Unit - III

Gauss's Divergence Theorem :- (G.d. The)

Statement :-

2m

(2) The normal surface integral of a vector point function F which is continuously differentiable over the boundary of closed region is equal to the volume integral of divergent F taken through out the region.

i.e) $$\boxed{\iint_S F.\hat{n}\, ds = \iiint_V \nabla.\bar{F}\, dv.}$$

where $\hat{n}$ is the unit outward normal verctor to S.

Corr : 1 $$\iiint_V \nabla u\, dv = \iint_S \hat{n}\, u\, ds$$

where u is a continuously differentiable scalar point function

Corr : 2

$$\iiint_V (\nabla \times \bar{F})\, dv = \iint_S \hat{n} \times \bar{F}\, ds$$

① verify Gauss's divergence theorem for $\bar{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the cube bounded by $x=0, x=1,$ $y=0, y=1, z=0, z=1$.

pf:

Gauss's divergence theorem is

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

Gnt $\bar{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\nabla\cdot\bar{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).$$

$$\cdot\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$\nabla\cdot\bar{F} = 4z - 2y + y = 4z - y$$

$$\iiint_V \nabla\cdot\bar{F}\,dv = \iiint (4z - y)\,dv$$

$dv = dx\,dy\,dz$

x, y, z varies from 0 to 1.

$$\therefore \iiint_V \nabla\cdot\bar{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (4zx - yx)_0^1\,dy\,dz$$

$$= \int_0^1 \int_0^1 (4z - y)\, dy\, dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2} \right]_0^1 dz$$

$$= \int_0^1 \left[4z - \frac{1}{2} - (0) \right] dz$$

$$= \int_0^1 \left(4z - \frac{1}{2} \right) dz$$

$$= \left[4\frac{z^2}{2} - \frac{1}{2} z \right]_0^1$$

$$= \left[\frac{4}{2} - \frac{1}{2} - (0) \right]$$

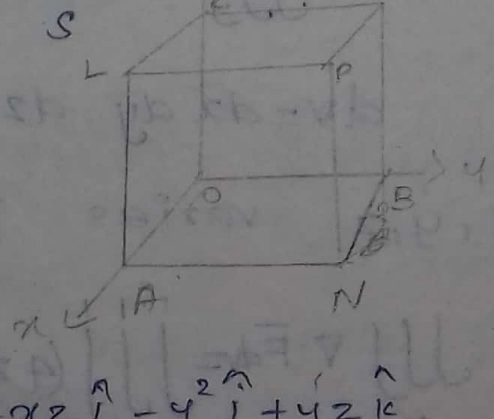
$$= \cancel{\frac{4-1}{2}}$$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{0}{2}} = \cancel{\ }\tfrac{1}{2}$$

$$\boxed{\iiint_V \nabla \cdot F\, dV = \frac{3}{2}} \quad \text{— (1)}$$

Now,

TO find $\iint_S F \cdot \hat{n}\, ds$.



Gnt $F = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$

and cube $x=0, x=1, y=0, y=1, z=0, z=1$

here we calculate $\iint_S F\cdot\hat{n}\, ds$ along 6 faces.

S_1 - face ANPL

S_2 - " OBMC

S_3 - " BMPN

S_4 - " OCLA

S_5 - " CMPL

S_6 - " OBNA.

In ~~for~~ S_1, ~~$\hat{n}$~~ the unit outward the normal to surface is $\hat{i}$

S_2, $\hat{n} = -\hat{i}$

S_3, $\hat{n} = \hat{j}$

S_4, $\hat{n} = -\hat{j}$

S_5, $\hat{n} = \hat{k}$

S_6, $\hat{n} = -\hat{k}$.

