UG / PG : PG

semester : IV

subject : Advanced Numerical Analysis

subject code : P16MA43

Name : T. KAMALA

Mobile no : 9788775671

ADVANCED NUMERICAL ANALYSIS

UNIT-V

Singlestep methods:-

Local Truncation Error

The true value $u(t_j)$ satisfies the equation

$$u(t_{j+1}) = u(t_j) + h\phi(t_{j+1}, t_j, u(t_{j+1}), u(t_j), h) + T_{j+1}$$

where T_{j+1} is called the local truncation error.

The truncation error is given by

$$T_{j+1} = u(t_{j+1}) - u(t_j) - h\phi(t_{j+1}, t_j, u(t_{j+1}), u(t_j), h)$$

1. Given the initial value problem $u' = t^2 + u^2$, $u(0) = 0$ determine the first three non-zero terms in the Taylor's series for $u(t)$ and hence obtain the value of $u(1)$. Also determine 't' when the error in $u(t)$ from the first two non-zero terms is to be less than 10^{-6} after rounding.

Soln:-

$$u' = t^2 + u^2, \quad u(0) = 0 \Rightarrow u = 0, t = 0$$

$$u'(0) = 0$$

$$u'' = 2t + 2uu'$$

$$u''(0) = 0$$

$$u''' = 2 + 2(u'u' + uu'')$$

$$u'''(0) = 2$$

$$u^{IV} = 2[u'u'' + u'u'' + uu''' + u''u']$$
$$= 2[uu''' + 3u'u'']$$
$$u^{IV}(0) = 0$$
$$u^{V} = 2[uu^{IV} + u'u''' + 3u'u''' + 3u''u'']$$
$$= 2[uu^{IV} + 4u'u''' + 3(u'')^2]$$
$$u^{V}(0) = 0$$
$$u^{6} = 2[uu^{5} + u^{4}u' + 4u'u^{4} + 4u''u''' + 6u''u''']$$
$$= 2[uu^{5} + 5u'u^{4} + 10u''u''']$$
$$u^{6}(0) = 0$$
$$u^{7} = 2[uu^{6} + u'u^{5} + 5u'u^{5} + 5u''u^{4} + 10u'''u''' + 10u''u^{4}]$$
$$= 2[uu^{6} + 6u'u^{5} + 15u''u^{4} + 10(u''')^2]$$
$$u^{7}(0) = 2[0 + 10(4)]$$
$$u^{7}(0) = 80$$
$$u^{8} = 2[uu^{7} + u'u^{6} + 6u'u^{6} + 6u''u^{5} + 15u'''u^{4} + 15u''u^{5} + 20u'''u^{4}]$$
$$= 2[uu^{7} + 7u'u^{6} + 21u''u^{5} + 35u'''u^{4}]$$
$$u^{8}(0) = 0$$
$$u^{9} = 2[uu^{8} + u'u^{7} + 7u'u^{7} + 7u^{6}u'' + 21u''u^{6} + 21u^{5}u''' + 35u'''u^{5} + 35u^{4}u^{4}]$$
$$= 2[uu^{8} + 8u'u^{7} + 28u''u^{6} + 56u'''u^{5} + 35(u^{4})^2]$$
$$u^{9}(0) = 0$$

$$u^{10} = 2[uu^{9} + u'u^{8} + 8u''u^{7} + 8u'u^{8} + 28u''u^{7} + 28u'''u^{6}$$
$$+ 56u^{4}u^{5} + 56u'''u^{6} + 70u^{4}u^{5}]$$
$$= 2[uu^{9} + 9u'u^{8} + 36u''u^{7} + 84u'''u^{6} + 126u^{4}u^{5}]$$

$$u^{10}(0) = 0$$

$$u^{11} = 2[uu^{10} + u'u^{9} + 9u''u^{8} + 9u'u^{9} + 36u'''u^{7}$$
$$+ 36u''u^{8} + 84u^{4}u^{6} + 84u'''u^{7} + 126u^{5}u^{5} + 126u^{4}u^{6}]$$
$$= 2[uu^{10} + 10u'u^{9} + 45u''u^{8} + 120u'''u^{7} + 210u^{4}u^{6}$$
$$+ 126(u^{5})^{2}]$$

