B.Sc computer Science [Numerical Analysis and statistics] (1)

UNIT - 3

Numerical solution of ODE

The Taylore's series method To find solution-

Solution- consider the first order differential Equation $x = x_0$, $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

$$y = y_0 + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y''.$$

1) The Taylor's series method to solve the equations:-

$$\frac{dy}{dx} = -xy, \quad y(0) = 1.$$

Soln:-

Given $y' = -xy \rightarrow (1)$

Now at $x = 0, y = 1$.

So $y'(0) = -(0)(1) = 0$.

differeniate eqn (1) repeatedlly and Put $y = 1$ at $x = a$.

$y'' = -xy' - y \Rightarrow y''(0) = -(0)(0) - 1 = -1$

$y''' = -xy'' - 2y' \Rightarrow y'''(0) = -(0)(-1) - 2(0)$
$= 0$

$y^{iv} = -xy''' - 3y'' \Rightarrow y^{iv}(0) = -(0)(0) - 3(-1)$
$= -3$.

$y^{v} = -xy^{iv} - 4y''' \Rightarrow y^{v}(0) = 0$

$y^{vi} = -xy^{v} - 5y^{iv} \Rightarrow y^{vi}(0) = -15$

Taylor's Series Expansion about $x = 0$
$(\because x_0 = 0)$

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots$$

$$y = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + 0 + \frac{x^4}{4!}(3) + 0 + \frac{x^6}{6!}(-15) + \cdots$$

$$\boxed{y = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots}$$

Eulers and Eulers modified Method.

Euler method is Runge kutta method of I^{st} order

Euler modified method is Runge kutta method of II^{nd} order

$$y_{n+1} = y_n + h f(x_n, y_n)$$

1) Solve $\frac{dy}{dx} = x+y$ with boundary condition $y=1$ at $x=0.1$

$x_0 = 0$	$y=1$ at $x=0.1$
$x_1 = 0.02$	$y_0 = 1$
$x_2 = 0.04$	$y_1 = 1.02$
$x_3 = 0.06$	$y_2 = 1.0408$
$x_4 = 0.08$	$y_3 = 1.0624$
$x_5 = 0.1$	$y_4 = 1.0848$
	$y_5 = 1.1081.$

$f(x,y) = x+y$

$$y_{n+1} = y_n + h(x_n + y_n)$$

Put $n=0$, $y_1 = y_0 + h(x_0 + y_0) = 1 + 0.02(0+1)$

$$\boxed{y_1 = 1.02.}$$

put $n=1$, $y_2 = y_1 + h(x_1 + y_1) = 1.02 + 0.02 (0.02 + 1.02)$

$\boxed{y_2 = 1.0468}$

$y_3 = y_2 + h(x_2 + y_2) = 1.0408 + 0.02 (0.04 + 1.0408)$

$\boxed{y_3 = 1.0624}$

put $n=3$, $y_4 = y_3 + h(x_3 + y_3)$

$= 1.0624 + 0.02 (0.0624 + 1.0624)$

$\boxed{y_4 = 1.0848}$

$n=4$, $y_5 = y_4 + h(x_4 + y_4)$

$= 1.0848 + 0.02 (0.08 + 1.0848)$

$\boxed{y_5 = 1.1081}$

Euler's modified method

Runga kutta method of IInd order.

$$y^*_{n+1} = y_n + h\, f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right)$$

1) Given $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$, find $y(0.02)$ & $y(0.04)$ By Euler's modified method.

Soln:

$f(x,y) = x^2 + y$, $x_0 = 0$, $y_0 = 1$

$x_1 = 0.02$, $y_1^* = 1.02$, $y_1 = 1.0202$

$h = 0.02$ $\quad$ $x_2 = 0.04$, $y_2^* = 1.0406$, $y_2 = 1.0408$

put $n=0$

$$y_1^* = y_0 + hf(x_0, y_0)$$

$$y_1^* = y_0 + h(x_0^2 + y_0)$$

$$= 1 + 0.02(0^2+1) = 1.02.$$

Put $n=0$,

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^*)]$$

$$y_1 = y_0 + \frac{h}{2}((x_0^2 + y_0) + (x_1^2 + y_1^*)]$$

$$y_1 = 1 + \frac{0.02}{2}[(0^2+1) + (0.02)^2 + 1.02]$$

$$\boxed{y_1 = 1.0202.}$$

again put $n=1$

$$y_2^* = y_1 + hf(x_1, y_1)$$

$$y_2^* = y_1 + h(x_1^2 + y_1) = 1.0202 + 0.02((0.02)^2 + 1.0202)$$

$$\boxed{y_2^* = 1.0408}$$

Again put $n=1$

$$y_2 = y_1 + \frac{h}{2} f(x_1, y_1) + f(x_2, y_2^*)]$$

$$y_2 = y_1 + \frac{h}{2}(x_1^2 + y_1) + (x_2^2 + y_2^*)]$$

$$= 1.0202 + \frac{0.02}{2}((0.02)^2 + 0.0202 + (0.04)^2 + (1.0406)$$

$$\boxed{y_2 = 1.0408}$$

Runga - kutta method: of fourth order

consider intial value.

