14/3/19
Thursday.

UNIT - V

2

Motion under the action of Genelar forus:

Introduction:-

We have consider Same particular Cases of motion of a partical in two dimantion. To fix the position of a plane in the be require to coordinates and to study the motion of the partical we required it's component velocity and acceleration in two. is naturally $\perp r$ velocity.

In this capter we Shall use polar coordinates

		Magnitude	Direction	Sense.
1.	Radial component of velocity.	$\dot{r}$	Along the radius vector	In the direction in which r incre ses.
2.	Transvers component of velocity.	$r\dot{\theta}$	$\perp r$ to the radius vetor	in the direct which θ increase
3.	Radial compnt of acceleration	$\ddot{r}-r\dot{\theta}^2$	Along the radius vector	in the direct which r increases
4.	Tranvers compnet of acceleration	$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$	$\perp r$ to the radius vetor	in the direct which θ increases.

Cor:-

Suppose the particle p is describing a circle of radius 'a'. then, $r=a$ hence $\ddot{r}=0$

and the radial acceleration $= \ddot{r}-r\dot{\theta}^2$

$$= 0 - a\dot{\theta}^2$$

The acceleration $\perp r$ to $OP = \frac{1}{r}\frac{d}{dr}(r^2\dot{\theta})$

$$= \frac{1}{a}\cdot\frac{d}{dt}(a^2\dot{\theta})$$

$= \frac{1}{a} \frac{a^2}{a} \dot{\theta}$

$= a\dot{\theta}$.

hence for a partical describing a circle of Radius 'a' the acceleration at any point p has the component $a\ddot{\theta}$ along the tangent at p and $a\dot{\theta}^2$ along the radius to the centre.

The magnitude of the resultant velocity of $P = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$

$= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$

And the magnitude of the resultant acceleration $= \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + \left[\frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta})\right]^2}$

Equation of motion in polar coordinates:

If R & S are components of are external force acting on partical mass m in the radius and transverse direction we have the equation

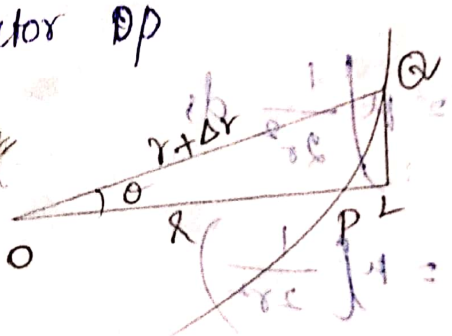
$$R = m(\ddot{r} - r\dot{\theta}^2) \quad \text{—} (1)$$

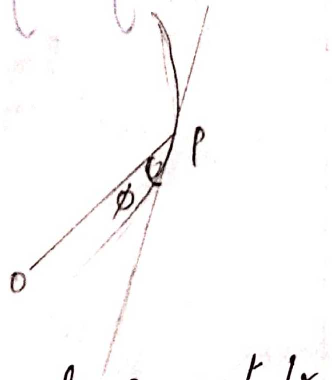
$$S = m \cdot \frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta}) \quad \text{—} (2)$$

If R & S are non functions of the coordinates R, θ and the time t, the diff/- equation ① & ② can be solved into find r & θ are functions are t and by eliminating t, polar equation to the path is got.

Note on the equiangular spiral:

Some questions this chapter on have related to the curve called the equiangular spiral. this curve is the important property that the tangent at any point P on it makes a constant angle with the radius vector OP





This velocity of a particle along and ⊥r to radius vector from a fixed origin are λr^2 and $\mu\theta^2$ where μ & λ are constants. S.T the equ/- to the path of the particle is

(A.) $\frac{\lambda}{\theta} + c = \frac{\mu}{2r^2}$ where c is constant. S.T the acceleration along ⊥r to the radius vector are $2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ and $\mu\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)$.