$$u^{11}(0) = 2[120(2)(80)]$$
$$= 38400$$

$$u(t) = \frac{t^{3}}{3!}u'''(0) + \frac{t^{7}}{7!}u^{7}(0) + \frac{t^{11}}{11!}u^{11}(0) + \cdots$$
$$= \frac{t^{3}}{3!}(2) + \frac{t^{7}}{7!}(80) + \frac{t^{11}}{11!}(38{,}400) + \cdots$$
$$= \frac{t^{3}}{3} + \frac{t^{7}}{63} + \frac{2t^{11}}{2079} + \cdots$$

$$u(1) = \frac{1}{3} + \frac{1}{63} + \frac{2}{2079}$$
$$= 0.3501683$$

(ii) The error is given from the next non-zero term

It is required error $< 10^{-6}$

$$T_{j+1} = \frac{2t^{11}}{2079} < 0.5 \times 10^{-7}$$

$$t^{11} < \frac{0.5 \times 10^{-7} \times 2079}{2}$$

$$t^{11} \simeq \left[\frac{0.5 \times 10^{-7} \times 2079}{2}\right]^{1/11}$$

$$\boxed{t^{11} \simeq 0.41}$$

2. Find three terms Taylor series solutions for the third Blasius equation $w''' + w''w = 0$, $w(0) = 0$, $w'(0) = 0$, $w''(0) = 1$ Find a bound on the error $t \in [0, 0.2]$

soln

$$w''' + ww'' = 0, \quad w(0) = 0, \quad w'(0) = 0, \quad w''(0) = 1$$

$$w''' = -ww''$$

$$w'''(0) = 0$$

$$w^4 = -[ww''' + w'w'']$$

$$w^4(0) = 0$$

$$w^5 = -[ww^4 + w'w''' + w'w''' + w''w'']$$

$$= -[ww^4 + 2w'w''' + (w'')^2]$$

$$w^5(0) = -[0 + 0 + 1]$$

$$= -1$$

$$w^6 = -[ww^5 + w'w^4 + 2w'w^4 + 2w''w''' + 2w''w''']$$

$$= -\left[w w^{5} + 3w' w^{4} + 4w'' w'''\right]$$

$$w^{6}(0) = 0$$

$$w^{7} = -\left[w w^{6} + w' w^{5} + 3w' w^{5} + 3w'' w^{4} + 4w'' w^{4} + 4w''' w'''\right]$$

$$= -\left[w w^{6} + 4w' w^{5} + 7w'' w^{4} + 4(w'')^{2}\right]$$

$$w^{7}(0) = 0$$

$$w^{8} = -\left[w w^{7} + w' w^{6} + 4w' w^{6} + 4w'' w^{5} + 7w'' w^{5} + 7w''' w^{4} + 8w''' w^{4}\right]$$

$$= -\left[w w^{7} + 5w w^{6} + 11w'' w^{5} + 15w''' w^{4}\right]$$

$$w^{8}(0) = -\left[11(1)(-1)\right]$$

$$= 11$$

$$w^{9} = -\left[w w^{8} + w' w^{7} + 5w' w^{7} + 5w'' w^{6} + 11w'' w^{6} + 11w''' w^{5} + 15w''' w^{5} + 15w^{4} w^{4}\right]$$

$$= -\left[w w^{8} + 6w' w^{7} + 16w'' w^{6} + 26w''' w^{5} + 15(w^{4})^{2}\right]$$

$$w^{9}(0) = 0$$

$$w^{10} = -\left[w w^{9} + w' w^{8} + 6w' w^{8} + 6w'' w^{7} + 16w'' w^{7} + 16w''' w^{6} + 26w''' w^{6} + 26w^{4} w^{5} + 30w^{4} w^{5}\right]$$

$$= -\left[w w^{9} + 7w' w^{8} + 22w'' w^{7} + 42w''' w^{6} + 56w^{4} w^{5}\right]$$

$$w^{10}(0) = 0$$

$$w^{11} = -\left[w w^{10} + w' w^{9} + 7w' w^{9} + 7w'' w^{8} + 22w'' w^{8} + 22w''' w^{7} + 42w''' w^{7} + 42w^{4} w^{6} + 56w^{4} w^{6} + 56w^{5} w^{5}\right]$$

