B$$
$$B = A+1 = \frac{3}{4}+1 = \frac{3+4}{4} = \frac{7}{4}$$
$$3x+1 = \frac{3}{4}(4x-1)+\frac{7}{4}$$

$$\int \frac{3x+1}{2x^2-x+5}dx = \int \left\{\frac{\frac{3}{4}(4x-1)+\frac{7}{4}}{2x^2-x+5}\right\}dx$$

$$= \frac{3}{4}\int \frac{4x-1}{2x^2-x+5}dx + \frac{7}{4}\int \frac{dx}{2x^2-x+5}$$

$$= \frac{3}{4}\log(2x^2-x+5) + \frac{7}{4}\int \frac{dx}{2\left(x^2-\frac{x}{2}+\frac{5}{2}\right)}$$

❑ PROPERTIES OF DEFINITE INTEGRALS

We know very well that $\int_a^b f(x)\,dx = F(b) - F(a)$ where

$$\int f(x)\,dx = F(x) + c$$

Now we will see some properties of definite integrals.

Property 1 : $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Proof :

$$\text{L.H.S.} = \int_a^b f(x)\, dx = [F(x)]_a^b$$

$$= F(b) - F(a) \qquad \ldots (1)$$

$$\text{R.H.S.} = -\int_b^a f(x)\, dx = -[F(x)]_b^a$$

$$= -[F(a) - F(b)] = F(b) - F(a) \qquad \ldots (2)$$

From (1) and (2) we get

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

E.g., $\int_1^2 x^2\, dx = -\int_2^1 x^2\, dx = \frac{7}{3}$

Property 2 : $\int_a^b f(x)\, dx = \int_a^b f(y)\, dy$

$$\text{L.H.S.} = \int_a^b f(x)\, dx = [F(x)]_a^b \qquad \ldots (1)$$

$$= F(b) - F(a)$$

$$\text{R.H.S.} = \int_a^b f(y)\, dy = -[F(y)]_b^a \qquad \ldots (2)$$

$$= F(b) - F(a) \qquad [\because \int f(y) = F(y) + c]$$

From (1) and (2) we get,

$$\int_a^b f(x)\, dx = \int_a^b f(y)\, dy$$

Thus the value of the definite integral depends only on the limits but not on the variable.

E.g., $\int_1^3 x^3\, dx = \int_1^3 y^3\, dy = \int_1^3 u^3\, du$.

Property 3 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where c is a point between a and b.

Proof :

$$\text{L.H.S.} = \int_a^b f(x)\,dx = [F(x)]_a^b$$

$$= F(b) - F(a) \qquad \ldots (1)$$

$$\text{R.H.S.} = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= [F(x)]_a^c + [F(x)]_c^b$$

$$= [F(c) - F(a)] + [F(b) - F(c)]$$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a) \qquad \ldots (2)$$

From (1) and (2) we get

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

E.g., $\int_1^3 (x + x^2)\,dx = \int_1^2 (x + x^2)\,dx + \int_2^3 (x + x^2)\,dx$ where 2 lies between 1 and 3.

Property 4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof :

$$\text{R.H.S.} = \int_0^a f(a-x)\,dx \qquad \ldots (1)$$

Put $a - x = y$ when $x = 0,\ y = a$

i.e., $-dx = dy$ when $x = a,\ y = 0$

$$(1) \Rightarrow \int_0^a f(a-x)\,dx = \int_a^0 f(y)(-dy) = -\int_a^0 f(y)\,dy$$

$$= \int_0^a f(y)\,dy \qquad \text{[Propery 1]}$$

$$= \int_0^a f(x)\,dx \qquad \text{[Property 2]}$$

Hence $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Note : This property is very important in solving of many difficult problems in integration.

Property 5 : $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx$

Proof : We know that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \ldots (1)$$

(By property 3)

In the first integral on the right, put $x = -y$, $dx = -dy$

when $x = -a$, $y = a$; $x = 0$, $y = 0$

$$\therefore \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y)(-dy) = -\int_{a}^{0} f(-y)\,dy$$

$$= \int_{0}^{a} f(-y)\,dy \quad \text{[By property (1)]}$$

$$= \int_{0}^{a} f(-x)\,dx \quad \text{[By property (2)]} \quad \ldots (2)$$

Substituting (2) in (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

Corollary 1 : If $f(x)$ is an odd function *i.e.,* $f(-x) = -f(x)$ then

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

[In the above result by replacing $f(-x)$ by $-f(x)$]

$$\therefore \int_{-a}^{a} f(x)\,dx = 0 \quad \text{[when } f(x) \text{ is odd]}$$

E.g. 1. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x\,dx = 0$ [$\because \sin^5 x$ is an odd function)

E.g. 2. $\int_{-1}^{1} x^3\,dx = 0$ [$\because x^3$ is an odd function]

Corollary 2 : If $f(x)$ is an even function *i.e.,* $f(-x) = f(x)$ then

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx.$$

[In the above result by replacing $f(-x)$ by $f(x)$]

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx \qquad \text{[when } f(x) \text{ is an even function]}$$

E.g. 1. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\, dx = 2\int_{0}^{\frac{\pi}{2}} \cos^4 x\, dx$ [$\because \cos^4 x$ is an even function]

E.g. 2. $\displaystyle\int_{-1}^{1} x^6\, dx = 2\int_{0}^{1} x^6\, dx$ [$\because x^6$ is an even function]

E.g. 3. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\, dx = 2\int_{0}^{\frac{\pi}{2}} \sin^4 x\, dx$ [$\because \sin^4 x$ is an even function]

[$\because \sin(-x) = -\sin x$; $\therefore \sin^4(-x) = (-\sin x)^4 = \sin^4 x$]

Property 6 : $\displaystyle\int_{0}^{2a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(2a - x)\, dx.$

Proof : We know that

$$\int_{0}^{2a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{a}^{2a} f(x)\, dx \qquad \ldots(1)$$

In the second integral on the right,

put $x = 2a - y$

$dx = -dy$

When $x = a$, $y = a$,

when $x = 2a$, $y = 0$

$$\therefore \int_{a}^{2a} f(x)\, dx = \int_{a}^{0} f(2a - y)(-dy) = -\int_{a}^{0} f(2a - y)\, dy$$

$$= \int_{0}^{a} f(2a - y)\, dy \qquad \text{[By property 1]}$$

$$= \int_{0}^{a} f(2a - x)\, dx \qquad \text{[By Property 2]}$$

$$\text{i.e.,} \quad \int_a^{2a} f(x)\,dx = \int_0^a f(2a-x)\,dx \qquad \dots (2)$$

Substituting (2) in (1) we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Corollary 1 : If $f(x) = f(2a-x)$ then the above integral becomes

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

Corollary 2 : $\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx$

Proof :

$$\int_0^{\pi} \sin^n x\,dx = \int_0^{2\pi/2} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx$$

$$[\because \sin^n(\pi - x) = \sin^n x]$$

Example 1 *Prove that* $\int_0^{\pi} \sin^6 x\,dx = 2\int_0^{\pi/2} \sin^6 x\,dx$.

Solution

Here

$$f(x) = \sin^6 x$$
$$f(\pi - x) = \sin^6(\pi - x) = \sin^6 x$$

i.e.,

$$f(x) = f\left(2\frac{\pi}{2} - x\right)$$

$\therefore$ By cor 1 we get $\int_0^{\pi} \sin^6 x\,dx = 2\int_0^{\pi/2} \sin^6 x\,dx$

Example 2 *Prove that* $\int_0^{\pi} \cos^4 x\,dx = 2\int_0^{\pi/2} \cos^4 x\,dx$

Solution

$$f(x) = \cos^4(\pi - x) = (-\cos x)^4 = \cos^4 x$$

i.e.,

$$f(x) = f(\pi - x)$$

$\therefore$ By cor 1, we get $\int_0^{\pi} \cos^4 x\,dx = 2\int_0^{\pi/2} \cos^4 x\,dx$

Note : In general $\int_0^{\pi} \cos^n x\,dx = 2\int_0^{\pi/2} \cos^n x\,dx$ when n is even.

Corollary 3 : If $f(x) = -f(2a - x)$, then the result

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$$

takes the form $$\int_0^{2a} f(x)\,dx = -\int_0^{a} f(2a - x)\,dx + \int_0^{a} f(2a - x)\,dx$$

i.e., $$\int_0^{2a} f(x)\,dx = 0$$

Example 3 *Show that* $\int_0^{\pi} \cos^3 x\,dx = 0$

Solution

$$f(x) = \cos^3 x$$

$$f(\pi - x) = \cos^3(\pi - x) = -\cos^3 x = -f(x)$$

i.e., $f(\pi - x) = -f(x)$ $\therefore \int_0^{\pi} \cos^3 x\,dx = 0$

Note : In general $\int_0^{\pi} \cos^n x\,dx = 0$, when n is odd.

Example 4 *Evaluate* $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$.

Solution

Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x\,dx}{\sin^n x + \cos^n x} \quad \ldots (1)$$

Also $$I = \int_0^{\frac{\pi}{2}} \frac{\sin^n\left(\frac{\pi}{2} - x\right)}{\sin^n\left(\frac{\pi}{2} - x\right) + \cos^n\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a - x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}\, dx \qquad \ldots (2)$$

Adding (1) and (2) we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}}$$

$$2I = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

Example 5 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\, dx$

Solution

Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\, dx \qquad \ldots (1)$$

By using the property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ we have

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx$$

Integrals of the form $\sqrt{(x-\alpha)(\beta-x)}$, $\dfrac{1}{\sqrt{(x-\alpha)(\beta-x)}}$ **and** $\sqrt{\dfrac{x-\alpha}{\beta-x}}$ **where** $\beta > \alpha$.

The above integrals can be easily integrated by the substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$.

Example 1

Evaluate $\int \sqrt{(x+1)(4-x)}\, dx$.

Solution

Here

$$\alpha = -1, \ \beta = 4$$

$$\beta > \alpha$$

$$\therefore \quad \text{Put } x = -1\cos^2\theta + 4\sin^2\theta$$

$$dx = (2\cos\theta\sin\theta + 8\sin\theta\cos\theta)\, d\theta$$

$$= 10\cos\theta\sin\theta\, d\theta$$

$$x+1 = -\cos^2\theta + 4\sin^2\theta + 1$$

$$= \sin^2\theta + 4\sin^2\theta \qquad [\because 1-\cos^2\theta = \sin^2\theta]$$

$$\therefore \quad x+1 = 5\sin^2\theta \qquad \ldots (1)$$

$$4-x = 4-(-\cos^2\theta + 4\sin^2\theta)$$

$$= 4+\cos^2\theta - 4\sin^2\theta$$

$$= 4(1-\sin^2\theta) + \cos^2\theta$$

$$= 4\cos^2\theta + \cos^2\theta$$

$$\therefore \quad 4-x = 5\cos^2\theta \qquad \ldots (2)$$

$$\therefore \int \sqrt{(x+1)(4-x)\,dx)} = \int \sqrt{5\sin^2\theta \cdot 5\cos^2\theta}\; 10\cos\theta\sin\theta\, d\theta$$

$$= \int 5\sin\theta\cos\theta\; 10\cos\theta\sin\theta\, d\theta$$

$$= 50\int (\sin\theta\cos\theta)^2\, d\theta$$

$$= 50\int \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{25}{2}\int \sin^2\theta\, 2\theta\, d\theta$$

$$= \frac{25}{2}\int\left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{25}{4}\left[\theta - \frac{\sin 4\theta}{4}\right] \qquad \ldots (3)$$

$$\left.\begin{aligned} &\text{From (1), we get} \quad \sin\theta = \sqrt{\frac{x+1}{5}} \\ &\text{From (2), we get} \quad \cos\theta = \sqrt{\frac{4-x}{5}} \end{aligned}\right\} \qquad \ldots (4)$$

Substituting (4) in (3), we get

$$\int \sqrt{(x+1)(4-x)}\, dx = \frac{25}{4}\left[\sin^{-1}\sqrt{\frac{x+1}{5}} - \frac{2\sin 2\theta\cos 2\theta}{4}\right]$$

$$= \frac{25}{4}\left[\sin^{-1}\sqrt{\frac{x+1}{5}} - \frac{1}{2}\cdot 2\sin\theta\cos\theta\,(2\cos^2\theta - 1)\right]$$

$$= \frac{25}{4}\left[\sin^{-1}\sqrt{\frac{x+1}{5}} - \sqrt{\frac{x+1}{5}}\sqrt{\frac{4-x}{5}}\left(\frac{2(4-x)}{5} - 1\right)\right]$$

$$= \frac{25}{4}\left[\sin^{-1}\sqrt{\frac{x+1}{5}} - \sqrt{\frac{(x+1)(4-x)}{25}}\left(\frac{8-2x-5}{5}\right)\right]$$

$$= \frac{25}{4}\left[\sin^{-1}\sqrt{\frac{x+1}{5}} - \frac{\sqrt{(x+1)(4-x)}}{25}\right](3-2x)$$

