Phys-453 Nuclear & Particle Physics

Handwritten Lecture Notes

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This is the set of lecture notes I prepared for teaching this undergraduate-level course at Bilkent University. None of the material here is original. Especially the theoretical part (second 2/3) closely follows Griffiths IEPP.

List of references (quasi-complete, in order of contribution):

- D. Griffiths, "Introduction to Elementary Particle Physics", 2nd Edition, Wiley, 2008.
- Wikipedia, especially for the illustrations, as noted in the text.
- H. D. Young and R. A. Freedman, "University Physics", Vol. III, 13th Edition, Addison Wesley, 2012.
- R. Eisberg and R. Resnick, "Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles", 2nd Edition, Wiley, 1985.
- K. S. Krane, "Introductory Nuclear Physics", Wiley, 1988.

Contents

Nuclear & Particle Physics

Length Scales

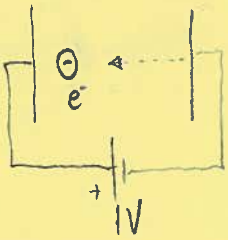
Atom — $\sim 1\,\mathring{A} = 10^{-10}\,m$

Nucleus — $\sim 10F = 10^{-14}\,m$

proton — $\sim 1F \equiv 10^{-15}\,m$

quark, $\bar{e}$ — $\leqslant 10^{-18}\,m$ (currently, point particles)

Energy Scales



Considering two conductors connected to a 1V battery, the energy gained by one $\bar{e}$ taken from anode (+ve plate) and placed to cathode is $1eV$.

↑ high potential for $\bar{e}$s

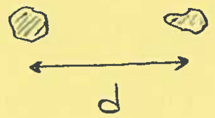
Atomic/Molecular/Solid State Physics $\sim$ eV

Nuclear Physics : keV

Particle Physics : MeV – GeV – TeV $\rightarrow \infty$

So, this realm of subatomic physics is also called the high energy physics. We need higher and higher energy accelerators, colliders to probe into smaller and smaller particles. Why?

Consider two pieces separated by a distance d

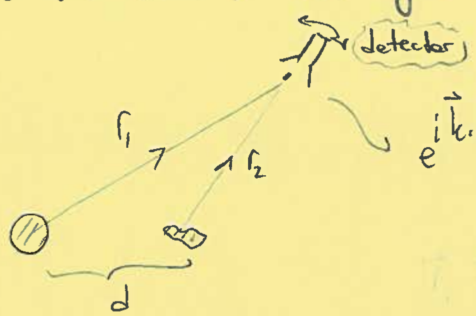


If you send a wave (EM or matter wave) of a wave length $\lambda \gg d$

$e^{-/+}$

then looking at the scattered wave, you cannot get a clue whether the scatterer is a single piece or multipieces.

However, if you decrease the wavelength to $\lambda \simeq d$, then by scanning over the solid $\measuredangle$ $d\Omega$, you will get an interference pattern



$$e^{i\vec{k}\cdot\vec{r}_1} + e^{i\vec{k}\cdot\vec{r}_2} = e^{i\vec{k}\cdot\vec{r}_1}\left[1 + e^{i\vec{k}\cdot(\vec{r}_2-\vec{r}_1)}\right]$$

$\frac{2\pi}{\lambda}$, $|\vec{r}_2-\vec{r}_1| \simeq d$

$$\theta = |\vec{k}\cdot(\vec{r}_2-\vec{r}_1)| \simeq 2\pi\frac{d}{\lambda}$$

Note that when $\lambda \gg d$, the phase difference is so tiny.

But when $\lambda \simeq d$ we can easily resolve from the resulting pattern that there actually two pieces separated by d.

Recall that $E = h\nu = \hbar c k = 2\pi \hbar c/\lambda$; so as $\lambda \to 0$, $E \to \infty$

<u>Lesson Learned</u>: Sending a higher energy probe (e^-, e^+, γ) we can resolve finer details in the "target".

More on Units

Particle physics - theoreticians often work in "natural" units where $\hbar = c = 1$. But following our text (Griffiths) we avoid this.

From $E = mc^2$, we shall quote masses in units of MeV/c^2 or GeV/c^2.

Likewise, momenta can be stated in MeV/c or GeV/c.

When it is obvious the c and c^2 terms in the units can also be suppressed, like a proton has a (rest) mass of 938.3 MeV

Standard Model

Current experimental status of particle physics is extremely well described by the so-called Standard Model. It is a combination of electroweak and strong forces. According to SM, the following particles are <u>fundamental</u>:

Fermions (spin 1/2 particles) Make up the Matter

	1st Gen.	2^{nd} Gen.	3^{rd} Generation		
	0.511 MeV	106 MeV	1777 MeV		
Leptons	e	μ	τ	$\rightarrow Q = -1$	
	ν_e	ν_μ	ν_τ	$\rightarrow Q = 0$	almost negligible masses
	~150 MeV	~1.5 GeV	~175 GeV		
Quarks	u	c	t	$\rightarrow Q = +2/3$	
	d	s	b	$\rightarrow Q = -1/3$	
	~150 MeV	~300 MeV	~4.5 GeV		

Bosons (spin-1 particles) Carry the forces

- γ — massless (as of now)
- $W^\pm, Z$ — $M_W = 80.42$ GeV, $M_Z = 91.2$ GeV
- g — massless (as of now)

the Higgs mechanism is believed to give rise to these non-zero masses

Antiparticles

One of the triumps of the Dirac equation (relativistic Schrödinger eqn. for spin-1/2 particles) is that it predicts the existence of antiparticles. So, each particle in the SM has an antiparticle. For some (like photon), its antiparticle is itself.

Mesons ($q\bar{q}$)

A quark and its antiparticle form nine possible pairs, known as mesons

$u\bar{u}$, $\underset{\pi^+,\,\rho^+}{u\bar{d}}$, $u\bar{s}$; $\underset{\pi^-,\,\rho^-}{d\bar{u}}$, $d\bar{d}$, $d\bar{s}$; $s\bar{u}$, $s\bar{d}$, $s\bar{s}$

These are bosons as they have integer spins.

There are other spin combinations like the singlet $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \rightarrow \pi^0$ (pion-zero)

There is actually a huge list of mesons that utilize s, b, t quarks.

Baryons (qqq)

These are fermions with spin 1/2 or 3/2

p: uud $\rightarrow$ 938.3 MeV/c^2

n: udd $\rightarrow$ 939.6 MeV/c^2

$$\left.\begin{array}{l} \Delta^{++}: uuu \\ \Delta^{+}: uud \\ \Delta^{-}: udd \\ \Delta^{--}: ddd \end{array}\right\} \quad 1232 \text{ MeV}/c^2$$

Fundamental Forces

The forces in the SM are carried by again (bosonic) particles.

Note that the gravitational force is not included in the SM, one of the most important flaws of SM. If we rank the four known forces in nature:

Force	Strength	Mediator	
Strong	10	Gluon, g	← Perturbation does not work!
EM	10^{-2}	Photon, γ	
Weak	10^{-13}	Intermediate Vector Bosons (Gauge), $W^{\pm}$, Z^{0}	
Gravitational	10^{-42}	Graviton	← No quantum theory available (superstring theory working on it)

* The particles with electric charge (other than neutrinos, gluons, Z^0) interact with EM force

* All particles in SM other than photon and gluons interact with weak force

* All quarks interact with strong force

Since photons carry no charge they do not interact directly with each other. However, both gluons (carrying color-anticolor) and gauge bosons (carrying weak charge) interact with one another!

[More on these, when revisit this subject in the 'Popular Part' of the Course]

Nuclear Physics – A Qualitative Treatment

NB: In this part of the course, we shall favor breadth over depth!

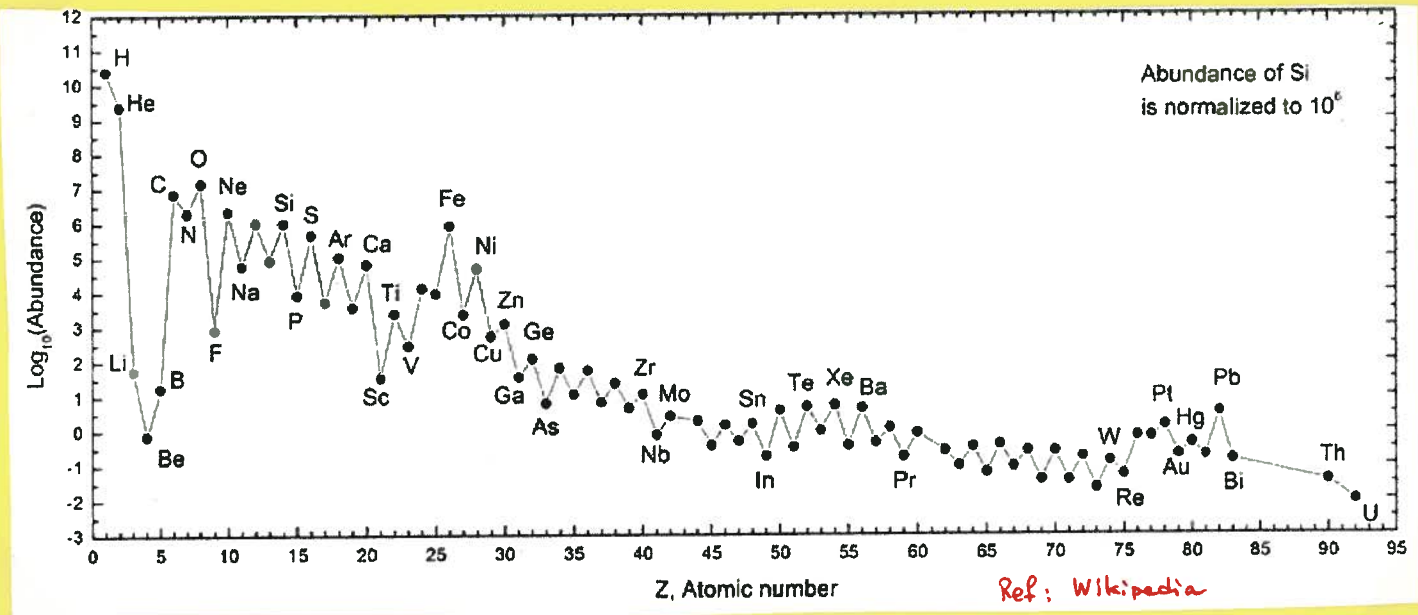
General Setting

The characteristic energy scale in nuclear physics is of the order of MeV. Compare this with that of atoms which is of the order of 1 eV, hence for the atoms, at room temperature ($kT = 25\,meV$) they can be easily excited, and they have little difficulty in combining to form molecules and solids.

For nuclei, the situation is quite different. Quoting from Weisskopf:

"In our immediate environment atomic nuclei exist only in their ground state; they affect the world in which we live only by their static properties charge, mass and not by their intricate dynamic properties. In fact, all interesting nuclear phenomena come into play only under conditions which we have created ourselves in accelerating machines. It is to some extent a man-made world.

It is not completely man made, however. The centers of all stars are regions of the universe where nuclear reactions go on, and thus where nuclear dynamics plays an essential role. Here, the nuclear phenomena are the basis of our energy supply on Earth, in reactors as well as in the Sun. But nuclear physics is even more important for the world in which we live from the point of view of the history of the universe. The composition of matter as we see it today is the product of nuclear reactions which have taken place a long time ago in the stars or in star explosions, where conditions prevailed $[kT \gtrsim MeV]$ which we simulate in a very microscopic way within our accelerating machines. I cannot better illustrate the interconnection of all facts of nature, the tightly woven net of the laws of physics, than pointing to the chart of abundances of elements in our part of the universe. Each maximum and minimum in the curve of abundances corresponds to some trait of nuclear dynamics, here a closed shell, there a strong neutron cross section, or a low binding energy. If the 7.65 MeV resonance in carbon did not exist then practically no carbon would have formed and we would probably not have evolved to contemplate these problems. Whenever we probe nature – be it studying the str. of nuclei, or by learning about macromolecules, or about elementary particles, or about the structure of solids – we always get some essential part of this universe."