Sol:- Radial velocity $\frac{dr}{dt} = \lambda r^2$ —— ①

$\therefore$ transverse velocity $\frac{r\,d\theta}{dt} = \mu\theta^2$ —— (2)

dividing (2) by (1)

$$\frac{\frac{dr}{dt}}{\frac{r\,d\theta}{dt}} = \frac{\lambda r^2}{\mu\theta^2} \qquad \frac{\frac{r\,d\theta}{dt}}{\frac{dr}{dt}} = \frac{\mu\theta^2}{\lambda r^2}$$

$$\frac{dr}{r\,d\theta} = \frac{\lambda r^2}{\mu\theta^2} \qquad \frac{r\,d\theta}{dr} = \frac{\mu\theta^2}{\lambda r^2}$$

$$\lambda \frac{d\theta}{\theta^2} = \frac{\mu}{r^3}\,dr$$

Integrating $\int \lambda \frac{d\theta}{\theta^2} = \int \frac{\mu}{r^3}\,dr$

$$\lambda \int \frac{1}{\theta^2} = \mu \int \frac{1}{2r^2}\,dr$$

$$\frac{\lambda}{\theta} = \mu\left(\frac{1}{2r^2}\right)$$

$$\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + c \quad \text{—— (3)}$$

Differentiating (1).

$$\frac{d^2r}{dt^2} = 2\lambda^2 r^3 \qquad 2\lambda r \cdot \lambda r^2 = 2\lambda^2 r^3.$$

Radius acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= 2\lambda^2 r^3 - r\left(\frac{\mu\theta}{r}\right)^2$$

$$= 2\lambda^2 r^3 - r\,\frac{\mu^2\theta^4}{r^2}$$

$$= 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$$

Transverse acceleration

$$= \frac{1}{r}\cdot\frac{d}{dt}(r^2\dot\theta)$$

$$= \frac{1}{r}\cdot\frac{d}{dt}\left(r^2\cdot\frac{d\theta}{dt}\right)$$

$$= \frac{1}{r}\cdot\frac{d}{dt}\left(r^2\cdot\frac{\mu\theta^2}{r}\right)$$

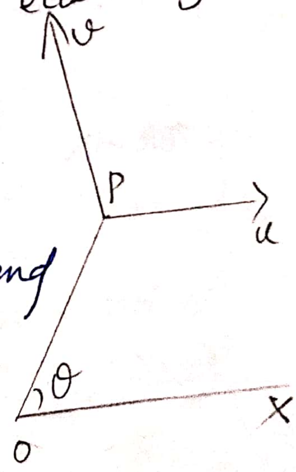
$$= \frac{1}{r}\cdot\frac{d}{dt}(r\cdot\mu\theta^2)$$

$$= \frac{\mu}{r}\frac{d}{dt}(r\theta^2)$$

$$= \frac{\mu}{r}\left[2\theta r\frac{d\theta}{dt} + \theta^2\frac{dr}{dt}\right]$$

$$= \frac{\mu}{r}\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)$$

S.T the path of a point P which passesses two constant velocities u & v, the first of which is in a fixed direction and the second of ⊥r to the radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is u/v.



Sol:

Take O as the pole and the line OX ||el to the given direction as the initial line.

P has two velocity ||lel to ox and v ⊥r to op.

Resolving the velocities along and ⊥r to op. we have.

$$\frac{dr}{dt} = u\cos\theta \rightarrow ①$$

$$r\,d\theta/dt = v - u\sin\theta \rightarrow ②$$

To get the equation to the path we have to eliminate t

$$②/① \Rightarrow r\frac{d\theta}{dt} = \frac{v - u\sin\theta}{u\cos\theta}$$

i.e) $\frac{u\cos\theta}{v - u\sin\theta}\,d\theta = \frac{dr}{r}$ or $\frac{d(u\sin\theta)}{v - u\sin\theta}$

$$= \frac{dr}{r}$$

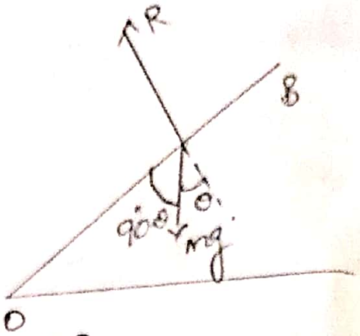
A Smooth Straight within tole with you with angular velocity ω in a vertical palne above 1 extremity which is fixed. if at zero time the tube be horizontal and a particle inside at a distance a from the fixed end and be moving with velocity B along the tube, S.T the distance at time t is

$$a\cosh\omega t + \left(\frac{v}{\omega} - \frac{g}{2\omega^2}\right)\sinh\omega t + \frac{g}{2\omega^2}\sin\omega t.$$

