Note-4:

$\langle u, 0\rangle = \langle 0, u\rangle = 0$.

For $\langle u, 0\rangle = \langle u, 0 0\rangle = 0\langle u, 0\rangle = 0$.

Similarly $\langle 0, u\rangle = 0$.

Examples:

5. $V_n(R)$ is a real inner product space with inner product defined by

$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ where

$x = (x_1, x_2 \dots x_n)$ and $y = (y_1, y_2 \dots y_n)$

This is called the Standard inner product on $V_n(R)$.

Proof:

Let $x, y, z \in V_n(R)$ and $\alpha \in R$.

(i) $\langle x+y, z\rangle = (x_1+y_1)z_1 + (x_2+y_2)z_2 + \dots + (x_n+y_n)z_n$

$= (x_1 z_1 + x_2 z_2 + \dots + x_n z_n) +$

$(y_1 z_1 + y_2 z_2 + \dots + y_n z_n)$

$= \langle x, z\rangle + \langle y, z\rangle$.

(ii) $\langle \alpha x, y\rangle = \alpha x_1 y_1 + \alpha x_2 y_2 + \dots + \alpha x_n y_n$

$= \alpha(x_1 y_1 + x_2 y_2 + \dots + x_n y_n)$

$= \alpha\langle x, y\rangle$.

(iii) $\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$.

$= y_1 x_1 + y_2 x_2 + \dots + y_n x_n$

$= \langle y, x\rangle$

(iv) $\langle x, x\rangle = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$ and

$\langle x, x\rangle = 0$ iff $x_1 = x_2 = \dots = x_n = 0$.

$\therefore \langle x, x \rangle = 0$ iff $x = 0$.

(b) $V_n(\mathbb{C})$ is a Complex Inner product space with Inner product defined by.

$\langle x, y \rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$ where

$x = (x_1, x_2 \dots x_n)$ and $y = (y_1, y_2 \dots y_n)$.

Proof :-

Let $x, y, z \in V_n(\mathbb{C})$ and $\alpha \in \mathbb{C}$

(i) $\langle x+y, z \rangle = (x_1+y_1)\bar{z}_1 + (x_2+y_2)\bar{z}_2 + \cdots + (x_n+y_n)\bar{z}_n$

$= (x_1\bar{z}_1 + x_2\bar{z}_2 + \cdots + x_n\bar{z}_n) + (y_1\bar{z}_1 + y_2\bar{z}_2 + \cdots + y_n\bar{z}_n)$

$= \langle x, z \rangle + \langle y, z \rangle$.

(ii) $\langle \alpha x, y \rangle = \alpha x_1\bar{y}_1 + \alpha x_2\bar{y}_2 + \cdots + \alpha x_n\bar{y}_n$

$= \alpha(x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n)$.

$= \alpha\langle x, y \rangle$.

(iii) $\overline{\langle y, x \rangle} = \overline{y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n}$

$= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$

$= x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$

$= \langle x, y \rangle$

(iv) $\langle x, x \rangle = x_1\bar{x}_1 + \cdots + x_n\bar{x}_n$

$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$

and $\langle x, x \rangle = 0$ iff $x = 0$.

(c) Let V be the set of all continuous real valued functio[ns] defined on the closed interval $[0, 1]$. V is a real inner product space with inner product defined by

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt.$$

Proof:

Let $f, g, h \in V$ and $\alpha \in \mathbb{R}$.

(i) $\langle f+g, h\rangle = \int_0^1 [f(t)+g(t)]h(t)\,dt$

$= \int_0^1 f(t)h(t)\,dt + \int_0^1 g(t)h(t)\,dt.$

$= \langle f, h\rangle + \langle g, h\rangle.$

(ii) $\langle \alpha f, g\rangle = \int_0^1 \alpha f(t)g(t)\,dt = \alpha \int_0^1 f(t)g(t)\,dt.$

$= \alpha\langle f, g\rangle$

(iii) $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt = \int_0^1 g(t)f(t)\,dt.$

$= \langle g, f\rangle.$

(iv) $\langle f, f\rangle = \int_0^1 [f(t)]^2\,dt \geq 0$ and

$\langle f, f\rangle = 0$ iff $f = 0$.