Example 2 Evaluate $\int \sqrt{\frac{5-x}{x-2}}\,dx$

Solution

Here $\alpha = 5,\ \beta = 2$

Put $x = 2\sin^2\theta + 5\cos^2\theta$

$$dx = (4\sin\theta\cos\theta - 10\cos\theta\sin\theta)\,d\theta$$
$$= -6\sin\theta\cos\theta\,d\theta$$

Now
$$5 - x = 5 - 2\sin^2\theta - 5\cos^2\theta$$
$$= 5 - 2\sin^2\theta - 5(1-\sin^2\theta)$$
$$= 5 - 2\sin^2\theta - 5 + 5\sin^2\theta$$
$$5 - x = 3\sin^2\theta$$
$$\sin\theta = \sqrt{\frac{5-x}{3}} \quad \ldots (1)$$

Now
$$x - 2 = 2\sin^2\theta + 5\cos^2\theta - 2$$
$$= 2\sin^2\theta + 5(1-\sin^2\theta) - 2$$
$$= 2\sin^2\theta + 5 - 5\sin^2\theta - 2$$
$$= -3\sin^2\theta + 3$$
$$= 3(1-\sin^2\theta)$$
$$x - 2 = 3\cos^2\theta$$
$$\Rightarrow \quad x - 2 = 3\cos^2\theta$$
$$\cos\theta = \sqrt{\frac{x-2}{3}} \quad \ldots (2)$$

$$\int\sqrt{\frac{5-x}{x-2}}\,dx = \int\sqrt{\frac{3\sin^2\theta}{3\cos^2\theta}} \times -6\sin\theta\cos\theta\,d\theta$$
$$= -6\int\frac{\sin\theta}{\cos\theta}\sin\theta\cos\theta\,d\theta = -6\int\sin^2\theta\,d\theta$$
$$= -6\int\frac{[1-\cos 2\theta]}{2}\,d\theta = -3\left[\theta - \frac{\sin 2\theta}{2}\right]$$
$$= -3\left[\sin^{-1}\left(\frac{\sqrt{5-x}}{3}\right) - \sin\theta\cos\theta\right]$$
$$= -3\left[\sin^{-1}\left(\frac{\sqrt{5-x}}{3}\right) - \frac{\sqrt{(5-x)(x-2)}}{3}\right]$$
$$= 3\sin^{-1}\left(\frac{\sqrt{5-x}}{3}\right) + \frac{\sqrt{(5-x)(x-2)}}{3}$$

Example 3

Evaluate $\displaystyle\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta < \alpha)$

Solution

Put

$$x = \alpha \sin^2 \theta + \beta \cos^2 \theta$$

$$dx = (2\alpha \sin\theta \cos\theta - 2\beta \cos\theta \sin\theta)\, d\theta$$

$$= 2(\alpha - \beta) \sin\theta \cos\theta \, d\theta \qquad \text{...(1)}$$

$$x - \alpha = \alpha \sin^2\theta + \beta \cos^2\theta - \alpha$$

$$= \alpha(\sin^2\theta - 1) + \beta\cos^2\theta$$

$$= \alpha(-\cos^2\theta) + \beta\cos^2\theta$$

$$\therefore x - \alpha = (\beta - \alpha)\cos^2\theta \qquad \text{...(2)}$$

$$\beta - x = \beta - \alpha\sin^2\theta - \beta\cos^2\theta$$

$$= \beta(1 - \cos^2\theta) - \alpha\sin^2\theta$$

$$= \beta\sin^2\theta - \alpha\sin^2\theta$$

$$\beta - x = (\beta - \alpha)\sin^2\theta \qquad \text{...(3)}$$

$$\therefore \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \int \frac{2(\alpha-\beta)\sin\theta\cos\theta\, d\theta}{\sqrt{(\beta-\alpha)\cos^2\theta\,(\beta-\alpha)\sin^2\theta}}$$

$$= -2\int d\theta$$

$$= -2(\theta)$$

$$= -2\sin^{-1}\sqrt{\frac{\beta - x}{\beta - \alpha}}$$

Type II : Integrals of th

❑ BERNOULLI'S FORMULA

If u and v are functions of x then

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\ldots$$

where $u, u', u'', u''', \ldots\ldots$ represent the successive derivative of u and $v_1, v_2, v_3, \ldots\ldots$ represent the repeated integrals of v.

Example 1

Evaluate $\int x^4 e^{2x}\,dx$

Solution

$$u = x^4 \qquad v = e^{2x}$$

$$u' = 4x^3 \qquad v_1 = \int e^{2x}\,dx = \frac{e^{2x}}{2}$$

$$u'' = 12x^2 \qquad v_2 = \frac{e^{2x}}{4}$$

$$u''' = 24x \qquad v_3 = \frac{e^{2x}}{2}$$

$$u^{iv} = 24 \qquad v_4 = \frac{e^{2x}}{16},\ v_5 = \frac{e^{2x}}{32}$$

By applying the above formula, we get

$$\int x^4 e^{2x}\,dx = \frac{x^4 \cdot e^{2x}}{2} - 4x^3 \cdot \frac{e^{2x}}{4} + 12x^2 \cdot \frac{e^{2x}}{8} - 24x \cdot \frac{e^{2x}}{16} + 24 \cdot \frac{e^{2x}}{32}$$

$$= \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3x^2}{2} e^{2x} - \frac{3x}{2} e^{2x} + \frac{3}{2} e^{2x}$$

Example 2

Evaluate $\int x^2 \sin^2 x\,dx.$

Solution

$$u = x^2; \qquad v = \sin 2x$$

$$u' = 2x \qquad v_1 = \frac{-\cos 2x}{2}$$

$$u'' = 2 \qquad v_2 = -\frac{\sin 2x}{4}, \qquad v_3 = \frac{\cos 2x}{8}$$

By using the above formula

$$\int x^2 \sin 2x\,dx = x^2\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + 2\left(\frac{\cos 2x}{8}\right)$$

$$= \frac{x^2 \cos 2x}{2} + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x$$

❑ REDUCTION FORMULA

A reduction formula is a linear relationship which connects the given integral with other integrals which is of the same type but a lower index than the given integral. The original integral or the given integral can be evaluated by successive applications of this reduction formula. This definition of reduction formula can be easily understood from the following examples.

Reduction formula for $\int \sin^n x \, dx$ (n being +ve integer)

Let
$$I_n = \int \sin^n x \, dx$$
$$= \int \sin^{n-1} x \sin x \, dx$$
$$= \int \sin^{n-1} x \, d(-\cos x) \qquad [\because d(-\cos x) = \sin x \, dx]$$
$$= -\sin^{n-1} x \cos x - \int -\cos x \, d(\sin^{n-1} x)$$
$$= -\sin^{n-1} x \cos x + \int \cos x \, (n-1) \sin^{n-2} x \cos x \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$
$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$
$$I_n (1 + n - 1) = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$
$$nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$
$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$$

This gives the reduction formula for $\int \sin^n x \, dx$

The ultimate integral of the original (or given) integral will finally depends on either I_1 or I_0 according as 'n' is odd or even.

The value of $I_1 = \int \sin x \, dx = -\cos x$ and $I_0 = \int \sin^0 x \, dx = \int dx = x$.

Example 1

Evaluate $\int \sin^7 x\, dx$.

Solution

Let $I_7 = \int \sin^7 x\, dx$

We know that $I_n = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{(n-1)}{n} I_{n-2}$ ...(1)

Putting $n = 7$ in (1) we get,

$$I_7 = -\frac{\sin^6 x \cos x}{7} + \frac{6}{7} I_5 \quad \text{... (2)}$$

Putting $n = 5$ in (1), we get

$$I_5 = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} I_3 \quad \text{... (3)}$$

Putting $n = 3$ in (1), we get

$$I_3 = -\frac{\sin^2 x \cos x}{3} + \frac{2}{3} I_1 \quad \text{... (4)}$$

Putting $n = 1$ in (1), we get

$$I_1 = \int \sin x\, dx = -\cos x \quad \text{... (5)}$$

$$I_1 = -\cos x$$

By substituting (5) in (4), we get

$$I_3 = \frac{-\sin^2 x \cos x}{3} - \frac{2}{3} \cos x$$

Substituting this I_3 in (3) we get

$$I_5 = \frac{-\sin^4 x \cos x}{5} - \frac{4}{5}\left(\frac{\sin^2 x \cos x}{3} + \frac{2}{3} \cos x\right)$$

Substituting this I_5 in (2), we get

$$I_7 = \frac{-\sin^6 x \cos x}{7} - \frac{6}{7}\left\{\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\left(\frac{\sin^2 x \cos x}{3} + \frac{2}{3} \cos x\right)\right\}$$

Example 2

Evaluate $\int \sin^6 x\, dx$.

Solution

Let $I_6 = \int \sin^6 x\, dx$

We know that

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2} \quad \text{... (1)}$$

Putting $n = 6$ in (1), we get

$$I_6 = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6} I_4 \quad \ldots (2)$$

Putting $n = 4$ in (1), we get

$$I_4 = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} I_2 \quad \ldots (3)$$

Putting $n = 2$ in (1), we get

$$I_2 = \frac{-\sin x \cos x}{2} + \frac{1}{2} I_0 \quad \ldots (4)$$

It is not possible to find I_0 from (1). But we know that

$$I_0 = \int \sin^0 x\, dx = \int dx = x \qquad [\because (\sin x)^0 = 1]$$

$$\therefore I_0 = x \quad \ldots (5)$$

By using (2), (3), (4) and (5) we get

$$I_6 = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6}\left[\frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left\{\frac{-\sin x \cos x}{2} + \frac{1}{2}x\right\}\right]$$

Example 3

Evaluate $\int_0^{\frac{\pi}{2}} \sin^n x\, dx$ *(n being a positive integer).*

Solution

Let
$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

We know that

$$\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} \int \sin^{n-2} x\, dx$$

$$\therefore I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\frac{\pi}{2}} + \frac{(n-1)}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

For both upper and lower limits the first term in R.H.S. become zero.

$$\therefore I_n = \frac{n-1}{n} I_{n-2} \quad \ldots (1)$$

Similarly
$$I_{n-2} = \frac{n-3}{n-2} I_{n-4} \quad \ldots (2)$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6} \quad \ldots (3)$$

....

Substituting (2), (3), etc. in (1), we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$$

Now there are two cases arises depends on the value of n.

Case (i) : When n is even integer, we have

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 \qquad \ldots (1)$$

Now $$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \qquad \ldots (2)$$

Substituting (2) in (1), we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Case (ii) : When 'n' is an odd integer, we have

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \qquad \ldots (3)$$

Now $$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1 \qquad \ldots (4)$$

Substituting (4) in (3) we get

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

Example 4

Evaluate $\int_0^{\frac{\pi}{2}} \sin^{10} x \, dx$.

Solution

Here $n = 10$ which is an even integer.

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{10} x \, dx = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 0.3863$$

(Using the formula under Case (*i*))

Example 5

Evaluate $\int_0^{\frac{\pi}{2}} \sin^9 x \, dx$

Solution

Here $n = 9$, which is an odd integer.

Using formula in Case (*ii*) above, we get

$$\therefore \int_0^{\frac{\pi}{2}} \sin^9 x\, dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{384}{945}$$

Reduction formula for $\int \cos^n x\, dx$ (n being +ve integer)

Let $\quad I_n = \int \cos^n x\, dx$

$$= \int \cos^{n-1} x \cos x\, dx$$

$$= \int \cos^{n-1} x\, d(\sin x) \qquad [\because \cos x\, dx = d(\sin x)]$$

$$= \cos^{n-1} x \sin x - \int \sin x\, d(\cos^{n-1} x)$$

Integrating by parts, we get,

$$= \cos^{n-1} x \sin x - \int \sin x\, (n-1) \cos^{n-2} x (-\sin x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^n x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\, I_{n-2} - (n-1)\, I_n$$

$$I_n + (n-1)\, I_n = \cos^{n-1} x \sin x + (n-1)\, I_{n-2}$$

$$I_n (1 + n - 1) = \cos^{n-1} x \sin x + (n-1)\, I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$$

This is the reduction formula for $\int \cos^n x\, dx$. The ultimate integral of the given integral will finally depends on either I_1 or I_0 according as n is odd or even. Now $I_1 = \int \cos x\, dx = \sin x$, $I_0 = \int \cos^0 x\, dx = \int dx = x$.

