# Analytical Geometry - Three dimensions.

UNIT - 5



## Condition for plane to touch the quadric cone :-

Condition for the plane $lx+my+nz=0$ to touch the quadric cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

Let $(x_1, y_1, z_1)$ be the point of contact.

The tangent plane at $(x_1, y_1, z_1)$ is

$$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0$$

This is identical with the plane $lx+my+nz=0$

$$\therefore ax_1+hy_1+gz_1-kl=0 \rightarrow (1)$$
$$hx_1+by_1+fz_1-km=0 \rightarrow (2)$$
$$gx_1+fy_1+cz_1-kn=0 \rightarrow (3)$$

Since $(x_1, y_1, z_1)$ lies on $lx+my+nz=0$

$$\therefore lx_1+my_1+nz_1=0 \rightarrow (4)$$

Eliminating $x_1, y_1, z_1$ from equations (1), (2), (3) and (4).

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Simplifying, we get

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0 \rightarrow (5)$$

where $A, B, C, F, G, H$ are the cofactors of $a, b, c, f, g, h$ in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

multiplying (1) by $A$ (2) by $H$ and (3) by $G$ and adding, we get

$$\Delta x_1 = k(Al+Hm+Gn)$$

Since $\Delta = Aa + Hh + Gg$.

$0 = Ah + Hb + Gf$.

$0 = Ag + Hf + Gc$.

lly $\Delta y_1 = k(Hl + Bm + Fn)$

$\Delta z_1 = k(Gl + Fm + Cn)$.

Hence the point of contact

$$\frac{x_1}{Al + Hm + Gn} = \frac{y_1}{Hl + Bm + Fn} = \frac{z_1}{Gl + Fm + Cn}.$$

$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ which is $\perp^r$ to the plane $lx + my + nz = 0$ at the origin, is a generator of the cone.

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \rightarrow (6)$$

$$\Delta' = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}, \text{ we get}$$

$A' = BC - F^2 = a\Delta'$, $F' = GH - AF = f\Delta'$.

$B' = CA - G^2 = b\Delta'$, $G' = HF - BG = g\Delta'$.

$C' = AB - H^2 = c\Delta'$, $H' = FG - CH = h\Delta'$.

Hence the $\perp^r$ to the tangent planes to the cone (6) generate cone.

$$A'x^2 + B'y^2 + C'z^2 + 2F'yz + 2G'zx + 2H'xy = 0$$

(ie) $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \rightarrow (7)$

The cones (6) and (7) are said to be reciprocal.

Example 1:-

Find the equation of the tangent planes to the cone $9x^2 - 4y^2 + 16z^2 = 0$ which contain the line $\frac{x}{32} = \frac{y}{72} = \frac{z}{27}$

The line is the intersection of the planes

$$72x - 32y = 0, \text{(ie) } 9x - 4y = 0$$

and $27y - 72z = 0$, (ie) $3y - 8z = 0$

Hence any plane passing through this line is of the form

$$9x - 4y + \lambda(3y - 8z) = 0.$$

$$\text{(ie) } 9x + y(3\lambda - 4) - 8\lambda z = 0 \rightarrow (1)$$

This line touches the cone.

$$9x^2 - 4y^2 + 16z^2 = 0 \rightarrow (2)$$

Hence the normal to the plane,

$$\frac{x}{9} = \frac{y}{3\lambda - 4} = \frac{z}{-8\lambda} \rightarrow (3)$$

is a generator of the reciprocal cone of the cone (2).

Equation of the reciprocal cone of (2) is

$$\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{16} = 0 \rightarrow (4)$$

(3) is a generator of cone (4).

$$\therefore \frac{9^2}{9} - \frac{(3\lambda - 4)^2}{4} + \frac{(-8\lambda)^2}{16} = 0.$$

we get $7\lambda^2 + 24\lambda + 20 = 0$

(ie) $\lambda = -2$ (or) $\lambda = \frac{-10}{7}$.