Def:

Let V be an inner product space and let $x \in V$. Then norm or length of x, denoted by $\|x\|$, is defined by $\|x\| = \sqrt{\langle x, x\rangle}$.

x is called a unit vector if $\|x\| = 1$.

8) Let V be the vector space of polynomials with inner product given by $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

Let $f(t) = t+2$ and $g(t) = t^2 - 2t - 3$

(ii) $\|f\|^2 = \langle f, f \rangle$

$$= \int_0^1 [f(t)]^2 \, dt = \int_0^1 (t+2)^2 \, dt$$

$$= \int_0^1 (t^2 + 4t + 4) \, dt = \left[\frac{t^3}{3} + 2t^2 + 4t \right]_0^1$$

$$= \frac{1}{3} + 2 + 4 = \frac{19}{3}$$

Proof :-

(i) $\|x\| = \sqrt{\langle x, x\rangle} \geq 0$ and $\|x\| = 0$ iff $x = 0$.

(ii) $\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle$

$= \alpha \langle x, \alpha x\rangle = \alpha\bar{\alpha}\langle x, x\rangle$.

$= |\alpha|^2 \|x\|^2$.

Hence $\|\alpha x\| = |\alpha| \, \|x\|$.

(iii) The inequality is trivially true when $x=0$ or $y=0$.
Hence let $x \neq 0$ and $y \neq 0$.

Consider $z = y - \dfrac{\langle y, x\rangle}{\|x\|^2} x$ Then $0 \leq \langle z, z\rangle$.

$$= \left\langle y - \frac{\langle y, x\rangle}{\|x\|^2} x, \; y - \frac{\langle y, x\rangle}{\|x\|^2} x \right\rangle.$$

$$= \langle y, y\rangle - \frac{\overline{\langle y, x\rangle}}{\|x\|^2} \langle y, x\rangle - \frac{\langle y, x\rangle}{\|x\|^2} \langle x, y\rangle + \frac{\langle y, x\rangle \overline{\langle y, x\rangle}}{\|x\|^2 \|x\|^2} \langle x, x\rangle$$

$$= \|y^2\| - \frac{\overline{\langle y, x\rangle} \langle y, x\rangle}{\|x\|^2} - \frac{\langle y, x\rangle \langle x, y\rangle}{\|x\|^2} + \frac{\langle y, x\rangle \overline{\langle y, x\rangle}}{\|x\|^2}.$$

$$= \|y^2\| - \frac{\overline{\langle x, y\rangle} \langle x, y\rangle}{\|x\|^2}.$$

$\therefore \; 0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2$.

$\therefore \; |\langle x, y\rangle| \leq \|x\| \|y\|$.

(iv) $\|x+y\|^2 = \langle x+y, x+y\rangle$.

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$.

$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$

$= (1, 2, -1) - (0, 2, 0)$

$= (1, 0, -1)$

$\therefore \|w_3\|^2 = 2$

$\therefore$ The Orthogonal basis is

$\{(1,0,1), (0,3,0), (1,0,-1)\}$.

Hence the Orthogonal basis is

$$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], (0,1,0), \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right] \right\}.$$

4. Problem:

Let V be the set of all polynomials of degree ≤ together with the zero polynomial. V is a real inner product space with inner product defined by $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$. Starting with the basis $\{1, x,$

Obtain an Orthonormal basis for V.

Soln:

Let $v_1 = 1$, $v_2 = x$ and $v_3 = x^2$.

Let $w_1 = v_1$

Then $\|w_1\|^2 = \langle w_1, w_1 \rangle = \int_{-1}^{1} 1\, dx = 2$.

Hence $\|w_1\| = \sqrt{2}$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$= x - \frac{1}{2}\int_{-1}^{1} x\,dx = x$

$\therefore \|w_2\|^2 = \langle w_2, w_2\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$.

Now, $w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2$

$= x^2 - \frac{1}{2}\int_{-1}^{1} x^2\,dx - \left[\frac{3x}{2}\right]\int_{-1}^{1} x^3\,dx$.

$= x^2 - \frac{1}{3}$

$\|w_3\|^2 = \langle w_3, w_3\rangle = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1}\left(x^4 + \frac{1}{9} - \frac{2}{3}x^2\right) dx$.