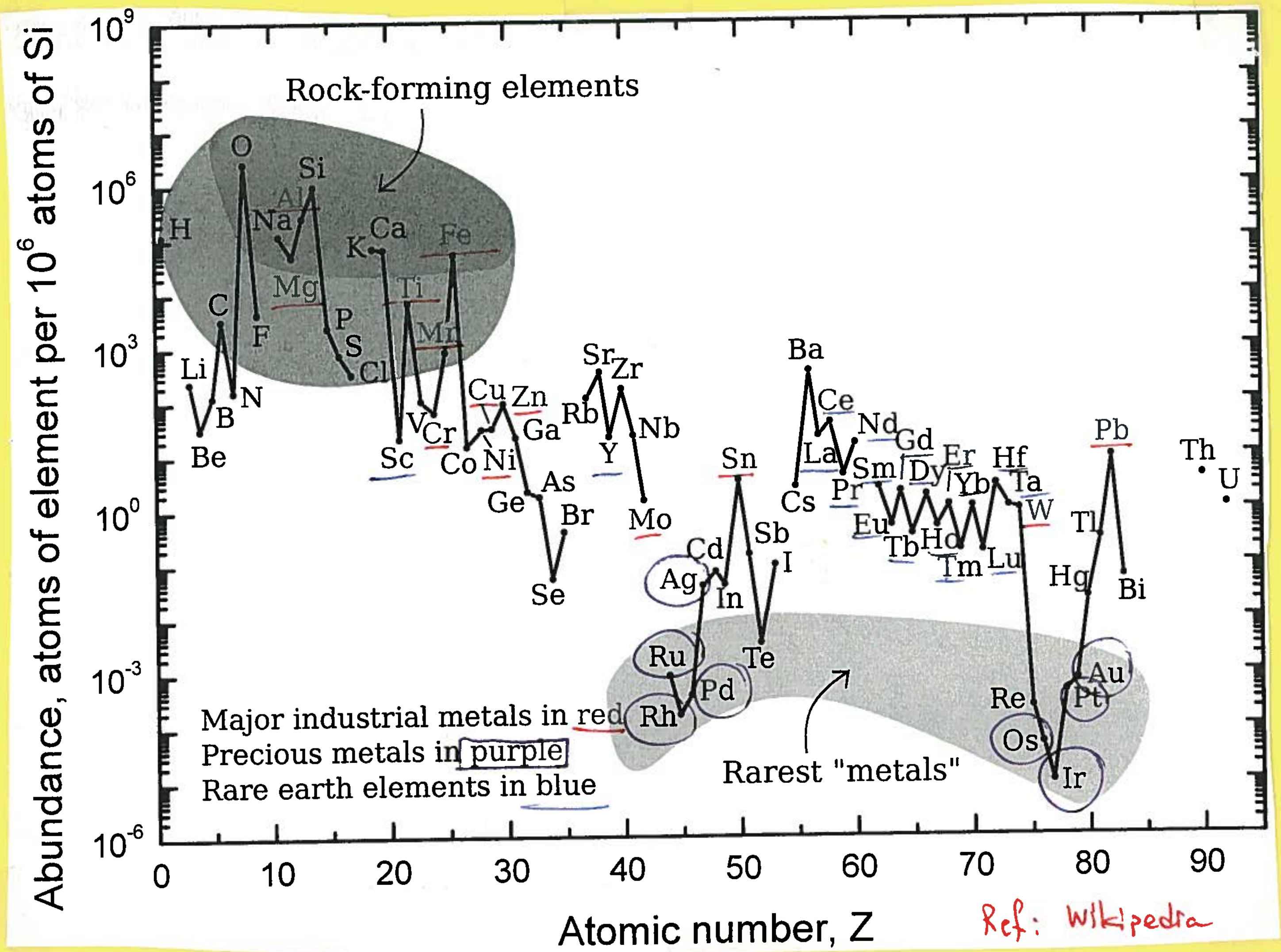
Ref: Wikipedia

Estimated abundance of chemical elements in our Solar system

Observe that H & He are most common, from the Big Bang. The next three elements (Li, Be, B) are rare because they are poorly synthesized in the Big Bang and in the stars. The two general trends in the remaining stellar-produced elements are:
(1) an alternation of abundance in elements as they have even or odd atomic numbers, and (2) a general decrease in abundance, as elements become heavier.

Earth's Crustal Elemental Abundance

O and Si are quite common elements, frequently combined with each other to form common silicate minerals. The Rare-Earths are not rare, this is a misnomer



Properties of Nuclei

Static Properties : Electric charge, radius, mass, binding energy, angular momentum, parity, magnetic dipole moment, electric quadrupole moment, energies of excited states.

Dynamic Properties : Decay & reaction probabilities, scattering cross sections.

We shall start with static properties.

Nucleons : neutrons and protons

The force that binds nucleons together is by far stronger (within the nucleus) than the EM force. So, we can non-destructively probe the nuclear properties with EM interaction (i.e., we do not seriously distort the object - nucleus we are trying to measure.) According to many scattering experiments, we can model a nucleus as a sphere with radius R that depends on the total # nucleons = A ← nucleon number.

$$R = \underbrace{R_0}_{1.2\,\mathrm{fm}} A^{1/3}$$

1.2 fm ... an experimentally def'd const.

A is also called the mass number because it is the nearest integer number to the mass of the nucleus measured in unified atomic mass units (u).

$$1\,u = 1.660538782(83) \times 10^{-27}\,\mathrm{kg}$$

NB: All masses in this course refer to rest masses.

Nuclear Density: The volume (V) of a nucleus is proportional to A. Dividing A (the approximate mass in u) by V, $\rho = \frac{M}{V}$ cancels out A. Thus all nuclei have approximately the same density. This is of crucial importance in nuclear structure.

NB: $\rho_{nuclear} = \frac{A}{V} = 2.84 \cdot 10^{17}\ kg/m^3$ compare with $\rho_{water} = 10^3\ kg/m^3$ ← Normal matter is full of empty space!

Example: Comparing H with U

Uranium has mass number 238 and hydrogen has 1, accordingly their nuclear diameters reflect this $\sqrt[3]{A_U/A_H} \simeq 6.2$ ratio.

Actual values are
$$\left.\begin{array}{l} D(\text{H nucleus}) = 1.75\ fm \\ D(\text{U nucleus}) = 15\ fm \end{array}\right\} \text{ratio} : 8.57$$

Compare this with their atomic diameters.

$$\left.\begin{array}{l} D(\text{H atom}) \simeq 2.54\ \text{Å} \\ D(\text{U atom}) \simeq 3.45\ \text{Å} \end{array}\right\} \text{ratio} : 1.36$$

So, note that even though nuclear diameters are quite different, the two atoms' diameters are very close! This is because of the contrast bet. the behavior of nuclear vs. EM forces. The latter is long range and does not saturate; the 92 protons of U exert a huge Coulomb force on the $\bar{e}$'s, reducing the diameter close to that of H.

Nuclides & Isotopes

The atomic particles have the following masses:

$$m_p = 1.007276\ u = 1.672622 \cdot 10^{-27}\ kg = 938.3\ MeV/c^2$$

$$m_n = 1.008665\ u = 1.674927 \cdot 10^{-27}\ kg = 939.6\ MeV/c^2$$

$$m_e = 0.000548580\ u = 9.10938 \cdot 10^{-31}\ kg = 0.511\ MeV/c^2$$

Note that m_n is slightly heavier than m_p so that n can decay into a p in its rest frame.

Z : Atomic Number = # protons in the nucleus (a neutral atom has Z e^-'s)
responsible for the chemical properties

N : Neutron Number = # neutrons " " "

$$A = Z + N$$

A single nuclear species having specific values of both Z and N is called a **nuclide**.

Nuclides with same Z but different N are called **isotopes** of that element.

Different isotopes usually have slightly different physical properties such as melting and boiling temperatures and diffusion rates. The two common isotopes of uranium with A = 235 and 238 are usually separated industrially by taking advantage of different diffusion rates of gaseous uranium hexafluoride UF_6 containing the two isotopes.

Notation:

$$ {}^{A}_{Z}\mathrm{El}_{N} $$

sometimes omitted

Eg. The isotopes of chlorine with $A = 35$ and 37 are written

$^{35}_{17}Cl$ and $^{37}_{17}Cl$ and pronounced chlorine-35 and chlorine-37, resp.

It is important to note that atomic masses are <u>less</u> than sum of the masses of their parts. This mass defect is responsible for nuclear binding → More later.

NB: Isotopes of hydrogen have their own names:

$^{1}_{1}H_0$: hydrogen, $\underbrace{^{2}_{1}H_1 \text{: deuterium}}$, $^{3}_{1}H_2$: tritium (unstable)

also denoted as D as in heavy water D_2O

We shall leave the discussion of nuclear spins and magnetic moments to the subject of nuclear magnetic resonance.

Nuclear Binding & Nuclear Structure

Because energy must be added to a nucleus to separate it into its individual constituents, (n's, p's), the total rest mass (and energy) E_0 of the separated nucleons is greater than the rest mass (energy) of the nucleus. Thus the rest energy of the nucleus is $E_0 - E_B$

binding energy ($\frac{E_B}{c^2}$: Mass defect)

For a nucleus with Z: protons, N: neutrons

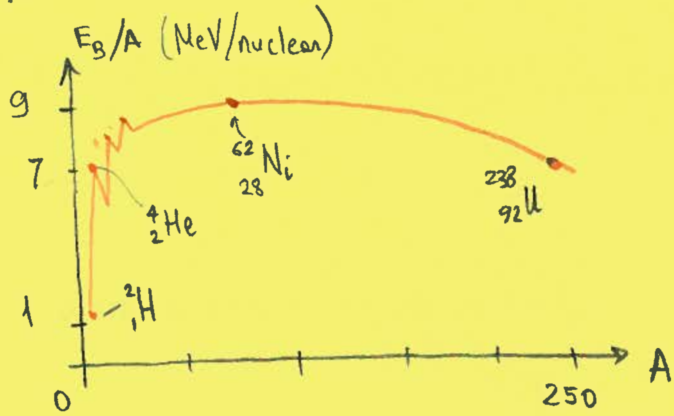
$$E_B = \left(Z M_H + N m_n - \underbrace{{}^{A}_{Z}M}_{\text{mass of neutral atom}}\right) c^2$$

Ex: The nucleus of deuterium ${}^{2}_{1}H$ is called deuteron: p n

$$E_B = \left(1.007825\,u + 1.008665\,u - 2.014102\,u\right) \frac{931.5\text{ MeV}}{u}$$

$$= 2.224 \text{ MeV}$$

$\frac{E_B}{A} = 1.112$ MeV ... binding energy per nucleon (a more universal parameter)



${}^{2}_{1}H$ has the lowest binding en. per nucleon of all nuclides, whereas nickel ${}^{62}_{28}Ni$ has the highest. α-particle also has quite high value!

Nuclear Force (No simple Coulomb's Law-like expression is currently available!)

The force that binds p's and n's together in the nucleus, despite the electrical repulsion of the p's is an example of the strong force.

Ex: Calculate the Coulomb energy bet. two p's separated by 1 fm.

$$E_C = e \cdot \frac{e}{4\pi\varepsilon_0 r} = \frac{\left(1.6\ 10^{-19}\right)^2}{\underbrace{4\pi \cdot 8.8}_{\simeq 100}\ 10^{-12}} \cdot 10^{15} \Big/ \underbrace{\left(1.6 \cdot 10^{-19}\right)}_{\text{to convert into eV}} \simeq 1.6\ \text{MeV}$$

Some Properties of the Nuclear Force:

1) It does not depend on the electric charge (same for n's & p's)

2) It has short range ~ a few fm. Otherwise the nucleus would grow by pulling in additional p's & n's. Within its range, nuclear force is much stronger than electrical forces; otherwise nucleus could be unstable

3) The nearly constant density of nuclear matter and the nearly const. E_B/A of larger nuclides show that a particular nucleon cannot interact simultaneously with all the other nucleons in a nucleus, but only with those few in its immediate vicinity. (This is in contrast with Coulomb force.) This limited number of interactions is called saturation (analogous to covalent bonding in molecules/solids).

4) Nuclear force favors binding of pairs of p's & n's with opposite spins and of pairs of pairs 

Hence, α-particle ^4_2He is an exceptionally stable nucleus for its mass number. (This pairing reminds the Cooper pairs in BCS theory)

Nuclear Models

In the nucleus three different kinds of force take part (EM, weak and strong forces). The combination of all three make the nuclear force quite complicated, yet to be fully understood. For this reason the analysis of nuclear structure is more complex than the analysis of many-electron atoms.

Several simple nuclear models exist which are of great help, each with different levels of success, in gaining some insight into nuclear structure.

Liquid-Drop Model

* First proposed by George Gamow in 1928, later expanded by N. Bohr.
* Based on the observation that all nuclei have nearly same density
* Quite successful in correlating nuclear masses, and understanding decay processes of unstable nuclides
* Other models are better suited for angular momentum and excited state properties.
* This is a phenomenological model which does not use a QM framework (unlike the Shell Model)

In the Liquid-Drop model, a nucleus with mass number A and atom number Z has the following form of binding energy.

$$E_B = \underbrace{C_1 A}_{\text{volume (saturation)}} - C_2 A^{2/3} - C_3 \frac{Z(Z-1)}{A^{1/3}} - C_4 \frac{(A-2Z)^2}{A} \pm C_5 A^{-4/3}$$

volume (saturation)

An individual nuclear only interacts with its nearest neighbors

$\Rightarrow E_B \propto A$

A: # nucleons

Surface

Nucleons on the surface are less tightly bound (no neighbors outside)

$-4\pi R^2$ term

$R \propto A^{1/3}$

$\Rightarrow -A^{2/3}$

Coulomb

every proton (Z of them) repels the others $(Z-1)$

$$-\frac{Z.(Z-1)}{R}$$

R: $A^{1/3}$

Balance

$$E_B \propto \frac{(N-Z)^2}{A}$$

... gives a good balance bet. the energies associated w/ n's & p's

pairing

Nuclear force favors pairing of p's & n's

Choose + sign: if both Z & N are even

Choose − sign: " " " " " odd

Set to zero: otherwise

Best fit is obtained with constants taken as:

$C_1 = 15.75$ MeV, $C_2 = 17.80$ MeV, $C_3 = 0.7100$ MeV, $C_4 = 23.69$ MeV, $C_5 = 39$ MeV

Combining this expression with $E_B = (Z M_H + N m_n - {}^A_Z M)c^2$ leads to

$${}^A_Z M = Z M_H + N m_n - \frac{E_B}{c^2}$$

... semiempirical mass formula

Example: E_B and M estimation for $^{62}_{28}Ni$ based on Liquid-Drop Model

$Z = 28$, $A = 62$, $N = 34$

The five terms in the LDM have the following contributions:

1. $C_1 A = 976.5$ MeV
2. $-C_2 A^{2/3} = -278.8$ MeV
3. $-C_3 \dfrac{Z(Z-1)}{A^{1/3}} = -135.6$ MeV
4. $-C_4 \dfrac{(A-2Z)^2}{A} = -13.8$ MeV
5. $+C_5 A^{-4/3} = 0.2$ MeV

dominant contributions

Both N & Z are even

$$\rightarrow E_B = 548.5 \text{ MeV} \xrightarrow[\text{LDM with}]{\text{compare}} 545.3 \text{ MeV (true value)}$$

LDM is only 0.6 % larger than the true value

Use E_B in semiempirical mass formula

$$^{62}_{28}M = 28(1.007825u) + 34(1.008665u) - \frac{548.5}{931.5} = 61.925u$$

only 0.005 % smaller than

61.928349 u (true value)

He Ne Ar Kr Xe Rn

Atomic Structure: $Z = 2, 10, 18, 36, 54$, and 86 ← particularly stable $\bar{e}$ config.

