The inverse Transforms:

Let the symbol $L^{-1}\{$ denote a function whose laplace Transform is $F(s)$.

Thus if $L\{f(t)\} = F(s)$

Then $f(t) = L^{-1}\{F(s)\}$

If $L\{f(t)\} = F(s)$ then

$L[e^{-at} t] = F(s+a)$.

Hence we get the result

$L^{-1} F(s+a) = e^{-at} f(t)$

$L^{-1}\{F(s+a)\} = e^{-at} L^{-1}\{F(s)\}$

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Find $L^{-1}\left[\frac{1}{(s+a)^2}\right]$

Soln:-

$$L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-at}\, t$$

Find $L^{-1}\left[\frac{1}{(s+2)^2+16}\right]$

Soln:-

$$L^{-1}\left[\frac{1}{(s+2)^2+16}\right] = e^{-2t} L^{-1}\left[\frac{1}{s^2+4^2}\right]$$

$$= \frac{1}{4} e^{-2t} L^{-1}\left[\frac{4}{s^2+4^2}\right]$$

$$= \frac{1}{4} e^{-2t} \sin 4t.$$

Find $L^{-1}\left[\frac{s}{s^2+2s+5}\right]$

Soln:-

$$L^{-1}\left[\frac{s}{\quad}\right] \quad L^{-1}\left[\frac{s}{\quad}\right]$$

$$= L^{-1}\left[\frac{s}{(s+1)^2+2^2}\right]$$

$$= L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+2^2}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] - L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^{-t} L^{-1}\left[\frac{s}{s^2+2^2}\right] - e^{-t} L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= e^{-t}\cos 2t - \tfrac{1}{2} e^{-t} L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= e^{-t}\cos 2t - \tfrac{1}{2} e^{-t}\sin 2t$$ //.

Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Soln:

Consider.

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + B(s+2)s + Cs(s+1)$$

$$1 = A(s+1)(s+2) + Bs(s+2) + C(s+1)s \longrightarrow (1).$$

Put $s=0$ in (1)

$$1 = A(0+1)(0+2) + B(0)(0+2) + C(0)(0+1)$$

$$1 = A(1)(2)$$

$$1 = 2A$$

$$A = \frac{1}{2}$$

Put $s=-1$ in (1)

$$1 = A(-1+1)(-1+2) + B(-1)(-1+2) + C(-1)(-1+1)$$

$$1 = B(-1)(1)$$

$$B = -1$$

Put $s=-2$ in (1)

$$1 = A(-2+1)(-2+2) + B(-2)(-2+2) + C(-2)(-2+1)$$

$$1 = C(-2)(-1)$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

$$= L^{-1}\left[\frac{1/2}{s} + \frac{(-1)}{s+1} + \frac{1/2}{s+2}\right]$$

$$= L^{-1}\left[\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2}(1) - e^{-t} L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2} e^{-2t} L^{-1}\left(\frac{1}{s}\right)$$

$$= \frac{1}{2} - e^{-t}(1) + \frac{1}{2} e^{-2t}(1)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$$

Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$

Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Find $L^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s+3)}\right]$

Find $L^{-1}\left[\frac{s}{(s+1)^5}\right]$

Soln:

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$$

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Consider.

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}$$

$$\frac{1}{\cancel{(s+1)(s^2+2s+2)}} = \frac{A(s^2+2s+2)+(Bs+C)(s+1)}{\cancel{(s+1)(s^2+2s+2)}}$$

$$1 = A(s^2+2s+2)+(Bs+C)(s+1) \longrightarrow ①.$$

Put $s=-1$ in ①

$1 = A((-1)^2+(-1)2+2)+(B(-1)+C)(-1+1)$

$1 = A(1+2-2)$

$1 = A(1)$

$A=1$

Put $s=0$ in ①

$1 = A(0+0+2)+(B(0)+C)(0+1)$

$1 = 2A+(C)(1)$

$1 = 2(1)+C$

$1 = 2+C$

$1-2 = C$

$C=-1$

Put $s=1$ in ①

$1 = A(1+2+2)+(B+C)(2)$

$1 = A(5)+(B+C)(2)$

$1 = 5+(B-1)(2)$

$1 = 5 + 2B - 2$

$1 = 5 - 4 + 2 + 2B$

$1 = 3 + 2B$

$-2 = 2B$

$\boxed{B = -1}$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = L^{-1}\left[\frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} + \frac{(-1)(s)+(-1)}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} + \frac{(-s-1)}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} - \frac{(s+1)}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s+1}{s^2+2s+1+2-1}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$$