$= \left(\frac{x^5}{5} + \frac{1}{9}x - \frac{2}{9}x^3\right)_{-1}^{1} = \left(\frac{1}{5} + \frac{1}{9} - \frac{2}{9} + \frac{1}{5} + \frac{1}{9} - \frac{2}{9}\right)$.

$= \frac{1}{5} - \frac{1}{9} + \frac{1}{5} - \frac{1}{9} = \frac{2}{5} - \frac{2}{9} = \frac{18-10}{45}$.

$= \frac{8}{45}$

Hence the orthogonal basis is $\{1, x, x^2 - \frac{1}{3}\}$.

$\therefore$ The required orthonormal basis is

$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{\sqrt{10}}{4}(3x^2 - 1)\right\}$.

15. Problem:

Find a vector of unit length which is orthogonal to $(1,3,4)$ in $V_3(\mathbb{R})$ with standard inner product.

Soln:

Let $x = (x_1, x_2, x_3)$ be any vector orthogonal to $(1,3,4)$. Then $x_1 + 3x_2 + 4x_3 = 0$. Any solution of this

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(ii) $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$

(iii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$, where $\overline{\langle v, u\rangle}$ is the complex conjugate of $\langle u, v\rangle$

(iv) $\langle u, v\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

A Vector space with an inner product defined on it is called an inner product space. An inner product space is called an Euclidean space or unitary space according as F is the field of real number or complex number.

Note - 1 :

If F is the field of real numbers then condition (iii) takes the form $\langle u, v\rangle = \langle v, u\rangle$. Further (iii) asserts that $\langle u, u\rangle$ is always real and hence (iv) is meaningful whether F is the field of real or complex numbers.

Note - 2 :

$$\langle u, \alpha v\rangle = \bar{\alpha}\langle u, v\rangle$$

For $\langle u, \alpha v\rangle = \overline{\langle \alpha v, u\rangle}$

$= \overline{\alpha \langle v, u\rangle}$

$= \bar{\alpha}\,\overline{\langle v, u\rangle}$

$= \bar{\alpha}\langle u, v\rangle$

Note - 3 :

$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$$

For $\langle u, v + w\rangle = \overline{\langle v + w, u\rangle}$

$= \overline{\langle v, u\rangle + \langle w, u\rangle}$

$= \overline{\langle v, u\rangle} + \overline{\langle w, u\rangle}$

$= \langle u, v\rangle + \langle u, w\rangle$

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Thus Continuing in this way we will ultimately obtain a non-zero orthogonal set $\{w_1, w_2 \ldots w_n\}$.

By Theorem 10. this set is linearly independent set and hence a basis.

To obtain an orthonormal basis we replace each w_i by $\frac{w_i}{\|w_i\|}$

13) Problem:

Apply Gram - Schmidt process to construct an orthonormal basis for $V_3(R)$ with the Standard Inner product for the basis $\{v_1, v_2, v_3\}$ where $v_1 = (1,0,1)$, $v_2 = (1,3,1)$ and $v_3 = (3,2,1)$.

Soln.

Take $w_1 = v_1 = (1,0,1)$

Then $\|w_1\|^2 = \langle w_1, w_1 \rangle = 1^2+0^2+1^2 = 1+1 = 2$.

and $\langle v_3, w_1 \rangle = 1+0+1 = 2$.

Put $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$.

$= (1,3,1) - (1,0,1)$.

$= (0,3,0)$

$\therefore \|w_2\|^2 = 9$.

Also $\langle w_2, w_3 \rangle = 0+6+0 = 6$ and $\langle w_1, w_3 \rangle = 3+0+1 = 4$

Now, $w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$

$= (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9}(0,3,0)$

$= (3,2,1) - 2(1,0,1) - \frac{2}{3}(0,3,0)$

$= (3,2,1) - (2,0,2) - (0,2,0)$

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Let $T_1, T_2 \in L(V,W)$ and $M(T_1) = (a_{ij})$ and $M(T_2) = (b_{ij})$

$$M(T_1) = (a_{ij}) \Rightarrow T_1(v_i) = \sum_{j=1}^{n} a_{ij} w_j .$$

$$M(T_2) = (b_{ij}) \Rightarrow T_2(v_i) = \sum_{j=1}^{n} b_{ij} w_j .$$

$$\therefore (T_1+T_2)(v_i) = \sum_{j=1}^{n} (a_{ij}+b_{ij}) w_j .$$

$$\therefore M(T_1+T_2) = (a_{ij}+b_{ij})$$
$$= (a_{ij}) + (b_{ij})$$
$$= M(T_1) + M(T_2) .$$

Similarly $M(\alpha T_1) = \alpha M(T_1)$.