Hence the equation of the planes are

$9x - 10y + 16z = 0$ and $63x - 58y + 80z = 0$

Ex:2
find the general equation to a cone which touches the co-ordinateplane.

If the co-ordinate planes touch a cone, the perpendiculars to co-ordinate plane touch the reciprocal cone.

Hence the cone touching the co-ordinate planes is reciprocal to the cone passing through the co-ordinate axes.

The direction cosines of the co-ordinate axes are $1, 0, 0$; $0, 1, 0$; $0, 0, 1$

The equation of the cone passing throug the axis is of the form

$$2fyz + 2gzx + 2hxy = 0.$$

The required cone is the reciprocal cone of this cone and its equation is

$$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0.$$

This equation can be put in the form

$$\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0.$$

The angle between the lines in which the plane $ux+vy+wz=0$ cuts the cone.

$$f(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

The plane meets the cone in two lines which pass through the origin and So the equation of the lines are of the form $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

②

The line lies in the plane and in the cone.

$\therefore ul + vm + wn = 0$ and $\rightarrow (1)$

$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \rightarrow (2)$

Eliminating between (1) and (2), we get.

$$l^2(cu^2 + aw^2 - 2gwu) + 2lm(hw^2 + cuv - fwv - gvw) + m^2(cv^2 + bw^2 - 2fvw) = 0 \rightarrow (3)$$

The direction cosines of the two lines satisfy the equation (3) and if they are $l_1, m_1, n_1$ and $l_2, m_2, n_2$ we have.

$$\frac{l_1}{m_1} + \frac{l_2}{m_2} = -\frac{2(hw^2 + cuv - fuw - gvw)}{cu^2 + aw^2 - 2gwu}$$

$$\frac{l_1 l_2}{m_1 m_2} = \frac{cv^2 + bw^2 - 2fvw}{cu^2 + aw^2 - 2gwv}$$

$$\therefore \frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1 m_2}{cu^2 + aw^2 - 2gwn}$$

$$= \frac{l_1 m_2 + l_2 m_1}{-2(hw^2 + cuv - fuw - gvw)}$$

$$= \frac{l_1 m_2 - l_2 m_1}{\pm\{(hw^2 + cuv - fuw - gvw)^2 - (bw^2 + cv^2 - 2fvw)(cu^2 + uw^2 - 2gwn)\}^{1/2}}$$

$$= \frac{l_1 m_2 - l_2 m_1}{\pm 2w(-Au^2 - Bv^2 \mp cw^2 - 2Fvw - 2Gwu - 2Huv)^{1/2}}$$

$$= \frac{l_1 m_2 - l_2 m_1}{\pm 2wP} \rightarrow (4)$$

where $p^2 = -(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv)$

and $A, B, C, F, G, H$ are the cofactors of $a, b, c, f, g, h$ in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

From the symmetry, we get the expression in (A) is equal to,

$$\frac{n_1 n_2}{av^2 + bu^2 - 2huv} = \frac{m_1 n_2 - m_2 n_1}{\pm 2up} = \frac{n_1 l_2 - n_2 l_1}{\pm 2vp} \rightarrow (5)$$

Each expression in (4) and (5) is.

$$\frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{bw^2 + cv^2 - 2fvw + cu^2 + aw^2 - 2gwu + av^2 + bu^2 - 2huv}$$

$$= \frac{\{\Sigma (m_1 n_2 - m_2 n_1)^2\}^{1/2}}{\pm 2(u^2 + v^2 + w^2)^{1/2} p}$$

If $\theta$ is the angle between the lines,

$$\frac{\cos\theta}{l_1 l_2 + m_1 m_2 + n_1 n_2} = \frac{\sin\theta}{(\Sigma (m_1 n_2 - m_2 n_1)^2\}^{1/2}}.$$

$$\therefore \frac{\cos\theta}{(a+b+c)(u^2+v^2+w^2) - f(u,v,w)}$$

$$= \frac{\sin\theta}{\pm 2(u^2+v^2+w^2)^{1/2} p} \rightarrow (6)$$

Condition that the cone has three mutually perpendicular generators:

The condition that the plane should cut the cone in perpendicular

generators is that $\theta = 90°$. In that case by (6) of the previous article $(a+b+c)(u^2+v^2+w^2)$
$= f(u,v,w)$.