$$= e^{-t} L^{-1}\left[\frac{1}{s}\right] - e^{-t} L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= e^{-t}(1) - e^{-t}\cos t$$

$$= e^{-t} - e^{-t}\cos t$$

$$= e^{-t}(1 - \cos t)$$

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12. Soln :-

ex

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{(s+2)-2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2}\right] - L^{-1}\left[\frac{2}{(s+2)^2}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{\cancel{s}}{s^{\cancel{2}}}\right] - e^{-2t} L^{-1}\left[\frac{2}{s^2}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{1}{s}\right] - 2e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-2t}(1) - 2e^{-2t}\, t$$

$$= e^{-2t}(1-2t).$$

13. Soln :-

ex

$$L^{-1}\left[\frac{1}{s(s+a)}\right]$$

Consider,

$$\frac{1}{s(s+a)} = \frac{A}{s} + \frac{B}{s+a}$$

$$\frac{1}{\cancel{s(s+a)}} = \frac{A(s+a)+B(s)}{\cancel{(s)(s+a)}}$$

$$1 = A(s+a) + B(s) \longrightarrow (1)$$

Put $s=-a$ in (1)

$1 = A(-a+a) + B(-a)$

$1 = B(-a)$

$B = \frac{1}{-a}$

Put, $s = 0$ in ①

$1 = A(0+a) + B(0)$

$1 = A(a)$

$A = \frac{1}{a}$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = L^{-1}\left[\frac{A}{s} + \frac{B}{s+a}\right]$$

$$= L^{-1}\left[\frac{\frac{1}{a}}{s} + \frac{-1/a}{s+a}\right]$$

$$= 1/a\, L^{-1}\left[\frac{1}{s}\right] - 1/a\, L^{-1}\left[\frac{1}{s+a}\right]$$

$$= 1/a(1) - 1/a\, e^{-at} L^{-1}\left[\frac{1}{s}\right]$$

$$= 1/a(1) - 1/a\, e^{-at}(1)$$

$$= 1/a - \frac{e^{-at}}{a}$$

$$= \frac{1 + e^{-at}}{a}$$

$$\left[(s+1)(s+2)(s+3) \right]$$

Consider :-

$$\frac{7s-1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$\frac{7s-1}{\cancel{(s+1)(s+2)(s+3)}} = \frac{A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)}{\cancel{(s+1)(s+2)(s+3)}}$$

$$7s-1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \longrightarrow (1)$$

Put $s = -1$ in (1)

$$7(-1)-1 = A(-1+2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1+2)$$

$$-7-1 = A(1)(2)$$

$$-8 = 2A$$

$$-8/2 = A$$

$$A = -4$$

Put $s = -2$ in (1)

$$7(-2)-1 = A(-2+2)(-2+3) + B(-2+1)(-2+3) + C(-2+1)(-2+2)$$

$-14-1 = B(-1)(1)$

$-15 = -B$

$B = 15$

Put $s = -3$ in ①

$7(-3) - 1 = A(-3+2)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3+2)$

$-21 - 1 = C(-2)(-1)$

$-22 = 2C$

$C = -11$

$$L^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s+3)}\right] = L^{-1}\left[\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}\right]$$

$$= L^{-1}\left[\frac{-4}{s+1} + \frac{15}{s+2} + \frac{(-11)}{s+3}\right]$$

$$= -4\, L^{-1}\left(\frac{1}{s+1}\right) + 15\, L^{-1}\left(\frac{1}{s+2}\right) - 11\, L^{-1}\left(\frac{1}{s+3}\right)$$

$$= -4\, e^{-t} L^{-1}\left(\frac{1}{s}\right) + 15\, e^{-2t} L^{-1}\left(\frac{1}{s}\right) - 11\, e^{-3t} L^{-1}\left(\frac{1}{s}\right)$$

$$= -4\, e^{-t} + 15 e^{-2t} - 11 e^{-3t}$$

we have

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$u = e^{-st}$ $\qquad dv = f'(t)\,dt$

$du = -se^{-st}\,dt$ $\qquad v = f(t)$

$$L\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\cdot(-se^{-st})\cdot dt$$

$$= sL\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

$$L\{f'(t)\} = \{F'(t)\} - F(0)$$

$$= sL\{F(t)\} - F(0)$$

$$= sL\{f'(t)\} - f'(0)$$

$$= s[sL\{f(t)\} - f(0)] - f'(0)$$

$$= s^2 L\{f(t)\} - sf(0) - f'(0)$$

Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$ given that

~~soln~~:- $y=0$ and $\frac{dy}{dt} = 0$ when $t=0$.