program $\frac{dy}{dx} = f(x,y)$

when $y(x_0) = y_0$

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + h/2, y_n + k_1/2)$

$k_3 = hf(x_n + h/2, y_n + k_2/2)$

$k_4 = hf(x_n + h/1, y_n + k_3)$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + 2k_4)$

$y_{n+1} = y_n + k$

(1) Given $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, find $y(0.2)$ when $h = 0.1$ using Rk of order form

Soln:

$x_1 = x_0 + h$

$x_0 = 0$ $y_0 = 1$

$x_1 = 0.1$ $y_1 = 1.1165$

$x_2 = 0.2$ $y_2 = 1.2737$

$f(x,y) = x + y^2$

put $n = 0$

$k_1 = hf(x_0, y_0)$

$k_1 = h(x_0 + y_0^2) = 0.1(0+1) = 0.1$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$.

$= h[(x_0 + h/2) + (y_0 + k_1/2)^2]$

$= 0.1[(0 + 0.1/2) + (1 + \frac{0.1}{2})^2]$

$\boxed{k_2 = 0.11525}$

⑥

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$= h[(x_0 + h/2) + (y_0 + k_2/2)^2]$

$= 0.1[(0 + \frac{0.1}{2}) + (1 + 0.11525/2)^2$

$\boxed{k_3 = 0.1169}$

$k_4 = hf(x_0+h, y_0+k_3) = h((x_0+h)$
$+ (y_0+k_3)^2]$

$= 0.1[(0+0.1) + (1+0.1169)^2]$

$= 0.13479.$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k = \frac{1}{6}[0.1 + 2(0.111525) + 2(0.11169)$
$+ 0.1347)$

$\boxed{k = 0.1165}$

$y_1 = y_0 + k = 1 + 0.1165 = 1.1165$

put $n = 1$

$k_1 = hf(x_1, y_1) = h(x_1 + y_1^2)$
$= 0.1[0.1 + (1.1165)^2]$

$\boxed{k_1 = 0.1347.}$

$k_2 = hf(x_1 + h/2, y_1 + k_1/2)$

$= h[(x_1 + h/2) + (y_1 + k_1 h)^2]$

$= 0.1[(0.1 + 0.1/2)$
$+ (1.1165 + 0.1347/2)^2]$

$$\boxed{k_2 = 0.1552}$$

$$k_3 = hf(x_1 + h/2,\ y_1 + k_2/2)$$

$$= h\left[(x_1 + h/2) + (y_1 + k_2/2)^2\right]$$

$$= 0.1\left[(0.1) + 0.1/2) + (1.1165 + 0.1552/2)^2\right]$$

$$\boxed{k_3 = 0.1576}$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= h\left[(x_0 + h) + (y_0 + k_3)^2\right]$$

$$= 0.1\left[(0.1) + (0.1) + [(1.1165) + 0.1576]\right]$$

$$\boxed{k_4 = 0.1823}$$

$$k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= \frac{1}{6}\left[0.1347 + 2(0.1552) + 2(0.1576) + 0.1823\right]$$

$$\boxed{k = 0.1572}$$

$$y_2 = y_1 + k = 1.1165 + 0.1572 = 1.2737.$$

$$\boxed{y_2 = 1.2737.}$$

Adams - predicted & corrected method.

predicted & corrector formula:.

$$y_{n+1,P} = y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$$

$$y_{n+1,c} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2})$$

① given $\frac{dy}{dx} = x^2(1+y)$ by adam's Bashforth method : $y(1)=1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$. find $y(1.4)$

Soln:

$f(x,y) = x^2(1+y)$

x	y	$f = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$f_0 = x_0^2(1+y_0) = 1(1+1) = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$f_1 = x_1^2(1+y_1) = (1.1)^2(1+1.233)^2$ $= 2.70193$
$x_2 = 1.2$	$y_2 = 1.548$	$f_2 = x_2^2(1+y_2)$ $= (1.2)^2(1+1.548)$ $= 3.69912$
$x_3 = 1.3$	$y_3 = 1.979$	$f_3 = x_3^2(1+y_3) = (1.3)^2(1+1.979)$ $= 5.03451$
$x_4 = 1.4$	$y_4^P = 2.5723$ $y_4^C = 2.5749$	$f_4^P = x_4^2(1+y_4^P) = (1.4)^2(1+2.5723)$ $= 7.0017$

$$y_4^P = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 1.979 + \frac{0.1}{24}[55(5.03451) - 59(3.69912) + 37(2.70193) - 9(2)]$$