Example 1 *Evaluate $\int \cos^6 x\, dx$ by using reduction formula.*

Solution

(1) Let $\qquad I_6 = \int \cos^6 x\, dx$

(2) We know that $\qquad I_n = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{(n-1)}{n} I_{n-2} \qquad \ldots (1)$

Putting $n = 6$, we get

$$I_6 = \frac{\cos^5 x \sin x}{6} + \frac{5}{6} I_4 \quad \dots (2)$$

Putting $n = 4$, we get

$$I_4 = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2 \quad \dots (3)$$

Putting $n = 2$, we get

$$I_2 = \frac{\cos x \sin x}{2} + \frac{1}{2} I_0 \quad \dots (4)$$

Where $\quad I_0 = \int \cos^0 x \, dx = \int dx = x \quad \dots (5)$

Substituting (5) in (4) we get

$$I_2 = \frac{\cos x \sin x}{2} + \frac{1}{2} x$$

Substituting this value of I_2 in (3) we get

$$I_4 = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left(\frac{\cos x \sin x}{2} + \frac{x}{2}\right)$$

Substituting this value of I_4 in (2), we get

$$I_6 = \frac{\cos^5 x \sin x}{6} + \frac{5}{6}\left[\frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left(\frac{\cos x \sin x}{2} + \frac{x}{2}\right)\right]$$

Example 2 *Evaluate $\int \cos^7 x \, dx$ using reduction formula.*

Solution

Let $\quad I_7 = \int \cos^7 x \, dx$

We know that $\quad I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2} \quad \dots (1)$

Putting $n = 7$ in (1), we get

$$I_7 = \frac{\cos^6 x \sin x}{7} + \frac{6}{7} I_5 \quad \dots (2)$$

Putting $n = 5$ in (1), we get

$$I_5 = \frac{\cos^4 x \sin x}{5} + \frac{4}{5} I_3 \quad \dots (3)$$

Putting $n = 3$ in (1), we get

$$I_3 = \frac{\cos^2 x \sin x}{3} + \frac{2}{3} I_1 \quad \dots (4)$$

$$I_1 = \int \cos x \, dx = \sin x \quad \dots (5)$$

By using (2), (3), (4) and (5) we get,

$$I_7 = \frac{\cos^6 x \sin x}{7} + \frac{6}{7}\left[\frac{\cos^4 x \sin x}{5} + \frac{4}{5}\left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x\right)\right]$$

Example 3

Evaluate $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$ ***(n being a positive integer)***

Solution

Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$

We know that

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n}\int \cos^{n-2} x\, dx$$

Here $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$

$$= \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

$$I_n = 0 + \frac{n-1}{n} I_{n-2}$$

[$\because$ For both upper and lower limits the first term become zero]

$$I_n = \frac{n-1}{n} I_{n-2} \qquad \ldots (1)$$

Similarly $I_{n-2} = \dfrac{n-3}{n-2} I_{n-4} \qquad \ldots (2)$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6} \qquad \ldots (3)$$

From (1), (2) and (3), we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$$

Now there are two cases arises depends on the value of n.

Case (i) : When n is an even integer, we have,

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots \frac{3}{4} \cdot \frac{1}{2} \cdot I_0$$

Now $I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x\, dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

Case (ii) : When n is an odd integer, we have

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} \cdot I_1$$

Now $$I_1 = \int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}} = 1$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1$$

Example 4 *Evaluate* $\int_0^{\frac{\pi}{2}} \cos^8 x \, dx$.

Solution

Here $n = 8$ is an even integer.
Hence using the formula under Case *(i)*, we get

$$\int_0^{\frac{\pi}{2}} \cos^8 x \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{105\pi}{768}$$

Example 5 *Evaluate* $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx$

Solution

Here $n = 7$ is an odd integer.
Hence using the formula under Case *(ii)* we get,

$$\int_0^{\frac{\pi}{2}} \cos^7 x \, dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{48}{105}$$

Reduction formula for $\int \sin^m x \cos^n x \, dx$ (m, n being positive integer)

We have two types of reduction formulae for $\int \sin^m x \cos^n x \, dx$.

1. By reducing the power of $\cos x$ (*i.e.*, reducing n)
2. By reducing the power of $\sin x$ (*i.e.*, reducing m)

Reduction formula for $\int \sin^m x \cos^n x \, dx$ by reducing n

1. Let $$I_{m,n} = \int \sin^m x \cos^n x \, dx .$$

$$= \int \sin^m x \cos^{n-1} x \cos x \, dx .$$

$$= \int \sin^m x \cos^{n-1} x \, d(\sin x)$$

$$= \int \cos^{n-1} x \, d\left(\frac{\sin^{m+1} x}{m+1}\right) \qquad \left[\because \int x^n \, dx = \int d\left(\frac{x^{n+1}}{n+1}\right)\right]$$

$$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} - \int \frac{\sin^{m+1} x}{m+1} \, d(\cos^{n-1} x)$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{(m+1)} - \frac{1}{m+1} \int \sin^{m+1} x \, (n-1) \cos^{n-2} x \, (-\sin x) \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{(m+1)} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{(m+1)} + \frac{n-1}{m+1} \int \sin^{m} x \sin^2 x \cos^{n-2} x \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{(m+1)} + \frac{n-1}{m+1} \int \sin^{m} x \, (1-\cos^2 x) \cos^{n-2} x \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{(m+1)} + \frac{n-1}{m+1} \int \sin^{m} x \cos^{n-2} x \, dx - \frac{n-1}{m+1} \int \sin^{m} x \cos^{n} x \, dx$$

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{(m+1)} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$I_{m,n}\left(1 + \frac{n-1}{m+1}\right) = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n}\left(\frac{m+1+n-1}{m+1}\right) = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n}\left(\frac{m+n}{m+1}\right) = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

This is the required reduction formula in which the power of $\cos x$ is reduced by 2.

2. Reduction formula for $\int \sin^m x \cos^n x \, dx$ by reducing 'm'

Let $\quad I_{m,n} = \int \sin^m x \cos^n x \, dx$

$$= \int \sin^{m-1} x \sin x \cos^n x \, dx$$

$$= \int \sin^{m-1} x \cos^n x \, d(-\cos x)$$

$$= \int \sin^{m-1} x \, d\left(\frac{-\cos^{n+1} x}{n+1}\right) \qquad \left[\because \int x^n \, dx = \int d\left(\frac{x^{n+1}}{n+1}\right)\right]$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} - \int \frac{-\cos^{n+1} x}{n+1}\, d(\sin^{m-1} x)$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{1}{n+1}\int \cos^{n+1} x\,(m-1)\sin^{m-2} x \cos x\, dx$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \cos^{n} x \cos^2 x \sin^{m-2} x\, dx$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \cos^{n} x\,(1-\sin^2 x)\sin^{m-2} x\, dx$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \cos^{n} x \sin^{m-2} x\, dx - \frac{m-1}{n+1}\int \sin^{m} x \cos^{n} x\, dx$$

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$I_{m,n}\left(1+\frac{m-1}{n+1}\right) = \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n}$$

$$I_{m,n}\left(\frac{m+n}{n+1}\right) = \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n}$$

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

This is another form of reduction formula in which the power of $\sin x$ *i.e.*, m, is reduced by 2.

Note 1 : Consider the formula

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

Here if n is odd, then the ultimate integral is $I_{m,1}$.

$$I_{m,1} = \int \sin^m x \cos^1 x\, dx = \int \sin^m x\, d(\sin x)$$

$$= \frac{\sin^{m+1} x}{m+1}$$

If n is even, then the ultimate integral is $I_{m,0}$.

Now $$I_{m,0} = \int \sin^m x \cos^0 x\, dx$$

$$= \int \sin^m x\, dx$$

This can be evaluated by using the reduction formula for $\int \sin^m x\, dx$.

Note 2 : Consider the formula

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

If m is odd, then the ultimate integral is $I_{1,n}$.

Now $\quad I_{1,n} = \int \sin x \cos^n x \, dx = \int \cos^n x \, d(-\cos x)$

$$= \frac{-\cos^{n+1} x}{n+1}$$

If m is even, then the ultimate integral is $I_{0,n}$.

Now $\quad I_{0,n} = \int \sin^0 x \cos^n x \, dx$

$$= \int \cos^n x \, dx$$

which can be evaluated by using the reduction formula for $\int \cos^n x \, dx$.

Note 3 : If both m and n are odd integer reduce the smaller index. Similarly if both m and n are even reduce the smaller index.

Example 1

Evaluate $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$.

Solution

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx.$$

We know that

$$\int \sin^m x \cos^n x \, dx = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx$$

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\frac{\pi}{2}} + \frac{n-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-2} x \, dx$$

$$I_{m,n} = 0 + \frac{n-1}{m+n} I_{m,n-2}$$

Similarly $\quad I_{m,n-2} = \frac{n-3}{m+n-2} I_{m,n-4}$

$$I_{m,n-4} = \frac{n-5}{m+n-4} I_{m,n-6}$$

when n is odd $\quad I_{m,n} = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \ldots I_{m,1} \ldots \text{(A)}$

when n is even $\quad I_{m,n} = \dfrac{n-1}{m+n} \cdot \dfrac{n-3}{m+n-2} \cdot \dfrac{n-5}{m+n-4} \cdots I_{m,0}$ ... (B)

Now, $\quad I_{m,1} = \displaystyle\int_0^{\frac{\pi}{2}} \sin^m x \cos x \, dx .$

$$= \int_0^{\frac{\pi}{2}} \sin^m x \, d(\sin x) = \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\frac{\pi}{2}}$$

$$\therefore I_{m,1} = \frac{1}{m+1} \quad \text{... (C)}$$

$$I_{m,0} = \int_0^{\frac{\pi}{2}} \sin^m x \, dx .$$

$$= \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \; (m \text{ is even}) \quad \text{... (D)}$$

$$= \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{2}{3} \; (m \text{ is odd}) \quad \text{... (E)}$$

Combining (A) and (C) we get

$$I_{m,n} = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

(n is odd, m may be odd or even)

Combining (B) and (D) we get

$$I_{m,n} = \left[\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{1}{m+2}\right] \left[\frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

(Both m and n are even)

Combining (B) and (E) we get

$$I_{m,n} = \left[\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{1}{m+2}\right] \left[\frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{2}{3}\right]$$

(n is even, m is odd)

Note : $\displaystyle\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx .$

$$= \frac{[(m-1)(m-3) \ldots \text{go on subtracting 2 till we get 1}]\;[(n-1)(n-3) \ldots \text{go on subtracting 2 till we get 1}]}{[(m+n)(m+n-2) \ldots \text{go on subtracting 2 till we get 1}]}$$

Example 2

Evaluate $\int_0^{\frac{\pi}{2}} \sin^8 x \cos^6 x \, dx$.

Solution

Here $m = 8$, $n = 6$ (both even) also $m + n = 8 + 6 = 14$.

Using the above note we get

$$\int_0^{\frac{\pi}{2}} \sin^8 x \cos^6 x \, dx \, . = \frac{[(8-1)(8-3)(8-5)(8-7)]\,[(6-1)(6-3)(6-5)]}{(14)(14-2)(14-4)(14-6)(14-8)(14-10)(14-12)} \times \frac{\pi}{2}$$

$$= \frac{7 \times 5 \times 3 \times 1 \times 5 \times 3 \times 2}{14 \times 12 \times 10 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{5\pi}{4096}$$

Example 3

Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x \, dx$.

Solution

Here $m = 7$ (odd), $n = 5$ (odd). $m + n = 7 + 5 = 12$

By using the above note we get,

$$\int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x \, dx = \frac{[(7-1)(7-3)(7-5)]\,[(5-1)(5-3)]}{[(12)(12-2)(12-4)\,(12-6)(12-8)(12-10)}$$

$$= \frac{6 \times 4 \times 2 \times 4 \times 2}{12 \times 10 \times 8 \times 6 \times 4 \times 2} \qquad \left[\text{no } \frac{\pi}{2} \text{ by note}\right]$$

$$= \frac{1}{120}$$

Example 4

Evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x \, dx$

Solution

Here $m = 5$ (odd), $n = 6$ (even); $m + n = 5 + 6 = 11$.