soln:-

Given $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$

$$y_0'' + 2y_0' - 3y = \sin t$$

$$L[y_0''] + L[2y'] - 3L[y] = L[\sin t]$$

$$[s^2L(y) - sy(0) - y'(0)] + 2[sL(y) - y(0)] - 3L(y) = \frac{1}{s^2+1}$$

$$[s^2L(y) - s(0) - (0)] + 2[sL(y) - 0] - 3L(y) = \frac{1}{s^2+1}$$

$$s^2L(y) + 2sL(y) - 3L(y) = \frac{1}{s^2+1}$$

$$L(y)[s^2+2s-3] = \frac{1}{s^2+1}$$

$$L(y) = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$y = L^{-1}\left[\frac{1}{(s^2+1)(s^2+2s-3)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

Consider

$$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+3}$$

$$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{(As+B)(s+3)(s-1) + C(s^2+1)(s+3) + D(s^2+1)(s-1)}{(s^2+1)(s-1)(s+3)}$$

$$1 = (As+B)(s+3)(s-1) + C(s^2+1)(s+3) + D(s^2+1)(s-1) \longrightarrow (1)$$

Put $s=-3$ in (1)

$$1 = (A(-3)+B)(-3+3)(-3-1) + C((-3)^2+1)(-3+3) + D((-3)^2+1)(-3-1)$$

$$1 = 0 + 0 + D(9+1)(-4)$$

$$1 = D(10)(-4)$$

$$1 = D(-40)$$

$$\boxed{D = -1/40}$$

Put $s=1$ in (1).

$$1 = (A(1)+B)(1+3)(1-1) + C(1+1)(1+3) + D(1^2+1)(1-1)$$

$$1 = 0 + C(2)(4) + 0$$

$$1 = 8C$$

$$\boxed{C = 1/8}$$

Put $s=0$ in (1)

$$1 = (A(0)+B)(0+3)(0-1) + C(0+1)(0+3) + D(0+1)(0-1)$$

$$1 = B(3)(-1) + C(1)(3) + D(1)(-1)$$

$$1 = -B3 + 3C \cancel{+} D$$

$$1 = -3B + 3(1/8) + 1/40$$

$$1 = -3B + \frac{3}{8} + \frac{1}{40}$$

$$1 + 3B = \frac{15+1}{40}$$

$$1 + 3B = \frac{\cancel{16}^{2}}{\cancel{40}_{5}}$$

$$1 + 3B = \frac{2}{5}$$

$$\cancel{B = \frac{2}{18}}$$

$$3B = \frac{2}{5} - 1$$

$$3B = \frac{2-5}{5}$$

$$3B = -\frac{3}{5}$$

$$B = -\frac{3}{5} \cdot \frac{1}{3}$$

$$\boxed{B = -\frac{1}{5}}$$

Put $s = -1$ in (1)

$$1 = (A(-1) + B)(-1+3)(-1-1) + C(1+1)(-1+3) + D(1+1)(-1-1)$$

$$1 = (-A+B)(2)(-2) + C(2)(2) + D(2)(-2)$$

$$1 = (-A+B)(-4) + 4C - 4D$$

$$1 = 4A - 4B + 4C - 4D$$

$1 = 4(A - B + C - D)$

$\frac{1}{4} = A + \frac{1}{5} + \frac{1}{8} + \frac{1}{40}$

$\frac{1}{4} = A + \frac{8+5+1}{40}$

$\frac{1}{4} = A + \frac{\cancel{14}\,7}{\cancel{40}\,20}$

$\frac{1}{4} = A + \frac{7}{20}$

$A = \frac{1}{4} - \frac{7}{20}$

$A = \frac{5-7}{20}$

$A = -\frac{\cancel{2}\,1}{\cancel{20}\,10}$

$$\boxed{A = -\frac{1}{10}}$$

$$= \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+3}$$

$$y = L^{-1}\left(\frac{1}{(s^2+1)(s-1)(s+3)}\right)$$

$$= L^{-1}\left[\frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+3}\right]$$

$$= L^{-1}\left[\frac{-\frac{1}{10}s - \frac{1}{5}}{s^2+1} + \frac{\frac{1}{8}}{s-1} - \frac{\frac{1}{40}}{s+3}\right]$$

