## (iv) Charpit's Method :

A general method of solving a partial differential equation is due to Charpit

Consider $F(x, y, z, p, q) = 0 \longrightarrow (1)$

(a) If we can find another relation between the variables and the differential coefficients, the two can be regarded as simultaneous equation in $p$ and $q$ and their values obtained in terms of $x, y$, $z$ explicitly. These values when substituted in $dz = p\,dx + q\,dy$ will easily integrable.

The integral thus obtained is a solution of (1)

Let the other relation be

$$f(x, y, z, p, q) = A \longrightarrow (2)$$

where $f$ is arbitrary and $A$ is an arbitrary constant.

Differentiating (1) and (2) partially with respect to $x$ and $y$,

$$F_x + F_z p + F_p \frac{\partial p}{\partial x} + F_q \frac{\partial q}{\partial x} = 0 \rightarrow (3)$$

$$f_x + f_z p + f_p \frac{\partial p}{\partial x} + f_q \frac{\partial q}{\partial x} = 0 \rightarrow (4)$$

$$F_y + F_z q + F_p \frac{\partial p}{\partial y} + F_q \frac{\partial q}{\partial y} = 0 \rightarrow (5)$$

$$f_y + f_z q + f_p \frac{\partial p}{\partial y} + f_q \frac{\partial q}{\partial y} = 0 \rightarrow (6)$$

Eliminating $\frac{\partial p}{\partial x}$ between (3) & (4), we get

$$(F_x f_p - F_p f_x) + p\,(F_z f_p - F_p f_z) + \frac{\partial q}{\partial x}(F_q f_p - F_p f_q) = 0 \rightarrow (7)$$

Eliminating $\frac{\partial q}{\partial y}$ between (5) & (6), we get

$$(F_y f_q - F_q f_y) + q\,(F_z f_q - F_q f_z) + \frac{\partial p}{\partial y}(F_p f_q - F_q f_p) = 0 \rightarrow (8)$$

As $\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x}$

Adding (7) & (8), the terms involving these quantities cancel out;

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$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q}$$

$$= \frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{df}{0}$$

Ex :1  Find complete solution of $zpq = p+q$ (4)

Soln:-

$$f(x,y,z,p,q) \equiv zpq - p - q = 0 \longrightarrow ①$$

The auxiliary equation is   $z(aq) - (aq) - q = 0$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$zaq - aq = q$

$aq(z-1) = q$

$$\Rightarrow \frac{dp}{0+p(pq)} = \frac{dq}{0+q(pq)} = \frac{dz}{---} = \frac{dx}{zq-1} = \frac{dz}{zp-1}$$

$$\Rightarrow \frac{dp}{p^2 \cancel{q}} = \frac{dq}{pq^2} = ----$$

$$\left( \because f_x = \frac{\partial f}{\partial x} = 0 \quad f_z = \frac{\partial f}{\partial z} = pq \quad f_y = \frac{\partial f}{\partial y} = 0 \right)$$

Take first two fractions

$$\frac{dp}{p} = \frac{dq}{q}$$

$\int$ing we get

$$\log p = \log q + \log a$$

$$\Rightarrow \cancel{\log} p = \cancel{\log}(aq)$$

$$\boxed{p = aq} \longrightarrow ② \Rightarrow p = a\left(\frac{1+a}{az}\right)$$

$$\boxed{p = \frac{1+a}{z}}$$

$zaq$

$zaq$

$q$

∴ (1) $\Rightarrow zpq - p - q = 0$

$\Rightarrow z\,aq \cdot q - \cancel{p}\,aq - q = 0$

$\Rightarrow z\,aq^2 - aq - q = 0$

$\Rightarrow z\,aq^2 = q\,(1+a)$

$\Rightarrow zaq = 1+a$

$\Rightarrow \boxed{q = \dfrac{1+a}{az}}$

W.K.T $dz = p\,dx + q\,dy$

$$dz = \left(\frac{1+a}{z}\right)dx + \left(\frac{1+a}{az}\right)dy$$

$$z\,dz = (1+a)\left[dx + \frac{1}{a}\,dy\right]$$

$\int$ing we get, $\dfrac{z^2}{2} = (1+a)\left[x + \dfrac{1}{a}y\right] + b$

$$z^2 = 2(1+a)\left(x + \frac{1}{a}y\right) + b$$

$aq - q$

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Ex: 2 Solve :- $(p^2+q^2)\,y = qz$