Hence M is the required isomorphism from

i.e) $L(V,W) \cong M_{m \times n}(F)$

$\therefore \dim L(V,W) = mn$.

INNER PRODUCT SPACES:

Def: INNER PRODUCT:

Let V be a vector space over F. An inner product on V is a function which assigns to each ordered pair of vectors u, v in V a scalar in F denoted by $\langle u, v \rangle$ satisfying the following conditions

(i) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

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For example (7, -5, 2) is one such vector (by cross multiplication method)

Hence {(1, 3, 4), (1, 1, -1), (7, -5, 2)} is an orthogonal basis containing (1, 3, 4)

ORTHOGONAL COMPLEMENT:

Def:

Let V be an inner product space. Let S be a subset of V. The orthogonal complement of S, denoted by $S^{\perp}$, is the set of all vectors in V which are orthogonal to every vector of S.

Examples:

1. $V^{\perp} = \{0\}$ and $\{0\}^{\perp} = V$ since 0 is the only vector which is orthogonal to every vector.

2) Let $S = \{(x, 0, 0) / x \in R\} \subseteq V_3(R)$ with standard inner product. Then

$S^{\perp} = \{(0, y, z) / y, z \in R\}$.

ie) The orthogonal complement of the x-axis is the yz plane.

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equation gives a vector orthogonal to (1, 3, 4).

For example $\alpha = (1, 1, -1)$ is orthogonal to (1, 3, 4).

Also $\|\alpha\| = \sqrt{3}$. Hence a unit vector orthogonal to (1, 3, 4)

is given by $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$

Note

The set of all vectors orthogonal to (1, 3, 4) are the points lying on the plane $x + 3y + 4z = 0$, which is a two dimensional subspace of $V_3(R)$.

15) Problem:

Find an orthogonal basis containing the vector (1, 3, 4) for $V_3(R)$ with the standard inner product.

Soln:

(1, 1, -1) is a vector orthogonal to (1, 3, 4)

(refer problem 14 above).

Now let $y = (y_1, y_2, y_3)$ be a vector orthogonal to both (1, 3, 4) and (1, 1, -1)

Then $y_1 + 3y_2 + 4y_3 = 0$.

$y_1 + y_2 - y_3 = 0$.

Any solution of this system of equations gives a vector orthogonal to (1, 3, 4) and (1, 1, -1).

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For example $(7,-5,2)$ is one such vector (by cross multiplication method).

Hence $\{(1,2,4),(1,1,-1),(7,-5,2)\}$ is an orthogonal basis containing $(1,2,4)$

ORTHOGONAL COMPLEMENT:

Def:

Let V be an inner product space. Let S be a subset of V. The orthogonal complement of S, denoted by $S^{\perp}$, is the set of all vectors in V which are orthogonal to every vector of S.

Examples:

1. $V^{\perp}=\{0\}$ and $\{0\}^{\perp}=V$ since 0 is the only vector which is orthogonal to every vector.

2) Let $S=\{(x,0,0)/x\in R\}\subseteq V_3(R)$ with standard inner product. Then

$S^{\perp}=\{(0,y,z)/y,z\in R\}$.

(ie) The orthogonal complement of the x-axis is the yz plane.

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$= \|x\|^2 + 2 Re \langle x, y \rangle + \|y\|^2$.

$\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2$.

$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ (by (iii))

$\leq (\|x\| + \|y\|)^2$

$\therefore \|x+y\| \leq \|x\| + \|y\|$.

ORTHOGONALITY:

Def: ORTHOGONAL:

Let V be an inner product space and let $x, y \in V$. x is said to be orthogonal to y if $\langle x, y \rangle = 0$.

Note 1:

x is orthogonal to $y \Rightarrow \langle x, y \rangle = 0$.

$\Rightarrow \overline{\langle x, y \rangle} = \overline{0}$

$\Rightarrow \langle y, x \rangle = 0$.

$\Rightarrow$ y is orthogonal to x.