Sol!.

let at time t p be the position the partical of m and μ OB the forces acting at pole

(i) it's weight mg vertically downwards and



(ii) Normal reaction $R \perp r$ to OB

let P be (r, θ)

Angular velocity $= \dot{\theta} = \frac{d\theta}{dt} = \omega$ given

Integrating $d\theta = \omega\, dt$

$$\int d\theta = \int \omega\, dt$$

$$\theta = \omega t + A$$

Initialy when $t=0$, $\theta = 0$ $\therefore A = 0$

$$\theta = \omega t + 0$$

$$\boxed{\theta = \omega t} \longrightarrow (1)$$

Resolving along the radius vector OB

$$m(\ddot{r} - r\dot{\theta}^2) = -mg\cos(90° - \theta)$$

$$= -mg\sin\theta$$

$$m(\ddot{r} - r\dot{\theta}^2) = -mg\sin\theta$$

$$(\ddot{r} - r\omega^2) = -g\sin\theta$$

$$\ddot{r} - r\omega^2 = -g\sin\omega t$$

(ie) $(D^2 - \omega^2) r = -g\sin\omega t \longrightarrow (2)$

$$D = d/dt$$

The complementry function Y is found such that

$(D^2-\omega^2)Y=0$.

The solution of diff equt is

$D^2-\omega^2=0$

$D^2=\omega^2$

$D=\pm\omega$

$Y=Ae^{m_1x}+Be^{m_2x}$

$=Ae^{\omega t}+Be^{-\omega t} \longrightarrow (3)$

where A & B are constant the particular integral 'u' of the equation (2) is given

$(D^2-\omega^2)u=-g\sin\omega t$

$u=\frac{-g}{D^2-\omega^2}\sin\omega t$

$=\frac{-g}{-\omega^2-\omega^2}\sin\omega t$

$=\frac{-g}{-2\omega^2}\sin\omega t$

$=\frac{g}{2\omega^2}\sin\omega t \longrightarrow (4)$

hence the General solution of (2) is

$Y+u$

$r=Y+u=Ae^{\omega t}+Be^{-\omega t}+\frac{g}{2\omega^2}\sin\omega t \longrightarrow (5)$

The initial condition are $t=0$, $r=a$ & $\dot{r}=B$

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$$= Ae^{\omega(0)} + Be^{-\omega(0)} + \frac{g}{2\omega^2} \sin\omega(0)$$

$$a = A+B$$

$$\boxed{A+B=a} \longrightarrow (6)$$

Diff. (5)

$$\dot{r} = A\omega e^{\omega t} - B\omega e^{-\omega t} + \frac{g}{2\omega^2} \cos\omega t(\omega)$$

$$\dot{r} = A\omega e^{\omega t} - B\omega e^{-\omega t} + \frac{g}{2\omega} \cos\omega t \longrightarrow (7)$$

Putting $t=0$, $\dot{r} = V$ in equ. (7)

$$V = A\omega e^{\omega(0)} - B\omega e^{-\omega(0)} + \frac{g}{2\omega} \cos\omega(0)$$

$$= A\omega - B\omega + \frac{g}{2\omega}$$

$$A\omega - B\omega + \frac{g}{2\omega} = V$$

$$A\omega - B\omega = V - \frac{g}{2\omega}$$

$$(A-B)\omega = V - g/2\omega$$

$$A - B = V/\omega - \frac{g}{2\omega^2} \longrightarrow (8)$$

Solving (6) & (8)

~~ABBOON~~ $A+B = a$

~~ABBO~~ $A - B = \frac{v}{\omega} - \frac{g}{2\omega^2}$

$$2A = a + \frac{v}{\omega} - \frac{g}{\omega^2}$$

$$\cancel{2}A = \frac{1}{2}\left[a + \frac{v}{\omega} - \frac{g}{2\omega^2}\right]$$

$$\frac{1}{2}\left[a + \frac{v}{\omega} - \frac{g}{2\omega^2}\right] + B = a$$

$$B = a - \frac{1}{2}\left[a + \frac{v}{\omega} - \frac{g}{2\omega^2}\right]$$

$$B = a - \frac{a}{2} - \frac{v}{2\omega} + \frac{g}{4\omega^2}$$

$$= \frac{1}{2}a - \frac{v}{2\omega} + \frac{g}{4\omega^2}$$

$$= \frac{1}{2}\left[a - \frac{v}{\omega} + \frac{g}{2\omega^2}\right]$$

Substituting value ⓔ.