O Ca Ni Sn Pb

Nuclear Structure: $\underbrace{Z = 2, 8, 20, 28, 50, 82,}_{\text{magic numbers}}$ and 126 ← unusually stable nuclei

has not yet been observed in nature

There are also doubly magic nuclides for which both Z and N are magic

$$^{4}_{2}He_{2} \qquad ^{16}_{8}O_{8} \qquad ^{40}_{20}Ca_{20} \qquad ^{48}_{20}Ca_{28} \qquad ^{208}_{82}Pb_{126}$$

both Z & N are magic

All these nuclides have substantially higher binding energy per nucleon than do nuclides w/ neighboring values for N or Z. They also have zero nuclear spin. The magic numbers correspond to filled-shell or -subshell configurations of nucleon energy levels with relatively large jump in energy to next allowed level. [Another model will be introduced in discussing nuclear moments.]

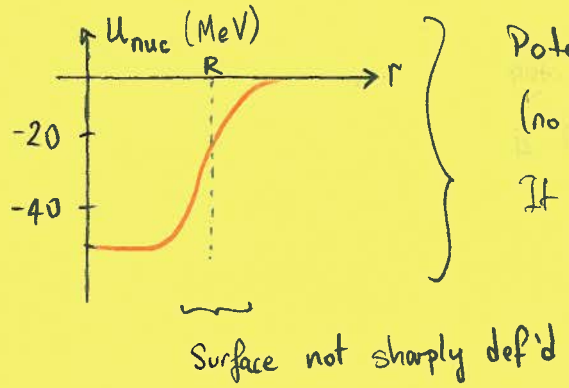
Nuclear Stability

* Among 2500 known nuclides, fewer than 300 are stable.
* Unstable nuclei decay by emitting particles + γ → α, e^+, e^-, heavier nuclei
* Time scale of decay μs – billions of years
* Stable nuclei show a pattern when plotted over a Z vs. N chart (called Segrè chart)

Shell Model

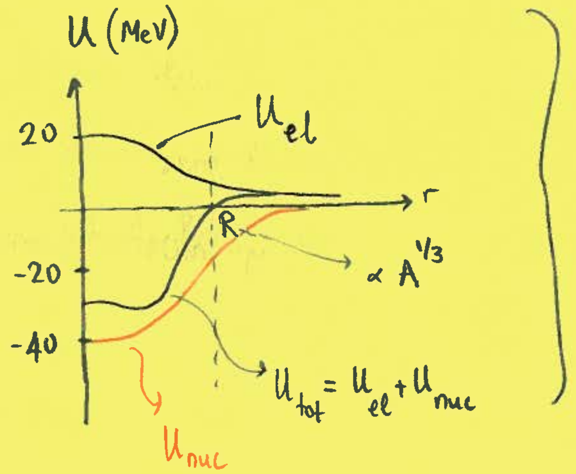
* Analogous to the central-field approximation in atomic physics.

As if, each nucleon is moving in a potential that represents averaged-out effect of all other nucleons.



Potential energy profile for neutrons (no Coulombic force contribution in this case)

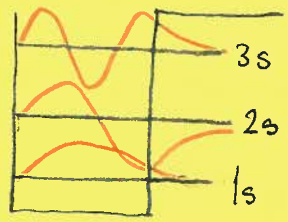
It is a rounded square-well potential



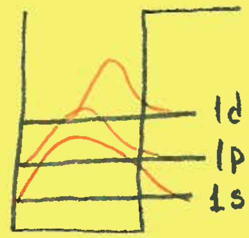
Potential energy profile for protons

In this case, we add the repulsive Coulomb energy among protons.

For any spherically symmetric potential energy (note that this only an approximation for the nuclear matter), the angular momentum states $Y_{lm}(\theta,\phi)$ are the same as for the $\bar{e}$'s in the central-field approximation in atomic physics. However, radial force in nuclear matter is different from the Coulomb potential; there is even a notational subtlety. To illustrate this, consider a square-well



Atomic Phys.



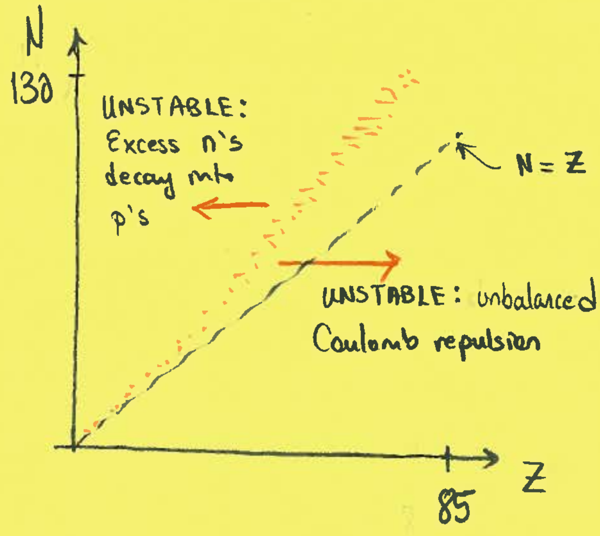
Nuclear Phys.

It should be noted that when using the radial node quantum number n of nuclear physics there is no restriction on the largest possible value of l for a given n. There is such a restriction in atomic physics because the quantum number n used there, called the principal q.#, is defined as:

$$n_{principal} = n_{radial} + l$$

Since the minimum of n_{radial} is 1, the largest value of l for a given $n_{principal}$ is $(n_{principal} - 1)$. So, then why $n_{principal}$ is used atomic physics? This is because $V(r)$ is an attractive Coulomb pot. $-1/r$, the way the energy of a level increases with increasing n_{radial} happens to be precisely the same way it increases with increasing l. Thus the energy of the levels of a Coulomb potential does not depend on both n_{radial} & l, but only on their sum $n_{principal}$. This gives yet another insight into the origin of the degeneracy of the H atom.

In nuclear physics n refers to n_{radial}. If you observe the plot on the right (top of page), the states with $n=1$ and $l=0, 1$ and 2 we observe the centrifugal effect that tends to prevent a nucleon from approaching $r=0$ as the orbital angular momentum l increases. Solving the Schrodinger eqn. with $\psi(r,\theta,\phi) = R_n(r)\, Y_{lm}(\theta,\phi)$ yields the same filled shells and subshells as in atomic physics, from which we can infer the stability of the nuclei.



General Trends:
- For low mass numbers, $N \approx Z$.
- N/Z gradually increases with A up to 1.6 at large mass (This is to compensate the increasing EM repulsion among p's)
- No nuclide with $A > 209$ or $Z > 83$ is stable

In Particular:
- Only 4 stable N=odd, Z=odd nuclides exist:

  ${}^{2}_{1}H$, ${}^{6}_{3}Li$, ${}^{10}_{5}B$, ${}^{14}_{7}N$

  This shows the power of pairing!

- Unusual stability of doubly magic ${}^{4}_{2}He$ nucleus, renders $A=5$ and $A=8$ unstable

  eg. ${}^{8}_{4}Be$ immediately splits into $\alpha + \alpha$

- There is no stable nuclide with $A = 43$ or 61

  43 → technetium, 61 → promethium

About 90% of the known nuclides are radioactive; they decay into other nuclides by emitting α or β particles.

α → ${}^{4}He$, β → e^-

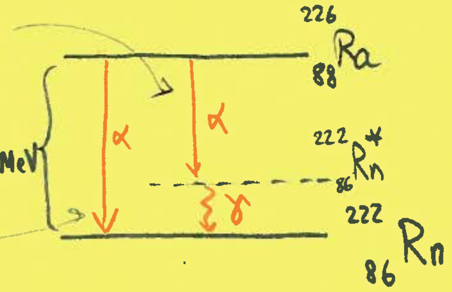
Alpha Decay $\;{}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-4}Y + \alpha$

α emission occurs principally with nuclei that are too large to be stable.

As $N \rightarrow N-2$ and $Z \rightarrow Z-2$ they move closer to stable territory on Segrè chart.

Ex: ${}^{226}_{88}Ra \rightarrow {}^{222}_{86}Rn^{*} + \alpha \rightarrow {}^{222}_{86}Rn + \gamma + \alpha$

or ${}^{226}_{88}Ra \rightarrow {}^{222}_{86}Rn + \alpha$



α-decay is possible whenever the mass of the original neutral atom is greater than the sum of the masses of the final neutral atom and the neutral ^{4}He atom

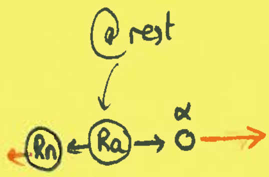
Let's see that this is the case for the above Radium α-decay.

The mass difference is:

$$226.025403\,u - (222.017571\,u + 4.002603\,u) = +\,0.005229\,u$$

$$E = \frac{0.005229\,u}{931.5\ MeV/u} = 4.871\ MeV$$

This is the total KE of Rn & α



Non-relativistic treatment can still be used at this energy.

Since Rn & α will have the same p (but opposite dir's), their velocity will be inversely prop. masses. Hence α gets $\frac{222}{222+4}$ of the total K.E $= 4.78$ MeV $\rightarrow v_\alpha = 0.05c$

Beta Decay

There exists three varieties of β decay:

1) β^- , 2) β^+, 3) Electron capture ← In all these types A remains constant

β^- decay:

How come a nucleus emit an $\bar{e}$ if there aren't any $\bar{e}$'s inside the nucleus?

Actually if we free a n from the nucleus, it decays in about 15 min.

$$n \longrightarrow p + \beta^- + \bar{\nu}_e$$

β^-: e^-

$\bar{\nu}_e$: electron antineutrino (more in particle phys.)

For β^- decay to occur, the mass of the original neutral atom should be larger than that of the final atom.

<u>Ex:</u> β decay of ${}^{60}_{27}Co$, an odd-odd unstable nucleus which is used in medical & industrial applications of radiation

$${}^{60}_{27}Co \longrightarrow {}^{60}_{28}Ni + \beta^- + \bar{\nu}_e$$

NB: Unlike the α-decay where there were two decay products, in the β-decay we cannot predict precisely how will the energy be shared among decay products

The β^- decay occurs with nuclides that have too large a N/Z ratio (obviously after decay we get $\frac{N}{Z} \to \frac{N-1}{Z+1}$)

β^+ decay

For nuclides having N/Z too small, they emit a e^+ for stability.

The basic process is:

$$p \rightarrow n + \beta^+ + \nu_e$$

β^+ decay can occur whenever the mass of the original neutral atom is at least two e^- masses larger than that of the final atom.

e^- capture

There are a few nuclides for which β^+ emission is not energetically possible but in which an orbital e^- (usually in the K shell) can combine with a proton in the nucleus to form a n and a ν_e. The n stays in the nucleus and ν_e is emitted. The basic process is:

$$p + \beta^- \rightarrow n + \nu_e$$

This reaction also helps to explain the formation of neutron stars.

e^- capture can occur whenever the mass of the original neutral atom is larger than that of the final atom.

Note that <u>all</u> these three β decays occur within a nucleus.

β^- decay can also occur outside the nucleus

β^+ decay is forbidden by conservation of mass-energy for a p outside nucleus.

The e^- capture can occur outside the nucleus only with the addition of some extra energy, in a collision.

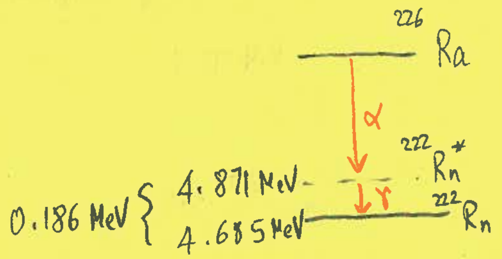
γ Decay

Nucleus being trapped in a potential well has quantized set of energy levels. In ordinary physical & chemical transformations the nucleus is always in its grand state. When a nucleus is placed in an excited state either by bombardment with high-energy particles or by radioactive transformation, it can decay into the ground state by emission of one or more photons with typical energies: 10 keV – 5 MeV, therefore they are in the gamma rays spectrum.

Ex: Recall ^{226}Ra α-decay, in one of the decay channels we had

$$^{226}Ra \rightarrow {}^{222}Rn^{*} + \alpha \rightarrow \underbrace{^{222}Rn}_{} + \alpha + \gamma$$

Radon (more on this later!)



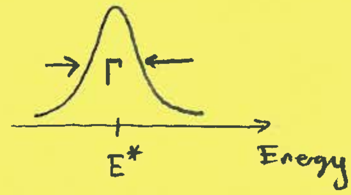
In the γ decay, the element does not change; the nucleus merely goes from an excited state to a less excited state.