S_1 - Face ANPL :-

Here the unit outward normal vector $\hat{n} = \hat{i}$ and $x = 1$.

$$F\cdot\hat{n} = (4xz\,\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{n}$$

$$= \cancel{yz}\ 4xz.$$

$$\iint_{S_1} F\cdot\hat{n}\, ds = \int_0^1\int_0^1 4xz \frac{dy\, dz}{|\hat{n}\cdot\hat{i}|}$$

$$= \int_0^1\int_0^1 4xz \frac{dy\, dz}{\hat{i}\cdot\hat{i}}$$

$$= \int_0^1\int_0^1 4xz\, dy\, dz$$

$$= \int_0^1\int_0^1 4(1)z\, dy\, dz = \int_0^1 (4yz)_0^1\, dz$$

$$= \int_0^1 (1-0)z\, dz = \int_0^1 4z\, dz$$

$$= 4\left(\frac{z^2}{2}\right)_0^1 = 4\left(\frac{1}{2} - 0\right)$$

$$\iint_{S_1} F\cdot\hat{n}\, ds = 2$$

S_2 - Face OMPC

here $\hat{n} = -\hat{i}$ and $x = 0$

$$F\cdot\hat{n} = 4xz = 4(0)z$$

$$= 0$$

$$\therefore \iint_{S_2} F\cdot\hat{n}\, ds = 0.$$

S_3 - Face BMPN

here $\hat{n} = \hat{j}$ and $y = 1$

$$F\cdot\hat{n} = (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\,\hat{j}$$

$$= -y^2$$

$$= -(1)^2 = -1$$

$$\iint_{S_3} F\cdot\hat{n}\, ds = \int_0^1\int_0^1 (-1)\frac{dx\,dz}{|\hat{j}\cdot\hat{n}|}$$

$$= -\int_0^1\int_0^1 \frac{dx\,dz}{|\hat{j}\cdot\hat{j}|}$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$= -\int_0^1 (x)_0^1\, dz = -\int_0^1 (1-0)\,dz$$

$$= -(z)_0^1 = -1$$

$$\therefore \iint_{S_3} F\cdot\hat{n}\, ds = -1.$$

S_4 - Face OCLA :-

here $\hat{n} = -\hat{j}$ and $y = 0$.

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$F \cdot \hat{n} = 0$

$$\iint_{S_4} F \cdot \hat{n}\, ds = 0.$$

S_5 - Face CMPL :-

here $\hat{n} = \hat{k}$ and $z = 1$

$$F \cdot \hat{n} = (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\hat{k}$$

$$= yz$$

$$= y$$

$$\iint_{S_5} F \cdot \hat{n}\, ds = \int_0^1\int_0^1 y \frac{dx\, dy}{|\hat{k} \cdot \hat{n}|}$$

$$= \int_0^1\int_0^1 y\, dx\, dy = \int_0^1 (xy)_0^1\, dy$$

$$= \int_0^1 y\, dy = \left(\frac{y^2}{2}\right)_0^1$$

$$= \frac{1}{2}$$

S_6 - Face OBNA :

here $\hat{n} = -\hat{k}$ and $z = 0$

$$F \cdot \hat{n} = yz = y(0)$$

$$= 0$$

$$\therefore \iint_{S_6} F \cdot \hat{n}\, ds = 0$$

$$\therefore \iint_S F \cdot \hat{n}\, ds = \iint_{S_1} F \cdot \hat{n}\, ds + \iint_{S_2} F \cdot \hat{n}\, ds + \iint_{S_3} F \cdot \hat{n}\, ds$$

$$+ \iint_{S_4} F \cdot \hat{n}\, ds + \iint_{S_5} F \cdot \hat{n}\, ds + \iint_{S_6} F \cdot \hat{n}\, ds.$$

$= 2 + 0 - 1 + 0 + \frac{1}{2} + 0$

$= 2 - 1 + 1/2 \quad = \quad 1 + 1/2$

$\iint_S F \cdot \hat{n} \, ds = 3/2$ —— (2)

from (1) and (2)

$$\iint_S F \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv.$$

2) Verify Gauss's divergence Thm for $F = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over rectangular parallelopiped $0 \le x \le a$ $0 \le y \le b$ $0 \le z \le c$.

(or)

(3) (1) $x = 0$ to $x = a$, $y = 0$ to $y = b$, $z = 0$ to $z = c$.