By using note we get,

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x \, dx = \frac{[(5-1)(5-3)] \, [(6-1)(6-3)(6-5)]}{11(11-2)(11-4)(11-6)(11-8)(11-10)}$$

$$= \frac{4 \times 2 \times 5 \times 3 \times 1}{11 \times 9 \times 7 \times 5 \times 3 \times 1} \quad \left(\text{no } \frac{\pi}{2} \text{ by note}\right)$$

$$= \frac{8}{693}$$

Reduction formula for $\int x^n e^{ax} dx$ (n being a positive integer)

Let $\quad I_n = \int x^n e^{ax} dx = \int x^n d\left(\frac{e^{ax}}{a}\right)$

$$= \frac{x^n e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot nx^{n-1} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} dx$$

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

This is the reduction formula for $\int x^n e^{ax} dx$.

Example 1 *Evaluate $\int x^5 e^{2x} dx$.*

Solution

Let $\quad I_5 = \int x^5 e^{2x} dx$

We know that $I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$ ... (1)

Putting $n = 5$ in (1), we get

$$I_5 = \frac{x^5 e^{2x}}{2} - \frac{5}{2} I_4 \quad \text{... (2)}$$

(For our problem $n = 5$, $a = 2$)

Putting $n = 4, 3, 2, 1$ successively in (1), we get respectively

$$I_4 = \frac{x^4 e^{2x}}{2} - \frac{4}{2} I_3 \qquad \dots (3)$$

$$I_3 = \frac{x^3 e^{2x}}{2} - \frac{3}{2} I_2 \qquad \dots (4)$$

$$I_2 = \frac{x^2 e^{2x}}{2} - \frac{2}{2} I_1 \qquad \dots (5)$$

$$I_1 = \frac{xe^{2x}}{2} - \frac{1}{2} I_0 \qquad \dots (6)$$

$$I_0 = \int x^0 e^{2x}x\, dx = \int e^{2x}\, dx = \frac{e^{2x}}{2} \qquad \dots (7)$$

From (2), (3), (4), (5), (6) and (7), we get

$$I_5 = \frac{x^5 e^{2x}}{2} - \frac{5}{2}\left[\frac{x^4 e^{2x}}{2} - 2\left\{\frac{x^3 e^{2x}}{2} - \frac{3}{2}\left(\frac{x^2 e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{1}{4}e^{2x}\right)\right\}\right]$$

Simplifying we get

$$= \frac{x^5 e^{2x}}{2} - \frac{5}{2}\left[\frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - \frac{3}{2} xe^{2x} + \frac{3}{4} e^{2x}\right]$$

Example 2

If $I_n = \int x^n e^{-x} dx$, n being a positive integer, show that $I_n = -x^n e^{-x} + nI_{n-1}$. Hence show that

$$\int_0^\infty x^n e^{-x} dx = n!$$

Solution

Given

$$I_n = \int x^n e^{-x} dx$$

$$= \int x^n\, d(-e^{-x})$$

$$= -x^n e^{-x} - \int -e^{-x}\, d(x^n)$$

$$= -x^n e^{-x} + \int e^{-x}\, nx^{n-1}\, dx$$

$$= -x^n e^{-x} + n\int e^{-x}\, x^{n-1}\, dx$$

$$I_n = -x^n e^{-x} + nI_{n-1} \qquad \dots (1)$$

From (1) we get,

$$\int x^n e^{-x} dx = -x^n e^{-x} + n\int x^{n-1} e^{-x} dx$$

Now $$I_n = \int_0^\infty x^n e^{-x} dx = [-x^n e^{-x}]_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx$$

i.e., $$I_n = 0 + nI_{n-1} \qquad \dots (2)$$

$$\therefore I_{n-1} = 0 + (n-1)\, I_{n-2} \qquad \dots (3)$$

$$I_{n-2} = (n-2)\, I_{n-3},\ etc. \qquad \dots (4)$$

From (2), (3), and (4) we get

$$I_n = n(n-1)(n-2)(n-3)\ldots I_0$$

where $$I_0 = \int_0^\infty x^0 e^{-x}\,dx = [-e^{-x}]_0^\infty = (0+1) = 1$$

$$\therefore I_n = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$$

$$I_n = n!$$

Example 3

If $I_n = \int_0^a x^n\, e^{-x}\,dx$, prove that

$$I_n - (n+a)\,I_{n-1} + a\,(n-1)\,I_{n-2} = 0.$$

Solution

$$I_n = \int_0^a x^n\, e^{-x}\,dx = \int_0^a x^n\, d(-e^{-x})$$

$$= [-x^n e^{-x}]_0^a - \int_0^a -e^{-x}\, d(x^n)$$

$$= (-a^n e^{-a} + 0) + \int_0^a n\, x^{n-1}\, e^{-x}\,dx$$

$$I_n = -a^n e^{-a} + nI_{n-1} \qquad \ldots (1)$$

Replacing n by $n-1$ in (1), we get,

$$I_{n-1} = -a^{n-1} e^{-a} + (n-1)\,I_{n-2}$$

i.e., $$aI_{n-1} = -a^n e^{-a} + a\,(n-1)\,I_{n-2} \qquad \ldots (2)$$

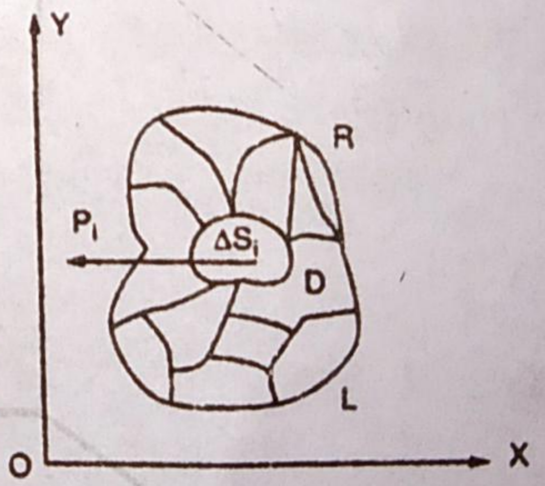
$(1)-(2) \Rightarrow$ $$I_n - aI_{n-1} = nI_{n-1} - a\,(n-1)\,I_{n-2}$$

i.e., $I_n - (a+n)\,I_{n-1} + a\,(n-1)\,I_{n-2} = 0$

Reduction formula for

MULTIPLE INTEGRALS (DOUBLE AND TRIPLE INTEGRALS)

Let us consider a closed region R bounded by a line L in the XY-plane. In the region R let $z = f(x, y)$ be a continuous function. Let us divide the region R into n parts as $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ are called subregions. To avoid confusion let us assume that $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ represents ***subregions as well as their areas.***



Let P_i be any point in the subregion Δs_i [whether P_i may lie inside or in boundary of Δs_i and $P_i = (x_i, y_i)$]

❑ EVALUATION OF DOUBLE INTEGRALS

We know that the double integral over the region R of a function $f(x, y)$ is

$$\iint_R f(x, y)\, dx\, dy \qquad \ldots (1)$$

Case (i) **: Now let R be the region bounded by the lines $x = c_1, x = c_2$, $y = c_3$ and $y = c_4$ where c_1, c_2, c_3, c_4 are constants. Clearly the region R is a rectangle ABCD as shown in figure. Since we know the region of integration R the double integral $\iint_R f(x, y)\, dx\, dy$ can be written as**

$$\int_{c_3}^{c_4}\int_{c_1}^{c_2} f(x, y)\, dx\, dy \qquad \ldots (2)$$

(Here we replace R by putting the limits for both x and y)

Example 1 Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$

Solution

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx = \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \left\{\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}\right\}_0^a$$

$$= \frac{a^2}{2}\sin^{-1}(1) = \frac{\pi a^2}{4}$$

Example 2

Evaluate $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\, dy\, dx$

Solution

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\, dy\, dx = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{2ax-x^2}} dx$$

$$= \int_0^{2a} x\left[\frac{2ax-x^2}{2}\right] dx$$

$$= \int_0^{2a}\left(ax^2 - \frac{x^3}{2}\right) dx$$

$$= \left[a\cdot\frac{x^3}{3} - \frac{x^4}{8}\right]_0^{2a} = \left[a\cdot\frac{8a^3}{3} - \frac{16a^4}{8}\right]$$

$$= \left[\frac{8}{3}a^4 - 2a^4\right]$$

$$= \frac{2a^4}{3}$$

Example 3

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$$

To evaluate $\displaystyle\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$

Put $a^2 - r^2 = t$	$r = 0, \quad t = a^2$
$-2r\, dr = dt$	$r = a\cos\theta, \quad t = a^2\sin^2\theta$
$r\, dr = \dfrac{-dt}{2}$	

$$= -\frac{1}{2}\left[\frac{r^{\frac{3}{2}}}{\frac{3}{2}}\right]_{a^2}^{a^2\sin^2\theta}$$

$$= -\frac{1}{3}\left[(a^2\sin^2\theta)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}\left[a^3\sin^3\theta - a^3\right]$$

$$\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta = -\frac{1}{3}\int_0^{\frac{\pi}{2}}\left[a^3(\sin^3\theta - 1)\right]d\theta$$

$$= \frac{-a^3}{3}\left[\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta - \int_0^{\frac{\pi}{2}}d\theta\right]$$

$$= \frac{-a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right]$$

$$= \frac{a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right]$$

Example 4

Evaluate $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\,dr\,d\theta$

Solution

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^{2\cos\theta}d\theta = \frac{1}{3}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta\,d\theta$$

$$= \frac{8}{3}\left[2\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta\right]$$

[$\cos^3\theta$ is an even function]

$$= \frac{16}{3}\left[\frac{2}{3}\right] = \frac{32}{9}$$

Example 5

Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Solution

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{y^2+\left(\sqrt{x^2+1}\right)^2}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{x^2+1}} \left\{\tan^{-1}\left[\frac{y}{\sqrt{x^2+1}}\right]\right\}_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{x^2+1}} \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$= \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx \quad = \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right)\right]$$

Example 6

Evaluate $\displaystyle\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Solution

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(x^{5/2} + \frac{x^{3/2}}{3}\right) - \left(x^3 + \frac{x^3}{3}\right)\right] dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\right) dx$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{3 \times 5/2} - \frac{4}{3}\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{7} \left[+\frac{2}{15} - \frac{1}{3} = \frac{3}{35}\right.$$

Example 7 Evaluate $\int_0^2 \int_x^{x^2} e^{y/x}\, dy\, dx$

Solution

$$\int_0^2 \int_x^{x^2} e^{y/x}\, dy\, dx = \int_0^2 \left(\frac{e^{y/x}}{1/x}\right)_0^{x^2} dx \qquad \left[\int e^{ax}\, dx = \frac{e^{ax}}{a}\right]$$

$$= \int_0^2 x e^{x^2/x}\, dx = \int_0^2 x e^x\, dx$$

$$= \left(xe^x - e^x\right)_0^2 = 2e^2 - e^2 + 1$$

$$= e^2 - 1$$

Example 8 Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$

Solution

$$\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy = \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy = \int_0^a y\,\frac{ay}{2}\, dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a = \frac{a}{6}\, a^3$$

$$= \frac{a^4}{6}$$

Example 9 Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(\theta + \phi)\, d\theta\, d\phi$

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(\theta + \phi)\, d\theta\, d\phi = \int_0^{\frac{\pi}{2}} [-\cos(\theta + \phi)]_0^{\frac{\pi}{2}}\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \left[-\cos\left(\frac{\pi}{2} + \phi\right) + \cos\phi\right] d\phi$$

$$= \int_0^{\frac{\pi}{2}} (\sin\phi + \cos\phi)\, d\phi$$

$$[\because \cos(90+\phi) = -\sin\phi]$$

$$= [-\cos\phi + \sin\phi]_0^{\frac{\pi}{2}}$$

$$= \left\{-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right\} - \{-\cos 0 + \sin 0\}$$

$$= 0 + 1 + 1 - 0 = 2$$

■ EVALUATION OF TRIPLE INTEGRALS ■

Example 10 Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\, dy\, dx}{(x+y+z+1)^3}$