$$= L^{-1}\left[\frac{-\frac{s}{10} - \frac{1}{5}}{s^2+1}\right] + \frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{40}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= -\frac{1}{10} L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{5} L^{-1}\left(\frac{1}{s^2+1}\right) + \frac{1}{8} L^{-1}\left(\frac{1}{s-1}\right)$$

$$-\frac{1}{40} L^{-1}\left(\frac{1}{s+3}\right)$$

$$= -\frac{1}{10}\cos t - \frac{1}{5}\sin t + \frac{1}{8} e^{t} L^{-1}\left(\frac{1}{s}\right)$$

$$-\frac{1}{40} e^{-3t} L^{-1}\left(\frac{1}{s}\right)$$

$$= -\frac{1}{10}\cos t - \frac{1}{5}\sin t + \frac{1}{8} e^{t}(1) - \frac{1}{40} e^{-3t}(1)$$

$$y = -\frac{1}{10}\cos t - \frac{1}{5}\sin t + \frac{1}{8} e^{t} - \frac{1}{40} e^{-3t}$$

Slove $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$ given that

$y = 0,\ y' = 2$ when $t = 0$.

Soln :-

Given $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$

$$y'' + 4y' - 5y = 5$$

$$L[y''] + 4L[y'] - L[5y] = L[5]$$

$$[s^2 L(y) - s y(0) - y'(0)] + 4[sL(y) - y(0)]$$

$$-5L[y] = 5/s$$

$$[s^2 L(y) - s(0) - 2] + 4[sL(y) - 0]$$

$$-5L[y] = 5/s$$

$s^2 L(y) - 2 + 4sL(y) - 5L(y) = 5/s$

$s^2 L(y) + 4sL(y) - 5L(y) = 5/s + 2$

$L(y)[s^2 + 4s - 5] = \frac{5+2s}{s}$

$L(y) = \frac{5+2s}{s(s^2+4s-5)}$

$L(y) = \frac{5+2s}{s(s-1)(s+5)}$

$y = L^{-1}\left[\frac{5+2s}{s(s-1)(s+5)}\right]$

Consider:

$\frac{5+2s}{s(s-1)(s+5)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{s+5}$

$\frac{5+2s}{\cancel{s(s-1)(s+5)}} = \frac{A(s-1)(s+5) + B(s+5)s + C(s)(s-1)}{\cancel{s(s+5)(s-1)}}$

$5+2s = A(s-1)(s+5) + B(s+5)(s) + C(s)(s-1) \longrightarrow (1)$

Put $s = 1$ in (1)

$5+2(1) = A(1-1)(1+5) + B(1+5)(1) + C(1)(1-1)$

$5+2 = B(6)(1)$

$7 = B(6)$

$B = 7/6$

Put $s = -5$ in ①

$5(-5)+8/=/A(($

$5+2(-5) = A(5-1)(-5+5) + B(-5+5)(-5) + C(-5)(-5-1)$

$5-10 = C(-5)(-6)$

$-5 = 30C$

$\frac{-\cancel{5}^{1}}{\cancel{30}_{6}} = C$

$C = -\frac{1}{6}$

Put $s=0$ in ①

$5+2(0) = A(0-1)(0+5) + B(0+5)(0) + C(0)(0-1)$

$5 = A(-1)(5)$

$5 = -5A$

$A = -1$

$\therefore y = L^{-1}\left(\frac{5+2s}{s(s-1)(s+5)}\right)$

$$y = L^{-1}\left[\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+5}\right]$$

$$= L^{-1}\left[\frac{-1}{s} + \frac{7/6}{s-1} + \frac{(-1/6)}{s+5}\right]$$

$$= L^{-1}\left(-\frac{1}{s}\right) + \frac{7}{6}L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{6}L^{-1}\left(\frac{1}{s+5}\right)$$

