B.Sc Mathematics 3^{rd} year -

Complex Analysis - unit 5 - Cauchy's Residue

Evaluation of definite integrals

Type 1. $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

To evaluate this type of integral we substitute $z = e^{i\theta}$. As θ varies from 0 to 2π, z describes the unit circle $|z| = 1$.

Also, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + z^{-1}}{2}$ and

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i}$

Substituting these values in the given integrand the integral is transformed into $\int_C \phi(z)\,dz$ where $\phi(z) = f\left[\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2i}\right]$ and C is the positively oriented unit circle $|z| = 1$, The integral $\int_C \phi(z)\,dz$ can be evaluated using the residue theorem

Problem: 1

1. Evaluate $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$.

Soln:

Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

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put z · z

put $z = e^{i\theta}$

$\therefore \; dz = ie^{i\theta} d\theta = iz\,d\theta$

$d\theta = \frac{dz}{iz}$.

Also $\sin\theta = \frac{z - z^{-1}}{2i}$

$I = \int_c \frac{dz}{iz\left[5 + 4\left(\frac{z - z^{-1}}{2i}\right)\right]}$

where c is the unit circle $|z| = 1$.

$= \int_c \frac{dz}{iz\left[5 + \frac{4}{2i}\left(z - \frac{1}{z}\right)\right]}$

$= \int_c \frac{dz}{iz\left[5 + \frac{2}{i}\left(\frac{z^2 - 1}{z}\right)\right]}$

$= \int_c \frac{dz}{iz\left[\frac{5iz + 2z^2 - 2}{iz}\right]}$

$= \int_c \frac{dz}{2z^2 + 5iz - 2}$

Let $f(z) = \frac{1}{2z^2 + 5iz - 2}$

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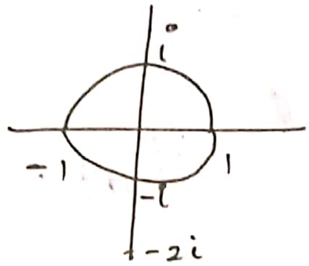
③

$$f(z) = \frac{1}{2[z-(-2i)][z-(-\frac{1}{2}i)]}$$

$-2i$ & $\frac{-i}{2}$ are simple poles of $f(z)$ and the pole $-i/_2$ lies inside c and $-2i$ lies outside c.



Roots of $2z^2+5iz-2=0$

$$z = \frac{-5i \pm \sqrt{25i^2+16}}{4}$$

$$= \frac{-5i \pm \sqrt{-25+16}}{4}$$

$$= \frac{-5i \pm \sqrt{-9}}{4}$$

$$= \frac{-5i \pm 3i}{4}$$

$$= \frac{-8i}{4}, \frac{-2i}{4}$$

$$= -2i, -i/_2$$

Also $Res\{f(z); -i/_2\}$

$$= \frac{1}{2[-\frac{i}{2}+2i]} = \frac{1}{2[\frac{-i+4i}{2}]}$$

$$= \frac{1}{3i}.$$

Hence by cauchy's Residue theorem,

$$I = 2\pi i\left(\frac{1}{SP}\right) = \frac{2\pi}{3}$$

problem 2 :-

P.T $I = \int_0^\pi \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{a^2+1}}$ $(a>0)$

soln :-

$$I = \int_0^\pi \frac{a\,d\theta}{a^2+\sin^2\theta} = \int_0^\pi \frac{a\,d\theta}{a^2+\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$= \int_0^\pi \frac{2a\,d\theta}{2a^2+1-\cos 2\theta}$$