Solution

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{-1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\, dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \{(2)^{-2} - (x+y+1)^{-2}\}\, dy\, dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{y}{4} + (x+y+1)^{-1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left\{\frac{1-x}{4} + 2^{-1} - (x+1)^{-1}\right\} dx = -\frac{1}{2}\int_0^1 \left(\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right) dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right) dx = \frac{-1}{2}\int_0^1 \left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right] dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2\right] = \frac{1}{16}[8\log 2 - 5]$$

Example 11 Evaluate $\int_0^4 \int_0^1 \int_0^1 (x+y+z)\, dz\, dy\, dx$

Solution

$$\int_0^4 \int_0^1 \int_0^1 (x+y+z)\, dz\, dy\, dx = \int_0^4 \int_0^1 \left(xz + yz + \frac{z^2}{2}\right)_0^1 dy\ dx$$

$$= \int_0^4 \int_0^1 \left(x + y + \frac{1}{2}\right) dy\ dx = \int_0^4 \left(xy + \frac{y^2}{2} + \frac{1}{2}\ y\right)_0^1 dx$$

$$= \int_0^4 \left(x + \frac{1}{2} + \frac{1}{2}\right) dx = \left[\frac{x^2}{2} + x\right]_0^4$$

$$= 12$$

Example 12 Evaluate $\int_1^3 \int_{\frac{1}{x}}^1 \int_0^{\sqrt{xy}} xyz\ dz\, dy\, dx$

Solution

$$\int_1^3 \int_{\frac{1}{x}}^1 \int_0^{\sqrt{xy}} xyz\ dz\, dy\, dx$$

$$= \int_1^3 \int_{\frac{1}{x}}^1 xy \left(\frac{z^2}{2}\right)_0^{\sqrt{xy}} dy\ dx$$

$$= \int_1^3 \int_{\frac{1}{x}}^1 xy\ \frac{xy}{2}\ dy\ dx = \frac{1}{2} \int_1^3 x^2 \left(\frac{y^3}{3}\right)_{\frac{1}{x}}^1 dx$$

$$= \frac{1}{6} \int_1^3 x^2 \left(1 - \frac{1}{x^3}\right) dx = \frac{1}{6} \int_1^3 \left(x^2 - \frac{1}{x}\right) dx$$

$$= \frac{1}{6} \left[\frac{x^3}{3} - \log x\right]_1^3 = \frac{1}{6}\left[9 - \log 3 - \frac{1}{3}\right]$$

$$= \frac{1}{6}\left[\frac{26}{3} - \log 3\right]$$

Example 13 Evaluate $\displaystyle\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy$

Solution

$$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy = \int_0^1 \int_{y^2}^1 x \, (z)_0^{1-x} \, dx \, dy$$

$$= \int_0^1 \int_{y^2}^1 x(1-x) \, dx \, dy = \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{y^2}^1 dy$$

$$= \int_0^1 \left\{\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right\} dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy = \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$= \left[\frac{1}{6} - \frac{1}{10} + \frac{1}{21}\right] = \frac{12}{105}$$

Example 14 Evaluate $\displaystyle\int_0^a \int_0^b \int_0^c (x+y+z) \, dz \, dy \, dx$

Solution

$$\int_0^a \int_0^b \int_0^c (x+y+z) \, dz \, dy \, dx = \int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2}\right)_0^c dy \, dx$$

$$= \int_0^a \int_0^b \left(cx + cy + \frac{c^2}{2}\right) dy \, dx$$

$$= \int_0^a \left(cxy + \frac{cy^2}{2} + \frac{c^2 y}{2}\right)_0^b dx = \int_0^a \left(bcx + \frac{b^2 c}{2} + \frac{c^2 b}{2}\right) dx$$

$$= \left[bc\frac{x^2}{2} + \frac{b^2 c}{2} x + \frac{c^2 b}{2} x\right]_0^a = \frac{a^2 bc}{2} + \frac{b^2 ca}{2} + \frac{ac^2 b}{2}$$

$$= \frac{abc}{2} \, (a + b + c)$$

Example 15 Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$

Solution

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy\,(1-x^2-y^2)\; dy\, dx$$

$$= \frac{1}{2}\int_0^1 x \int_0^{\sqrt{1-x^2}} (y - x^2y - y^3)\; dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{(1-x^2)}{2} - \frac{(1-x^2)^2}{4}\right] dx$$

$$= \frac{1}{8}\int_0^1 x\,(1+x^4-2x^2)\; dx = \frac{1}{8}\left[\frac{x^2}{2} + \frac{x^6}{6} - \frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{1}{2} + \frac{1}{6} - \frac{1}{2}\right]$$

$$= \frac{1}{48}$$

Example 16 Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dx\, dy$

Solution

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dx\, dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dx\, dy$$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\; dx\, dy$$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\; dx\, dy$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} \right.$$

$$\left. + \left(\frac{a^2-y^2}{2}\right) \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{\pi}{4}\int_0^a (a^2-y^2)\; dy = \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi}{4}\,\frac{2a^3}{3} = \frac{\pi a^3}{6}$$

CHANGE OF ORDER OF INTEGRATION

The evaluation of some double integrals may be very difficult. In this case, we may evaluate it easily by changing the order of integration in a given double integral. When we change the order of integration the limits are also changed. The following points are very important when the change of order of integration takes place.

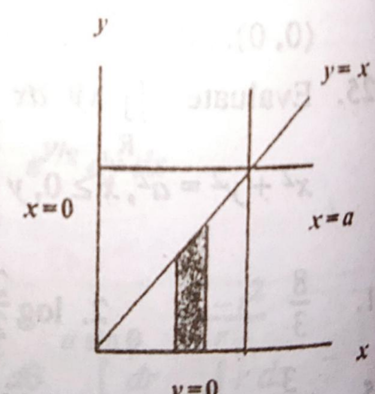
1. If the limits of the inner integral is a function of x (or function of y) then the first integration should be w.r.t. y (or w.r.t. x)
2. Draw the region of integration by using the given limits. (Here the knowledge of parabola, ellipse, hyperbola, circle, straight lines are required).
3. If the integration is first w.r.t. x keeping y as a constant then consider the horizontal strip and find the new limits accordingly.
4. If the integration is first w.r.t. y, keeping x as a constant then consider the vertical strip and find the new limits accordingly.
5. After finding the new limits evaluate the inner integral first and then the outer integral.

Example 1 *Change the order of integration in* $\int_0^a \int_y^a \frac{x\, dy\, dx}{x^2+y^2}$ *and hence evaluate it.*

Solution

First we have to rewrite the given problem as follows.

$$\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2+y^2}$$



The limits for x varies from $x = y$ to $x = a$ and the limits for y varies from $y = 0$ to $y = a$.

The region of integration is enclosed between the following curves.

$y = 0$ (x - axis)

$y = x$ (a line which bisects the angle between x-axis and y-axis)

$x = a$ (a line parallel to y - axis passing at a distance of 'a' unit from y - axis)

$y = a$ (a line parallel to x - axis passing at a distance of 'a' unit from x - axis)

This region of integration is shown in the figure.

Now we have to change the order of integration. That is we have to first integrate w.r.t. y and then w.r.t. x. Since the first integration is w.r.t. y we have to consider vertical strip.

For this vertical strip (see figure) the limits for 'y' varies from 0 to x.

Now we move this vertical strip horizontally from $x = 0$ to $x = a$.

i.e., the limits for x is $x = 0$ to $x = a$.

$$\text{Hence} \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a \int_0^x \frac{x\,dy\,dx}{x^2+y^2} = \int_0^a \left(\int_0^x \frac{x\,dy}{y^2+x^2} \right) dx$$

$$= \int_0^a x \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x dx$$

$$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right]$$

$$= \int_0^a \left[\tan^{-1}\left(\frac{x}{x}\right) - \frac{1}{x} \tan^{-1}\left(\frac{0}{x}\right) \right] dx$$

$$= \int_0^a \tan^{-1}(1)\,dx = \frac{\pi}{4} \int_0^a dx$$

$$= \frac{\pi}{4} [x]_0^a = \frac{\pi}{4} [a - 0]$$

$$= \frac{\pi a}{4}$$

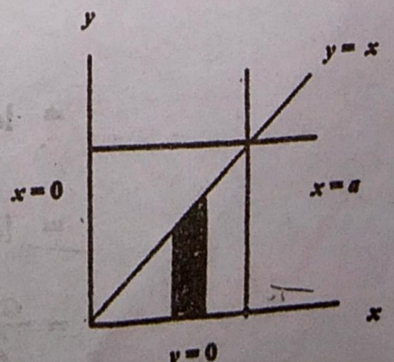
Example 2 *Change the order of integration in* $\displaystyle\int_0^a \int_y^a \frac{x\,\cancel{dy\,dx}}{\sqrt{x^2+y^2}}$ dx dy

and hence evaluate it.

Solution

First we have to rewrite the given problem as

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$



The limits for x varies from $x = y$ to $x = a$ and the limits for y varies from $y = 0$ to $y = a$.

The region of integration is the region enclosed by the four lines $y = 0, y = a, x = a, x = y$. This region of integration is shown in the figure.

Now we have to change the order of integration. That is our first integration should be w.r.t. y and then w.r.t. x. Since the first integration is w.r.t. y we have to consider vertical strip.

The limits for this vertical strip are $y = 0$ to $y = x$. Then we slide this strip horizontally from $x = 0$ to $x = a$. *i.e.*, the limits for x is $x = 0$ to $x = a$.

Hence $\int_0^a \int_y^a \frac{x\, dx\, dy}{\sqrt{x^2+y^2}} = \int_0^a \int_0^x \frac{x\, dy\, dx}{\sqrt{y^2+x^2}}$

$$= \int_0^a x \left[\int_0^x \frac{dy}{\sqrt{y^2+x^2}} \right] dx$$

$$= \int_0^a x \left[\log(y + \sqrt{y^2+x^2}) \right]_0^x dx$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left[x + \sqrt{x^2+a^2}\right] \right]$$

$$= \int_0^a x \left\{ \log\left[x + \sqrt{x^2+x^2}\right] - \log\left(\sqrt{x^2}\right) \right\} dx$$

$$= \int_0^a x \left\{ \log\left[x + \sqrt{2x^2}\right] - \log(x) \right\} dx$$

$$= \int_0^a x \left\{ \log x\left[1 + \sqrt{2}\right] - \log(x) \right\} dx$$

$$= \int_0^a x \left[\log x + \log(1+\sqrt{2}) - \log x \right] dx$$

$$= \int_0^a x \log(1+\sqrt{2})\, dx$$

$$= \log(1+\sqrt{2}) \int_0^a x\, dx$$

$$= \log(1+\sqrt{2}) \left[\frac{x^2}{2}\right]_0^a$$

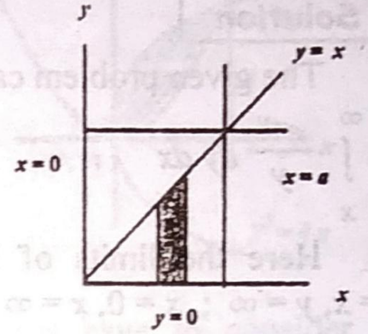
$$= \log(1+\sqrt{2}) \left[\frac{a^2}{2} - 0\right]$$

$$= \frac{a^2}{2} \log(1+\sqrt{2})$$

Example 3

Change the order of integration in $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$ ***and then evaluate it.***

Solution



The limits for x varies from $x = y$ to $x = a$ and the limits for y varies from $y = 0$ to $y = a$.

The region of integration is enclosed between the four lines $x = y$, $x = a$, $y = 0$, $y = a$. This region of integration is shown in the figure.

Now we have to change the order of integration. That is first we have to integrate w.r.t. y and then w.r.t. x. Since the first integration is w.r.t. y we have to consider the vertical strip. Along vertical strip the limits for 'y' varies from $y = 0$ to $y = x$.