$B = 7/6$

$$y = -1 + \frac{7}{6} e^{t}(1) - \frac{1}{6} e^{-5t}(1)$$

$$y = -1 + \frac{7}{6} e^{t} - \frac{1}{6} e^{-5t}.$$

$$y = -1 + \frac{7e^{t} - e^{-5t}}{6}$$

$$y = \frac{-6 + 7e^{t} - e^{-5t}}{6}$$

Sol:-

$$L^{-1}\left[\frac{s}{(s+1)^5}\right] = L^{-1}\left[\frac{(s+1)-1}{(s+1)^5}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^5}\right] - L^{-1}\left[\frac{1}{(s+1)^5}\right]$$

$$= e^{-t} L^{-1}\left[\frac{s}{s^{4}}\right] - e^{-t} L^{-1}\left[\frac{1}{s^5}\right]$$

$$= e^{-t} L^{-1}\left[\frac{1}{s^4}\right] - e^{-t} L^{-1}\left[\frac{1}{s^5}\right]$$

$$= e^{-t} \frac{t^3}{3!} - e^{-t} \frac{t^4}{4!}$$

Solve the $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 4e^{-t}$ given that $y=0$ and $\frac{dy}{dt}=0$ when $t=0$

Soln:

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 4e^{-t}$$

$$y'' + 2y' + 5y = 4e^{-t}$$

$$L[y''] + 2L[y'] + 5L[y] = 4L[e^{-t}]$$

$$[s^2L(y) - sy(0) - y'(0)] + 2[sL(y) - y(0)] + 5L[y] = 4\left(\frac{1}{s+1}\right)$$

$$[s^2L(y) - 0 - 0] + 2[sL(y) - 0] + 5L(y) = \frac{4}{s+1}$$

$$s^2L(y) + 2sL(y) + 5L(y) = \frac{4}{s+1}$$

$$L(y)[s^2 + 2s + 5] = \frac{4}{s+1}$$

$$L(y) = \frac{4}{(s+1)(s^2+2s+5)}$$

$$y = L^{-1}\left[\frac{4}{(s+1)(s^2+2s+5)}\right]$$

$\cancel{(s+1)(s^2+2s+5)}$ $\cancel{(s+1)(s^2+2s+5)}$

$$4 = A(s^2+2s+5) + (Bs+C)(s+1) \longrightarrow ①$$

Put $s = -1$ in ①

$$\cancel{4 = A(0+0+5) + (B(0)+C)(0+1)}$$

$$\cancel{4 = A(5)+C(1)}$$

$$4 = A(1+(-2)+5) + (B(-1)+C)(0)$$

$$4 = A(1-2+5)$$

$$4 = A(-3+5)$$

$$4 = A(4)$$

$$A = 1$$

Put $s = 0$ in ①

$$4 = A(0+0+5) + (B(0)+C)(0+1)$$

$$4 = A(5) + C(1)$$

$$4 = (1)5 + C$$

$$4 = 5 + C$$

$$\boxed{C = -1}$$

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Put $s=1$ in ①

$4 = A(1+2+5)+(B(1)+C)(1+1)$

$4 = A(8)+(B+C)(2)$

$4 = A8+2B+2C$

$4 = 8+2B+2(-1)$

$4 = 8-2+2B$

$4 = 6+2B$

$-2 = 2B$

$\boxed{B=-1}$

$$y = L^{-1}\left[\frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} + \frac{(-1)s-1}{s^2+2s+5}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} - \frac{s+1}{s^2+2s+5}\right]$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{s+1}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)$$

$$= e^{-t}L^{-1}\left(\frac{1}{s}\right) - \cancel{L^{-1}}\, e^{-t}L^{-1}\left(\frac{s}{s^2+2^2}\right)$$

$$= e^{-t} - e^{-t}\cos 2t.$$

$$= (1-\cos 2t)e^{-t}$$

$L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Slove: $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 24y = 24x$ given that $y=0,\ \frac{dy}{dx}=0$ when $x=0$.

Soln:-

Given $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 24y = 24x$

$y'' - 10y' + 24y = 24x$

$L[y''] - 10L[y'] + 24L[y] = 24L[x]$

$[s^2L(y) - sy(0) - y'(0)] - 10[sL(y) - y(0)]$

$+24L[y] = 24\left[\frac{1}{s^2}\right]$

$[s^2L(y) - 0 - 0] - 10[sL(y) - 0] + 24L(y) = 24/s^2$

$$s^2 L(y) - 10 s L(y) + 24 L(y) = \frac{24}{s^2}$$

$$L(y)\left[s^2 - 10s + 24\right] = \frac{24}{s^2}$$

$$L(y) = \frac{24}{s^2(s^2 - 10s + 24)}$$

-10; $-6 \mid -4$; 24

$$y = L^{-1}\left[\frac{24}{s^2(s^2 - 10s + 24)}\right]$$

$$= L^{-1}\left[\frac{24}{s^2(s-6)(s-4)}\right]$$

Consider :-

$$\frac{24}{s^2(s-6)(s-4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-6} + \frac{D}{s-4}$$

$$\frac{24}{\cancel{s^2(s-6)(s-4)}} = \frac{A(s^2)(s-6)(s-4) + B(s-6)(s-4) + C(s)(s^2)(s-4) + D(s)(s^2)(s-6)}{\cancel{s^2(s-6)(s-4)}}$$

$$24 = A(s^2)(s-6)(s-4) + B(s-6)(s-4) + C(s)(s-4)(s^2) + D(s)(s^2)(s-6) \rightarrow (1)$$