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Put $2\theta = \phi$

$2d\theta = d\phi$

when $\theta = 0$, $\phi = 0$

when $\theta = \pi$, $\phi = 2\pi$

$$\therefore I = \int_0^{2\pi} \frac{a\,d\theta}{2a^2+1-\cos\phi}$$

Put $z = e^{i\phi}$

$$\cos\phi = \frac{z+z^{-1}}{2}$$

$z = e^{i\phi}$

$dz = ie^{i\phi}\,d\phi = iz\,d\phi$

$$\Rightarrow d\phi = \frac{dz}{iz}$$

$$I = \int_C \frac{a\,dz}{iz\left[2a^2+1-\left(\frac{z+z^{-1}}{2}\right)\right]}$$ where c is unit circle $|z|=1$.

$$= \frac{a}{i}\int_C \frac{dz}{z\left[2a^2+1-\frac{1}{2}\left(z+\frac{1}{z}\right)\right]}$$

$$= \frac{ai}{i^2}\int_C \left[\frac{dz}{z\left[\frac{(2a^2+1)-z^2-1}{2z}\right]}\right]$$

$$= \frac{-2ai}{(-1)}\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

$$I = 2ai\int_C f(z)\,dz \rightarrow (1)$$

where $f(z) = \dfrac{1}{z^2-2(2a^2+1)z+1}$

Poles of $f(z)$ are the roots of

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$$z^2 - 2(2a^2+1)z + 1 = 0$$

$$z = \frac{2(2a^2+1) \pm \sqrt{4(2a^2+1)^2 - 4}}{2}$$

$$= \frac{2(2a^2+1) \pm \sqrt{4(4a^4+1+4a^2) - 4}}{2}$$

$$= \frac{2(2a^2+1) \pm \sqrt{16a^4+4+16a^2-4}}{2}$$

$$= \frac{2(2a^2+1) + 4\sqrt{a^2(a^2+1)}}{2}$$

$$= 2\left[\frac{(2a^2+1) \pm 2a\sqrt{a^2+1}}{2}\right]$$

$$z = (2a^2+1) \pm 2a\sqrt{a^2+1}$$

$$z_1 = (2a^2+1) + 2a\sqrt{a^2+1} \quad ; \quad z = (2a^2+1) - 2a\sqrt{a^2+1}$$

clearly $|z_1| > 1$ and $|z_1 z_2| = 1$

so that $|z_2| \not< 1$

Hence the only pole inside C is

$z = z_0$

$$\text{Res}\{f(z); z_2\} = \lim_{z \to z_2} (z - z_0)\frac{1}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{z - z_1}$$

$$= \frac{1}{(2a^2+1) - 2a\sqrt{a^2+1} - (2a^2+1) - 2a\sqrt{a^2+1}}$$

$$= \frac{1}{-4a\sqrt{a^2+1}}$$

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from (1) $I = 2\pi i \left[\frac{2ai}{-4a\sqrt{a^2+1}}\right]$

$$I = +\frac{\pi}{\sqrt{a^2+1}} \quad (\because i^2 = -1)$$

$$\boxed{I = \frac{\pi}{\sqrt{a^2+1}}}$$

Type 2 :-

$\int_{-\infty}^{\infty} f(x)\,dx$ where $f(x) = \frac{g(x)}{h(x)}$ and $g(x), h(x)$ are polynomials in x and the degree of $h(x)$ exceeds that of $g(x)$ by at least two.

To evaluate this type of integral we have $f(z) = \frac{g(z)}{h(z)}$.

The poles of $f(z)$ are determined by the zeros of the equation $h(z) = 0$

Case (i) No pole of $f(z)$ lies on the real axis.

We choose the curve C consisting of the intervals $[-r, r]$ on the real axis and the semi circle $|z|=r$ lying in the upper half of the plane.

Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C. then we have

$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz$$

where C_1 is the semi circle.

since $\deg h(x) - \deg f(x) \geq 2$ it follows that $\int_{C_1} f(z)dz \to 0$ as $r \to \infty$ and

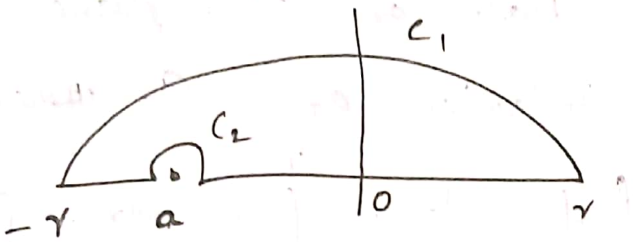
hence $\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx$.