Now sliding this strip horizontally x varies from $x = 0$ to $x = a$.

$$\text{Hence } \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy = \int_0^a \int_0^x \frac{x^2}{\sqrt{x^2+y^2}}\, dy\, dx$$

$$= \int_0^a x^2 \left[\log \left\{y + \sqrt{x^2+y^2}\right\}\right]_0^x dx$$

$$= \int_0^a x^2 \left[\log \left(x + \sqrt{2x^2}\right) - \log x\right] dx$$

$$= \int_0^a x^2 \left[\log \left\{x\,(1+\sqrt{2})\right\} - \log x\right] dx$$

$$= \int_0^a x^2 \left[\log x + \log(1+\sqrt{2}) - \log x\right] dx$$

$$= \int_0^a x^2 \log (1+\sqrt{2})\, dx$$

$$= \log (1+\sqrt{2}) \int_0^a x^2\, dx$$

$$= \log (1+\sqrt{2}) \left[\frac{x^3}{3}\right]_0^a$$

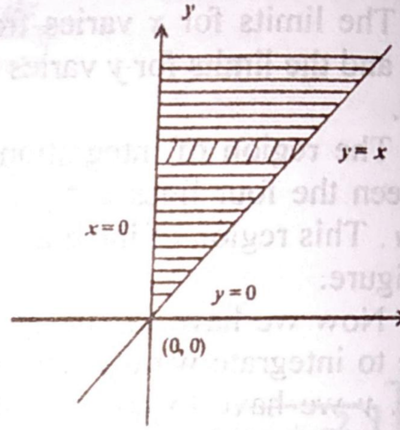
$$= \frac{a^3}{3} \log (1+\sqrt{2})$$

Example 4 *Evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, \cancel{dx\,dy}$ dydx *by changing the order of integration.*

Solution

The given problem can be written as

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$



Here the limits of integration are $y = x, y = \infty$; $x = 0, x = \infty$

The region of integration is shown in the figure.

To change the order means we have to first integrate w.r.t. 'x' and then w.r.t. 'y'.

Since the first integration is w.r.t. 'x' we have to consider horizontal strip. Along this horizontal strip x varies from $x = 0$ to $x = y$. Now sliding this strip vertically, y varies from $y = 0$ to $y = \infty$.

$$\text{Hence} \quad \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \frac{e^{-y}}{y}\, (x)_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \times y\, dy = \int_0^\infty e^{-y}\, dy$$

$$= \left(-e^{-y}\right)_0^\infty = -e^{-\infty} + e^{-0}$$

$$= 1$$

Example 5 *Evaluate by changing the order of integration in* $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\, dx.$

Solution

The region of integration is bounded by the following curves.

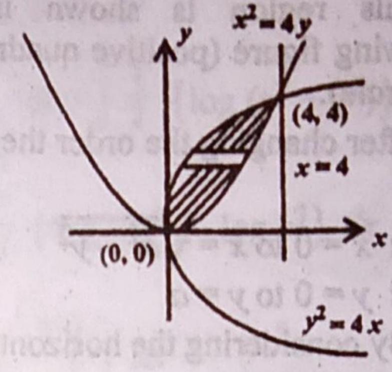
$$y = \frac{x^2}{4} \text{ (or) } x^2 = 4y \quad \text{(Parabola; open upward)}$$

$$y = 2\sqrt{x} \text{ (or) } y^2 = 4x \quad \text{(Parabola; open rightward)}$$

$x = 0$ ($y-$axis)

$x = 4$ (line parallel to $y-$axis)

The enclosed region is as shown is figure.



The point of intersection of the parabolas $x^2 = 4y$ and $y^2 = 4x$ is (4, 4). The line $x = 4$ is also passing through the point (4, 4).

After changing the order of integration the first integration should be w.r.t. 'x' and then w.r.t. 'y'.

Since the first integration is w.r.t. 'x' we have to consider the horizontal strip. Along the horizontal strip 'x' varies from $x = \frac{y^2}{4}$ to $x = 2\sqrt{y}$.

Then sliding this strip vertically we have y varies from $y = 0$ to $y = 4$.

$$\therefore \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\,dx = \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\,dy = \int_0^4 (x)_{\frac{y^2}{4}}^{2\sqrt{y}}\,dy$$

$$= \int_0^4 \left(2\sqrt{y} - \frac{y^2}{4}\right) dy = \left[\frac{2y^{3/2}}{\frac{3}{2}} - \frac{y^3}{12}\right]_0^4$$

$$= \frac{4}{3}(4)^{\frac{3}{2}} - \frac{64}{12} = \frac{4 \times 8}{3} - \frac{16}{3}$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3}$$

Example 6

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$ ***by changing the order of integration.***

Solution

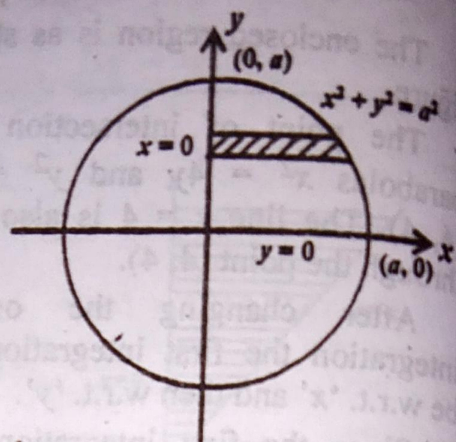
The region of integration is bounded by the following curves.

$y = 0$ ($x-$axis)

$y = \sqrt{a^2 - x^2}$ (or) $x^2 + y^2 = a^2$ (Circle)

$$x = 0 \quad (y-\text{axis})$$
$$y = a \quad (\text{line parallel to } x-\text{axis})$$

This region is shown in the following figure (positive quadrant of the circle).



After changing the order the limits for

$x : x = 0$ to $x = \sqrt{a^2 - y^2}$
$y : y = 0$ to $y = a$

[By considering the horizontal strip]

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\, dx\, dy$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a y\,(a^2 - y^2)\, dy = \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}$$

Example 7

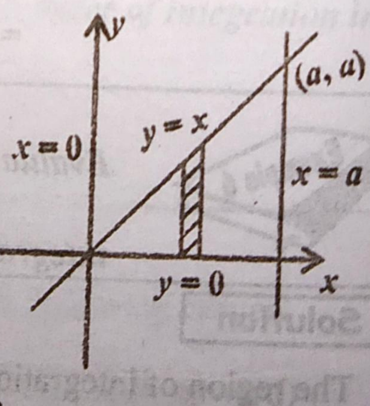
Change the order of integration in the integral and hence find the value of $I = \int_0^a \int_y^a \frac{x+y}{x^2+y^2}\, dx\, dy$

Solution

The region of integration is bounded by the following curves:

$x = y$ (a line which bisect the angle of xoy)
$x = a$ (a line parallel to $y-$ axis)
$y = 0$ ($x-$ axis)
$y = a$ (a line parallel to $x-$ axis)

This region is shown in the figure.



After changing the order of integration the limits for

$x : x = 0$ to $x = a$
$y : y = 0$ to $y = x$

(By considering horizontal strip).

$$\therefore \int_0^a \int_y^a \frac{x+y}{x^2+y^2}\, dx\, dy = \int_0^a \int_0^x \left(\frac{x+y}{x^2+y^2}\right) dy\, dx$$

$$= \int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx + \int_0^a \int_0^x \frac{y}{x^2+y^2}\, dy\, dx$$

$$= \int_0^a x \left\{ \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right\}_0^x dx + \int_0^a \frac{1}{2} \left\{ \log (x^2+y^2) \right\}_0^x dx$$

$$= \int_0^a \left(\frac{\pi}{4} - 0\right) dx + \int_0^a \frac{1}{2} \left\{ \log (2x^2) - \log x^2 \right\} dx$$

$$= \frac{\pi}{4} (x)_0^a + \frac{1}{2} \int_0^a \log\left(\frac{2x^2}{x^2}\right) dx$$

$$= \frac{\pi a}{4} + \frac{1}{2} \log 2 \times a$$

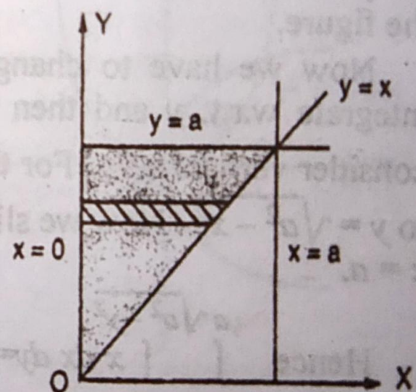
$$= \frac{\pi a}{4} + \frac{a}{2} \log 2$$

Example 8 *Change the order of integration in* $\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$ *and hence evaluate it.* *[Apr. 86, Nov. 90]*

Solution

Here the limits for y varies from $y = x$ to $y = a$ and the limits for x varies from $x = 0$ to $x = a$. Our region of integration is the area enclosed between these four lines viz. $y = x$, $y = a$, $x = 0$ and $x = a$.

The region of integration is shown in figure.



The given integral is $\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$.

Now we have to change the order of integration. *i.e.*, our integration should be first w.r.t. x and then w.r.t. y. When the first integration is w.r.t. x, we have to keep y as constant. In other words first integration is along x direction keeping y as constant. Whenever the integration is w.r.t. x we take horizontal strips. From figure for the horizontal strips x varies from 0 to y. Now we slide this strip along vertical direction. When we move this strip along vertical direction y varies from 0 to a.

Hence $\quad \int_0^a \int_x^a (x^2+y^2)\, dy\, dx = \int_0^a \int_0^y (x^2+y^2)\, dx\, dy$

$$= \int_0^a \left[\frac{x^3}{3} + y^2 \cdot x\right]_0^y dy = \int_0^a \left(\frac{y^3}{3} + y^3\right) dy$$

$$= \left[\frac{y^4}{12} + \frac{y^4}{4}\right]_0^a = \frac{a^4}{12} + \frac{a^4}{4} = \frac{a^4}{3}$$

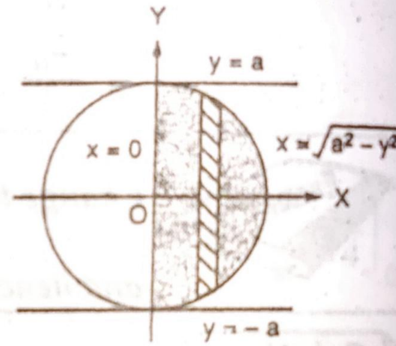
Example 9 *Change the order of integration in* $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} x \, dx \, dy$.

[Nov. 88]

Solution

The limits for x varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and the limits for y varies from $y = -a$ to $y = a$.

The region of integration is the region enclosed between the curves $x = 0$ (Y axis), $x = \sqrt{a^2 - y^2}$ (*i.e.*, $x^2 + y^2 = a^2$ which is a circle whose centre is at $(0, 0)$ and radius a), $y = -a$ (a line parallel to X axis passing through $(0, -a)$), $y = a$ (a line parallel to X axis passing through the point $(0, a)$). This region is shown in the figure.



Now we have to change the order of integration. That is we first integrate w.r.t. y and then w.r.t. x since first integration is w.r.t. y we consider vertical strip. For this vertical strip y varies from $y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$. Now we slide this vertical strip horizontally from $x = 0$ to $x = a$.

Hence $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} x \, dx \, dy = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x \, dy \, dx = 2\int_0^a \int_0^{\sqrt{a^2-x^2}} x \, dy \, dx$

$$= 2\int_0^a x \, [y]_0^{\sqrt{a^2-x^2}} dx = 2\int_0^a x\sqrt{a^2 - x^2} \, dx$$

Put $\quad x = a \sin \theta$

$\quad dx = a \cos \theta \, d\theta$

when $x = 0, \theta = 0$

when $x = a, \theta = \dfrac{\pi}{2}$

$$= 2\int_0^{\frac{\pi}{2}} a \sin\theta \sqrt{a^2 - a^2 \sin^2\theta} \quad a\cos\theta \, d\theta$$

$$= 2a^3 \int_0^{\frac{\pi}{2}} \sin\theta \cos^2\theta \, d\theta = 2a^3 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d(-\cos\theta)$$

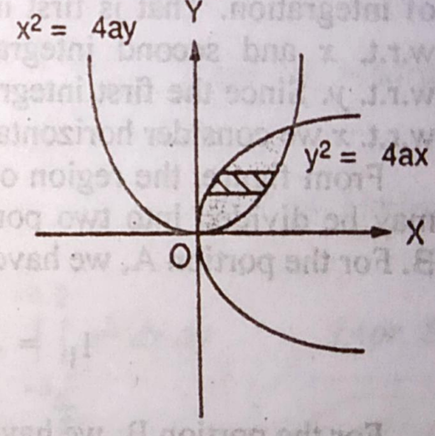
$$= 2a^3 \left[\frac{-\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{2a^3}{3}$$

Example 10 *Change the order of integration and hence evaluate it* $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy \, dy \, dx.$

Solution

The limits for y varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and the limits for x varies from $x = 0$ to $x = 4a$.

The region of integration is enclosed between the following curves viz. $x^2 = 4ay$ (parabola) $y^2 = 4ax$ (parabola), $x = 0$ (Y axis), and $x = 4a$ (a line parallel to Y axis passing through the point $(4a, 0)$).