$$= -\left[ww^{10} + 8w'w^{9} + 29w''w^{8} + 64w'''w^{7} + 98w^{4}w^{6} + 56(w^{5})^{2}\right]$$

$$w^{11}(0) = -\left[29(1)(11) + 56(-1)^{2}\right]$$

$$= -375$$

using Taylor series with non-zero term

(i) $$w(t) = \frac{t^2}{2!}w''(0) + \frac{t^5}{5!}w^{5}(0) + \frac{t^8}{8!}w^{8}(0) + \frac{t^{11}}{11!}w^{11}(0)$$

$$= \frac{t^2}{2!}(1) + \frac{t^5}{5!}(-1) + \frac{t^8}{8!}(11) + \frac{t^{11}}{11!}(-375)$$

$$= \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{11t^8}{8!} + E8 - - -$$

(ii) To find a bound on the error

$$E8 \leq \max_{0<t<0.2} w^{(9)}(t)\,\frac{t^9}{9!}$$

$$w^{9}(t) = \frac{d^9}{dt^9}\left[\frac{-375t^{11}}{11!}\right]$$

$$= \frac{-375t^2}{2}$$

$$\max_{0<t<0.2} w^{9}(t) = \frac{375}{2}(0.2)^2$$

$$= 7.5$$

$$E8 \leq \frac{(7.5)(0.2)^9}{9!} = 1.0582 \times 10^{-11}$$

$$= 1.06 \times 10^{-11}$$

Runge-kutta Methods

Euler cauchy or Heun method:

$$u_{j+1} = u_j + \frac{h}{2}[k_1 + k_2]$$

$$k_1 = hf(t_j, u_j)$$

$$k_2 = hf(t_j + h, u_j + k_1)$$

Ex:3

Given the initial Value problem $u' = -2tu^2$ $u(0)=1$ Estimate $u(0.4)$ using (i) Modified Euler cauchy method (ii) Heun method with $h=0.2$ compare the results with the exact solution.

soln:

$$u(t) = \frac{1}{1+t^2}$$

(i) Modified Euler cauchy method

$$u_{j+1} = u_j + k_2$$

$$k_1 = hf(t_j, u_j) = 0.2[-2t_j u_j^2]$$

$$= -0.4 t_j u_j^2$$

$$k_2 = hf(t_j + h/2, u_j + k_1/2)$$

$$= -0.4(t_j + 0.1)(u_j + \tfrac{1}{2}k_1)^2$$

For $j=0$

$t_0 = 0$, $u_0 = 1$, $k_1 = 0$

$K_2 = -0.4\,(0.1)\,(1)^2$

$K_2 = -0.04$

$u(0.2) = u_1 = u_0 + K_2$

$= 1 - 0.04$

$u(0.2) = 0.96$

For $j = 1$

$t_1 = 0.2\ ,\quad u_1 = 0.96$

$K_1 = -0.4\, t_j\, (u_j)^2 = -0.4\,(0.2)\,(0.96)^2$

$K_1 = -0.073728$

$K_2 = -0.4\,(0.2 + 0.1) = \left(0.96 - \frac{0.073728}{2}\right)^2$

$= -0.102262$

$u(0.4) = u_2 = u_1 + K_2 = 0.96 - 0.102262$

$u(0.4) = 0.857738$

(ii) The Heun method is given by

$u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2)$

$K_1 = hf(t_j, u_j)$

$= -0.4\, t_j\, u_j^2$

$K_2 = hf(t_j + h,\ u_j + K_1)$

$= -0.4\,(t_j + 0.2)\,(u_j + K_1)^2$

for $j = 0$

$t_0 = 0,\quad u_0 = 1$

$$K_1 = -0.4\,(t_0)\,(u_0)^2$$
$$= -0.4\,(0)\,(1)$$
$$K_1 = 0$$
$$K_2 = -0.4\,(t_0+0.2)\,(u_0+K_1)^2$$
$$= -0.4\,(0+0.2)\,(1+0)^2$$
$$K_2 = -0.08$$
$$u(0.2) = u_1 = u_0 + \tfrac{1}{2}(K_1+K_2)$$
$$= 1 + \tfrac{1}{2}(0-0.08)$$
$$u(0.2) = 0.96$$