Put $s = 6$ in (1)

$$24 = A(6)^2(6-6)(6-4) + B(6-6)(6-4) + C(6)(6-4)(36) + D(6)(6^2)(6-6)$$

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$24 = 0 + 0 + C(\cancel{4})(2)(\cancel{36}) + D(0)$

$(2)(36)$

$24 = C(\cancel{4 \times 2 \times 2})\,72$

$C = \dfrac{24}{\cancel{432}\ 72}$ $\quad {}^{12}\ \ {}^{36}\ 3$

$C = \frac{1}{3}$

Put $s = 4$ in ①

$24 = A(4)^2(4-6)(4-4) + B(4-6)(4-4)$

$+ C(4-4)(4)^2 + D(4)^2(4-6)$

$24 = 0 + 0 + 0 + D(16)(-2)$

$24 = -32D$

$D = -\dfrac{\cancel{24}}{\cancel{32}}$ $\quad {}^{12}\ 3$, ${}^{16}\ 4$

$D = -\frac{3}{4}$

Put $s = 0$ in ①

$24 = A(0)(0-6)(0-4) + B(0-6)(0-4)$

$+ C(0-4)(0) + D(0)(0-6)$

$24 = 0 + B(-6)(-4) + 0 + 0$.

$24 = +24B$

$B = 1$

Put $s = 1$ in ①

$24 = A(1)(1-6)(1-4) + B(1-6)(1-4)$

$+ C(1-4)(1) + D(1)(1-6)$

$24 = A(1)(-5)(-3) + B(-5)(-3)$

$+ C(-3)(1) + D(-5)(1)$

$24 = A(+15) + B(15) - C(3) - 5D$

$24 = 15A + 15B - 3C - 5D$

$24 = 15A + 15(1) - 3(1/3) - 5(-3/4)$

$24 = 15A + 15 - 1 + 15/4$

$24 = 15A + 14 + 15/4$

$15A = 24 - 14 - 15/4$

$15A = 10 - 15/4$

$15A = \frac{40-15}{4}$

$15A = 25/4$

$\cancel{A = 24/4 \times 1/15}$

$\cancel{A = 8/2}$

$15A = 25/4$

$A = \frac{25}{4} \times \frac{1}{15}$

$A = 5/12$

$$y = L^{-1}\left[\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-6} + \frac{D}{s-4}\right]$$

$$= L^{-1}\left[\frac{5/12}{s} + \frac{1}{s^2} + \frac{1/3}{s-6} - \frac{3/4}{s-4}\right]$$

$$= 5/12\, L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2}\right] + 1/3\, L^{-1}\left[\frac{1}{s-6}\right]$$

$$- 3/4\, L^{-1}\left[\frac{1}{s-4}\right]$$

$$= 5/12\,(1) + x + e^{6x}\cdot 1/3\, L^{-1}\left(\frac{1}{s}\right)$$

$$- 3/4\, e^{4x}\, L^{-1}\left(\frac{1}{s}\right)$$

$$= 5/12 + x + \frac{e^{6x}}{3} - \frac{3e^{4x}}{4}.$$

84) Determine y which satisfies the equation $\frac{dy}{dt} + 3y + 2\int_0^t y\,dt = t$ for which $y(0) = 0$

$L(t) = 1/s^2$

Soln:

Given

$$\frac{dy}{dt} + 3y + 2\int_0^t y\,dt = t$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$y' + 3y + 2\int_0^t y\,dt = t$$