Width vs. Lifetime

An excited (nuclear) state has a finite lifetime within which it decays to a lower state. According to the energy-time uncertainty principle, if an average nucleus survives in an excited state only for the lifetime τ of the state, then its energy in the state can be specified only within an energy range Γ,

$$\Gamma = \frac{\hbar}{\tau}$$

Excited states are, therefore, not perfectly sharp.



Ex: A typical γ-decaying state has a lifetime $\tau \sim 10^{-10}$ s.

$$\Gamma = \frac{\hbar}{\tau} \sim \frac{10^{-15}\ \text{eV.s}}{10^{-10}} = 10^{-5}\ \text{eV}$$

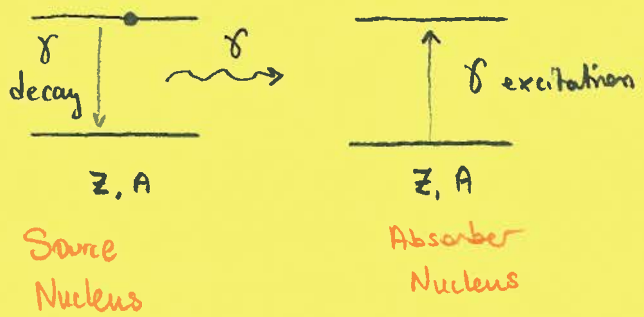
In comparison to the typical energy $E = 1$ MeV of such a state, Γ is extremely small.

$$\frac{\Gamma}{E} \sim 10^{-11}$$

This minute value of the ratio will lead to a remarkably accurate solid-state spectroscopic technique, Mössbauer spectroscopy with a sensitivity of a few parts in 10^{11} (which follows from above consideration).

Mössbauer Effect

There are two ingredients in this effect. First one is resonant absorption of a radiated photon by a source (nucleus) captured by an absorber (nucleus)



This emission and resonant absorption had been observed for X-rays (that are produced by electronic transitions) for gases. However, attempts to observe γ-ray resonance in gases failed primarily due to much more significant energy being lost to recoil that down shifts the radiated photon. There is a huge advantage to extend resonant absorption to γ-ray wavelengths as typical width/Energy (Γ/E) ratios lie in 10^{-11} range that has the potentiality as an ultrahigh precision energy spectrometer. So, let's first see the associated recoil energy problem.

Iridium, $^{191}_{77}Ir$ has an energy of 0.129 MeV as its first excited nuclear state, which has a measured lifetime of $T = 1.4 \cdot 10^{-10}$ s

As the mass M of the nucleus is large, we can use classical expressions

$P_n = \sqrt{2MK}$ for its recoil momentum

K.E. of nuclear recoil

Cons. of linear momentum: $P_n = P_\gamma = \frac{E_\gamma}{c}$ of the radiated γ-ray

$$\Rightarrow \quad K = \frac{E_\gamma^2}{2Mc^2}$$

Cons. of energy: $E_{decay} = K + E_\gamma$

0.129 MeV (transition energy of the source nucleus)

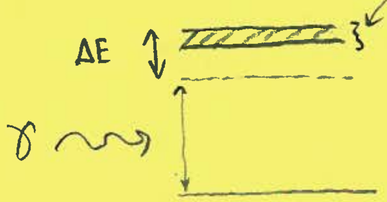
So, the downward shift ΔE in the energy of the γ-ray due to recoil is

$$\Delta E = -K = -\frac{E_\gamma^2}{2Mc^2} \overset{\text{approx.}}{\simeq} -\frac{(0.129)^2\,\text{MeV}^2}{2 \times 191 \times 931\ \text{MeV}} = -4.7 \cdot 10^{-2}\ \text{eV}$$

NB: The same result could be obtained by considering the γ-ray to be emitted from a moving source, the recoiling nucleus, and using the longitudinal Doppler shift formula to evaluate the downward shift in its frequency, or energy.

Now, considering the broadening (width) of the $^{191}_{77}Ir$ first excited state

$$\Gamma = \frac{\hbar}{T} = \frac{6.6 \cdot 10^{-16}\ \text{eV.s}}{1.4 \cdot 10^{-10}\ \text{s}} = 4.7 \cdot 10^{-6}\ \text{eV}$$



$\Delta E \gg \Gamma$ ($\exists$ factor of 10^4)

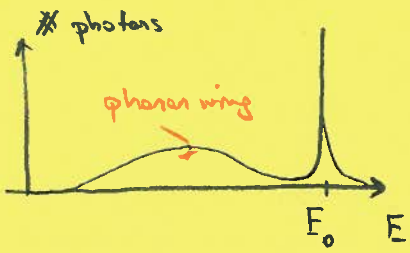
so, due to energy lost to recoil, this γ-ray cannot be reabsorbed by another Ir nucleus!

awarded Nobel prize in 1961

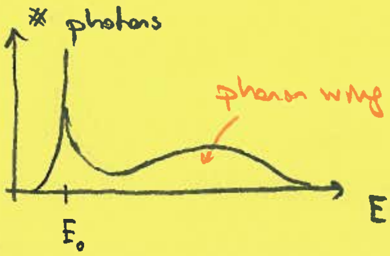
In 1958, a graduate student Rudolf Mössbauer was able to observe resonance in solid iridium, which raised the question of why γ-ray resonance was possible in solids, but not in gases. Mössbauer proposed that, for the case of atoms bound into a solid, under certain circumstances (zero phonon emission) a fraction of the nuclear events could essentially occur w/o recoil. He attributed the observed resonance to this recoil-free fraction of nuclear events.

The classical analogy for illustrating to lay people is this: imagine jumping from a boat to shore, and imagine that the distance from boat to shore is the largest you can jump (on land). If the boat is floating in water, you will fall <u>short</u> b/c some of your energy goes into pushing the boat back. If the water is frozen solid, however, you will be able to make it!

So, the second ingredient is the recoil-free emission/absorption. In solids, even though the nucleus is bound to a macroscopic body (so that $M_{nuc} \rightarrow M_{solid}$ diminishing the recoil momentum/energy), there is the added complication of phonons. During this process <u>no</u> phonons should be involved!



Emission spectrum



Absorption spectrum

Note that emission and absorption spectra overlap only for the Mössbauer (zero phonon) events and a few events involving low energy phonons.
Most Mössbauer effects are performed at cryogenic (liquid He) temp's where zero-phonon processes are enhanced.

Mössbauer Spectroscopy

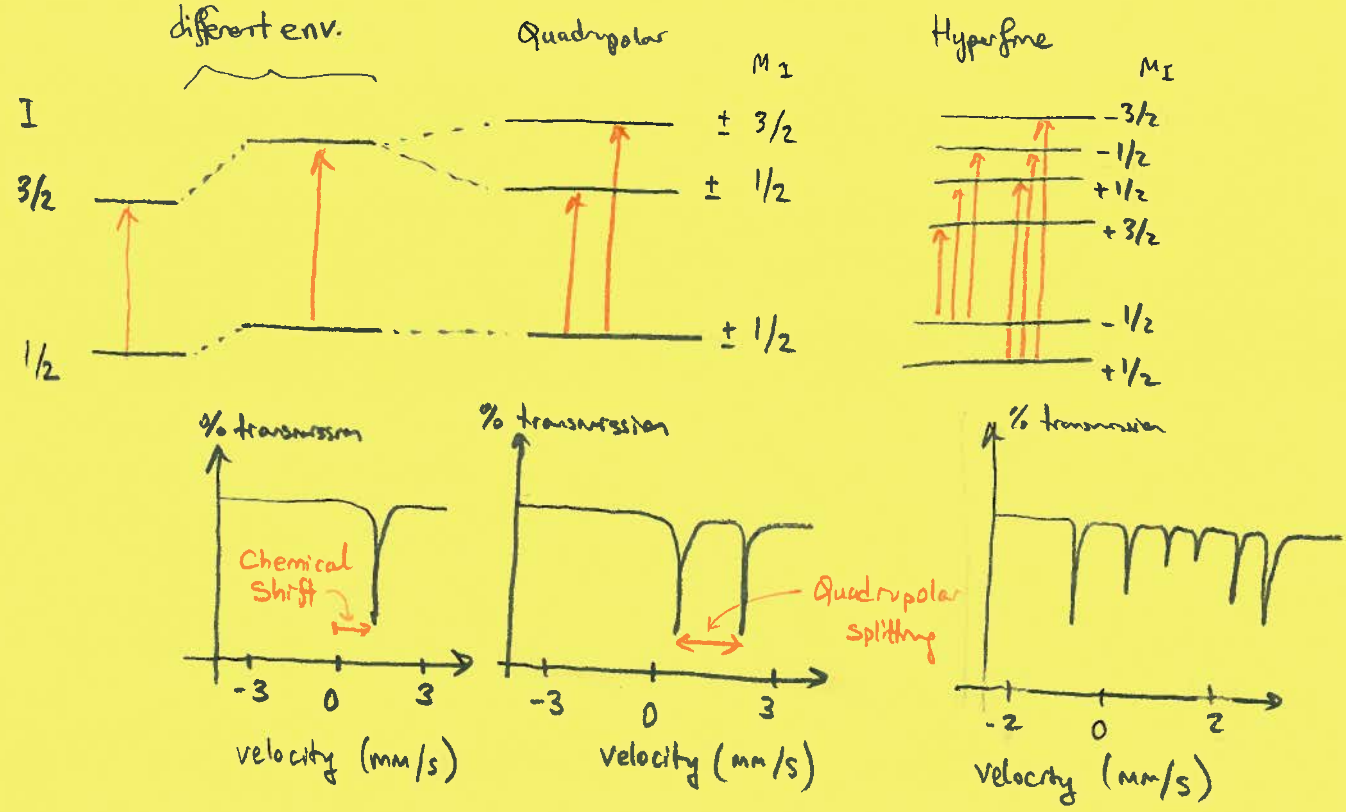
Mössbauer spectroscopy probes tiny changes in the energy levels (a few parts per 10^{11}) of an atomic nucleus in response to its environment.
Typically, three types of nuclear interaction may be observed:

1) Chemical Shift (Isomer shift)

The s electrons which have finite probability at $r=0$ where there is the nucleus. So, MS is sensitive to variations of the s $\bar{e}$ density within the sample (but also, indirectly to p and d $\bar{e}$'s, as they influence s $\bar{e}$ density through screening)

2) Quadrupole Splitting

Nuclei with spin > 1/2 have electric quadrupole moments.
Due to any electric field gradient at such a nuclear site, the spectra is further split. E.g. For ^{57}Fe with $I = 3/2$, there appear two peaks in MS corresponding to $m_I = \pm 1/2$ and $m_I = \pm 3/2$



3) Hyperfine Splitting

If there is an ext. B field, spectra split into $2I+1$ Zeeman sublevels.

The Mössbauer peak is scanned by placing the emitter and absorber in different solids and moving them relative to each other. The photon energy in ref. frame becomes $E = E_0 \left(1 + \frac{v}{c}\right)$

Typical doppler shifts $\Delta E / E \sim 10^{-11}$ require relative velocities in the range $\pm$ 10 mm/s

MS is limited by the need for a suitable γ-ray source.

The source for ^{57}Fe consists of ^{57}Co which decays by $\bar{e}$ capture to an excited state of ^{57}Fe, then subsequently decays to its ground state by emitting the desired γ-ray. So, most common element used in this technique is ^{57}Fe. Recently used for understanding the structure of iron containing enzymes. Also, in the field of geology, it is used for identifying the composition of iron-containing specimens including meteors and moon rocks. A miniaturised Mössbauer spectrometer was used by NASA's Mars Exploration Rovers (2004) where there are iron-rich rocks.

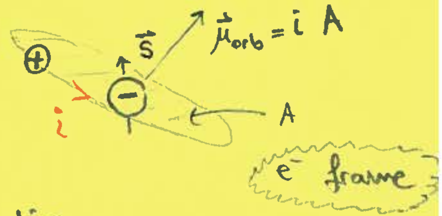
Other common elements studied using this technique are Iodine (^{129}I), tin (^{119}Sn) and antimony (^{121}Sb).

How do we learn about nuclear spin?

One means is to take a look at the atomic spectra somewhat closely. If you examine an atomic spectra quite crudely, you only realize the orbital levels of the electrons. A closer look reveals that there is a so-called fine structure which arises from the coupling of the spin and orbital degrees of freedom of the electrons.

How do spin & orbit know about each other?

In the e^- rest frame the nucleus appears to be moving, which creates an orbital magnetic moment, $\vec{\mu}_{orb}$. The two magnetic moments have dipole-dipole coupling $\vec{S} \cdot \vec{\mu}_{orb}$, spin-orbit coupling.



This coupling exerts an orbital-dependent shift on the atomic levels

$$\vec{L}^e + \vec{S}^e = \vec{J}^e$$

(because of this coupling, neither $\vec{L}^e$ nor $\vec{S}^e$ are cons'd)

At a higher resolution, one finds that even the fine structure components split into subcomponents, so-called hyperfine structure which arises from the again magnetic interaction bet. nuclear magnetic moment $\vec{\mu}_N$, with the electronic magnetic moment $\vec{\mu}_J$

HFS energy splitting

$$\Delta E_{HFS} = \frac{A}{2}\left[F(F+1) - J^e(J^e+1) - I(I+1)\right] \; ; \; \vec{F} = \vec{J}^e + \vec{I}$$

total ang. mom. of the atom; total electronic ang. mom.; total nuclear ang. mom.