For $j = 1$

$$t_1 = 0.2\,,\quad u_1 = 0.96$$
$$K_1 = -0.4\,(t_1)\,(u_1)^2$$
$$= -0.4\,(0.2)\,(0.96)^2$$
$$K_1 = -0.073728$$
$$K_2 = -0.4\,(t_1+0.2)\,(u_1+K_1)^2$$
$$= -0.4\,(0.2+0.2)$$
$$= (0.96-0.073728)^2$$
$$= -0.125676$$
$$u(0.4) = u_2 = u_1 + \tfrac{1}{2}(K_1+K_2)$$
$$= 0.96 + \tfrac{1}{2}(-0.073728-0.125676)$$
$$u(0.4) = 0.860298$$

The exact solution is $u(0.2) = 0.961538$

$u(0.4) = 0.862069$

The absolute errors in the Numerical solutions

(i) Modified Euler cauchy method

$\varepsilon(0.2) = 0.001538$, $\varepsilon(0.4) = 0.004331$

(ii) Heun method $\varepsilon(0.2) = 0.001538$

$\varepsilon(0.4) = 0.001771$

Taylor series method:-

We write in vector form

$$u_{j+1} = u_j + h u_j' + \frac{h^2}{2!} u_j'' + \cdots \frac{h^p}{p!} u_j^{(p)}, \quad j = 0, 1, 2 \ldots N-1$$

$$u_j^{(k)} = \begin{bmatrix} u_{1,j}^{(k)} \\ u_{2,j}^{(k)} \\ \vdots \\ u_{k,j}^{(k)} \end{bmatrix} = \begin{bmatrix} \dfrac{d^{k-1}}{dt^{k-1}} f_1(t_j, u_{1,j}, u_{2,j} \ldots u_{n,j}) \\ \dfrac{d^{k-1}}{dt^{k-1}} f_n(t_j, u_{1,j}, u_{2,j} \ldots u_{n,j}) \end{bmatrix}$$

The Euler method can be written as

$$u_{j+1} = u_j + h u_j', \quad j = 0, 1, 2 \ldots N-1$$

System of equations:-

1. Compute an approximation to $u(1)$, $u'(1)$ and $u''(1)$ with the Taylor series method of second order and step length $h=1$, for the initial value problem. $u''' + 2u'' + u' - u = \cos t$, $0 \le t \le 1$, $u(0)=0$, $u'(0)=1$, $u''(0)=2$ after reducing it to a system of first order equation.

Soln

$$u = V_1, \quad V_1' = V_2, \quad V_2' = V_3$$

$V_2 = V_1' = u'$, $V_3 = V_2' = u''$ and $V_3' = V_2'' = u'''$

The system of equation is

$$V_1' = V_2 \qquad V_1(0) = 0$$

$$V_2' = V_3 \qquad V_2(0) = 1$$

$$V_3' = \cos t - 2V_3 - V_2 + V_1, \quad V_3(0) = 2$$

$$V' = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}' = \begin{bmatrix} V_2 \\ V_3 \\ \cos t - 2V_3 - V_2 + V_1 \end{bmatrix}$$

$$V(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Taylor series method

$$V(t_0+h) = V_0 + hV_0' + \frac{h^2}{2!} V_0''$$

$$= V_0 + V_0' + \tfrac{1}{2} V_0''$$

put $h = 1$

$$V_0' = \begin{bmatrix} V_2(0) \\ V_3(0) \\ 1 - 2V_3(0) - V_2(0) + V_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$V'' = \begin{bmatrix} V_2' \\ V_3' \\ -\sin t - 2V_3' - V_2' + V_1' \end{bmatrix}$$

$$V_0'' = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

$$V(1) = V_0 + V_0' + \tfrac{1}{2} V_0''$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix}$$

$\therefore$ $u(1) = 2$, $u'(1) = 1$, $u''(1) = 3/2$

Solve the system of equations $u' = -3u + 2v$ $u(0) = 0$, $v' = 3u - 4v$, $v(0) = 0.5$ with $h = 0.2$ on the interval $[0, 0.4]$ use the (i) Euler-cauchy method (ii) classical Runge kutta fourth order method.