$y_4^p = 2.5723$

$y_4^c = y_3 + \frac{h}{24}\left[9f_4^p + 19f_3 - 5f_2 + f_1\right]$

$y_4^c = 1.979 + \frac{0.1}{24}(9(7.0017)$

$+19(5.03451)$

$-5(3.69912) + 2.70193)$

$y_4^c = 2.5749$

Formula :-

Adam's Bash forth predictor formula is

$y_{n+1,p} = y_n + \frac{h}{24}(55y_n' - 59y_{n-1}'$

$+37y_{n-2}' - 9y_{n-3}')$

Putting $n=3$, we get,

$y_{4,p} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1'$

$-9y_0')$

Adam's Bash forth corrector formula is

$y_{n+1,c} = y_n + \frac{h}{24}(9y_{n+1}' + 19y_n' - 5y_{n-1}'$

$+y_{n-2}')$

Putting $n=3$, we get

$y_{4,c} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2'$

$+y_1')$

Examples:-

(1) $y' = \frac{x+y}{2}$

x_i	0	0.5	1	1.5
y_i	2	2.636	3.595	4.968

find $y(2)$ by Adams bashforth predic tor method:-

Soln:-

$y' = \frac{x+y}{2}$

Adam's Bashforth Predictor formula is,

$$y_{n+1,p} = y_n + \frac{h}{24}(55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}')$$

putting $n=3$, we get

$$y_{4,p} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0') \to ②$$

We have given that,

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$

$y_0 = 2$, $y_1 = 2.636$, $y_2 = 3.595$,

$y_3 = 4.968$.

$y' = \frac{x+y}{2}$

$y_0' = \frac{x+y}{2} = 1$ (where $x=0, y=2$)

$y_1' = \frac{x+y}{2} = 1.568$ (where $x = 0.5$, $y = 2.636$)

$y_2' = \frac{x+y}{2} = 2.2975$ (where $x = 1, y = 3.595$)

$y_3' = \frac{x+y}{2} = 3.234$ (where $x = 1.5$, $y = 4.968$)

putting the values in (2), we get,

$y_{4,P} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0')$

$y_{4,P} = 4.968 + \frac{0.5}{24}(55(3.234) - 59(2.2975) + 37(1.568) - 9(1))$

$y_{4,P} = 4.968 + \frac{0.5}{24}(177(87) - 135(5525) + 58(016) - 9)$

$y_{4,P} = 4.968 + \frac{0.5}{24}(91(3335))$

$\boxed{y_{4,P} = 6.8708}$

predicted Value $= 6.8708$.

$y_4' = \frac{x+y}{2} = 4.4354$ ($x = 2, y = 6.8708$)

Adam's Bash forth Corrector formula is,

$y_{n+1,C} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_3 - 5y'_2 + y'_1)$

$n=3$

$y_{4,C} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$

$y_{4,C} = 4.968 + \frac{0.5}{24}(9(4.4354)$
$+19(3.234)$
$-5(2.2975)$
$+1.568)$

$y_{4,C} = 4.968 + 1.9051$

$y_{4,C} = 6.8731$

$\boxed{\therefore y(2) = 6.8731.}$

Milne's predictor & corrector method to solution of ODE Numerical method:

consider $\frac{dy}{dx} = f(x,y)$

$y(x_0) = y_0$

$y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$ where x_0, x_1, x_2, x_3 are equi distance.

Value of x with step size h.

Milne's predictor formula,

$y_4^P = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

Milne's corrector formula.

$y_4^C = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^P)$

① using milne's predictor - corrector method. (13)

find y when $x = 0.8$ given

$$\frac{dy}{dx} = x - y^2$$

$y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$

$y(0.6) = 0.1762$, $y(0.8) = ?$

Soln:.

$f(x,y) = x - y^2$

x	y	f
$x_0 = 0$	$y_0 = 0$	$f_0 = x_0 - y_0^2 = 0 - 0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = x_1 - y_1^2 = (0.2) - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = x_2 - y_2^2 = (0.4) - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = x_3 - y_3^2 = (0.6) - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4^p = 0.3049$	
$h = 0.2$	$y_4^c = 0.3046$	

Milne predictor formula:.

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 0 + 4(0.2)[2(0.1996) - (0.3937) + 2(0.5689)]$$

$$\boxed{y_4^p = 0.3049}$$

14

Milne corrector formula,

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p)$$

$$y_4^c = 0.0795 + \frac{0.2}{3}(0.3937$$

$$+ 4(0.5689) + 0.707$$

$$\boxed{y_4^c = 0.3046}$$

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