$$= \frac{1}{2}\left(a + \frac{v}{\omega} - \frac{g}{2\omega^2}\right)e^{\omega t} +$$

$$\frac{1}{2}\left[a - \frac{v}{\omega} - \frac{g}{2\omega^2}\right] + \frac{g}{2\omega}\sin\omega t$$

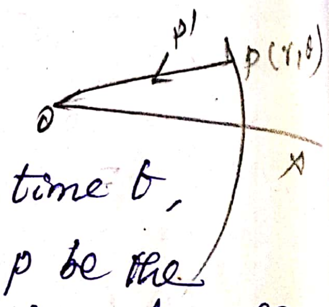
Motion under a Central Force:

The fixed point known as the Centre of the force usually the magnitude of the Central attraction F is a function only of the distance 'r' of the particle from 'O'. In such a motion the particle must be always moving only in the plane containing 'O' and the tangent at any point of this path. Since there is no component of attraction $\perp r$ to the above plane hence a central orbit must be a plane curve.

Differential equt of Central orbits:

A particle moves on a plane with an acceleration which is always directed to a fixed point 'O' in the plane to obtain the diff equt of its path.

Take 'O' as the pole and a fixed line through 'O' as the initial line.



Let $P(r, \theta)$ be the polar coordinates of the particle at time t, and 'm' be its mass, also let P be the magnitude of the Central acceleration along PO. The equation of motion of the particle are

$$m(\ddot{r} - r\dot{\theta}^2) = -mP$$

$$\ddot{r} - r\dot{\theta}^2 = -P \longrightarrow (1)$$

and $\frac{m}{r} \cdot \frac{d}{dt}(r^2\theta^{\cdot}) = 0$

$\frac{1}{r} \cdot \frac{d}{dt}(r^2\theta^{\cdot}) = 0 \longrightarrow (2)$

Equation (2) follows from the fact that as there ~~there~~ is no force at right angle to OP, transt component of the acceleration is 0. around the motion.

from (2) $r^2\theta^{\cdot} = \text{constant}$

$r^2\theta^{\cdot} = h \longrightarrow (3).$

We get the polar equation of the path we to eliminate the element of time b/w eqn (1) & (3)

For this purpose it's found convenient to put $u = \frac{1}{r}$ & ~~or~~ work with u as the dependent variable.

from (3). $\theta^{\cdot} = \frac{h}{r^2} = hu^2$

Also $\dot{r} = \frac{dr}{dt}$

$= d/dt\,(r)$

$= d/dt\,(1/u)$

$= -\frac{1}{u^2}\frac{d}{dt}$

$= -\frac{1}{u^2}\frac{du}{d\theta} \cdot \frac{d\theta}{dt}$

$= -\frac{1}{u^2}\frac{du}{d\theta} \cdot hu^2$

$= -h\frac{du}{d\theta}$

$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$

$$= -h \frac{d^2u}{d\theta^2} hu^2$$

$$= -h^2u^2 \frac{d^2u}{d\theta^2}$$

Substituting r & θ in (1)

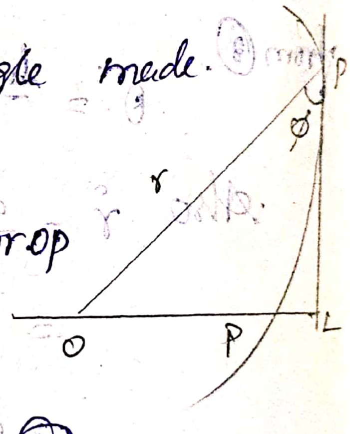
$$-h^2u^2 \frac{d^2u}{d\theta^2} - \frac{1}{u} h^2u^4 = -p$$

$$h^2u^2 \left(\frac{d^2u}{d\theta^2} + u\right) = p \text{ (or) } u + \frac{d^2u}{d\theta^2} = \frac{p}{h^2u^2} \rightarrow$$

This is the diff. equ. of central orbit; polar co-ordinates //.