soln

(i) Euler - cauchy method

$$u_{j+1} = u_j + \frac{1}{2}(k_1 + k_2), \quad j = 0, 1$$

$$k_1 = hf(t_j, u_j)$$

$$k_2 = hf(t_j + h, u_j + k_1)$$

$j = 0$

$$t_0 = 0, \quad v_0 = 0.5, \quad u_0 = 0$$

$$k_{i1} = hf_i[t_j, u_{1,j}, u_{2,j} \ldots u_{n,j}]$$

$$k_{11} = hf_1(t_0, u_0, v_0)$$

$$= 0.2[-3u_0 + 2v_0]$$

$$= 0.2[-3(0) + 2(0.5)]$$

$$k_{11} = 0.2$$

$$k_{21} = hf_2[t_0, u_0, v_0]$$

$$= 0.2[3u_0 - 4v_0]$$

$$= 0.2[3(0) - 4(0.5)]$$

$$= -0.4$$

$i = 1, 2.$ $\quad j = 0$

$k_{i2} = hf_i(t_j + h, u_{1,j} + k_{i1}, u_{2,j} + k_{2i} \cdots u_{n,j} + k_{ni})$

$k_{12} = hf_1(t_0 + h, u_0 + k_{11}, v_0 + k_{21})$

$= 0.2[-3(u_0 + k_{11}) + 2(v_0 + k_{21})]$

$= 0.2[-3(0.2) + 2(0.5 - 0.4)]$

$= -0.08$

$k_{22} = hf_2(t_0 + h, u_0 + k_{11}, v_0 + k_{21})$

$= 0.2(3(u_0 + k_{11}) - 4(v_0 + k_{21}))$

$= 0.2(3(0 + 0.2) - 4(0.5 - 0.4))$

$= 0.04$

$u(0.2) = u_{j+1} = u_j + \frac{1}{2}(k_1 + k_2)$

$u_1 = u_0 + \frac{1}{2}(k_{11} + k_{12})$

$= 0 + \frac{1}{2}(0.2 - 0.08)$

$\boxed{u(0.2) = 0.06}$

$v(0.2) = v_1 = v_0 + \frac{1}{2}(k_{21} + k_{22})$

$= 0.5 + \frac{1}{2}(-0.4 + 0.04)$

$\boxed{v(0.2) = 0.32}$

$\underline{j=1}$

$t_1 = 0.2, \; V_1 = 0.32, \; u_1 = 0.06$

$k_{11} = h f_1 (t_1, u_1, v_1)$

$= 0.2(-3u_1 + 2v_1) = 0.2[-3(0.06) + 2(0.32))$

$\boxed{k_{11} = 0.092}$

$k_{21} = h f_2 (t_1, u_1, v_1)$

$= 0.2(3u_1 - 4v_1) = 0.2(3(0.06) - 4(0.32))$

$\boxed{k_{21} = -0.22}$

$k_{12} = h f_1 (t_1 + h, u_1 + k_{11}, v_1 + k_{21})$

$= 0.2(-3(u_1 + k_{11}) + 2(v_1 + k_{21}))$

$= 0.2(-3(0.06 + 0.092) + 2(0.32 - 0.22))$

$\boxed{k_{12} = 0.0512}$

$k_{22} = h f_2 (t_1 + h, u_1 + k_{11}, v_1 + k_{21})$

$= 0.2(3(u_1 + k_{11}) - 4(v_1 + k_{21}))$

$= 0.2(3(0.06 + 0.092) - 4(0.32 - 0.22))$

$= 0.0112$

$u(0.4) = u_2 = u_1 + \frac{1}{2}(k_{11} + k_{12})$

$= 0.06 + \frac{1}{2}(0.092 - 0.0512)$

$= 0.0804$

$v(0.4) = v_2 = v_1 + \frac{1}{2}(k_{11} + k_{22})$

$\boxed{v(0.4) = 0.2156}$

(ii) classical Runge-kutta fourth order method:-

For $\underline{j=0}$ $\quad t_0=0,\ u_0=0,\ v_0=0.5$

$$k_{i1} = hf_i(t_j, u_{1,j}, u_{2,j}, \ldots u_{n,j})$$

$$k_{11} = hf_1(t_0, u_0, v_0)$$

$$= 0.2(-3(u_0)+2(v_0))$$

$$= 0.2(-3(0)+2(0.5))$$

$$k_{11} = 0.2$$

$$k_{21} = hf_2(t_0, u_0, v_0)$$

$$= 0.2(3u_0 - 4v_0)$$

$$= 0.2(3(0)-4(0.5))$$

$$k_{21} = -0.4$$

$$k_{12} = hf_1\left(t_0 + h/2,\ u_0 + k_{11}/2,\ v_0 + \frac{k_{21}}{2}\right)$$