$\therefore \int_{-\infty}^{\infty} f(x)dx$ can be evaluated by evaluating $\int_C f(z)dz$ which in turn can be evaluated by using Cauchy's residue theorem.

case(ii) $f(z)$ has poles lying on the real axis.

suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi-circle

C_2 of radius ϵ with centre a lying in the upper half plane where ϵ is chosen to be sufficiently small.



such an indenting must be done for every pole of $f(z)$ lying on the real axis.

It can be proved that $\int_{C_2} f(z)dz = -\pi i \operatorname{Res}\{f(z); a\}$ By taking limit as $r \to \infty$ and $\epsilon \to 0$ we obtain the value of $\int_{-\infty}^{\infty} f(x)dx$.

① use contour integration method to evoluate $\int_0^\infty \frac{dx}{1+x^4}$.

soln:

Let $f(z) = \frac{1}{1+z^4}$

The poles of $f(z)$ are given by the roots of the equation

$z^4+1=0$

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$z^4 = -1 \Rightarrow z = (-1)^{1/4}$ which are four forth roots of -1.

By De moivre's thm they are given by $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$ and all are simple poles.

We choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi circle $|z| = r$ which is denoted by C_1

since $e^{in\pi} = \cos n\pi + i \sin n\pi$

when n is odd.

$\therefore$ 4 roots, $e^{i\pi}, e^{i3\pi}, e^{i5\pi}, e^{i7\pi}$

$$\therefore \int f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \oint_{C_1} f(z)\,dz \rightarrow (1)$$

The poles of $f(z)$ lying inside the contour C are $e^{i\pi/4}, e^{i3\pi/4}$ only

[Since $e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$ are lying below the upper semicircle in lower semicircle]

$$\therefore \text{Res}\{f(z); e^{i\pi/4}\} = \frac{h(e^{i\pi/4})}{k'(e^{i\pi/4})}$$

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where $h(z) = 1$ and $k(z) = z^4+1$

so that $k'(z) = 4z^3$

$$\therefore \operatorname{Res}\{f(z), e^{i\pi/4}\} = \frac{1}{4(e^{i\pi/4})^3}$$

$$= \frac{e^{-i3\pi/4}}{4}$$

Similarly $\operatorname{Res}\{f(z), e^{i3\pi/4}\}$

$$= \frac{e^{-i9\pi/4}}{4}$$

By Residue theorem:

$\int_C f(z)dz = 2\pi i$ [sum of the Residue]

$$= 2\pi i\left[\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4}\right]$$

$$= \frac{\pi i}{2}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{\pi i}{2}\left[\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

$$= \frac{\pi i}{2}\left(\frac{-2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

from (1),

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$$\frac{\pi}{\sqrt{2}} = \int_{-r}^{r} \frac{dx}{1+x^4} + \int_{C_1} f(z)\,dz \rightarrow (2)$$

As $r \rightarrow \infty$, $\int_{C_1} f(z)\,dz \rightarrow 0$

from (2)

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Type 3

$$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax\,dx \text{ (or) } \int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax\,dx$$

where $g(x)$ and $h(x)$ are real polynomials such that degree of $h(x)$ exceeds that of $g(x)$ by at least one and $a>0$.

Case (i)

$h(x)$ has no zeros on the real axis.

In this case take

$$f(z) = \frac{g(z)}{h(z)} e^{iaz}.$$

$\therefore f(z)$ has no poles on the real axis.

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choose the contour as in Type 2 and proceeding as in Type 2 we get the value of $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iax} dx$.

Taking the real and imaginary parts of $\frac{g(x)}{h(x)} e^{iax} dx$ we obtain the values of $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax \, dx$

and $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax \, dx$.