⊥r from the pole on the tangent formula in polar co-ordinates:-

let ϕ be the angle made by the tangent at p. with he radius vector op



W.K.T

$$\tan\phi = r. \frac{d\theta}{dr} \rightarrow (1).$$

from o draw OL ⊥r to the tangent at p. and let OL = p

$$\text{then } \sin\phi = \frac{OL}{op} = \frac{p}{r}$$

$$R = r\sin\phi \rightarrow (2)$$

from ② $\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$

$$= \frac{1}{r^2} \operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] \text{ from ②.}$$

$$u, \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[\frac{dr}{d\theta}\right]^2 \longrightarrow ③$$

using $r = \frac{1}{u}$, $\frac{dr}{d\theta} = \frac{dr}{du} \times \frac{du}{d\theta}$

$$= -\frac{1}{u^2}\frac{du}{d\theta}$$

hence (3). becomes

$$\frac{1}{p^2} = \frac{1}{u^2} + u^4 - \frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{u^2} + \left(\frac{du}{d\theta}\right)^2 \quad — ④.$$

Pedal equation of the central orbit:

In certain curve the relation b/w p (the ⊥r from the pole on the tangent) and r (radius vector) is very the simple such a relation is called the pedal eqn of the (p, r), equ to the curve

We can get the (p, r) equ to the central orbit as follows.

In the usual notation we have from.

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad — ①$$

Diff/. both side.

$$-\frac{2}{p^3}\cdot\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}$$

$$= 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right) \longrightarrow (2)$$

But the Diff/. equ is polar in $u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

hence (2) $\Rightarrow$ $-\frac{1}{p^3}\cdot\frac{dp}{d\theta} = \frac{P}{h^2u^2}\cdot\frac{du}{d\theta}$

$$-\frac{1}{p^3}dp = \frac{P}{h^2u^2}du$$

$$= \frac{P}{h^2}r^2\, d(1/r)$$

$$= \frac{Pr^2}{h^2}\times\left(-1/r^2\, dr\right)$$

$$= -P/h^2\, dr$$

$$\text{or } \frac{h^2}{p^3}\cdot\frac{dp}{dr} = P \longrightarrow (3).$$

(3) is the (p, r) equ on the pedal equ to the Central orbit.

18/3/19.
Monday

Velocities in a Central orbit:

In every central orbit the areal velocity is constant. and the linear velocity varies inversely as the ⊥r from the centre upon the tangent to the path.

The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle.

$r^2\dot\theta = h$, h = twice the areal velocity and as h is a constant the areal velocity is constant.

In other words equal areas are described by the radius vector in equal times.

V.V.I Imp. Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described.

Soln:

$r^n = a^n \cos n\theta$

Since $r = \frac{1}{u}$; the equation is

$$u^n a^n \cos n\theta = 1 \longrightarrow (1)$$

Taking logarithms

$$\log u^n \log a^n \log \cos n\theta = \log(1)$$

$$n \log u + n \log a + \log \cos n\theta = 0 \longrightarrow (2)$$

Diff. (2) w.r.to θ.

$$n.\frac{1}{u}.\frac{du}{d\theta} - \frac{n \sin n\theta}{\cos n\theta} = 0$$

$$\frac{du}{d\theta} = u \tan\theta \longrightarrow (3)$$

Diff w.r.to θ to (3).

$$\frac{d^2u}{d\theta^2} = un \sec^2 n\theta + \tan n\theta . \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta \quad \text{(using (3))}$$

areal velocity of the parti...

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V.V.I.
(X) Imp.
AH

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Diff. (2) w.r.to θ.

$$n.\frac{1}{u}.\frac{du}{d\theta} - \frac{n \sin n\theta}{\cos n\theta} = 0$$

$$\frac{du}{d\theta} = u \tan\theta \longrightarrow (3)$$

Diff w.r.to θ to (3).

$$\frac{d^2u}{d\theta^2} = un \sec^2 n\theta + \tan n\theta . \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta$$ (using (3))

areal velocity of the particle.

$r^2\dot{\theta} = h$, h = twice the areal velocity and as h is a constant the areal velocity is constant.

In other words equal areas are described by the radius vector in equal times.

Imp. Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described.