$$= 0.2\left(-3\left(u_0 + k_{11}/2\right) + 2\left(v_0 + k_{21}/2\right)\right)$$

$$= 0.2\left(-3\left(0 + 0.2/2\right) + 2\left(0.5 - \frac{0.4}{2}\right)\right)$$

$$k_{12} = 0.06$$

$$k_{22} = hf_2\left(t_0 + h/2,\ u_0 + k_{11}/2,\ v_0 + \frac{k_{21}}{2}\right)$$

$$= 0.2\left(3\left(u_0 + k_{11}/2\right) - 4\left(v_0 + \frac{k_{21}}{2}\right)\right)$$

$$= 0.2\left(3\left(0 + \frac{0.2}{2}\right) - 4\left(0.5 - \frac{0.4}{2}\right)\right)$$

$$k_{22} = -0.18$$

$$K_{13} = hf_1\left(t_0 + h/2,\ u_0 + \frac{K_{12}}{2},\ v_0 + \frac{K_{22}}{2}\right)$$

$$= 0.2\left(-3\left(u_0 + \frac{K_{12}}{2}\right) + 2\left(v_0 + \frac{K_{22}}{2}\right)\right)$$

$$= 0.2\left(-3\left(0 + \frac{0.06}{2}\right) + 2\left(0.5 - \frac{0.18}{2}\right)\right)$$

$$K_{13} = 0.146$$

$$K_{23} = hf_2\left(t_0 + h/2,\ u_0 + \frac{K_{12}}{2},\ v_0 + \frac{K_{22}}{2}\right)$$

$$= 0.2\left[3\left(u_0 + \frac{K_{12}}{2}\right) - 4\left(v_0 + \frac{K_{22}}{2}\right)\right]$$

$$= 0.2\left[3\left(0 + \frac{0.06}{2}\right) - 4\left(0.5 - \frac{0.18}{2}\right)\right]$$

$$K_{23} = -0.31$$

$$K_{14} = hf_1(t_0 + h,\ u_0 + K_{13},\ v_0 + K_{23})$$

$$= 0.2\left[-3(u_0 + K_{13}) + 2(v_0 + K_{23})\right]$$

$$= 0.2\left[-3(0 + 0.146) + 2(0.5 - 0.31)\right]$$

$$K_{14} = -0.0116$$

$$K_{24} = hf_2(t_0 + h,\ u_0 + K_{13},\ v_0 + K_{23})$$

$$= 0.2\left[3(u_0 + K_{13}) - 4(v_0 + K_{23})\right]$$

$$= 0.2\left[3(0 + 0.146) - 4(0.5 - 0.31)\right]$$

$$K_{24} = -0.0644$$

$$u(0.2) = u_1 = u_0 + 1/6\,(K_{11} + 2K_{12} + 2K_{13} + K_{14})$$

$$= 0 + 1/6\,(0.2 + 0.12 + 0.292 - 0.0116)$$

$$\boxed{u(0.2) = 0.1001}$$

$$V(0.2) = V_1 = V_0 + 1/6\,(k_{21} + 2k_{22} + 2k_{23} + k_{24})$$

$$= 0.5 + 1/6\,(-0.4 - 0.36 - 0.62 - 0.0644)$$

$$V(0.2) = 0.2593$$

For $\underline{j=1}$ $\quad t_1 = 0.2,\ u_1 = 0.1001,\ V_1 = 0.2593$

$$k_{11} = hf_1(t_1, u_1, v_1)$$

$$= 0.2(-3u_1 + 2V_1)$$

$$= 0.2\left(-3(0.1001) + 2(0.2593)\right)$$

$$k_{11} = 0.0437$$

$$k_{21} = hf_2(t_1, u_1, v_1)$$

$$= 0.2\left[3u_1 - 4V_1\right]$$

$$= 0.2\left[3(0.1001) - 4(0.2593)\right]$$

$$= -0.1474$$

$$k_{12} = hf_1\left[t_1 + h/2,\ u_1 + k_{11}/2,\ V_1 + \frac{k_{21}}{2}\right]$$