HF Constant

$$A = \frac{g_N \mu_N B_J}{\sqrt{J^e(J^e+1)}}$$

nuclear magnetic moment

nuclear g-factor

field produced by the e^- @ nuclear site

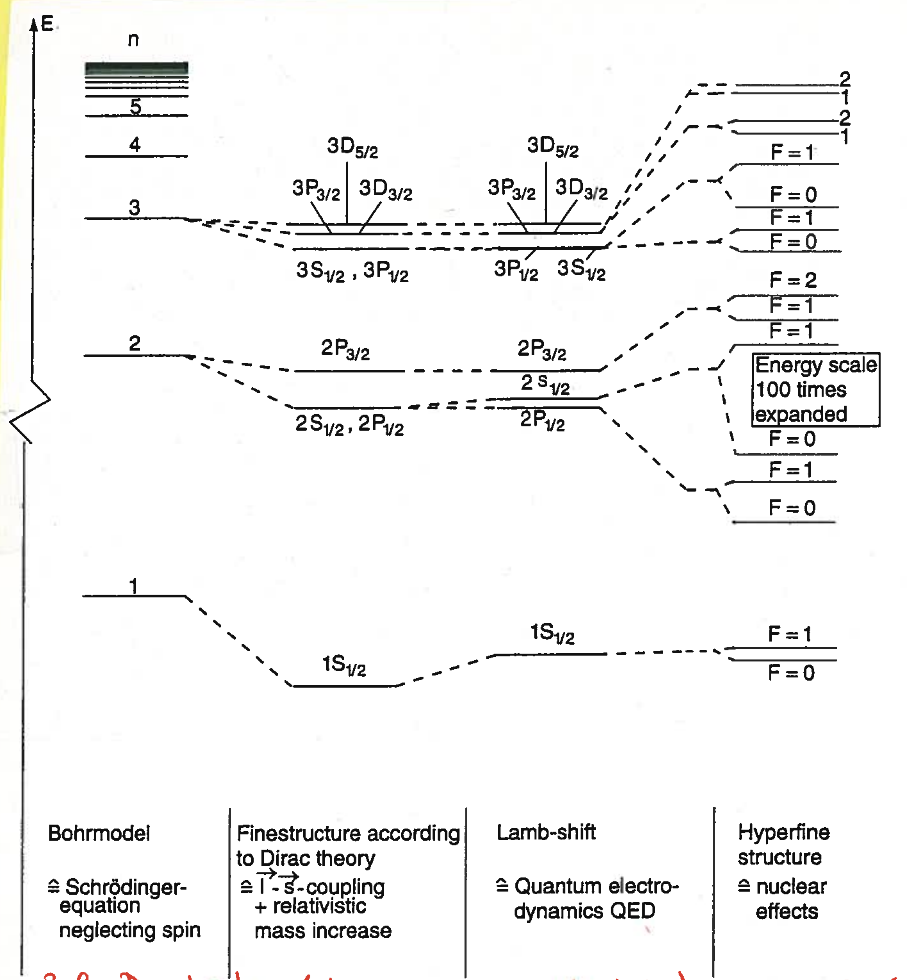
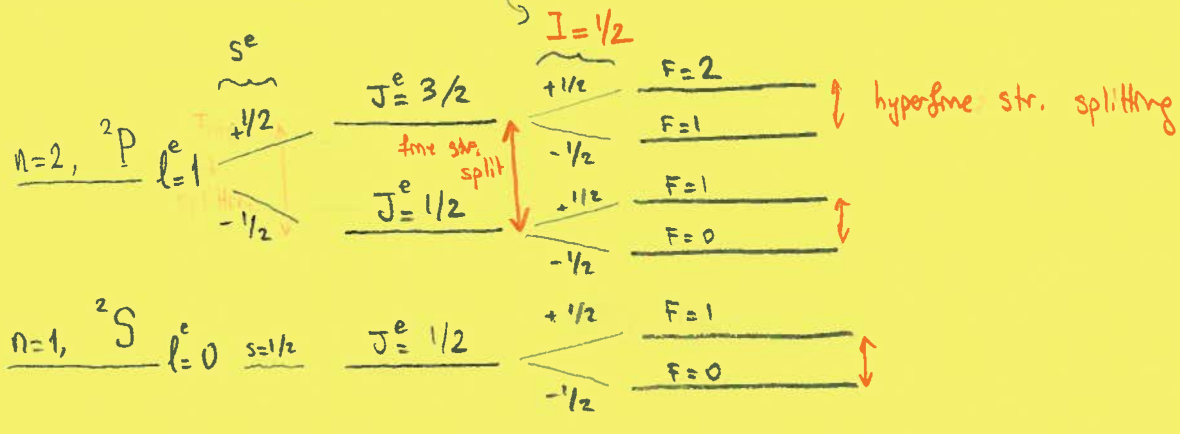


Fig. 5.33. Complete level scheme of the H atom including all interactions known so far. *Note*: The fine, HFS structure, and the Lamb shift are not drawn to scale. They are exaggerated in order to illustrate the splittings and shifts

Ref: Demtröder, 'Atoms, Molecules and Photons' Springer, 2006.

A simpler illustration for the same Hydrogen splittings is given below (not to scale)



Hence, from the hyperfine splitting multiplet $(2I+1)$ we can infer the value of I as well as the nuclear magnetic moment μ_N

Nuclear Spin & Moments

Each nuclear state is assigned a unique "spin" quantum number I, representing the <u>total</u> angular momentum (orbital + intrinsic) of all the nucleons in the nucleus.

$$\vec{I} = \sum_{i=1}^{A} \vec{l}_i + \vec{s}_i$$

$$= \vec{L} + \vec{S} = \sum_{i=1}^{A} \vec{j}_i$$

either of these decompositions can be used (if need be). No dominance bet. LS and jj couplings.

The quantum number $\vec{I}$ satisfies the usual angular momentum relations:

$$|\vec{I}| = \sqrt{I(I+1)}\,\hbar$$

$$I_z = m_I \hbar , \qquad m_I = -I, -I+1, \dots , I-1, I$$

It may not be apparent why we can neglect the complicated internal structure of the nucleus and treat it as if nucleus were an elementary particle with a single quantum number, representing the intrinsic angular momentum of the "particle". This is possible only because the interactions to which we subject the nucleus (eg. EM fields) are not sufficiently strong to change the internal str. or break the coupling of the nucleons that is responsible for $\vec{I} = \sum_i \vec{l}_i + \vec{s}_i$.

Likewise, for the electronic matter: $\vec{J}^e = \sum_{i=1}^{Z} \vec{l}_i^{(e)} + \vec{s}_i^{(e)}$

total electronic angular momentum

Finally, there are cases in which it is most appropriate to deal with the total (nuclear + electronic) angular momentum, usually called $\vec{F}$:

$$\vec{F} = \vec{I} + \vec{J}^e$$

The quantum numbers I and J may be either integral → A = even — nuclear or Z = even — electronic

or half-integral → A = odd — nuclear or Z = odd — electronic

A	Z	I		J		F
Even	Even	Integer	+	Integer	=	Integer
Odd	Even	Half-Integer	+	Integer	=	Half-int.
Even	Odd	Integer	+	Half-Int.	=	Half-int.
Odd	Odd	Half-int.	+	Half-int.	=	Integer

For nuclear ground states, there are several rules for determining spins.

1. All even-Z, even-N nuclei have $I = 0$ (based on pairing) eg. ${}^{28}_{14}Si_{14}$ has $I=0$

2. If Z & N are both odd, then $I \in \mathbb{Z} > 0$

e.g. ${}^{2}_{1}H_{1} \to I=1$, ${}^{10}_{5}B_{5} \longrightarrow I=3$, ${}^{14}_{7}N_{7} \longrightarrow I=1$

From now on, the ground state nuclear "spin" is simply called "nuclear spin".

Parity & Multipole Moments

Each electromagnetic multipole moment has a parity, determined by the behaviour of the multipole operator, $\hat{M}$ under, $\vec{r} \longrightarrow -\vec{r}$

$$\text{The parity of } \begin{cases} \text{electric moments}: (-1)^L \\ \text{magnetic moments}: (-1)^{L+1} \end{cases} \quad \begin{matrix} L=0 \to \text{monopole} \\ L=1 \to \text{dipole} \\ L=2 \to \text{quadrupole} \\ \vdots \end{matrix}$$

When we compute the expectation value of a moment, we must evaluate an integral of the form $\int \psi^* \underbrace{\hat{M}}_{\text{appropriate EM multipole operator}} \psi \, dv$

NB: The parity of ψ is not important; because it appears twice in the integral. $\psi^*\psi$ will always have even parity. But, for this, nuclear state must have a definite parity, and indeed this has been verified to one part in 10^7.

If, however, $\hat{M}$ has odd parity, then the integrand is an odd fn. and must vanish identically.

Accordingly, electric dipole, magnetic quadrupole, electric octupole (L=3) must all vanish for any nucleus under ordinary circumstances.

$\vec{\mu} = I\vec{A}$

Nuclear Magnetic Moment

The magnetic dipole moment $\vec{\mu}$ arises from electric currents (orbital) as well as intrinsic spin of the particle. Note that neutron even though is overall charge neutral it is actually a composite particle made up of charged quarks, so not surprisingly it has a nuclear "spin". The leading term in magnetic multipole expansion is the dipole moment, recall from classical electrodynamics that

$$\vec{\mu} = \frac{1}{2} \int_{Vol.} \vec{r}\,' \times \underbrace{\vec{j}(\vec{r}\,')}_{\rho(\vec{r}\,')\vec{v}(\vec{r}\,')} \, dv' \quad , \quad \vec{r}\,' \times \vec{v}\,' = \frac{\vec{l}}{m}$$

particle w/ mass m

$\rho(\vec{r}\,') \to e\,\psi^*(\vec{r}\,')\,\psi(\vec{r}\,')$

Hence, the quantum mechanical expression (at this level, for the orbital contribution) is given by

$$\vec{\mu} = \frac{e}{2m} \int \psi^*(\vec{r}\,')\, \vec{l}\, \psi(\vec{r}\,')\, dv'$$

If the wf ψ corresponds to a state of definite l_z, then only the z component of the integral is nonvanishing, and

$$\mu_z = \frac{e}{2m} \int \psi^*(\vec{r}\,')\, l_z\, \psi(\vec{r}\,')\, dv' = \frac{e\hbar}{2m} m_l \quad , \quad m_l = -l, -l+1, \ldots, l$$

$m_l \hbar$

What we observe in an experiment as <u>the</u> magnetic moment is <u>defined</u> to be the value of μ_z corresponding to the maximum possible value of the

z component of the angular momentum. The quantum number m_ℓ has a maximum value of $+\ell$, and thus the magnetic moment μ is

$$\mu = \frac{e\hbar}{2m}\,\ell$$

magneton

use proton mass

use electron mass

$$\mu_N = \frac{e\hbar}{2m_p} = 3.15245 \cdot 10^{-8}\ \frac{eV}{T} \qquad\qquad \mu_B = \frac{e\hbar}{2m_e} = 5.78838 \cdot 10^{-5}\ \frac{eV}{T}$$

nuclear magneton

Bohr magneton

⇒ Observe that due to 10^3 mass contrast, electronic magnetism is about 10^3 times larger than the nuclear magnetism!

Now, if we further include the intrinsic spin of a <u>nucleon</u>, we extent as

$$\vec{\mu} = \left(g_\ell\,\vec{\ell} + g_s\,\vec{s}\right)\frac{\mu_N}{\hbar}$$

orbital & intrinsic $\vec{g}$ - factors

For a proton : $g_\ell = 1$, $\quad g_s = 5.5856912 \rightarrow \vec{\mu} \parallel \vec{s}$ (as expected from a +ve charge)

For a neutron : $g_\ell = 0$, $\quad \underbrace{g_s = -3.8260837}$

no net charge

$\vec{\mu}$ and $\vec{s}$ are anti-// as would be the case for a negative charge distribution!

In real nuclei, we must make a modification to allow for the effects of all the nucleons:

$$\vec{\mu} = \sum_{i=1}^{A} \left[g_{\ell,i} \, \vec{l}_i + g_{s,i} \, \vec{s}_i \right] \frac{\mu_N}{\hbar}$$

just like the case in $\vec{I}$. However, such a calculation is beyond the current capabilities, as the interactions bet. the nucleons are strong and the relative spin orientations are not sufficiently well known.

So, what is done is to measure it, and overall relation between the magnetic moment and the angular momentum of a nucleus is given by the gyromagnetic ratio (which is an important parameter for NMR):

$$\vec{\mu} = \gamma \, \vec{I}, \qquad [\gamma] \xrightarrow{SI} \text{rad } s^{-1} \, T^{-1}$$

The gyromagnetic ratio may have either sign.

$\gamma > 0$ (most of the atomic nuclei) $\quad \vec{\mu} \parallel \vec{I}$

$\gamma < 0$ (e^-, n, and a few nuclei) $\quad \vec{\mu}$ anti-$\parallel$ $\vec{I}$

As discussed, the (anomalous) magnetic moment of the e^- as predicted by QED agrees with experiment to the accuracy of 11 significant digits.

On the other hand, the magnetic moments of quarks, nucleons, and nuclei are not not understood on this level of theoretical detail. But they are known experimentally:

Isotope	Ground-state spin	Natural Abundance (%)	γ (10^6 rad s^{-1} T^{-1})	NMR freq. @ 11.74 T ($\omega^0/2\pi$) MHz
^1_1H	1/2	~100	267.522	- 500.000
^2_1H	1	0.015	41.066	- 76.753
^{13}C	1/2	1.1	67.283	- 125.725
^{14}N	1	99.6	19.338	- 36.132
^{15}N	1/2	0.37	- 27.126	+ 50.684
^{29}Si	1/2	4.7	- 53.190	+ 99.336
^{12}C	0	98.9		
^{16}O	0	~100		
^{28}Si	0	95.3		

all I = 0 nuclei are NMR silent!