The third generator is $\perp^r$ to these two generators. Hence it is normal to the plane containing these perpendicular generators.

If the normal to the plane $ux+vy+wz=0$ lies on the cone, we have $f(u,v,w)=0$

$\therefore a+b+c=0.$

**Example:-**

Find the equation to the cone through the co-ordinate axes and the lines in which the plane $lx+my+nz=0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

Let the equation of the cone passing through the co-ordinates axes by $Fyz+Gzx+Hxy=0$.

Eliminating between $lx+my+nz=0$ and

$$ax^2+by^2+cz^2+2fyz+2g\cancel{z}x+2hxy=0,$$

we get, $$ax^2+by^2+\frac{c(lx+my)^2}{n^2}-\frac{2fy(lx+my)}{n}$$
$$-\frac{2gx(lx+my)}{n}+2hxy=0.$$

(ie) $x^2(an^2+cl^2-2gln)+\dots+y^2(cm^2+bn^2-2fmn)$
$=0.$

$ll^y$ eliminates between $lx+my+nz=0$

and $Fyz+Gzx+Hxy=0.$

$$\therefore -\frac{Fy(lx+my)}{n}-\frac{Gx(lx+my)}{n}+Hxy=0$$

(i.e) $Glx^2+\dots+Fmy^2=0.$

Since the two cones have common generators we get,

$$\frac{an^2+cl^2-2gln}{Gl} = \frac{cm^2+bn^2-2fmm}{Fm}$$

similarly eliminating $gx$, we get the condition,

$$\frac{bl^2+am^2-2hml}{Hm} = \frac{an^2+cl^2-2gnl}{Gn}$$

$$\therefore \frac{an^2+cl^2-2gln}{Gnl} = \frac{bl^2+am^2-2hlm}{Hlm}$$

$$= \frac{cm^2+bn^2-2fmn}{Fmn}$$

Hence

$$\frac{F}{l(cm^2+bn^2-2fmn)} = \frac{G}{m(cl^2+an^2-2gln)}$$

$$= \frac{H}{n(am^2+bl^2-2hln)}$$

Hence the equation of the required Cone is,

$$l(cm^2+bn^2-2fmn)yz+m(cl^2+an^2-2gln)zx$$
$$+n(am^2+bl^2-2hlm)xy=0.$$

**central qudries:**

definition: If $p(x_1, y_1, z_1)$ lies on the surface.

$$Ax^2+By^2+Cz^2=1 \rightarrow (1).$$

$Q(-x_1, -y_1, -z_1)$ also lies on the Surface and $O$ the origin is the mid-point of $PQ$.

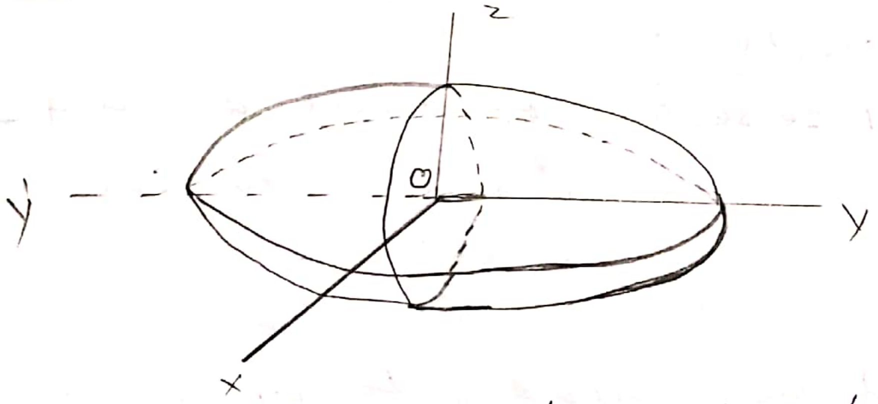
Hence all chords of (i) which pass through O are bisected at O. For this reason (i) is Called a central quadric, O is Called its centre and a chord through O is called a diameter.