The two curves $x^2 = 4ay$ and $y^2 = 4ax$ meet at the point $(4a, 4a)$. Also the line $x = 4a$ passes through the point of intersection of the two parabolas. The region of integration is shown in figure.

Now we have to change the order of integration. That is first we integrate w.r.t. x and then w.r.t. y. Since first integration is w.r.t. x, we consider horizontal strip. For the horizontal strip (see fig.) the limits for x varies from $x = \frac{y^2}{4a}$ to $x = \sqrt{4ay}$. Then we move this strip vertically from $y = 0$ to $y = 4a$. Hence

$$\int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ax}} xy \, dy \, dx = \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy \, dx \, dy = \int_0^{4a} \left[\frac{x^2}{2} \cdot y\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left[\frac{4ay}{2} \cdot y - \frac{y^4}{2 \cdot 16a^2} \cdot y\right] dy$$

$$= \left[2a \cdot \frac{y^3}{3}\right]_0^{4a} - \left[\frac{1}{32 \cdot a^2} \cdot \frac{y^6}{6}\right]_0^{4a}$$

$$= \frac{2a}{3}(64a^3) - \frac{1}{32 \times 6 \times a^2}(256 \times 16a^6)$$

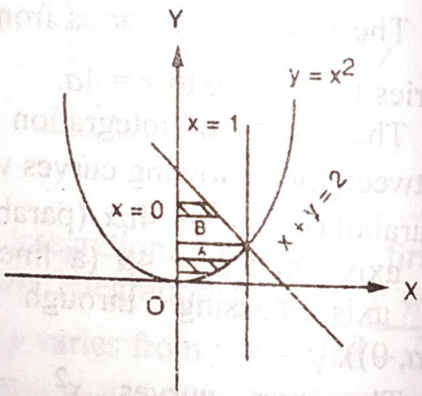
$$= \frac{128a^4}{3} - \frac{64a^4}{3} = \frac{64a^4}{3}$$

Example 11 *Change the order of integration and then evaluate* $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$.

Solution

Here the limits for y varies from $y = x^2$ to $y = 2 - x$ and the limits for x varies from $x = 0$ to $x = 1$.

The region of integration is shown in figure. Now we have to change the order of integration. That is first integration is w.r.t. x and second integration should w.r.t. y. Since the first integration should w.r.t. x we consider horizontal strips.



From figure, the region of integration may be divided into two portions A and B. For the portion A, we have

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy$$

For the portion B, we have

$$I_2 = \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

Hence $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy + \int_1^2 \int_0^{2-y} xy\, dx\, dy$

$$= \int_0^1 \left[\frac{x^2\, y}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 \left[\frac{x^2\, y}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{2}\, dy + \int_1^2 \frac{(2-y)^2 \cdot y}{2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[4\cdot\frac{y^2}{2} + \frac{y^4}{4} - 4\,\frac{y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{3}{8}$$

4. $\int_0^a f(x).dx = \int_0^a f(a-x).dx$

5. $\int_{-a}^a f(x).dx = \int_0^a f(x).dx + \int_0^a f(-x).dx$

6. If $f(x)$ is an odd function,

i.e., $f(-x) = -f(x)$ then

$$\int_{-a}^a f(x).dx = -\int_0^a f(x).dx + \int_0^a f(x).dx$$

7. If $f(x)$ is an even function

i.e., $f(-x) = f(x)$ then

$$\int_{-a}^a f(x).dx = \int_0^a f(x).dx + \int_0^a f(x).dx$$

$$= 2\int_0^a f(x).dx$$

8. $\int_0^{2a} f(x).dx = \int_0^a f(x).dx + \int_0^a f(2a-x).dx$

9. If $f(x) = f(2a-x)$,

$$\int_0^{2a} f(x).dx = 2\int_0^a f(x).dx$$

Fourier series

Definition: (periodic function)

A function $f(x)$ is said to be Periodic if $f(x+T) = f(x)$, for all x and for some T. The least positive real value of T is said to a period of $f(x)$

Example :

$$f(x) = \sin x$$

We take $T = 2\pi$

$\therefore f(x+T) = \sin(x+2\pi)$

$= \sin x \cos 2\pi + \cos x \sin 2\pi$

$= \sin x \cdot 1 + \cos x \cdot 0$

$= \sin x$

$f(x+T) = f(x)$

Ill'y it is time for $4\pi, 6\pi, 8\pi$

But 2π is Least value

$\Rightarrow f(x) = \sin x$ is a periodic function with the period 2π.

$\Rightarrow f(x) = \cos x$ is a periodic function with the period 2π

$\Rightarrow f(x) = \tan x$ is the period function with period π

Fourier series in $(0, 2l)$

Fourier series of the function $f(x)$ in the interval $(0, 2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\, dx$$

and

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\, dx$$

Here a_0, a_n and b_n are called Euler's constants.

Fourier series in $(0, 2\pi)$:

Fourier series of the function $f(x)$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \cdot dx$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \cdot dx$$

Here a_0, a_n, b_n are Euler's constant.

Fourier series in $(-l, l)$

Fourier series of the function $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) \cdot dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \cdot dx$$

and

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \cdot dx$$

Here a_0, a_n, b_n are Euler's constant.

Fourier series in $(-\pi, \pi)$

Fourier series of the function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{n} \int_{-\pi}^{\pi} f(x) \cos nx \cdot dx$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cdot f(x) \sin nx \cdot dx$$

Here a_0, a_n, and b_n are called Euler's constant

Condition for ~~A~~ fourier expansion [Dirichlet's condition]

Any function $f(x)$ can be developed as a fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0, a_n, b_n are constant, provided.

i) $f(x)$ is periodic, singled valued and finite

ii) $f(x)$ has a finite number of finite discontinuities in any one period and no infinite discontinuity

iii) $f(x)$ has at the most a finite number of maxima and minima

DISCONTINOUS

Continuous	End Point	Middle.
Substitute the value directly	Average value at end Point	Left hand limit + right hand limit 2
$f(x) = x$ in $(0, 2\pi)$ $f(x)$ at $x = \pi/2$, $f(\pi/2) = \pi/2$	$f(x) = x$ in $(0, 2\pi)$ $f(x)$ at $(x=0) = \frac{f(0)+f(2\pi)}{2}$ $= \frac{0+2\pi}{2} = \pi$ $f(x)$ at $(x = 2\pi) = \frac{f(0)+f(2\pi)}{2}$ $\frac{0+2\pi}{2}$ $= \pi$	$f(x) = x \quad 0 < x < \pi$ $= x^2 \quad \pi < x < 2\pi$ $f(x)$ at $(x=\pi) = \frac{f(\pi-)+f(\pi+)}{2}$ $f(\pi-) = \lim_{h \to 0} f(\pi-h) = \lim_{h \to 0}(\pi-h) = \pi$ $f(\pi+) = \lim_{h \to 0} f(\pi+h) = \lim_{h \to 0}(\pi+h) = \pi$ $\therefore f(x)$ at $(x=\pi) = \frac{\pi+\pi^2}{2}$.

14.10.19

Important formulas

1. Bernoulli's formulas:

$$\int uv.dx = uv_1 - u'v_2 + u''v_3 - \dots,$$

where

$$u' = \frac{du}{dx}, \quad u'' = \frac{d^2u}{dx^2} \dots$$

$$v_1 = \int v.dx, \quad v_2 = \int v_1.dx.$$

2. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$

3. $\int e^{ax} \cos bx . dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$

4. $(-1)^{n+1} = (-1)^{n-1}$

5. $\cos n\pi = (-1)^n, \quad \cos 2n\pi = (-1)^{2n} = 1$

$\sin n\pi = 0, \quad \sin 2n\pi = 0,$ if n is an integer

1. If $f(x) = \frac{1}{2}(\pi - x)$, find the Fourier series of period 2π in the interval $(0, 2\pi)$

Sol:

Given $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x).dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x).dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi^2 - \frac{4\pi^2}{2} - (0-0)\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \overset{u}{(\pi - x)}\, \overset{v}{\cos nx}\, dx. \rightarrow \text{Bernoullis formula}$$

$$\int uv.dx = uv_1 - u'v_2 \ldots$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[0 - \frac{1}{n^2} - \left(0 - \frac{1}{n^2}\right)\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx.dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(-\pi) - \frac{-1}{n} - 0\right] - \left[\pi\left(-\frac{1}{n}\right) - 0\right]$$

$$b_n = \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$b_n = \frac{1}{n}$$

$$\therefore f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} (0) \cos n x + \sum_{n=1}^{\infty} \frac{1}{n} \sin n x$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

2. Determine the fourier series for the function $f(x) = x^2$ of period 2π in $0 < x < 2\pi$.

Sol

$$f(x) = x^2$$

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,.dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 . dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 0\right]$$

$$a_0 = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx . dx$$

$$\frac{1}{\pi}\left[\int_0^{2\pi} x^2 \cos nx . dx\right.$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0+\frac{4\pi}{n^2}-0\right)-\left(0+0-0\right)\right]$$

$$\boxed{a_n = \frac{4}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cancel{\cos nx}\,\sin nx\cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cdot \cancel{\cos nx}\,\sin nx\cdot dx$$

$$\cancel{= \frac{1}{\pi}\left[x^2\cdot\frac{-\cancel{\sin nx}\,\cos nx}{n} - 2x\,\frac{-\cos nx}{n^2}\right]_0^{2\pi}}$$

$$\cancel{= \frac{1}{\pi}\left[0+\frac{4\pi^2}{n}\right.}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-4\pi^2}{n}+0+\frac{2}{n^3}\right)-\left(-0+0+\frac{2}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n}+\frac{2}{n^3}-\frac{2}{n^3}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\cancel{\frac{-4\pi^2}{n}}+\cancel{\frac{4}{n^3}}\right]$$

$$\boxed{b_n = \frac{-4\pi}{n}}$$

$$f(x) = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty}\left(\frac{-4\pi}{n}\right)\sin nx$$

$$f(x) = \frac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

3. Find fourier series of period $2l$ for function $f(x) = (l-x)^2$ in range $(0, 2l)$

Soln

$f(x) = (l-x)^2$ intervals $(0, 2\pi)$

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x).dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2.dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^3}{3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{(l-2l)^3}{3} - \frac{(l-0)^3}{3}\right]$$

$$= \frac{1}{3l}\left[(-l)^3 - (l)^3\right] = -\frac{1}{3l}\left[-2l^3\right]$$

$$\boxed{a_0 = \frac{2l^2}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}.dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \cos\frac{n\pi x}{l}.dx$$

$$= \frac{1}{l}\left[(l-x)^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2(l-x)(-1)\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right.$$

$$+ 2 \left. \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3} \right]_0^{2l}$$

$$= \frac{1}{l}\left[(l-x)^2 \frac{l}{n\pi} \sin\frac{n\pi x}{l} - 2(l-x)\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} \right.$$

$$\left. - \frac{2l^3}{n^3\pi^3} \sin\frac{n\pi x}{l} \right]_0^{2l}$$

$$= \frac{1}{l}\left(\left[0 + \frac{2l^3}{n^2\pi^2} + 0\right] - \left(0 - \frac{2l^3}{n^2\pi^2} - 0\right)\right)$$

$$= \frac{1}{l}\left(\frac{2l^3}{n^2\pi^2} + \frac{2l^3}{n^2\pi^2}\right)$$

$$-2(l-2l)\frac{l^2}{n^2\pi^2}\cos\frac{n\pi 2l}{l}$$

$$+2\frac{l^3}{n^2\pi^2}(1)$$

$$= \frac{1}{l}\left(\frac{4l^3}{n^2\pi^2}\right)$$

$$\boxed{a_n = \frac{4l^2}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x).\sin\frac{n\pi x}{l}.dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \sin\frac{n\pi x}{l}.dx$$

$$= \frac{1}{l}\left[(l-x)^2\left(\frac{-\cos n\pi x/l}{n\pi/l}\right) - 2(l-x)(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) \right.$$

$$\left. + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-(l-x)^2 \frac{l}{n\pi}\cos\frac{n\pi x}{l} - 2(l-x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} \right.$$

$$\left. + \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\left(\frac{-l^3}{n\pi} - 0 + \frac{2l^3}{n^3\pi^3}\right) - \left(\frac{-l^3}{n\pi} - 0 + \frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{l}{l}[0] = 0$$

$\therefore \boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{\left(\frac{2}{3}\right)l^2}{2} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l} + 0$$

$$f(x) = \frac{1}{3}l^2 + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \cos\frac{n\pi x}{l}\right]$$