Note that proton (i.e., ^{1}H nucleus) is one of the most magnetic ones among the nuclei. For this reason proton-NMR is of prime importance in organic chemistry with its less common alternative, ^{13}C-NMR.

Nuclear Zeeman Splitting

A nuclear state with spin I is (2I+1) - fold degenerate. Degeneracies are an outcome of symmetries in the Hamiltonian (here, space isotropy). If an external magnetic field is applied, this degeneracy is broken.

$$\hat{H}_z = -\hat{\vec{\mu}} \cdot \vec{B}_0$$

$$\left.\begin{aligned} \hat{H}_z &= -\gamma \hat{\vec{I}} \cdot \vec{B}_0 \\ \text{Let} \quad \vec{B} &= \hat{z}\, B_0 \end{aligned}\right\} \quad \hat{H}_z = -\gamma B_0 \hat{I}_z$$

ω_0: Larmor Frequency

$$\langle m | \hat{H}_z | m \rangle = m\hbar\omega_0 \quad ; \quad m = -I, \dots, I \qquad (2I+1) \text{ states}$$

with $\Delta E_z = \hbar|\omega_0|$ ← Level splitting

The splitting bet. the nuclear spin levels is called the nuclear Zeeman splitting. NMR is the spectroscopy of the nuclear Zeeman sublevels.

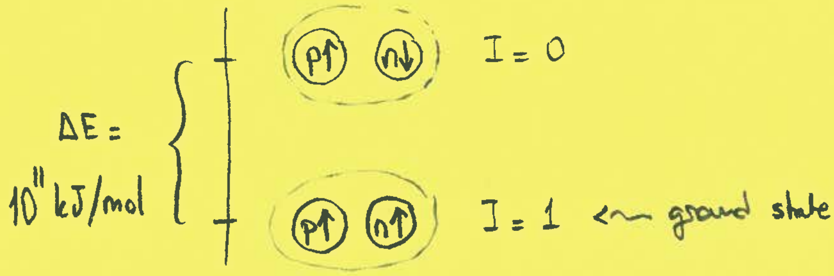
H is 10 times more magnetic than N

6 levels $= 2 \cdot \frac{5}{2} + 1$

The Zeeman splitting <u>within</u> the nuclear ground state must not be confused with the enormously large splitting bet. nuclear spin ground state and the nuclear spin excited states. The Zeeman splittings are far smaller than thermal energies (unless at ultracold T's).

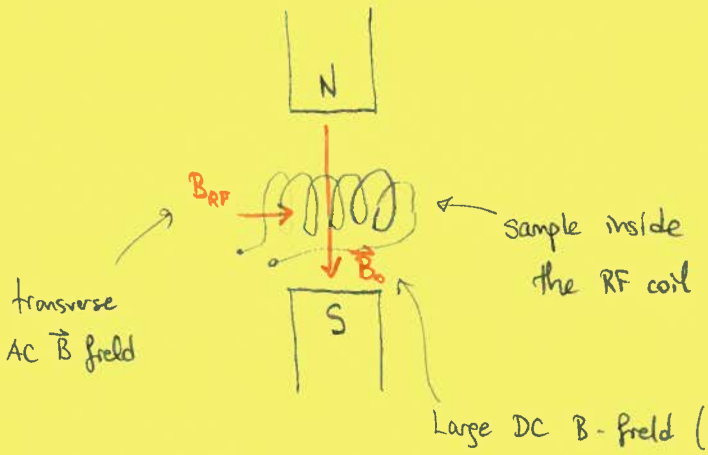
As an example consider the deuteron $^{2}_{1}H$, having $p+n$. Its "spin", meaning ground state total angular momentum is $I = 1$ (i.e, triplet).

If the "spin" of say the neutron flips to make it a singlet state, i.e., $I = 0$, due to associated spin-dependent nuclear forces, this configuration has an energy 10^{11} kJ/mol higher $\xrightarrow{\text{compare}}$ $E_{thermal}(300K) = 2.5$ kJ/mol



Nuclear Magnetic Resonance (NMR)

NMR was first described and measured in molecular beams by Rabi in 1938. In 1946 Bloch, Purcell and Pound expanded the technique for use on liquids and solids. In 1950 Hahn (as a PhD student) discovered the spin echo. NMR has turned into a crucial analytical tool for the diagnostics of inorganic and organic mat'l as well as in medicine. Its basic construction is simple.



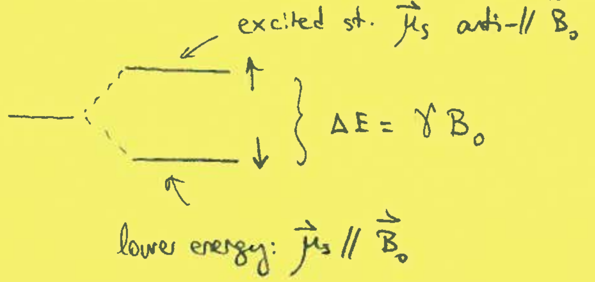
In some applications instead of the large DC $\vec{B}$ field, the Earth's magnetic field is used – Earth Field NMR or Zero field NMR. $B_{Earth} \simeq 30 - 60\,\mu T$ (30 @ equator, 60 @ poles)

The drawback is that it is weak so that for a proton the resonance frequencies fall into audio range ~ 2 kHz.

In most cases a several Tesla DC B field is used with its uniformity assured by the so-called gradient coils that correct any imperfections.

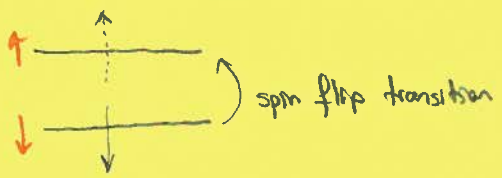
How NMR Operates?

1) The large DC $\vec{B}$ field polarizes the nuclear spins in the specimen, by introducing a Zeeman splitting of the previously degenerate nuclear spin states. For the case of a proton:

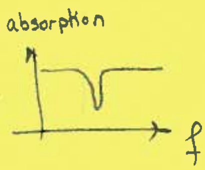


Because of this splitting, the population of the // nuclear spin state slightly exceeds the anti-// spin state. This imbalance enables a net absorption under an RF excitation.

2) An RF excitation pulse resonant only with the targetted nuclei's Zeeman splitting causes a net up-transition of the nuclear spins.



From the absorption signal one can in principle probe the existance of targetted nuclei



Compared to detecting the absorption, it is more practical to wait for the decay of the nuclei from their excited state while emitting the so-called free-induction decay signal, which can be picked up by same coil that is used for RF excitation.

Ex Let's work out the population imbalance achieved by a 1 Tesla magnet with the specimen being protons (H - nuclei).

At equilibrium each state will be populated according to Boltzmann distr.

$$n_i = e^{-E_i/k_BT}$$

$$\frac{n_\downarrow - n_\uparrow}{n_\downarrow} = 1 - \frac{n_\uparrow}{n_\downarrow} = 1 - e^{-\Delta E/k_BT} \simeq \frac{\Delta E}{k_BT}$$

$$\Delta E = \hbar \gamma B_0 = \left(267.522\ 10^{6}\ \text{rad.}\ s^{-1}\ T^{-1}\right)(1T)\left(0.6582.10^{-15}\ \frac{eV.s}{rad}\right)$$

$$\simeq 1.76\ 10^{-7}\ eV$$

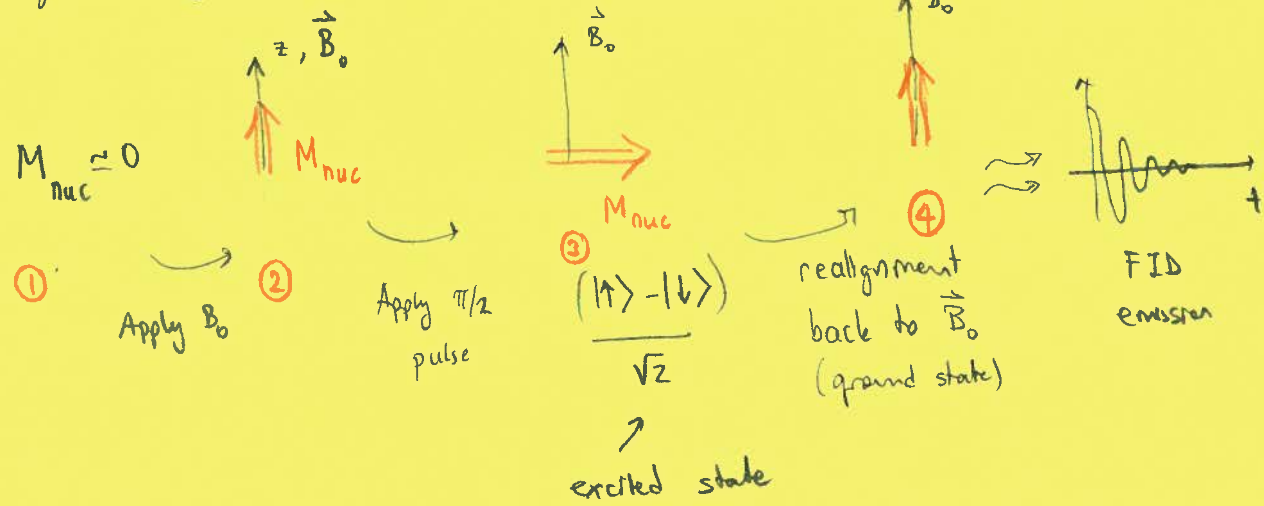
$$k_BT\Big|_{T=300K} \simeq 25\ meV$$

So, the fraction of excess spins is : $\dfrac{n_\downarrow - n_\uparrow}{n_\downarrow + n_\uparrow} \simeq \dfrac{n_\downarrow - n_\uparrow}{2n_\downarrow} \simeq \dfrac{\Delta E}{2k_BT} = 3.5 \times 10^{-6}$

This seems to be such a tiny imbalance, but, considering only one mole of such H nuclei, we get $\Delta N = N_\downarrow - N_\uparrow = 2.10^{18}/mol$

Hence their resonant absorption and subsequent FID signal can be easily picked up.

There is one technical detail that remains. Since there's a large DC field, it is quite impractical to detect the additional magnetization along that direction. The desirable orientation is to have the RF coil axis to be $\perp$ to DC B field. Therefore, nuclear spins are rotated by $90°$ through applying a so-called $\pi/2$-pulse after the nuclear spins are "aligned" along $\vec{B}_0$ dir. So, the procedure becomes:

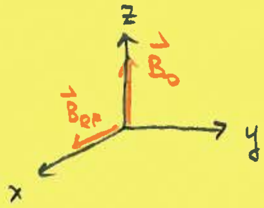


So, using NMR, we have a means to take the "attendance" of each nucleus that has non-zero nuclear spin $(I \neq 0)$. Thanks to fact that excitation is resonant only with the targetted transition tuned to that nuclei. (The $\pi/2$-pulse, for instance, will rotate by $90°$ only those specific nuclei and will not have a net effect on the other species.

Next, we shall illustrate how a $\pi/2$-pulse rotates nuclear spins $\perp$ $\vec{B}_0$.

Spin Rabi Oscillations

Consider a spin-1/2 particle (can be a neutron, an atom or an $\bar{e}$), placed in an external $\vec{B}$ field which has a strong DC part and a weak AC part $\perp$ to the DC.



$$\vec{B} = \hat{z} B_0 + \hat{x} B_{RF} \cos\omega t$$

with $B_0 \gg B_{RF}$

previously denoted as $\hat{H}_z$: Zeeman splitting

The interaction Hamiltonian will be: $\hat{H}' = \hat{H}_0 + \hat{H}_{RF}$

$$\hat{H}_0 \to -\vec{\mu}\cdot\vec{B}_0 \qquad \hat{H}_{RF} \to -\vec{\mu}\cdot\vec{B}_{RF}\cos\omega t$$

The Larmor spin precession frequency around z-axis is $\omega_0 = -\gamma B_0$ and around x axis is $-\gamma B_{RF} \cos\omega t$.

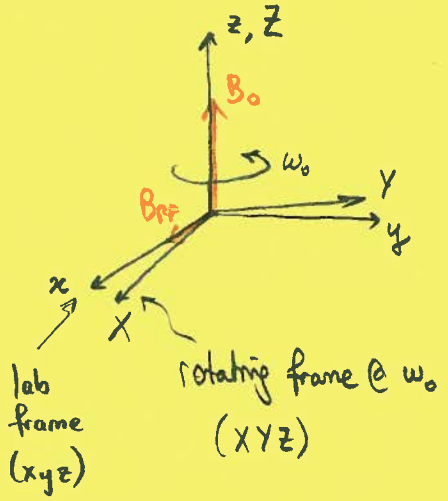
Decompose the AC part which is linearly pol'd into two circular pol's:

$$\hat{x} B_{RF} \cos\omega t = B_{RF}\,\mathrm{Re}\left\{\hat{x}\, e^{i\omega t} + \hat{y}\,\frac{i}{2}\, e^{i\omega t} - \hat{y}\,\frac{i}{2}\, e^{i\omega t}\right\}$$

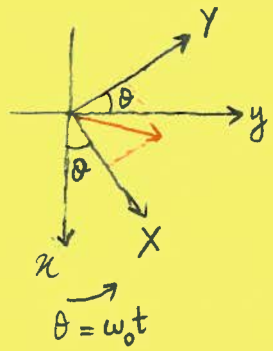
$$\Rightarrow \left\{\frac{1}{2}(\hat{x} \pm i\hat{y})\, e^{i\omega t}\right\}$$ counter rotating circularly pol'd fields

Since $B_0 \gg B_{RF}$, if we switch to the rotating frame around the z-axis with Larmor freq. ω_0, then these two components of the AC field will be transformed as: i) Slowly rotating field at $\omega - \omega_0$

ii) Fast rotating " " $\omega + \omega_0$



Top View (along z axis)



$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Since $\vec{B}_{RF} = \hat{x}\, B_{RF} \cos\omega t$

$$= \hat{X} \underbrace{\cos\omega_0 t \cos\omega t}_{\frac{1}{2}\left[\cos(\omega_0+\omega)t + \cos(\omega_0-\omega)t\right]} - \hat{Y} \underbrace{\sin\omega_0 t \cos\omega t}_{\frac{1}{2}\left[\sin(\omega_0+\omega)t + \sin(\omega_0-\omega)t\right]}$$

Technical Detail:

Actually, in the so-called rotating wave approximation (RWA), we discard the very fast rotating component (averaging out to zero).