Gauss's divergence Theorem.

$$\iint_S F \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv.$$

R.H.S:

$\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}.$

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$

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$$\nabla \cdot F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left((x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\right).$$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy).$$

$$= 2x + 2y + 2z.$$

$$\nabla \cdot \bar{F} = 2(x+y+z)$$

$$\iiint \nabla \cdot \bar{F}\, dv = \iiint 2(x+y+z)\, dv$$

$dv = dx\, dy\, dz$

x varies from 0 to a.

y " " 0 to b

z " " 0 to c.

$$\iiint_V \nabla \cdot \bar{F}\, dv = \int_0^a \int_0^b \int_0^c 2(x+y+z)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b \left[2xz + 2yz + \frac{2z^2}{2}\right]_0^c dy\, dx.$$

$$= \int_0^a \int_0^b \left[2x(c) + 2y(c) + (c)^2 - 0\right] dy\, dx.$$

$$= \int_0^a \int_0^b (2cx + 2cy + c^2)\, dy\, dx$$

$$= \int_0^a \left[2cxy + \frac{2cy^2}{2} + c^2 y\right]_0^b dx.$$

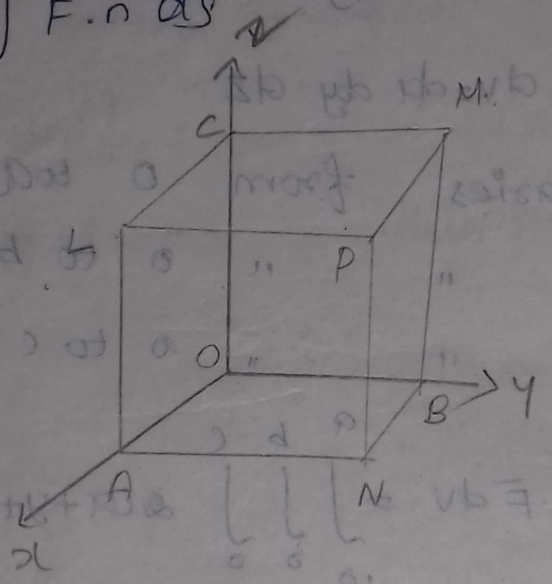
$$= \int_0^a (2cxb + cb^2 + c^2 b)\, dx$$

$$= \left[2cb\frac{a^2}{2} + cb^2 x + c^2 b x \right]_0^a$$

$$= cba^2 + cb^2 a + c^2 b a$$

$$\iiint_V F\,dv = abc(a+b+c) \quad \text{— (A)}$$

Now, To find $\iint_S F.\hat{n}\, ds$



Gn. $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$.

and $x=0,\ x=a,\ y=0,\ y=b,\ z=0,\ z=c$

here we calculate along

6 - faces. S_1 - Face ANPL. for $S_1, \hat{n} = \hat{i}$
S_2 " OBMC $S_2, \hat{n} = -\hat{i}$
S_3 - " BMPN S_3, " $= \hat{j}$
S_4 - " OCLA S_4, $\hat{n}$ $= -\hat{j}$
S_5 - " CMPL S_5, " $= \hat{k}$

S_6 - " OBNA. for S_6 - $\hat{n} = -\hat{k}$.

S_1-Face ANPL :- Here the unit outward normal vector : $\hat{n} = \hat{i}$, $x = a$ here

$$F\cdot\hat{n} = \left((x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\right)\cdot(\hat{i})$$

$$= x^2 - yz$$

$$\iint_{S_1} F\cdot\hat{n}\,ds = \int_0^b\int_0^c (x^2-yz)\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

$$= \int_0^b\int_0^c (x^2-yz)\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

$$\cancel{= \int_0^a\int_0^b \left(x^2y - \frac{zy^2}{2}\right)_0^b dz}$$

$$= \int_0^b\int_0^c (x^2-yz)\,dz\,dy$$

$$= \int_0^b \left(x^2z - \frac{yz^2}{2}\right)_0^c dy$$

$$= \int_0^b \left(x^2c - y\frac{c^2}{2}\right) dy$$

$$= \left[x^2yc - \frac{y^2}{2}\cdot\frac{c^2}{2}\right]_0^b$$

$$= x^2bc - \frac{b^2c^2}{4} \qquad [\because x = a]$$

$$\boxed{\iint_{S_1} F\cdot\hat{n}\,ds = \frac{4a^2bc - b^2c^2}{4}}$$

S_2 - Face OBMC.