Soln :-

$$f(x,y,z,p,q) = (p^2+q^2)\,y - qz = 0 \rightarrow (1)$$

The A.E is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$f_x = 0$, $\cancel{f_y = 2py + 2qy}$, $f_z = -q$, $f_p = 2py$

$f_q = 2qy - z$, $f_y = p^2+q^2$

$$\frac{dp}{0 + p(-q)} = \frac{dq}{(p^2+q^2)+q(-q)} = \frac{dz}{-p(2py)-q(2qy-z)}$$

$$= \frac{dq}{p^2+\cancel{q^2}-\cancel{q^2}}$$

$$= \frac{dx}{-2py} = \frac{dy}{-(2qy-z)}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2(p^2+q^2)y+qz} = \frac{dx}{-2py}$$

$$= \frac{dy}{z-2qy}$$

From the first two fractions we get

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$$p\,dp = -q\,dq \Rightarrow p\,dp + q\,dq = 0$$

$\int$ing we get, $p^2+q^2=a^2 \longrightarrow (2)$

From (1) & (2) we get $qz = a^2y$

$$\boxed{q = \frac{a^2y}{z}} \longrightarrow (3)$$

Substituting the value of $q$ in (2), we get

$$p^2 + \left(\frac{a^2y}{z}\right)^2 = a^2$$

$$p^2 = a^2 - \frac{a^4y^2}{z^2}$$

$$= \frac{a^2z^2 - a^4y^2}{z^2}$$

$$p^2 = \frac{a^2}{z^2}\left(z^2 - a^2y^2\right)$$

$$p = \pm \frac{a}{z}\sqrt{z^2-a^2y^2}$$

$\rightarrow (1)$

$\frac{dy}{-f_q}$

$y$

$-z)$

$$dz = p\,dx + q\,dy$$

$$dz = \pm \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$z\,dz - a^2y\,dy = a\sqrt{z^2 - a^2y^2}\,dx$$

$$d\left(\frac{z^2 - a^2y^2}{2}\right) = a\sqrt{z^2 - a^2y^2}\,dx$$

$$\frac{d\left(z^2 - a^2y^2\right)}{\sqrt{z^2 - a^2y^2}} = 2a\,dx \Rightarrow \frac{d\sqrt{(z^2 - a^2y^2)}}{\sqrt{(z^2 - a^2y^2)}}$$

$$d\left(z^2 - a^2y^2\right)^{1/2} = 2a\,dx$$

$\int$ing we get , $\left(z^2 - a^2y^2\right)^{1/2} = 2ax + c$

$$z^2 - a^2y^2 = (2ax + c)^2$$

$$z^2 = (2ax + c)^2 + a^2y^2$$

**Ex: 3** Solve the following PDE by charpit's method

$p^2 - xp - q = 0.$

**Soln:**

$$f(x, y, z, p, q) = 0 \Rightarrow p^2 - xp - q = 0 \rightarrow (1)$$

The A.E is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$f_x = -p, \quad f_y = 0, \quad f_z = 0, \quad f_p = 2p - x, \quad f_q = -1$$

$$\frac{dp}{-p + p(0)} = \frac{dq}{0 + q(0)} = \frac{dz}{-p(2p-x) - q(-1)}$$

$$= \frac{dx}{-(2p-x)} = \frac{dy}{-(-1)}$$

$$\frac{dp}{-p} = \frac{dq}{0} = \frac{dz}{-2p^2 + xp + q} = \frac{dx}{-2p + x} = \frac{dy}{1}$$