Soln:-

$$r^n = a^n \cos n\theta$$

Since $r = \frac{1}{u}$, the equation is

$$u^n a^n \cos n\theta = 1 \longrightarrow (1)$$

Taking logarithms

$$\log u^n \log a^n \log \cos n\theta = \log (1)$$

$$n \log u + n \log a + \log \cos n\theta = 0 \longrightarrow (2)$$

Diff (2) w.r.to θ.

$$n \cdot \frac{1}{u} \cdot \frac{du}{d\theta} - \frac{n \sin n\theta}{\cos n\theta} = 0$$

$$\frac{du}{d\theta} = u \tan \theta \rightarrow (3)$$

Diff w.r.to θ to (3).

$$\frac{d^2u}{d\theta^2} = un \sec^2 n\theta + \tan n\theta \cdot \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta \quad \text{(using (3))}$$

$$u+\frac{d^2u}{d\theta^2} = u + nu\sec^2 n\theta + u\tan^2 n\theta$$

$$= nu\sec^2 n\theta + u(1+\tan^2 n\theta)$$

$$= nu\sec^2 n\theta + u\sec^2 n\theta$$

$$= (n+1)u\sec^2 n\theta$$

$$= (n+1)u \cdot u^{2n}a^{2n}$$ (using (... Substituting Sec

$$= (n+1)a^{2n}u^{2n+1}$$

$$P = h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$= h^2u^2(n+1)a^{2n} \cdot u^{2n+1}$$

$$= (n+1)a^{2n}h^2u^{2n+3}$$

$$= (n+1)a^{2n} \cdot h^2 \cdot \frac{1}{r^{2n+3}}$$

ie) $P \propto \frac{1}{r^{2n+3}}$

which means the central acceleration var inversely as the $(2n+3)$ power of the distan

Note:

from equt (1) P is positive only i $n+1>0$ ie) $n>-1$ for values are $n<-1$, P will be negative and in such cases central force will be a repulsive one.

The above case is a comprehensive o giving the law of force for describing following values of n.

(i) when $n=1$, the equation is $r=a\cos\theta$ the curve is a circle and p directly proposnal to $1/r^5$, when $n=2$.

$$r^n = a^n \cos n\theta$$

$$r^2 = a^2 \cos 2\theta$$

$$p \propto \frac{1}{r^{2(2)+3}}$$

$$p \propto \frac{1}{r^7}.$$

when $n=1/2$. $\theta \Rightarrow r^{1/2} = a^{1/2}\cos\frac{1}{2}\theta \Rightarrow r = a\cos\theta/2.$ $\sqrt{r^{1/2}} = \sqrt{a^{1/2}\cos\frac{1}{2}\theta}$ Taking square root both side. $r = \frac{a}{2}(1+\cos\theta)$

$$p \propto \frac{1}{r^{2(1/2)+3}}$$

$$p \propto \frac{1}{r^4}$$

when $n = -1/2$.

$$r^{-1/2} = a^{-1/2}\cos(-1/2)\theta$$

$$r^{-1/2} = a^{-1/2}\cos\theta/2$$

(i) $a^{1/2} = r^{1/2}\cos\theta/2$

$$\text{so } r = \frac{a}{\cos^2\theta/2}.$$

$$= \frac{2a}{1+\cos\theta}$$

(i) $\frac{2a}{r} = 1+\cos\theta$

This is a parabola & $p \propto \frac{1}{r^2}$

when $n = -2$ the equ is $r^{-2} = a^{-2}\cos 2\theta$

(i) $r^2\cos 2\theta = a^2.$

This is a Rectangular hyperbola. in this Case the actual value of $P = -a^{-4} h^2 r$. The $-ve$ sign of P S.T the central force is repulsive one:

Hence the Result//

Find the law of force to an internal point under which a body will describe a circle.

Soln:

The pedal eqn of a circle for a general position of pole is, $c^2 = r^2 + a^2 - 2ap \rightarrow ①$

diffL w. r. to 'r'

$$0 = 2r - 2a\frac{dp}{dr}$$

$$\text{ie) } \frac{dp}{dr} = \frac{r}{a}$$

Now the central acceleration

$$P = \frac{h^2}{p^3}\frac{dp}{dr}$$

$$= \frac{h^2}{p^3 a}.$$

$$= \frac{8h^2a^2r}{(r^2+a^2-c^2)^3}$$

Substituting for p from ①.