<u>case (ii)</u>

$h(x)$ has zeros of order one on the real axis.

Take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$. we notice that $f(z)$ has real poles.

suppose a is a real zero of $h(x)$ on the real axis. In this case we indent the real axis as Case (ii) of Type 2 and evaluate the integral.

To prove that the integrals over the upper semicircle tends to zero as $r \to \infty$, we use the

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following lemma.

Jordan's lemma:

Let $f(z)$ be a function of the complex variable z satisfying the following conditions.

(i) $f(z)$ is analytic in upper half plane except at a finite number of poles.

(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ with $0 \leq \arg z \leq \pi$

(iii) a is a positive integer

Then $\lim_{r \to \infty} \int_C f(z) e^{iaz} dz = 0$ where C is the semi circle with centre at the origin and radius r.

1) P.T $\int_0^\infty \frac{\cos\theta}{1+x^2} dx = \frac{\pi}{2e}$

Soln:

Let $f(z) = \frac{e^{iz}}{1+z^2}$ [Here $\cos x$ $\therefore a = 1$]

The poles of $f(x)$ are the roots of $1+z^2=0$

$z = i, -i$.

Choose the contour C as shown in the figure (In type-2)

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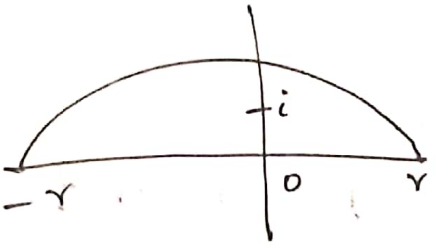


$\therefore$ i is the only pole lying in C.

Hence by residue Theorem.

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}\{f(z); i\}$$

$$= 2\pi i \frac{h(i)}{k'(i)} \quad \text{where } h(z_0) = e^{iz}$$

$$k(z) = 1+z^2$$

$$k'(z) = 2z$$

$$= 2\pi i \cdot \frac{e^{i\cdot i}}{2i}$$

$$= \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}$$

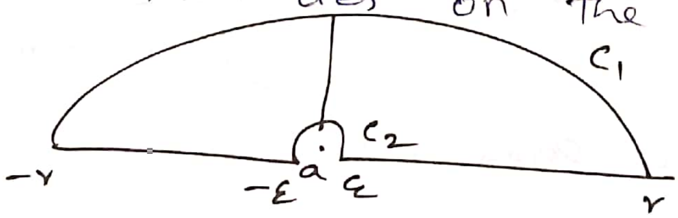
problem 6

P.T $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Soln:.

Let $f(z) = \dfrac{e^{iz}}{z}$

The only singular pole of $f(z)$ is 0 which lies on the real axis.



Then $\displaystyle\int_C f(z)\,dz = \int_{-r}^{-\varepsilon} f(x)\,dx + \int_{C_2} f(z)\,dz + \int_{\varepsilon}^{r} f(z)\,dz + \int_{C_1} f(z)\,dz$ $\rightarrow (1)$

since $f(z)$ is analytic within C,

$$\int_C f(z)dz = 0 \rightarrow (2)$$

Also $$\int_{C_2} f(z)dz = -\pi i \operatorname{Res}\{f(z); 0\}$$

$$= -\pi i \lim_{z \to 0} (z-0) \cdot \frac{e^{iz}}{(z-0)}$$

$$= -\pi i e^{i0}$$

$$= -\pi i \rightarrow (3)$$

further the integral over C_1 tends to 0 as $r \to \infty$

Hence using (2) & (3) in (1) and taking limit as $r \to \infty$ we get

$$0 = \int_{-\infty}^{0} f(x)dx - \pi i + \int_{0}^{\infty} f(x)dx.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}\,dx = \pi i$$

Equating imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi$$ [$\because$ since $\frac{\sin x}{x}$ is an even function]

$$\Rightarrow \boxed{\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}}$$

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