here $\hat{n} = -\hat{i}$ and $x = 0$

$$\bar{F}\cdot\hat{n} = -(x^2 - yz)$$

$$\iint_{S_2} F\cdot\hat{n}\,ds = -\int_0^b\int_0^c (x^2 - yz)\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= -\int_0^b\int_0^c (0 - y.z)\frac{dy\,dz}{|-\hat{i}\cdot\hat{i}|}$$

$$= \int_0^b\int_0^c y.z\,dy\,dz$$

$$= \int_0^b\int_0^c yz\,dz\,dy = \int_0^b\left(y\frac{z^2}{2}\right)_0^c dy$$

$$= \int_0^b y\frac{c^2}{2}\,dy = \left[\frac{c^2}{2}\cdot\frac{y^2}{2}\right]_0^b$$

$$\boxed{\iint_{S_2} F\cdot\hat{n}\,ds = \frac{c^2 b^2}{4}}$$

S_3 - Face BMPN :- here $\hat{n} = \hat{j}$ and $y = b$

$$\iint_{S_3} \quad \bar{F}\cdot\hat{n} = y^2 - zx$$

$$\bar{F}\cdot\hat{n} = b^2 - zx$$

$$\iint_{S_3} F\cdot\hat{n}\,ds = \int_0^a\int_0^c (b^2 - zx)\frac{dx\,dz}{|\hat{j}\cdot\hat{n}|}$$

$$= \int_0^a\int_0^c (b^2 - zx)\frac{dz\,dx}{|\hat{j}\cdot\hat{j}|}$$

$$= \int_0^a \left(b^2 z - \frac{z^2}{2}x\right)_0^c dx$$

$$= \int_0^a \left(b^2 c - \frac{c^2}{2} x\right) dx$$

$$= \left[b^2 c\, x - \frac{c^2}{2} \cdot \frac{x^2}{2} \right]_0^a$$

$$= b^2 c a - \frac{c^2 a^2}{4}$$

$$\boxed{\iint_{S_3} F \cdot \hat{n}\, ds = \frac{4 b^2 c a - c^2 a^2}{4}}$$

S_4 - Face OCLA :-

here $\hat{n} = -\hat{j}$ and $y = 0$.

$$F \cdot \hat{n} = -(y^2 - zx)$$

$$= -(0 - zx) = zx.$$

$$\iint_{S_4} F \cdot \hat{n}\, ds = \int_0^a \int_0^c (zx) \frac{dx\, dz}{|\hat{j} \cdot \hat{n}|}$$

$$= \int_0^a \int_0^c (zx) \frac{dx\, dz}{|\hat{j} \cdot -\hat{j}|}$$

$$= \int_0^a \int_0^c zx\, dz\, dx$$

$$= \int_0^a \left(x \frac{z^2}{2}\right)_0^c dx \qquad = \int_0^a \left(x \frac{c^2}{2} - 0\right) dx$$

$$= \left[\frac{c^2}{2} \cdot \frac{x^2}{2} \right]_0^a$$

$$\boxed{\iint_{S_4} F \cdot \hat{n}\, ds = \frac{c^2 a^2}{4}}$$

S_5 - Face CMPL :-

here $\hat{n} = \hat{k}$ and $z = c$

$\bar{F}\cdot\hat{n} = z^2 - xy$

$\bar{F}\cdot\hat{n} = c^2 - xy$

$$\iint_{S_5} \bar{F}\cdot\hat{n}\, ds = \int_0^a \int_0^b (c^2 - xy) \frac{dx\, dy}{|\hat{k}\cdot\hat{n}|}$$

$$= \int_0^a \int_0^b (c^2 - xy) \frac{dy\, dx}{|\hat{k}\cdot\hat{k}|}$$

$$= \int_0^a \left[c^2 y - \frac{xy^2}{2} \right]_0^b dx$$

$$= \int_0^a \left(c^2 b - \frac{xb^2}{2} \right) dx$$

$$= \left[bc^2 x - \frac{x^2}{2}\cdot\frac{b^2}{2} \right]_0^a$$

$$= \left[bc^2 a - \frac{a^2 b^2}{4} \right]$$

$$\boxed{\iint_{S_5} \bar{F}\cdot\hat{n}\, ds = ba c^2 - \frac{a^2 b^2}{4}.}$$

S_6 - Face OBNa :-

here $\hat{n} = -\hat{k}$ and $z = 0$.