Hence the Result//

A particle moves in a ellipse under a force which is always directed towards of forces. Find the law of force, the velocity at any point of the path & its periodic time.

Soln: The polar equt to the ellipse is

$$\frac{l}{r} = 1 + e\cos\theta. \longrightarrow (1)$$

where e is eccentricity & l is semi latus rectum.

from (1), $u = \frac{1}{r} = \frac{1 + e\cos\theta}{l}$

Hence $\frac{du}{d\theta} = -\frac{e\sin\theta}{l}$ & $\frac{d^2u}{d\theta^2} = -\frac{e\cos\theta}{l}$

$$u + \frac{d^2u}{d\theta^2} = \frac{1 + e\cos\theta}{l} - \frac{e\cos\theta}{l}$$

$$= \frac{1}{l}$$

W.K.T, $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$

$$= \frac{1}{l}$$

Hence $P = \frac{h^2u^2}{l}$

$= \frac{\mu}{r^2}$, where $\mu = h^2/l$.

i.e) The force varies inversely as the square of distance from the pole. we have the results.

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$= \left[\frac{1 + e\cos\theta}{l}\right]^2 + \left(\frac{e\sin\theta}{l}\right)^2$$

$$= \frac{1 + 2e\cos\theta + e^2}{l^2}$$

Also $h = pv$. where v is the linear velocity.

Hence $v^2 = \frac{h^2}{p^2} = \frac{h^2(1+2e\cos\theta+e^2)}{l^2}$

$= \frac{\mu l}{l^2}\left(1+e^2+2\left(\frac{l}{r}-1\right)\right)$

Sub for $e\cos\theta$ from ①

$= \frac{\mu}{l}\left(e^2+\frac{2l}{r}-1\right)$

$= \frac{\mu}{l}\left[\frac{2l}{r}-(1-e^2)\right]$

$= \mu\left[\frac{2}{r}-\frac{(1-e)^2}{l}\right] \rightarrow$ ②

Now, if a & b are the semi-axis of the ellipse. W.K.T

$$l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$

Hence putting $l = a(1-e^2)$ in ②

$v^2 = \mu\left[\frac{2}{r}-\frac{1}{a}\right]$ giving the ve[locity]

∴ Areal velocity in the orbit $= \frac{1}{2}h$ and [it] is constant. The total area of the ellipse = [πab]

periodic time $T = \frac{\pi ab}{(\frac{1}{2}h)} = \frac{2\pi ab}{h}$

$= \frac{2\pi ab}{\sqrt{\mu l}}$, since $\mu = \frac{h^2}{l}$

$= \frac{2\pi ab}{\sqrt{\mu}\cdot b}\cdot\sqrt{a}$ since $l = \frac{b^2}{a}$.

$= \frac{2\pi}{\sqrt{\mu}}\, a^{3/2}$.

Hence the Result//

A particle moves in a curve under a central acceleration so that its velocity at any point is equal to that in a circle at the same distance and under the same direction. S.T the law of force is that of the inverse cube.

Soln:

Let the central acceleration be P. If v is the velocity in a circle at a distance r under the normal acceleration P, then

$$\frac{v^2}{r} = P \quad \text{i.e.)} \quad v^2 = Pr \longrightarrow ①$$

Since v is also the velocity in the central orbit $h = pv$ or $v = h/p$.

putting this is ①,

$$\frac{h^2}{p^2} = Pr \longrightarrow ②$$

W.K.T $$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr} \longrightarrow ③$$

Subsituting ③ in ②,

$$\frac{h^2}{p^2} = \frac{h^2}{p^3} \cdot \frac{dp}{dr} r \quad \text{i.e.)} \quad \frac{dp}{p} = \frac{dr}{r}$$

Integrating, $\log p = \log r + \log A$ i.e.) $p = Ar \longrightarrow ④$

④ is clearly the (p, r) equation to an equiangular spiral.

from ④ $\frac{dp}{dr} = A$ Subst this in ③

$$P = \frac{h^2}{p^3} \cdot A = \frac{A h^2}{A^3 r^3} \quad \text{using ④}$$

$$= \frac{h^2}{A^2}\left(\frac{1}{r^3}\right) \quad \text{i.e.)} \quad P \propto \frac{1}{r^3}$$