The strong static field causes the Zeeman splitting of the spin-1/2 levels separated by $\hbar\omega_0$ (forming the stationary states)

———— $|e\rangle = |-1/2\rangle$

———— $|g\rangle = |+1/2\rangle$

$$\hat{H}_0 = \underbrace{-\frac{\hbar\omega_0}{2}}_{E_g} |g\rangle\langle g| + \underbrace{\frac{\hbar\omega_0}{2}}_{E_e} |e\rangle\langle e|$$

The AC field (essentially, its slowly varying part) which is weak, will cause transitions bet. the stationary states $|g\rangle \leftrightarrow |e\rangle$

$$\hat{H}'(t) = -\vec{\mu} \cdot \vec{B}_{RF} \cos\omega t$$

To solve the Schrödinger eqn. $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$.

we can write the solution formally as: $|\psi(t)\rangle = C_g(t)\, e^{-iE_g t/\hbar} |g\rangle + C_e(t)\, e^{-iE_e t/\hbar} |e\rangle$

the unknown coef's $C_g(t)$ and $C_e(t)$ can be solved by inserting into the Schrödinger eqn. subject to $C_g(t=0) = 1$, $C_e(t=0) = 0$ (spins start in ground st.)

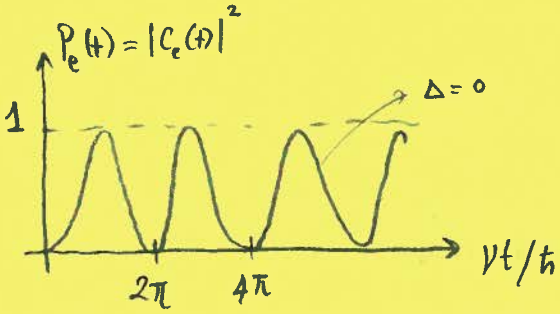
Under the RWA, we get coupled 1^{st} order d.e.

$$\left.\begin{aligned} \dot{C}_g &= -\frac{i}{2\hbar}\, \nu\, e^{i(\omega-\omega_0)t}\, C_e \\ \dot{C}_e &= -\frac{i}{2\hbar}\, \nu\, e^{-i(\omega-\omega_0)t}\, C_g \end{aligned}\right\} \quad \nu \equiv -\langle e|\vec{\mu}|g\rangle \cdot \vec{B}_{RF} \;\ldots\; \text{Rabi Freq.}$$

With the solutions:

$$\left.\begin{aligned} C_e(t) &= i\,\frac{\nu}{\Omega_R \hbar}\, e^{i\Delta t/\hbar} \sin\frac{\Omega_R t}{2} \\ C_g(t) &= e^{i\Delta t/2} \left\{ \cos\left(\frac{\Omega_R t}{2}\right) - i\,\frac{\Delta}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) \right\} \end{aligned}\right\} \quad \begin{aligned} &\Delta \equiv \omega_0 - \omega \;\ldots\; \text{detuning freq.} \\ &\Omega_R \equiv \sqrt{\Delta^2 + \frac{\nu^2}{\hbar^2}} \;\ldots\; \text{Generalized Rabi Freq.} \end{aligned}$$

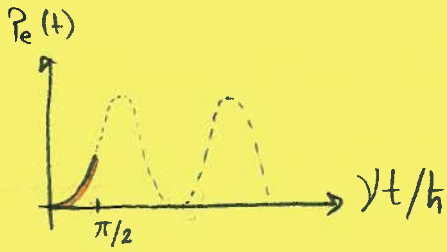
So, the spin system undergoes so-called Rabi oscillations between $|g\rangle \leftrightarrow |e\rangle$



For the so-called $\frac{\pi}{2}$-pulse,

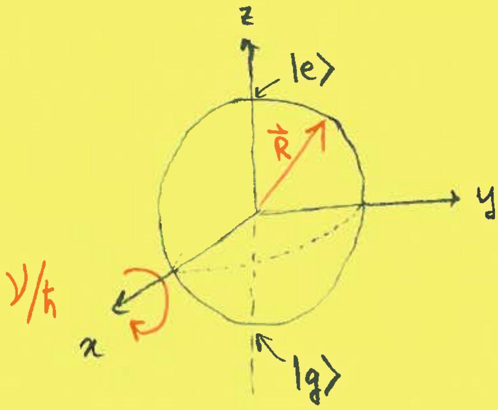
(a quarter-cycle of Rabi osc. – not $B_{RF}(t)$)

the spin rotates from $|g\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|g\rangle + |e\rangle\right)$



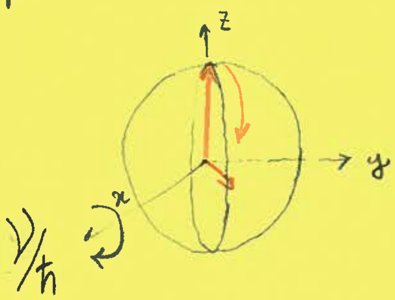
Bloch Sphere

Rabi oscillations are best illustrated on the so-called Bloch sphere

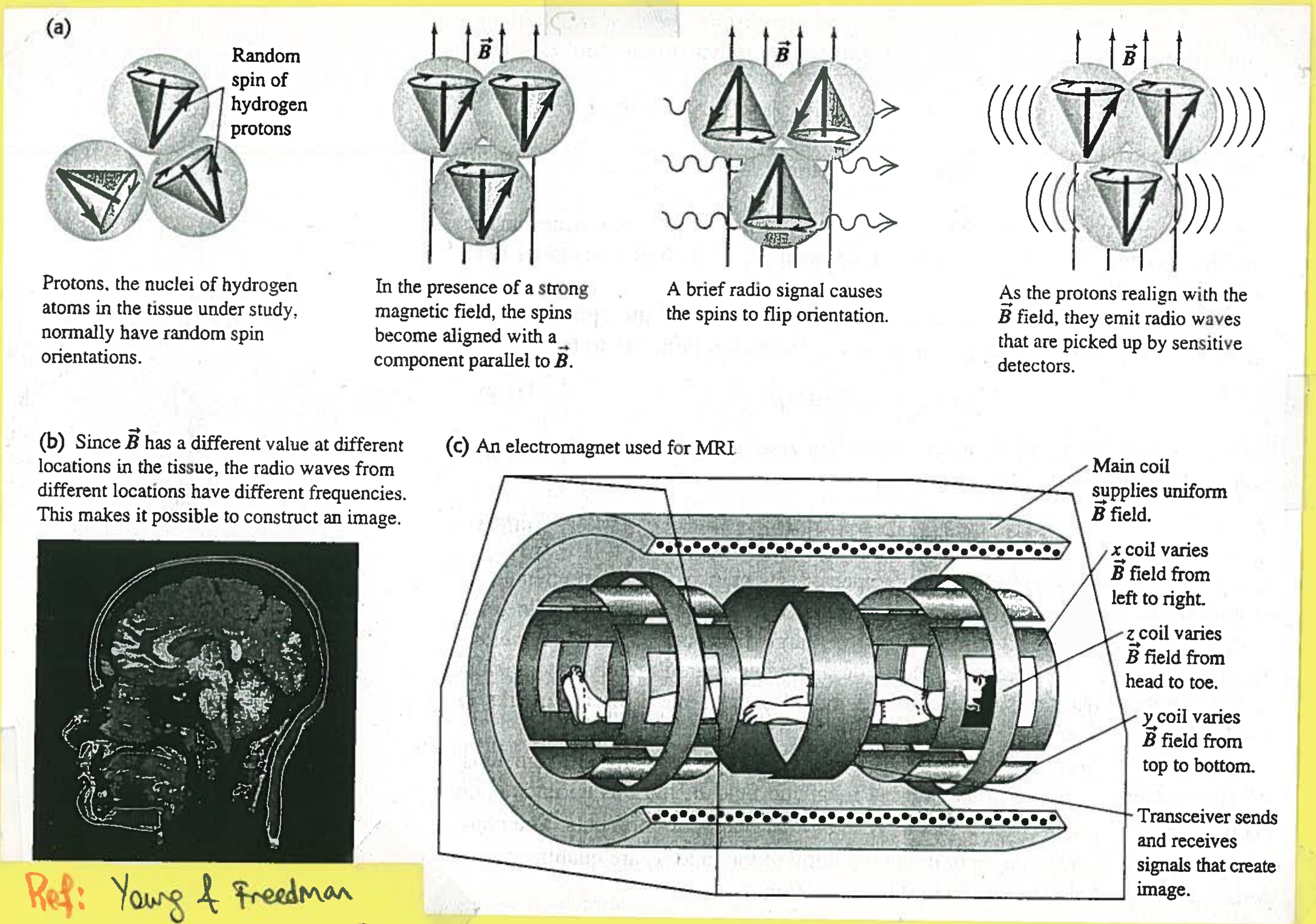


When the RF magnetic field is resonant with Zeeman splitting ie. $\omega = \omega_0$,

a $\frac{\pi}{2}$-pulse rotates the spin from North pole to the equator



As the transition energy (i.e. Zeeman splitting) is controllable by the DC B field, we can make use of this to gather position information of the nuclei, by making DC field position-dependent. So, a shifted FID frequency will tell us where that nucleus is positioned. In medical magnetic resonance Imaging (MRI) there are three orthogonal coils (x/y/z) that create these positional gradients independently.

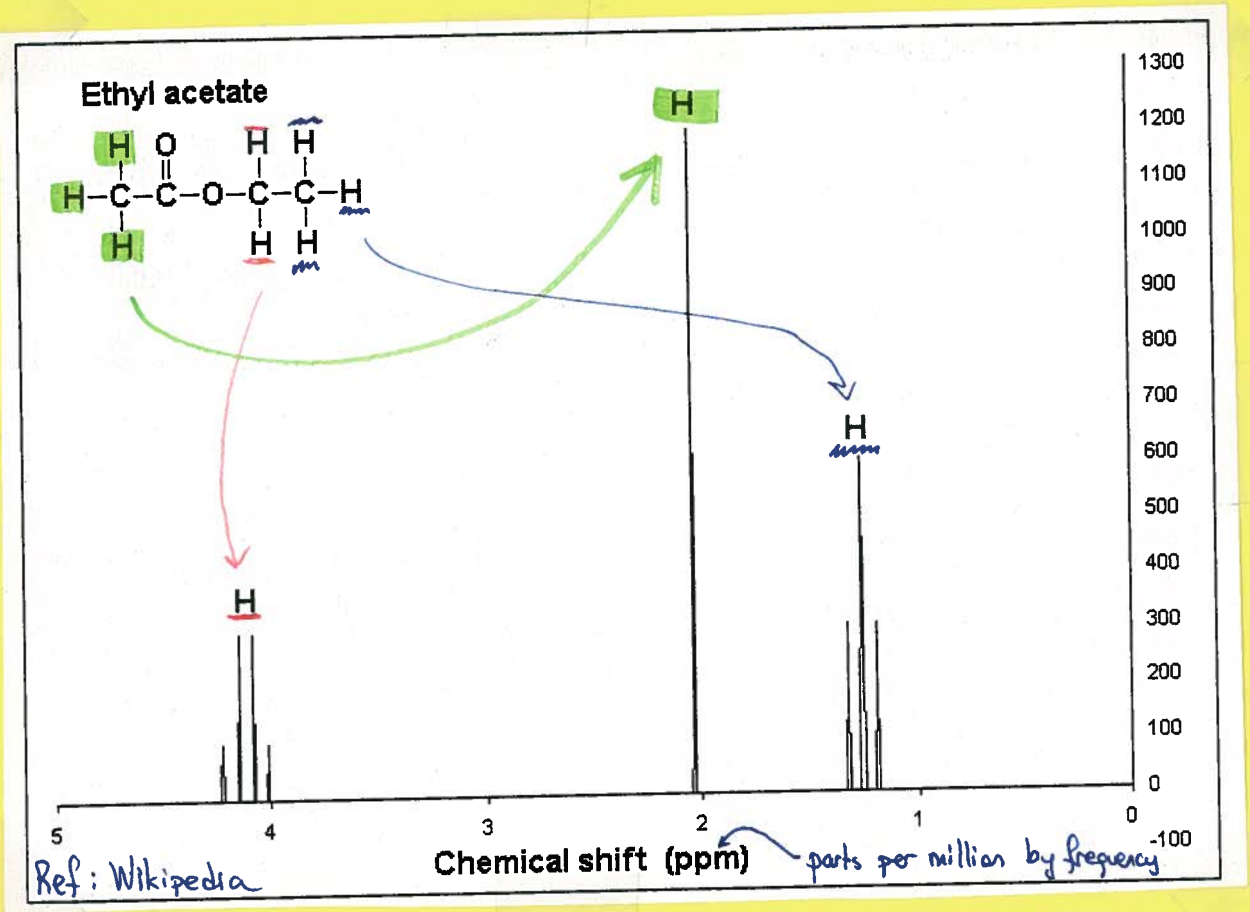


Ref: Young & Freedman 'University Physics' Vol. III.