$$= 0.2\left[-3(u_1 + k_{11}/2) + 2\left(V_1 + \frac{k_{21}}{2}\right)\right]$$

$$= 0.2\left[-3\left(0.1001 + \frac{0.0437}{2}\right) + 2\left(0.2593 - \frac{0.1474}{2}\right)\right]$$

$$= 0.0010$$

$$k_{22} = hf_2\left(t_1 + h/2,\ u_1 + \frac{k_{11}}{2},\ V_1 + \frac{k_{21}}{2}\right)$$

$$= 0.2\left(3\left(u_0 + \frac{k_{11}}{2}\right) - 4\left(V_0 + \frac{k_{21}}{2}\right)\right)$$

$$= 0.2\left[3\left(0.1001 + \frac{0.0437}{2}\right) - 4\left(0.2593 - \frac{0.1474}{2}\right)\right]$$

$$= -0.0753$$

$$K_{13} = 0.2\left[-3\left(u_1+\frac{K_{12}}{2}\right)+2\left(v_1+\frac{K_{22}}{2}\right)\right]$$

$$=0.2\left[-3\left(0.1001+\frac{0.0010}{2}\right)+2\left(0.2593-\frac{0.0753}{2}\right)\right]$$

$$=0.0283$$

$$K_{23} = 0.2\left[3\left(u_1+\frac{K_{12}}{2}\right)-4\left(v_1+\frac{K_{22}}{2}\right)\right]$$

$$=0.2\left[3\left(0.1001+\frac{0.0012}{2}\right)-4\left(0.2593-\frac{0.0753}{2}\right)\right]$$

$$=-0.1170$$

$$K_{14} = 0.2\left[-3(u_1+K_{13})+2(v_1+K_{23})\right]$$

$$=0.2\left[-3(0.1001+0.0283)+2(0.2593-0.1170)\right]$$

$$=-0.0201$$

$$K_{24} = 0.2\left[3(u_1+K_{13})-4(v_1+K_{23})\right]$$

$$=0.2\left[3(0.1001+0.0283)-4(0.2593-0.1170)\right]$$

$$=-0.0368$$

$$u(0.4) = u_2 = u_1 + \tfrac{1}{6}(K_{11}+2K_{12}+2K_{13}+K_{14})$$

$$=0.1001+\tfrac{1}{6}(0.0437+0.0020+0.0566-0.0201)$$

$$=0.1138$$

$$v(0.4) = v_2 = v_1 + \tfrac{1}{6}(K_{21}+2K_{22}+2K_{23}+K_{24})$$

$$=0.2593+\tfrac{1}{6}(-0.1474-0.1506-0.2340-0.0368)$$

$$\boxed{v(0.4) = 0.1645}$$

Implicit Runge-kutta second order methods:-

$$u_{j+1} = u_j + k_1$$

$$k_1 = hf(t_j + h/2, u_j + k_1/2)$$

1. Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with $h = 0.2$ on the interval $[0, 0.4]$. Use the second order implicit Runge-kutta method

Soln

The second order implicit Runge-kutta method

$$u_{j+1} = u_j + k_1 \quad , j = 0, 1$$

$$k_1 = hf(t_j + h/2, u_j + k_1/2)$$

$$k_1 = h\left[-2(t_j + h/2)(u_j + k_1/2)^2\right]$$

$$= -h\left((2t_j + h)(u_j + k_1/2)^2\right]$$

This is an implicit equation in k_1 and can be solved by using an iterative method

Use the Newton-Raphson method

$$F(k_1) = k_1 + h(2t_j + h)(u_j + k_1/2)^2$$

$$= k_1 + 0.2(2t_j + 0.2)(u_j + k_1/2)^2$$

$$F'(k_1) = 1 + h(2t_j + h)(u_j + k_1/2)$$

$$= 1 + 0.2(2t_j + 0.2)(u_j + k_1/2)$$

Newton-Raphson method

$$k_1^{(s+1)} = k_1^{(s)} - \frac{F(k_1^{(s)})}{F'(k_1^{(s)})}, \quad s = 0, 1 \ldots$$