case (i) Let A, B, C be all +ve

Put $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$ and $C = \frac{1}{c^2}$ and the equation (i) becomes,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \rightarrow (ii)$$

$z = k$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$, $z = k$ and these are the equation of an ellipse when $k^2 \le c^2$. when $k^2 > c^2$, the plane does not cut the surface in real points.




case (ii):- Let A and B be positive and C be negative

$A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$ and $C = \frac{-1}{c^2}$ and then the equation (i) Becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \quad z = k.$$

and for all values of k, this is an ellipse. This Surface is called a hyperboloid of one Sheet.

case (iii) Let $c$ be positive and $A$ and $B$ negative.

put $c = \frac{1}{c^2}$, $A = \frac{-1}{a^2}$, $B = \frac{-1}{b^2}$ and the equation (i) becomes:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The equations of the section of this surface by the plane $z = k$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1, \quad z = k.$$

These are the equations of an ellipse when $k^2 \geq c^2$.

when $k^2 < c^2$ the plane does not cut the surface in real points. Sections parallel to the $yz$ and $zx$ planes are hyperbolas.

This surface is called a hyperboloid of two sheets.

The intersection of a line and a quadric:-

Let the equation of the straight line be $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$.

and the quadric be $ax^2 + by^2 + cz^2 = 1$.

The co-ordinates of any point on the line are of the form,

$$(x_1 + lr, y_1 + mr, z_1 + nr)$$

If this point lies on the quadric.

$$a(x_1 + lr)^2 + b(y_1 + mr)^2 + c(z_1 + nr)^2 = 1.$$

(ie) $r^2(al^2 + bm^2 + cn^2) + 2r(alx_1 + bmy_1 + cnz_1)$
$+ ax_1^2 + by_1^2 + cz_1^2 - 1 = 0 \rightarrow (1)$

Tangents and tangents planes:.

Any line through $p(x_1, y_1, z_1)$ is of the form,

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \rightarrow (1)$$

$$(al^2+bm^2+cn^2)r^2 + 2r(alx_1+bmy_1+cnz_1) + ax_1^2+by_1^2+cz_1^2-1=0 \rightarrow (2).$$

If $(x_1, y_1, z_1)$ lies on the conicoid,

$$ax_1^2+by_1^2+cz_1^2=1.$$

Then the equation (2) becomes.

$$(al^2+bm^2+cn^2)r^2+2r(alx_1+bmy_1+cnz_1) = 0 \rightarrow (3)$$

If line (1) is a tangent to the conicoid, line (1) will meet the conicoid in two coincident points.

Hence equation (3) has two zero roots,

$$\therefore alx_1 + bmy_1 + cnz_1 = 0 \rightarrow (4).$$

Hence all tangent lines at $P$ lie in a plane through $P$ perpendicular to this direction,

This plane is known as the tangent plane at $p$ and its equation is,

$$ax_1(x-x_1) + by_1(y-y_1) + cx_1(z-z_1) = 0$$

(ie) $axx_1 + byy_1 + czz_1 = ax_1^2 + by_1^2 + cz_1^2.$

(ie) $axx_1 + byy_1 + czz_1 = 1.$

The condition for the plane $lx+my+nz=p$ to touch the conicoid $ax^2+by^2+cz^2=1$.

Let the plane touch the conicoid at $(x_1, y_1, z_1)$

The equation of the tangent plane at $(x_1, y_1, z_1)$ is

$$a x x_1 + b y y_1 + c z z_1 = 1 \rightarrow (1)$$

This plane is also represented by the equation

$$lx + my + nz = p. \rightarrow (2)$$

$$\frac{a x_1}{l} = \frac{b y_1}{m} = \frac{c z_1}{n} = \frac{1}{p}.$$

(ie) $x_1 = \frac{l}{ap}$, $y_1 = \frac{m}{bp}$, $z_1 = \frac{n}{cp}$.