Taking laplace tranfrom,

$$L\left[\int_0^t y\,dt\right] = \frac{L[y]}{s}$$

$$L[y'] + 3L[y] + 2L\left[\int_0^t y\,dt\right] = L[t]$$

$$[sL(y) - y(0)] + 3L[y] + 2/s\, L[y] = \frac{1}{s^2}$$

$$sL(y) + 3L(y) + 2/s\, L(y) = 1/s^2$$

$$L(y)\,[s + 3 + 2/s] = 1/s^2$$

$$L[y]\left[\frac{s^2+3s+2}{s}\right] = 1/s^2$$

$$L[y] = 1/s^{\cancel{2}} \times \frac{\cancel{s}}{s^2+3s+2}$$

$$L[y] = \frac{1}{s(s^2+3s+2)}$$

$$y = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$$

Consider:-

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\frac{1}{\cancel{s(s+1)(s+2)}} = \frac{A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)}{\cancel{s(s+1)(s+2)}}$$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1) \rightarrow ①$$

Put $s = -1$ in ①

$$1 = A(-1+1)(-1+2) + B(-1)(-1+2) + C(-1)(-1+1)$$

$$1 = 0 + B(-1)(1) + 0$$

$$1 = -B$$

$$B = -1$$

Put $s = -2$

$1 = A(-2+1)(-2+2) + B(-2)(-2+2) + C(-2)(-2+1)$

$1 = 0 + 0 + C(-2)(-1)$

$1 = 2C$

$C = \frac{1}{2}$

Put $s = 0$

$1 = A(0+1)(0+2) + B(0)(0+2) + C(0)(0+1)$

$1 = A(1)(2)$

$A = \frac{1}{2}$

$$y = L^{-1}\left[\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - 1\, L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} - e^{-t}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{2}e^{-2t}L^{-1}\left[\frac{1}{s}\right]$$

$$y = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

(85) Slove the simmultaneons equation

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$$

$$\frac{dx}{dt} + 4\frac{dy}{dx} + 3y = 0$$

Given that $x = 0 = y$ at $t = 0$.

$$L[f(t)] = \begin{cases} 1 & 0 < t < b \\ -1 & b < t < 2b \end{cases}$$

$$L[f(t)] = \int_0^b e^{-st}(1)\,dt + \int_b^{2b} e^{-st}(-1)\,dt$$

$$= \int_0^b e^{-st}\,dt - \int_b^{2b} e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^b - \left[\frac{e^{-st}}{-s}\right]_b^{2b}$$

$$= \left[\frac{e^{-s(b)}}{-s} + \frac{e^{-s(0)}}{s}\right] - \left[\frac{e^{-s(2b)}}{-s} + \frac{e^{-s(b)}}{s}\right]$$

$$= \frac{e^{-bs}}{-s} + \frac{1}{s} + \frac{e^{-2sb}}{s} - \frac{e^{-bs}}{s}$$

$$= -\frac{e^{-bs}}{s} - \frac{e^{-bs}}{s} + \frac{1}{s} + \frac{e^{-2bs}}{s}$$

$$= \frac{-2e^{-bs}}{s} + \frac{1+e^{-2bs}}{s}$$

$$= \frac{-2e^{-bs}+1+e^{-2bs}}{s}.$$

$$= \int_s^{\infty} \frac{s}{s^2+a^2}\,ds - \int_s^{\infty} \frac{s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log(\infty^2+a^2) - \log(\infty^2+b^2)\right.$$

$$\left. - \left[\log(s^2+a^2) - \log(s^2+b^2)\right]\right]$$

$$= \frac{1}{2}\left[-\log(s^2+a^2) + \log(s^2+b^2)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

$$= \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

$$\left[\frac{}{s^2 a^2 + b^2} \right]$$

Soln :-

$$L^{-1}\left[\frac{s}{s^2 a^2 + b^2}\right]$$

We know that

$$L^{-1}[F(s)] = f(t)$$

$$\frac{s}{s^2a^2+b^2} = \frac{1}{a} \cdot \frac{sa}{s^2a^2+b^2}$$

$$= \frac{1}{a} F_1(sa)$$

Where $F(sa) = \dfrac{sa}{s^2a^2+b^2}$

$$F(s) = \frac{s}{s^2+b^2}$$

$$L^{-1}\left[\frac{s}{s^2a^2+b^2}\right] = \frac{1}{a} L^{-1}\left[\frac{sa}{s^2a^2+b^2}\right]$$

vs

$$= \frac{1}{a} \cdot F(sa)$$

$$= \frac{1}{a} \cdot \frac{1}{a} F(t/a)$$

$$f(t) = L^{-1}\{f(s)\} = L^{-1}\left(\frac{s}{s^2+b^2}\right)$$

$$f(t/a) = \cos(bt/a) = \cos bt$$

$$= 1/a \cdot \cos\{bt/a\}.$$

8(a) Show that the solution of the differential equation

8(a) Show that the Solution of the differential equation

$$\frac{d^2y}{dt^2} + 4y = A \sin kt$$ which is such that $y = 0$

and $\frac{dy}{dt} = 0$ when $t = 0$ is

$$y = \frac{A}{4-k^2}\left(\sin kt - \frac{k}{2}\sin 2t\right)$$ if $k \neq 2$ if $k = 2$