$\bar{F}\cdot\hat{n} = -(z^2 - xy)$

$= -(0 - xy)$

$= xy$

$$\iint_{S_6} \bar{F}\cdot\hat{n}\, ds = \int_0^a \int_0^b xy \frac{dx\, dy}{|\hat{k}\cdot\hat{n}|}.$$

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$$= \int_0^a \int_0^b xy \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

$$= \int_0^a \int_0^b xy\, dy\, dx = \int_0^a \left[x\frac{y^2}{2} \right]_0^b dx$$

$$= \int_0^a \left(x\frac{b^2}{2} \right) dx = \left[\frac{b^2}{2} \cdot \frac{x^2}{2} \right]_0^a$$

$$\boxed{\iint_{S_6} F\cdot\hat{n}\, ds = \frac{a^2b^2}{4}}$$

$$\iint_S F\cdot\hat{n}\, ds = \iint_{S_1} F\hat{n}\, ds + \dots + \iint_{S_6} F\cdot\hat{n}\, ds.$$

$$= \left\{ \frac{4a^2bc - b^2c^2}{4} + \frac{c^2b^2}{4} + \frac{4b^2ca - c^2a^2}{4} + \frac{c^2a^2}{4} + bac^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4} \right\}$$

$$= \frac{4a^2bc}{4} - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{c^2b^2}{4}} + \frac{4b^2ca}{4} - \cancel{\frac{c^2a^2}{4}} + \cancel{\frac{c^2a^2}{4}} + ba^{\cancel{2}}c^2$$

$$= a^2bc + b^2ca + bac^2$$

$$\iint_S F\cdot\hat{n}\, ds = abc(a+b+c). \quad \text{— (B)}$$

from A and B.

$$\iint_S F\cdot\hat{n}\, ds = \iiint_V \nabla\cdot F\, dV$$

Verify d.T for

$\bar{F} = 2x^2\hat{i} - y^2\hat{j} + 4xz\hat{k}$ 36π, 36

in the first octant bounded by

$y^2 + z^2 = 9$, $x = 0, x = 2$.

(1) If V is the volume n closed by a closed surface S and

$F = x\hat{i} + 2y\hat{j} + 3z\hat{k}$ show that

$\iint_S F \cdot \hat{n}\, ds = 6V.$

Pf: W.k.T Gauss's divergence Theorem

$$\iint_S F \cdot \hat{n}\, ds = \iiint_V \nabla \cdot F\, dV$$

Gnt $\bar{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$.

$$\nabla \cdot \bar{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x\hat{i} + 2y\hat{j} + 3z\hat{k}\right)$$

$$= \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} 2y + \frac{\partial}{\partial z} 3z$$

$$= 1 + 2 + 3$$

$$\nabla \cdot \bar{F} = 6$$

$$\therefore \iiint_V \nabla\cdot F \, dv = \iiint_V (6) \, \cancel{dx\,dy\,dz} \, dv$$

$$\boxed{\iiint_V \nabla\cdot F \cdot dv = 6V}$$

$$\therefore \iint_V F\cdot\hat{n}\, ds = 6V$$

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(2) Evaluate $\iint_S F\cdot\hat{n}\, ds$ when $F = 4xy\,\hat{i} + yz\,\hat{j} - xz\,\hat{k}$ and S is the surface of cube bounded by the planes $x=0,\ x=2,\ y=0,\ y=2$, $z=0,\ z=2$.

H.W

(II) unit

Q.paperla

W.k.T Gauss's divergence theorem.