$\because (x_1, y_1, z_1)$ lies on the conicoid

$$a x_1^2 + b y_1^2 + c z_1^2 = 1.$$

$$\therefore a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right)^2 = 1.$$

(i.e) $p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}$.

Cor 1:-

There are two tangent planes to a conicoid parallel to the plane $lx + my + nz = 0$ and their equation are,

$$lx + my + nz = \pm \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)^{1/2}.$$

Ex: 1

If OD is the diameter parallel to a secant APQ through A meeting the coincoid at P and Q show that $\frac{AP \cdot AQ}{OD^2}$ is constant.

Let the conicoid be $ax^2 + by^2 + cz^2 = 1$, A be $(\alpha, \beta, \gamma)$ and the direction cosines of the lines APQ be $l, m, n$.

The equation APQ is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

The co-ordinates of a point at a distance $r$ from A are $(\alpha+lr, \beta+mr, \gamma+nr)$.

This point lies on the conicoid.

$\therefore a(\alpha+lr)^2 + b(\beta+mr)^2 + c(\gamma+nr)^2 = 1$.

(ie) $r^2(al^2+bm^2+cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$.

$$\therefore AP \cdot AQ = \frac{a\alpha^2 + b\beta^2 + c\gamma^2 - 1}{al^2 + bm^2 + cn^2}.$$

The direction cosines of the line OD are also $l, m, n$.

D is the point $(lk, mk, nk)$, where $k = OD$.

Since D lies on the conicoid $al^2k^2 + bm^2k^2 + cn^2k^2 = 1$.

$$\therefore k^2 = \frac{1}{al^2+bm^2+cn^2}.$$

Hence $\frac{AP \cdot AQ}{OD^2} = \frac{AP \cdot AQ}{k^2} = a\alpha^2 + b\beta^2 + c\gamma^2 - 1$

$= \text{constant}$.

Ex: 2

Find the equation of the tangent planes to $x^2+y^2+4z^2=1$ which intersect in the line whose equation are $12x - 3y - 5 = 0$, $z = 1$.

Any plane which passes through the line is given by,

$12x - 3y - 5 + \lambda(z-1) = 0$

(ie) $12x - 3y + \lambda z - (\lambda+5) = 0 \rightarrow (1)$

Let this plane touch the conicoid at $(x_1, y_1, z_1)$.

The equation of the tangent plane at $(x_1, y_1, z_1)$ is.

$$x x_1 + y y_1 + 4 z z_1 = 1 \rightarrow (2).$$

Equations (1) and (2), represent the same plane,

$$\therefore \frac{x_1}{12} = \frac{y_1}{-3} = \frac{4z_1}{\lambda} = \frac{1}{\lambda+5}$$

$$\therefore x_1 = \frac{12}{\lambda+5}, \quad y_1 = \frac{-3}{\lambda+5}, \quad z_1 = \frac{\lambda}{4(\lambda+5)}$$

since $(x_1, y_1, z_1)$ lies on the conicoid,

$$x_1^2 + y_1^2 + 4z_1^2 = 1.$$

$$\therefore \left(\frac{12}{\lambda+5}\right)^2 + \left(\frac{-3}{\lambda+5}\right)^2 + 4\left\{\frac{\lambda}{4(\lambda+5)}\right\}^2 = 1.$$

(ie) $3\lambda^2 + 40\lambda - 512 = 0$.

$$\therefore \lambda = 8 \text{ (or) } \lambda = -\frac{64}{3}.$$

Hence the equation of the tangent planes are.

$$12x - 3y - 5 + 8(z-1) = 0.$$

and $12x - 3y - 5 - \frac{64}{3}(z-1) = 0$.

(ie) $12x - 3y + 8z - 13 = 0$ and $36x - 9y - 64z + 49 = 0$