Thus x and y are orthogonal iff $\langle x, y \rangle = 0$.

Note 2:

x is orthogonal to $y \Rightarrow \alpha x$ is orthogonal to y.

Note-3:

x_1 and x_2 are orthogonal to $y \Rightarrow x_1 + x_2$ is orthogonal

Note-4:

0 is orthogonal to every vector in V and is the vector with is property.

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Def: Orthogonal Set:

Let V be an inner product space. A set S of vector in V is said to be an orthogonal set if any two distinct vectors in S are orthogonal.

Def: ORTHONORMAL SET:

S is said to be an orthonormal set if S is orthogonal and $\|x\| = 1 \;\forall\; x \in S$.

Example:

The standard basis $\{e_1, e_2 \dots e_n\}$ in R^n or C^n is an orthogonal set with respect to the standard inner product.

(o) Theorem:

Let $S = \{v_1, v_2 \dots v_n\}$ be an orthogonal set of non zero vectors in an inner product space V. Then S is linearly independent.

Proof:

Let $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$

Then $\langle \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, v_1 \rangle = \langle 0, v_1 \rangle = 0$

$\therefore \alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \langle v_2, v_1 \rangle + \dots + \alpha_n \langle v_n, v_1 \rangle = 0$.

$\therefore \alpha_1 \langle v_1, v_1 \rangle = 0$ ($\because S$ is orthogonal).

$\therefore \alpha_1 = 0$ ($\because v_1 \neq 0$).

Illy $\alpha_2 = \alpha_3 \dots = \alpha_n = 0$.

Hence S is linearly independent.

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11. Theorem:

Let $S = \{v_1, v_2 \dots v_n\}$ be an orthogonal set of non-zero vectors in V. Let $v \in V$ and $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$. Then $\alpha_k = \frac{\langle v, v_k \rangle}{\|v_k\|^2}$

Proof:

$\langle v, v_k \rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, v_k \rangle$.
$= \alpha_1 \langle v_1, v_k \rangle + \alpha_2 \langle v_2, v_k \rangle + \dots + \alpha_n \langle v_n, v_k \rangle$.
$= \alpha_k \langle v_k, v_k \rangle$ ($\because S$ is orthogonal).
$= \alpha_k \|v_k\|^2$

$\therefore \alpha_k = \frac{\langle v, v_k \rangle}{\|v_k\|^2}$.

12. Theorem:

Every finite dimensional inner product space has an orthogonal basis.

Proof:

Let V be an finite dimensional inner product space. Let $\{v_1, v_2 \dots v_n\}$ be a basis for V. From this basis we shall construct an orthonormal basis $\{w_1, w_2 \dots w_n\}$ by means of a construction known as Gram-Schmidt orthogonalisation process.

First we take $w_1 = v_1$

Let $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$.

We claim that $w_2 \neq 0$. For if $w_2 = 0$ then v_2 is a scalar multiple of w_1 and hence of v_1 which is a contradiction since v_1, v_2 are linearly independent.

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Also, $\langle w_2, w_1 \rangle = \langle v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1, w_1 \rangle$.

$= \langle v_2 - \frac{\langle v_2, w_1 \rangle}{\|v_1\|^2} v_1, v_1 \rangle$ $(\because w_1 = v_1)$.

$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_1 \rangle$.

$= \langle v_2, v_1 \rangle - \langle v_2, v_1 \rangle$.

$= 0$.

Now suppose that we have constructed non-zero, orthogonal vectors $w_1, w_2 \dots w_k$. Then put,

$$w_{k+1} = v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} w_j .$$

We claimed that $w_{k+1} \neq 0$. For if $w_{k+1} = 0$, then v_{k+1} is a linear combination of $w_1, w_2 \dots w_k$ and hence is a linear combination of $v_1, v_2 \dots v_k$ which is a contradiction since $v_1, v_2 \dots v_{k+1}$ are linearly independent.

Also,

$$\langle w_{k+1}, w_i \rangle = \langle v_{k+1}, w_i \rangle - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} \langle w_j, w_i \rangle$$

$= \langle v_{k+1}, w_i \rangle - \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} \langle w_i, w_i \rangle$.

$= \langle v_{k+1}, w_i \rangle - \langle v_{k+1}, w_i \rangle$.

$= 0$.