Consider $\dfrac{dp}{-p} = \dfrac{dy}{1} \Rightarrow \dfrac{dp}{p} = -dy$

$\int$ing we get, $\log p = -y + \log a$

$$dz = p\,dx + q\,dy$$

$$dz = \pm \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dx$$

$$d\left(\frac{z^2-a^2y^2}{2}\right) = a\sqrt{z^2-a^2y^2}\,dx$$

$$\frac{d\left(z^2-a^2y^2\right)}{\sqrt{z^2-a^2y^2}} = 2a\,dx \Rightarrow d\,\frac{\sqrt{(z^2-a^2y^2)}\sqrt{(}}{\sqrt{\cancel{(z^2}-a^2y^2)}}$$

$$d\left(z^2-a^2y^2\right)^{1/2} = 2a\,dx$$

$\int$ing we get, $\left(z^2-a^2y^2\right)^{1/2} = 2ax + c$

$$z^2 - a^2y^2 = (2ax+c)^2$$

$$z^2 = (2ax+c)^2 + a^2y^2$$

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Ex. 4 Solve $pxy + pq + qy = yz$

Soln:

Given $pxy + pq + qy = yz \longrightarrow (1)$

Let $F = pxy + pq + qy - yz$

$\therefore F_x = py, \quad F_z = -y, \quad F_y = px + q - z,$

$F_p = xy + q, \quad F_q = p + y$

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\frac{dp}{py + p(-y)} = \frac{dq}{px + q - z + q(-y)} = \frac{dz}{-p(xy+q) - q(p+y)}$$

$$= \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}$$

The subsidiary equations are $\frac{dp}{0} = \ldots$

$\therefore p = a$ (Constant)

$(1) \Rightarrow pxy + pq + qy = yz$

$axy + aq + qy = yz$

$q(a+y) = yz - axy$

$$dz = p\,dx + q\,dy$$

$$dz = \pm \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dx$$

$$d\left(\frac{z^2-a^2y^2}{2}\right) = a\sqrt{z^2-a^2y^2}\,dx$$

$$\frac{d\left(z^2-a^2y^2\right)}{\sqrt{z^2-a^2y^2}} = 2a\,dx \Rightarrow \frac{d\sqrt{(z^2-a^2y^2)}}{\sqrt{(z^2-a^2y^2)}}$$

$$d\left(z^2-a^2y^2\right)^{1/2} = 2a\,dx$$

$\int$ ing we get, $\left(z^2-a^2y^2\right)^{1/2} = 2ax + c$

$$z^2-a^2y^2 = (2ax+c)^2$$

$$z^2 = (2ax+c)^2 + a^2y^2$$

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$F_x = p^2 \,,\; F_y = -pq + 3y^2q - 2yz \,,$

$F_z = -y^2 \,,\; F_p = 2px - yq \,,\; F_q = -yp + y^3$

$$\frac{dp}{p^2 + p(-y^2)} = \frac{dy}{-(-yp + y^3)} = \ldots\ldots$$

$$\frac{dp}{p^2 - py^2} = \frac{dy}{py - y^3} = \ldots$$

$$\frac{dp}{p(p - \cancel{y^2})} = \frac{dy}{y(p - \cancel{y^2})} = \ldots$$

$$\frac{dp}{p} = \frac{dy}{y} \;;\; \text{hence } \log p = \log y + \log a$$

$$\log p = \log(ay)$$

$$\boxed{p = ay}$$

putting this value of $p$ in the given equation,

$$① \Rightarrow x(ay)^2 - y(ay)q + y^3q - y^2z = 0$$

$$a^2xy^2 - ay^2q + y^3q - y^2z = 0$$

$$y^2(a^2x - z) + y^3q - ay^2q = 0$$

$$q(y^3 - ay^2) = -y^2(a^2x - z)$$

$$q y^2 (y-a) = y(z-ax)$$

$$q(y-a) = z - a^2 x$$

$$\boxed{q = \frac{z-a^2x}{y-a}}$$

$$dz = p\,dx + q\,dy$$

$$dz = ay\,dx + \frac{z-a^2x}{y-a}\,dy \longrightarrow (2)$$

Consider the total differential equation neglect dy for the present ; integrating,

$$\int dz = \int ay\,dx$$

$$z = axy + f(y) \Rightarrow \boxed{z - axy = f(y)}$$

Differentiating totally and Comparing with (2)

$$\frac{z-axy}{y-a} = \frac{df}{dy} \Rightarrow \frac{df}{f} = \frac{dy}{y-a}$$

$$\therefore \quad f = b(y-a)$$

The complete integral is $z = axy + b(y-a)$

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