Show that $y = \frac{A(\sin 2t - 2t\cos 2t)}{8}$

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Soln :

Given

$$\frac{d^2y}{dt^2} + 4y = A \sin kt$$

$$y'' + 4y = A \sin kt$$

Applying laplace transformes, we get,

$L(y'') + 4L(y) = AL(\sin kt)$

$s^2 L(y) - sy(0) - y'(0) + 4L(y) = A\left(\frac{k}{s^2+k^2}\right)$

$s^2 \bar{y} + 4\bar{y} = A\left(\frac{k}{s^2+k^2}\right)$ $\quad (\because \bar{y} = L(y))$

$\bar{y}(s^2+4) = A\left(\frac{k}{s^2+k^2}\right)$

$$\bar{y} = A\left(\frac{k}{(s^2+k^2)(s^2+4)}\right)$$

$$L(y) = \frac{Ak}{(s^2+4)(s^2+k^2)}$$

$$y = Ak\, L^{-1}\left[\frac{1}{(s^2+4)(s^2+k^2)}\right]$$

Case (i)

If $k \neq 2$

$$y = Ak\, L^{-1}\left[\frac{\frac{1}{s^2+4} - \frac{1}{s^2+k^2}}{k^2-4}\right]$$

$$y = \frac{Ak}{k^2-4}\left[L^{-1}\left(\frac{1}{s^2+4}\right) - L^{-1}\left(\frac{1}{s^2+k^2}\right)\right]$$

$$y = \frac{Ak}{k^2-4}\left[\frac{\sin 2t}{2} - \frac{\sin kt}{k}\right]$$

$$= \frac{A}{k^2-4}\left[\frac{k\sin 2t}{2} - \frac{\cancel{k}\sin kt}{\cancel{k}}\right]$$

$$= \frac{A}{4-k^2}\left[\sin kt - \frac{k\sin 2t}{2}\right] \quad \text{if } k \neq 2$$

Case (ii)

If $k = 2$

$$y = 2A\left[L^{-1}\left(\frac{1}{s^2+4}\right)\left(\frac{1}{s^2+4}\right)\right]$$

$$= 2A\left[L^{-1}\left(\frac{1}{(s^2+4)^2}\right)\right]$$

$$\therefore L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at)$$

$$y = 2A\left[L^{-1}\left(\frac{1}{(s^2+2^2)^2}\right)\right] \Rightarrow y = 2A \cdot \frac{1}{2(2)^3}(\sin 2t - 2t\cos 2t)$$

$$y = \cancel{2}A\,\frac{1}{8\times\cancel{2}}(\sin 2t - 2t\cos 2t) \Rightarrow y = \frac{A}{8}(\sin 2t - 2t\cos 2t)$$

$\therefore$ Hence proved.

81) Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Soln:-

$$F'(s) = \frac{s}{(s^2+a^2)^2}$$

$$F(s) = \int \frac{s}{(s^2+a^2)^2}\,ds$$

$$= \frac{1}{2}\int \frac{2s}{(s^2+a^2)^2}\,ds$$

$t = s^2+a^2$

$dt = 2s \cdot ds$

$$= \frac{1}{2}\int \frac{db}{t^2}$$

$$= \frac{1}{2}\int t^{-2}\, dt$$

$$= \frac{1}{2}\left[\frac{t^{-2+1}}{-2+1}\right]$$

$$= -\frac{1}{2}\left[\frac{-1}{t}\right]$$

$$F(s) = \frac{-1}{2(s^2+a^2)}$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -t\, L^{-1}\left[\frac{-1}{2(s^2+a^2)}\right]$$

$$= \frac{t}{2}\, L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{t}{2a}\, L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{t}{2a}\sin at.$$

(10) Find $L^{-1}\left[\frac{s}{s^2+k^2}\right]$.

Ex

Soln:

$$L^{-1}\left[\frac{s}{s^2+k^2}\right] = \frac{d}{dt}\, L^{-1}\left[\frac{1}{s^2+k^2}\right]$$

$$= \frac{d}{dt}\, \frac{\sin kt}{k}$$