$$\iint_S F\,\hat{n}\,ds = \iiint_V \nabla\cdot F\, dv \quad -(1)$$

Gnt $F = 4xy\,\hat{i} + yz\,\hat{j} - xz\,\hat{k}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x} 4xy + \frac{\partial}{\partial y} yz + \frac{\partial}{\partial z}(-xz)$$

$$= 4y + z - x$$

$$\iiint_V \nabla\cdot F\, dv = \int_0^2\int_0^2\int_0^2 (4y + z - x)\, dx\, dy\, dz$$

$$= \int_0^2\int_0^2 \left[4xy + zx - \frac{x^2}{2} \right]_0^2 dy\, dz$$

$$= \int_0^2 \int_0^2 \left[4(2)y + 2z - \frac{(2)^2}{2} - (0) \right] dy\,dz$$

$$= \int_0^2 \int_0^2 (8y + 2z - 2)\, dy\, dz$$

$$= \int_0^2 \left[\frac{8y^2}{2} + 2zy - 2y \right]_0^2 dz$$

$$= \int_0^2 \left[8\frac{(2)^2}{\cancel{2}} + 2z(2) - 2(2) - (0) \right] dz$$

$$= \int_0^2 (16 + 4z - 4)\, dz$$

$$= \int_0^2 (12 + 4z)\, dz = \left[12z + \frac{4z^2}{2} \right]_0^2$$

$$= \left[12z + 2z^2 \right]_0^2$$

$$= \left[12(2) + \frac{4(2)^2}{2} - (0) \right] = 12(2) + 2(2)^2 = 24 + 2(4) = 24 + 8$$

$$= 24 + 8$$

$$\iiint_V \nabla \cdot \bar{F}\, dV = 32$$

$$\therefore \boxed{\iint_S \bar{F} \cdot \hat{n}\, ds = 32}$$ (by ①)

① Evaluate $\iint_S \bar{r} \cdot \hat{n}\, ds$ where S is the closed surface

Sol:

Given $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

W.K.T Gauss's divergence theorem is

$$\iint_S \bar{r}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\bar{r}\, dV \quad \text{—} \ (1).$$

now,

$$\iiint_V \nabla\cdot\bar{r}\, dV = \iiint \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x\hat{i} + y\hat{j} + z\hat{k})\, dV$$

$$= \iiint \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right) dV$$

$$= \iiint (1+1+1)\, dV$$

$$= \iiint 3\, dV$$

$$\iiint_V \nabla\cdot\bar{r}\, dV = 3V \quad \text{—} \ (2)$$

$$\iint_S \bar{r}\cdot\hat{n}\, ds = 3V \quad (\text{from } (1)).$$

(2) S.T $\iint_S \text{curl } \bar{F}\cdot\hat{n}\, ds = \iint_S \nabla\cdot(\nabla\times\bar{F})\, dV = 0$

$(\nabla\times\bar{F})$ (or)

<u>Sol</u>: Gauss's divergence theorem is,

$$\iint_S F\cdot\hat{n}\, ds = \iiint_V \nabla\cdot F\, dV.$$

To prove $\iint_S (\nabla\times F)\hat{n}\, ds = \iint_S \nabla\cdot(\nabla\times F)\, dV$ — (1)

$$\iint_S (\nabla\times F)\hat{n}\, ds = \iiint_V \nabla\cdot(\nabla\times F)\, dV.$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\nabla \times \bar{F} = \left\{ \hat{i}\left(\partial/\partial y\,(f_3) - \frac{\partial}{\partial z}(f_2) \right) - \hat{j}\left(\partial/\partial x\,(f_3) - \partial/\partial z\,(f_1) \right) + \hat{k}\left(\partial/\partial x\,(f_2) - \partial/\partial y\,(f_1) \right) \right\}$$

$$\nabla \times \bar{F} = 0$$

$$\nabla \cdot (\nabla \times \bar{F}) = 0.$$

$$\therefore \cancel{\iint_S \nabla \cdot (\nabla \times \bar{F})\hat{n}\,ds = \iiint}$$

$$\iiint_V \nabla \cdot (\nabla \times \bar{F})\,dv = 0.$$

$$\iint_S (\nabla \times \bar{F})\hat{n}\,ds = \iiint_V \nabla \cdot (\nabla \times F)\,dv$$

$$\therefore \iint_S (\nabla \times \bar{F})\hat{n}\,ds = 0.$$

$$\therefore \iint_S (\nabla \times \bar{F})\hat{n}\,ds = \iiint_V \nabla \cdot (\nabla \times \bar{F})\,dv = 0.$$

Hence proved

Gauss's divergence theorem

3) Using G.D.T evaluate : $\iint_S \nabla \cdot r^2 \, \hat{n} \, ds$.