Assume that $k_1^{(0)} = hf(t_j, u_j)$, $j = 0, 1$

$\underline{j=0}$ $\quad t_0 = 0, \; u_0 = 1$

$$k_1^{(0)} = h(-2t_0u_0^2)$$

$$= 0.2\,(2(0)(1)^2)$$

$$k_1^{(0)} = 0$$

$$F(k_1^{(0)}) = k_1^{(0)} + 0.2\,[2t_0 + 0.2]\,[u_0 + \tfrac{1}{2}k_1^{(0)}]^2$$

$$= 0 + 0.2\,[2(0) + 0.2]\,[1 + 0/2]^2$$

$$= 0.04$$

$$F'(k_1^0) = 1 + 0.2\,(2t_0 + 0.2)\,(u_0 + \tfrac{1}{2}k_1^0)$$

$$= 1 + 0.2\,(2(0) + 0.2)\,(1 + 0/2)$$

$$= 1 + 0.2\,(0.2)(1)$$

$$= 1.04$$

$$k_1^{(1)} = k_1^{(0)} - \frac{F(k_1^0)}{F'(k_1^0)} = 0 - \frac{0.04}{1.04}$$

$$= -0.03846150$$

$$F(k_1^{(1)}) = k_1^{(1)} + 0.2\,[2t_0 + 0.2]\,[u_0 + \frac{k_1^{(1)}}{2}]^2$$

$$= -0.03846150\,[0.2(0.2)(1 - \frac{0.03846150}{2})]^2$$

$$= 0.00001483$$

$$F'(k_1^{(1)}) = 1 + 0.2\,(2t_0 + 0.2)\left(u_0 + \frac{k_1^{(1)}}{2}\right)$$

$$= 1 + 0.2\,(0.2)\left(1 - \frac{0.03846150}{2}\right)$$

$$= 1.03923077$$

$$k_1^{(2)} = k_1^{(1)} - \frac{F(k_1^{(1)})}{F'(k_1^{(1)})}$$

$$= -0.03846150 - \frac{0.00001483}{1.03923077}$$

$$= -0.03847567$$

$$F(k_1^{(2)}) = 0.30 \times 10^{-8}$$

$$k_1 = k_1^{(2)} = -0.03847567$$

$$u(0.2) = u_1 = u_0 + k_1 = 1 - 0.03847567$$

$$u(0.2) = 0.96152433$$

<u>$j = 1$</u>. $t_1 = 0.2$, $u_1 = 0.96152433$

$$k_1^{(0)} = -h\,(2t_1 u_1^2) = -0.2\,(2\,(0.2)\,(0.96152433)^2)$$

$$k_1^{(0)} = -0.07396231$$

$$F(k_1^{0}) = k_1 + 0.2\,(2t_1 + 0.2)\left(u_1 + \tfrac{1}{2}k_1\right)$$

$$= -0.07396231 + 0.2\,(2(0.2) + 0.2)\left(0.96152433 - \frac{0.0739623}{2}\right)$$

$$= 0.02861128$$

$$F'(k_1^{(0)}) = 1 + 0.2\,(2t_1 + 0.2)\left(u_1 + \tfrac{1}{2}k_1\right)$$

$$= 1 + 0.2\left(2(0.2) + 0.2\right)\left(0.96152433 - \frac{0.07396231}{2}\right)$$

$$F'(k_1^{0}) = 1.11094517$$

$$k_1^{(1)} = -0.07396231 - \frac{0.02861128}{1.11094517}$$

$$k_1^{(1)} = -0.09971631$$

$$F(k_1^{(1)}) = 0.00001989$$

$$F'(k_1^{(1)}) = 1.10939993$$

$$k_1^{(2)} = -0.09971631 - \frac{0.00001989}{1.10939993}$$

$$k_1^{(2)} = -0.09973423$$

$$F(k_1^{(2)}) = 0.35 \times 10^{-7}$$

$$F'(k_1^{(2)}) = 1.10939885$$

$$k_1^{(3)} = -0.099773420$$

$$k_1 = k_1^{(3)} = -0.09973420$$

$$u(0.4) = u_2 = u_1 + k_1$$

$$= 0.96152433 - 0.09973420$$

$$u(0.4) = 0.86179013$$