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4. Find Fourier series for $f(x) = x$ in $(0, 2l)$

Sol

Given $f(x) = x$ in $(0, 2l)$

W.K.T

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} x\,dx$$

$$= \frac{1}{l}\left[\frac{x^2}{2}\right]_0^{2l} = \frac{1}{l}\left[\frac{4l^2}{2} - 0\right]$$

$\boxed{a_0 = 2l}$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} x \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[0 + \frac{1}{n\pi/l} - \left(0 + \frac{1}{n\pi/l}\right)\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} x \cdot \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (1)\,\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_0^{2l}$$

$$= \frac{1}{l}\left[2l\left(\frac{-1}{n\pi/l}\right) + 0 - (0+0)\right]$$

$$= \frac{1}{l}\left(\frac{-2l^2}{n\pi}\right)$$

$$\boxed{b_n = -\frac{2l}{n\pi}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{2l}{2} + \sum_{n=1}^{\infty}(0)\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\left(\frac{-2l}{n\pi}\right)\sin\frac{n\pi x}{l}$$

$$f(x) = l - \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$$

5. Find the Fourier series of

$$f(x) = \begin{cases} x/l & 0 < x < l \\ \frac{2l-x}{l} & l < x < 2l \end{cases}$$ in range $(0, 2l)$

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,.dx$$

$$= \frac{1}{l}\left[\int_0^{l} \frac{x}{l}\,.dx + \int_l^{2l} \frac{2l-x}{l}\,.dx\right]$$

$$= \frac{1}{l^2}\int_0^{l} x\,.dx + \int_l^{2l}(2l-x)\,.dx$$

$$= \frac{1}{l^2}\left[\left(\frac{x^2}{2}\right)_0^{l} + \left(2lx - \frac{x^2}{x}\right)_l^{2l}\right]$$

$$= \frac{1}{l^2}\left[\left(\frac{l^2}{2}-0\right) + \left(4l^2 - \frac{4l^2}{2}\right) - \left(2l^2 - \frac{l^2}{2}\right)\right]$$

$$= \frac{1}{l^2}\left[\left(\frac{l^2}{2}\right)\left(\frac{4l^2}{2}\right) - \left(\frac{3l^2}{2}\right)\right]$$

$$= \left[\left(\frac{1}{2}\right) + \left(\frac{4}{2}\right) + \left(\frac{3}{2}\right)\right] = \left(\frac{5}{2} - \frac{3}{2}\right)$$

$$\boxed{a_0 = 1}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\,.\cos\frac{n\pi x}{l}\,.dx$$

$$= \frac{1}{l}\left[\int_0^{l} \frac{x}{l}\cos\frac{n\pi x}{l}\,.dx + \int_l^{2l}\frac{2l-x}{l}\cos\frac{n\pi x}{l}\,.dx\right]$$

$$= \frac{1}{l^2}\left[\int_0^{l} x\cos\frac{n\pi x}{l}\,.dx + \int_l^{2l}(2l-x)\cos\frac{n\pi x}{l}\,.dx\right]$$

$$= \frac{1}{l^2}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - (1)\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^{l}$$

$$+ \frac{1}{l^2}\left[(2l-x)\frac{\sin\frac{n\pi x}{l}}{n\pi/l} - (-1)\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_l^{2l}$$

$$= \frac{1}{\ell^2}\left[x\frac{1}{n\pi}\sin\frac{n\pi x}{\ell} + \frac{\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell}\right]_0^{\ell}$$

$$\frac{1}{\ell^2}\left[(2\ell - x)\frac{1}{n\pi}\sin\frac{n\pi x}{\ell} - \frac{\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell}\right]_{\ell}$$

$$= \frac{1}{\ell^2}\left[\left(0 + \frac{\ell^2}{n^2\pi^2}(-1)^n\right) - \left(0 + \frac{\ell^2}{n^2\pi^2}\right) + \left(0 - \frac{\ell^2}{n^2\pi^2}(-1)^n\right.\right.$$

$$\left.0 - \frac{\ell^2}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{\ell^2}\left[\frac{\ell^2}{n^2\pi^2}(-1)^n - \frac{\ell^2}{n^2\pi^2} - \frac{\ell^2}{n^2\pi^2} + \frac{\ell^2}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{\ell^2}\left[\frac{2\ell}{n^2\pi^2}(-1)^n\right] - \frac{2\ell^2}{n^2\pi^2}$$

$$= \frac{1}{\ell^2}\left(\frac{2\ell^2}{n^2\pi^2}\right)\left[(-1)^n - 1\right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \dfrac{-4}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{1}{\ell}\left[\int_0^{\ell}\frac{x}{\ell}\sin\frac{n\pi x}{\ell} + \int_{\ell}^{2\ell}\frac{2\ell - x}{\ell}\sin\frac{n\pi x}{\ell}\,dx\right]$$

$$= \frac{1}{\ell^2}\int_0^{\ell} x\sin\frac{n\pi x}{\ell}\,dx + \int_{\ell}^{2\ell}(2\ell - x)\cdot\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{1}{\ell^2}\left[x\left(\frac{-\cos\frac{n\pi x}{\ell}}{n\pi/\ell}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{\ell}}{n^2\pi^2/\ell^2}\right)\right]_0^{\ell}$$

$$+\frac{1}{l^2}\left[(2l-x)\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}-(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_l^{2l}$$

$$=\frac{1}{l^2}\left[-x\frac{1}{n\pi}\cos\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$+\frac{1}{l^2}\left[(2l-x)\frac{1}{n\pi}\cos\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]$$

$$=\frac{1}{l^2}\left[\left(\frac{-l^2}{n\pi}(-1)^n+0\right)(0-0)-(0+0)-\frac{l^2}{n\pi}(-1)^n+0\right]$$

$$=\frac{1}{l^2}\left[\frac{-l^2}{n\pi}(-1)^n+\frac{l^2}{n\pi}(-1)^n\right]$$

$$\boxed{b_n=0}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\cdot dx+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$=\frac{1}{2}+\sum_{n=odd}^{\infty}-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{l}+0$$

$$=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}$$

$$f(x)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)n\pi}{l}$$

16.10.19

Odd function

A function is said to be an odd function if

$f(x) \neq f(-x)$

$$f(-x) = -f(x)$$

Example

$f(x) = x$

$f(-x) = -x$

$= -f(x)$

$f(-x) = -f(x)$

$f(x)$ is odd

Even function

A function $f(x)$ is said to be an even function if

$$f(-x) = f(x)$$

Example

$f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

$f(-x) = f(x)$

$\therefore f(x)$ is even.

Note - 1

Suppose $f(x)$ is an odd function in $(-\pi, \pi)$, then

$a_0 = 0$, $a_n = 0$

and $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \cdot dx$

Fourierseries is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Note - 2

Suppose $f(x)$ is an even function in $(-\pi, \pi)$ then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

and $b_n = 0$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Note - 3

Suppose $f(x)$ is an odd function in $(-l, l)$ then

$a_0 = 0$, $a_n = 0$

and $b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l}\,dx$

Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Note 4 ;

Suppose $f(x)$ is a even function in $(-l, l)$ then

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos nx \cdot dx$$

and $b_n = 0$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

10 Find Fourier series for $f(x) = x$ in $(-\pi, \pi)$

Sol

Given $f(x) = x$ in $(-\pi, \pi)$

$$\therefore f(-x) = -x$$
$$= -f(x)$$
$$f(-x) = -f(x)$$

$\therefore$ $f(x)$ is odd

$\therefore$ $a_0 = 0$ and $a_n = 0$

$$\therefore f(x) = \sum b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \cdot dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cdot \sin nx \cdot dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\left(\frac{-(-1)^n}{n}\right) + 0 - (0+0)\right]$$

$$= \frac{2}{\pi}\cdot\left(\frac{\pi\,(-1)^{n+1}}{n}\right)$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$$

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$

2. Find Fourier series for $f(x) = x^2$ in $-\pi \le x \le \pi$

Sol.

Given $f(x) = x^2$, $f(-x) = (-x)^2$

$f(x) = x^2$

Therefore $f(x)$ is an even function in $(-\pi, \pi)$

The fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,.\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,.\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{\pi^{\cancel{3}}}{3}\right]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,.\,\cos nx\,.\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,.\,\cos nx\,.\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,.\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right)\right] - 2\left(\frac{\sin nx}{n^3}\right)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{2\pi(-1)^n}{n^2} - 0\right) - (0+0-0)\right]$$

$$a_n = \frac{2}{\pi}\left(\frac{2\pi(-1)^n}{n^2}\right)$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos nx.$$

3. If $f(x) = x$ is defined in $-l < x < l$ with Period $2l$. find the fourier expansion of $f(x)$

Given

$$f(x) = x$$

$$f(-x) = -x$$

$$f(-x) = -f(x)$$

$\therefore f(x)$ is a odd function, $a_0 = 0$ $a_n = 0$

Fourier series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} \cdot dx$$

$$= \frac{2}{l}\int_0^l x \sin\frac{n\pi x}{l} \cdot dx$$

$$= \frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (1)\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

$$= \frac{2}{l}\left[-x\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l^2}{\pi x}\cos n\pi + 0 - (0+0)\right]$$

$$= \frac{2}{l}\left[\frac{-l^2}{\pi x}\cos n x\right]$$

$$\therefore b_n = \frac{-2l(-1)^n}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-2l(-1)^n}{n\pi} \sin\frac{n\pi x}{l}$$

$$f(x) = \frac{-2l}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{l}$$

4. find the fourier series of $f(x) = x+x^2$ in $(-\pi, \pi)$ of periodicity 2π.

Soln

$$f(x) = x + x^2$$

$$f(-x) = -x + (-x)^2$$

$$f(-x) \neq f(x) \Rightarrow f(-x) \neq -f(x)$$

$\therefore f(x)$ is neither even nor odd in $(-\pi, \pi)$.

Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2).dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}\right] - \left[\frac{\pi^2}{2}-\frac{\pi^3}{3}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx \cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx \cdot dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx \cdot dx$$

Since,

$x\cos nx \rightarrow$ odd function

$x^2\cos nx \rightarrow$ even function

$$\int_{-\pi}^{\pi} x\cos nx \cdot dx = 0, \quad \int_{-\pi}^{\pi} x^2\cos nx \cdot dx = 2\int_0^{\pi} x^2\cos nx \cdot dx$$

$$a_n = \frac{1}{\pi}[0] + \frac{1}{\pi}2\int_0^{\pi} x^2\cos nx \cdot dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{x} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} - 0 - (0+0+0)\right]$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx \cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin nx \, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \cdot dx$$

Since,

$x \cdot \sin nx \rightarrow$ even function

$x^2 \sin x \rightarrow$ odd function

$$\int_{-\pi}^{\pi} x \sin nx \cdot dx \rightarrow 2\int_{0}^{\pi} x\cdot\sin nx \cdot dx$$

$$\int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0$$

$$b_n = \frac{1}{\pi}\, 2\int_{0}^{\pi} x \cdot \sin nx \cdot dx$$

$$= \frac{2}{\pi}\left[x\cdot\left(\frac{-\cos nx}{n}\right) - (1)\,\frac{-\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-x\cdot\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi(-1)^n}{n} + 0\right] - (-0+0)$$

$$\boxed{b_n = -\frac{2}{n}(-1)^n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi^3}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx +$$

$$+ \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

Find b_n in the expansion of x^2 as a Fourier series in $(-\pi, \pi)$

Given $f(x) = x^2$ is an even function in $(-\pi, \pi)$ $\therefore b_n = 0$

If $f(x)$ is an odd function defined in $(-l, l)$ what value of a_0 and a_n?

Given $f(x)$ is an odd function interval $(-l, l)$ $a_0 = 0$, $a_n = 0$

Find the fourier constant b_n for $x \sin x$ in $(-\pi, \pi)$

Given $f(x) = x \sin x$ in $(-\pi, \pi)$

$$f(-x) = (-x) \sin(-x)$$
$$= (-x)[-\sin x]$$
$$= x \sin x$$
$$= f(x)$$
$$f(-x) = f(x)$$

$\therefore f(x)$ is an even function

$b_n = 0$

Determine the value of a_n in fourier series $f(x) = x^3$ in $-\pi < x < \pi$

Given $f(x) = x^3$

$$f(-x) = (-x)^3$$
$$f(-x) = -x^3$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function

$a_0 = 0$ and $a_n = 0$