Sol: W.K.T G.D.T is,

$$\iint_S \bar{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv.$$

To prove,

$$\iint (\nabla \cdot r^2) \hat{n} \, ds = \iiint \nabla \cdot (\nabla \cdot r^2) \, dv \quad \text{—} ①$$

$$\nabla \cdot r^2 = \left(\partial/\partial x \, \hat{i} + \partial/\partial y \, \hat{j} + \partial/\partial z \, \hat{k} \right)$$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2+y^2+z^2}$

$r = (x^2+y^2+z^2)$

$r^2 = (x^2+y^2+z^2)$

$$r^2 \left(x^2 \hat{i} + y^2 \hat{j} + z^2 \hat{k} \right)$$

$$= \partial/\partial x \, (x^2)\hat{i} + \partial/\partial y \, (y^2)\hat{j} + \partial/\partial z \, (z^2)\hat{k}$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla \cdot (\nabla r^2) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \partial/\partial z \, \hat{k} \right)$$

$$(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$= \partial/\partial x \, (2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z)$$

$$= 2 + 2 + 2$$

$$= 6.$$

$$\iiint_V \nabla \cdot (\nabla \cdot r^2) \, dv = \iiint_V 6 \, dv$$

$$= 6v$$

$$\therefore \iint_S (\nabla \cdot r^2) \hat{n} \, ds = 6v \quad (\text{from } ①)$$

$$\iint_S (\nabla \cdot r^2)\hat{n}\, ds = \iiint \nabla\cdot(\nabla \cdot r^2)\, dv$$

Hence proved.

4) $\iint_S \bar{F}\cdot\hat{n}\, ds$ where, $\bar{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and S is the surface of the sphere having the centre $(3,-1,2)$ and the radius 3.

Sol: W.K.T G.D.T is,

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\bar{F}\, dv.$$

Given $\bar{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$.

To find, $\iiint_V \nabla\cdot\bar{F}\, dv$.

$$\nabla\cdot\bar{F} = \left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left((2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}\right)\right\}$$

$$= \left\{\frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}(-(xz+y)) + \frac{\partial}{\partial z}(y^2+2z)\right\}$$

$$= 2 - 1 + 2$$

$$\boxed{\nabla\cdot\bar{F} = 3}$$

$$\iiint_V \nabla\cdot \bar{F}\, dv = \iiint_V 3\, dv$$

$$= 3V.$$

Gnf S is the ~~Sphere~~ surface of the sphere with radius 3.

V = volume of the sphere.

$$= \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi (3)^3$$

$$V = 36\pi$$

$$\iiint_V \nabla\cdot \bar{F}\, dv = 3(36)\pi$$

$$= 108\pi.$$

$$\iiint_V \nabla\cdot \bar{F}\, dv = 108\pi.$$

$$\iint_S F\cdot \hat{n}\, ds = \iiint_V \nabla\cdot \bar{F}\, dv$$

5) P.T $\iint_S \frac{\bar{r}}{r^2}\cdot \hat{n}\, ds = \iiint_V \frac{dv}{r^2}$

Sol: W.K.T G.D.T is

$$\iint_S F\cdot \hat{n}\, ds = \iiint_V (\nabla\cdot F)\, dv.$$

$$\bar{F} = \frac{\bar{r}}{r^2} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right).$$

$$= \frac{(x^2+y^2+z^2) - x(2x)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)(1) - y(2y)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)(1) - z(2z)}{(x^2+y^2+z^2)^2}$$

$$\nabla\cdot\bar{F} = \frac{3(x^2+y^2+z^2) - 2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{(x^2+y^2+z^2)^1}{(x^2+y^2+z^2)^2} = \frac{1}{(x^2+y^2+z^2)^2\,(x^2+y^2+z^2)^{-1}}$$

$$= \frac{1}{(x^2+y^2+z^2)} = \frac{1}{r^2}$$

$$\iiint_V \nabla\cdot\bar{F}\, dv = \iiint_V \frac{1}{r^2}\, dV$$

$$= \iiint_V \frac{dv}{r^2}$$

$$\therefore \iint_S \frac{\bar{r}}{r^2}\, \hat{n}\, ds = \iiint_V \frac{dv}{r^2}$$