The MRI imaging (which is exactly NMR) has the advantage over X-ray, γ-ray, or positron emission tomography (PET-scan) in that the patient is not exposed to ionizing radiation. In medical applications it is the water (H_2O) or rather its proton distribution is imaged over the body.

Applications of NMR in Chemistry & Materials Science

The role of gradient coils that differentiate slightly the target nuclei, is taken over by the so-called chemical shift in the case of chemical and materials investigations. The local chemical environment the targetted nucleus is in exerts a shift in its Larmor frequency, mainly due to ē distribution and its screening of the nuclear moments. This is illustrated in the following figure, where there are three different chemical shifts for the H nuclei:



As H is one of the highest magnetic nuclei, the proton-NMR is very commonly used. Moreover most organic compounds have H terminations.

In cases when the solvents' protons are desired to be masked, D_2O (heavy water) can be used where $^2_1H \equiv D$ has a different gyromagnetic ratio than ^1_1H. For organic compounds that are deficient of H, the ^{13}C-NMR can be used

Electron Spin Resonance (ESR) – a.k.a. electron paramagnetic resonance

This is just like NMR with the main difference being the use of electron spins (as opposed to nuclear spins). As the electronic magnetization is 10^3 times stronger than nuclear magnetism (m_e vs. m the signals are much stronger. The downside is that even though most nuclei (or at least some of their stable isotopes) have nonzero spins, only a few materials have unpaired electronic spins (free radicals, paramagnets).

Under the same $\vec{B}_0$ field, if the nuclear resonances are in the MHz range, than the electronic ones are in GHz range.

Biological Effects of Radiation

Ionizing Radiation: α, β, and neutrons and EM waves in the X-ray and γ-ray energies, as they pass through matter, they lose energy and break molecular bonds and creating ions. These interactions are extremely complex and heavily dose-dependent, ranging from (mild) causing burn (as in sunburn) to (severe) alterations of genetic material and the destruction of the components in bone marrow that produce red blood cells.

Radiation Doses

The SI unit of radioactivity is becquerel (Bq)

$$1\ Bq = 1\ \text{nuclear decay}/s$$

This is an extremely small unit of radioactivity.

The absorbed dose of radiation is defined as the energy delivered to the tissue per unit mass. SI unit is gray (Gy)

$$1\ Gy = 1\ J/kg$$

However, the absorbed dose by itself is not an adequate measure of biological effect because equal energies of different kinds of radiation cause different extents of biological effect. This variation is described by a numerical factor called relative biological effectiveness (RBE).

Hence, the biological effect is described in SI units by sievert (Sv)

$$\text{Equivalent dose (Sv)} = \text{RBE} \times \text{Absorbed dose (Gy)}$$

To put this into some perspective, let's list some radiation exposure values:

* A typical chest X-ray delivers a dose of 60 μSv

* Workers with occupational exposure to radiation are permitted 20 mSv/year

* For an average person the average annual radiation dose is 3.6 mSv.

  This is divided as follows (per year)

Radon:	2.0 mSv
X-rays:	0.4
food:	0.4
Earth:	0.3
Cosmic Rays:	0.3
Medical:	0.1

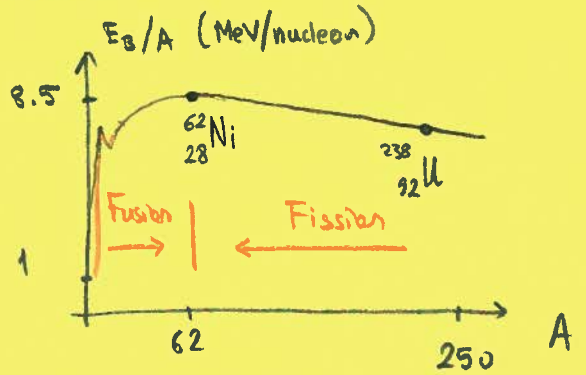
Radon

decay of ^{226}Ra continuously produces ^{222}Rn which is an inert, odorless radioactive gas. ^{226}Ra is found in minute quantities in the rocks and soil on which some houses are built.

* In the 2011 Fukushima accident highest reported level was 433 mSv/h

Nuclear Energy

The binding energy per nucleon curve illustrates the main trends in nuclear reactions.



For light mass number nuclides fusion is energetically favorable and for heavy nuclides their fission becomes spontaneous (or almost spontaneous)

Nuclear Fission

* Fission was discovered in 1938 through the experiments of Otto Hahn and Fritz Strassman in Germany where they achieved the fission of uranium. Lise Meitner helped them to identify that the products were Kr and Ba.

* Both the common isotope (99.3 %) ^{238}U and the uncommon isotope (0.7 %) ^{235}U can easily be split by neutron bombardment:

^{235}U : slow neutrons w/ K.E < 1eV

^{238}U : fast n's K.E > 1 MeV

This is called induced fission. Some nuclides can also undergo spontaneous fission w/o initial n absorption, but this is quite rare.

Typical ^{235}U fission reactions:

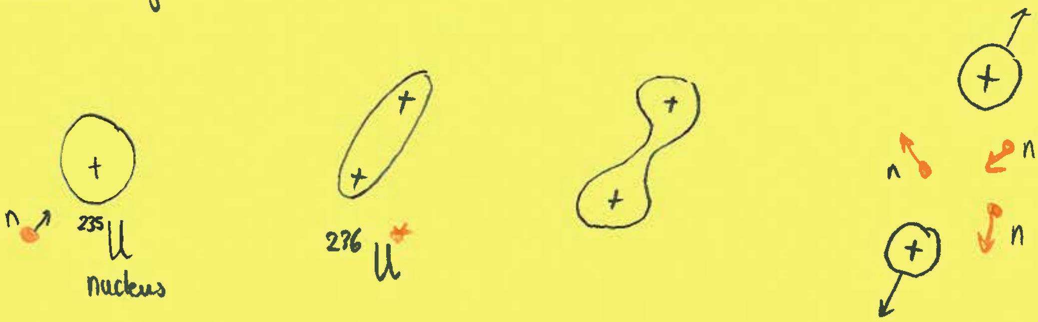
excited st.

$$^{235}_{92}U + {}^{1}_{0}n \rightarrow {}^{236}_{92}U^{*} \rightarrow {}^{144}_{56}Ba + {}^{89}_{36}Kr + 3\,{}^{1}_{0}n$$

$$^{235}_{92}U + {}^{1}_{0}n \rightarrow \underbrace{{}^{236}_{92}U^{*}}_{\text{unstable}} \rightarrow {}^{140}_{54}Xe + {}^{94}_{38}Sr + 2\,{}^{1}_{0}n$$

on avg. 2.5 n's

Note that the first reaction produces more neutrons which is more favorable in sustaining a chain reaction.



Typical Energy in Fission:

The binding energy per nucleon for heavy nuclei (mass ~ 240): ~ 7.2 MeV

" " " " " " intermediate mass : ~ 8.2 MeV

For a total of 240 nucleons, the energy released in a fission

$$Q = 240 \cdot (8.2 - 7.2) = 240 \text{ MeV}/\text{atom}$$

Compare this with a chemical reaction, burning of uranium:

$$U + O_2 \rightarrow \overbrace{UO_2}^{\text{uranium dioxide}} \quad \text{which releases} \quad 11 \text{ eV}/\text{atom}$$

So, nuclear reaction is nearly 20 million times as much energy!

Nuclear Reactors

Uranium that is used in reactors is often "enriched" by increasing the proportion of ^{235}U above the natural value of 0.7%, typically to 3% or so, by isotope-separation process. The weapon-grade uranium contains 80% or more ^{235}U.

In a reactor, a controlled chain reaction is needed. On average each fission of a ^{235}U nucleus produces 2.5 free n's, so 40% of n's are needed to sustain a chain reaction. In a reactor the high-energy n's (> MeV) are slowed down (so that they are more likely to cause fission) by collisions w/ nuclei in the surrounding material, called moderator (often, water or graphite).

The rate of reactions is controlled by inserting/withdrawing control rods made of elements such as Cd or B, whose nuclei absorb n's w/o undergoing any additional reaction.

A typical nuclear plant has an electric-generating capacity of 1 GW.

In modern nuclear plants the overall efficiency is about 1/3, so 3 GW of thermal power from the fission reaction is needed to generate 1 GW of electric power.

A simple calculation shows that Earth uranium reserves and costs do not add up to a uranium-based sustainable energy supply. Thorium in this respect has much better propects, both energetically & environmentally.

Nuclear Fusion

* When two nuclei combine to form a heavier nuclei below $A = 56$, energy is released. The process is the opposite of fission.

* This is the main radiation mechanism for the stars. Recall that we considered CNO cycle and p-p chain.
* Most promising reactions for fusion reactors are:

$$^{2}_{1}D + ^{2}_{1}D \longrightarrow ^{3}_{2}He + ^{1}_{0}n \qquad Q = 3.27 \text{ MeV}$$

$$^{2}_{1}D + ^{2}_{1}D \longrightarrow ^{3}_{1}T + ^{1}_{1}H \qquad Q = 4.03 \text{ MeV}$$

$$^{2}_{1}D + ^{3}_{1}T \longrightarrow ^{4}_{2}He + ^{1}_{0}n \qquad Q = 17.59 \text{ MeV}$$

* Deuterium is abundant on Earth, but tritium is radioactive with $T_{1/2} = 12.3$ yr. through β-decay to ^{3}He. So $^{3}_{1}T$ is rare on Earth.

* One of the major problems to achieve fusion reactors is to give to the nuclei enough K.E. to overcome the repulsive Coulomb force.

* The largest nuclear (fusion) reactor in "town" is the Sun!

* An international nuclear fusion project, 'International Thermonuclear Experimental Reactor – ITER is under construction in France, to be completed in 2019. The fuel will be mixed deuterium and tritium to be heated to 150 million °C, forming a hot plasma; strong $\vec{B}$ fields will confine the plasma away from the walls.

Isospin (From: Griffiths)

Shortly after the discovery of neutron in 1932, Heisenberg observed that, apart from the obvious fact that it carries no charge, it is almost identical to the proton. In particular, their masses are astonishingly close.

$M_p = 938.28 \text{ MeV}/c^2$, $M_n = 939.57 \text{ MeV}/c^2$, Heisenberg proposed that we regard them as two 'states' of a single particle, the nucleon. If we could somehow 'turn off' all electric charge, the proton and neutron would, according to Heisenberg, be indistinguishable. Or, to put it more prosaically, the strong forces experienced by p's and n's are identical.

Accordingly, we write the nucleon as a two-component column matrix

$$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{with} \quad p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By direct analogy with spin, $\vec{S}$, we are led to introduce 'isospin' (also called isobaric spin by nuclear physicists), $\vec{I}$. Unlike spin, $\vec{S}$, $\vec{I}$ is not a vector in ordinary space, with components along the coord. dir's x, y, and z, but rather dimensionless and in an abstract 'isospin space', with components I_1, I_2 and I_3. On this understanding, we may borrow the entire apparatus of angular momentum. The nucleon carries isospin $1/2$

$$p = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle, \quad n = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

I_3: isospin up $\qquad$ I_3: isospin down

Strong interactions are invariant under rotations in isospin space, so by Noether's theorem, isospin is conserved in all strong interactions.

Since isospin pertains only to the strong forces, it is not a relevant quantity for leptons. For consistency, all leptons and mediators are assigned isospin zero. Coming to quarks, u and d flavors have (like the proton & neutron):

$$u = \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle, \quad d = \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

and all other flavors carry isospin zero.

Dynamical Implications of Isospin

For two nucleons, from the rules for addition of angular momenta we know that combination will be either isospin 1 or 0.

$$\left.\begin{aligned} |1\ 1\rangle &= pp \\ |1\ 0\rangle &= \frac{1}{\sqrt{2}}(pn + np) \\ |1\ -1\rangle &= nn \end{aligned}\right\} \text{symmetric isotriplet} \qquad |0\ 0\rangle = \frac{1}{\sqrt{2}}(pn - np) \ \text{antisymm. isosinglet}$$

Experimentally, n & p form a single bound state, the deuteron; there is no bound state for two protons or two neutrons. Thus the deuteron must be an isosinglet. Evidently, there is a strong attraction in the $I=0$ channel, but not in the $I=1$ channel. Isospin invariance has further implications for nucleon-nucleon scatterings ...

Symmetries & Conservation Laws in Particle Physics

Noether's theorem states that there is a one-to-one correspondence bet. conservation laws and differentiable (i.e., continuous) symmetries of physical systems. Invariance under:

* Spatial translations → linear momentum
* Time displacements → mass-energy
* Angular rotations → angular momentum
* Lorentz transformations → Lorentz covariance

Kinematic conservation laws: applies to all interactions – strong, EM weak, gravity and anything to be found in the future

There are also "internal" symmetry transformations that do not mix fields with different spacetime properties, that is, transformations that commute with the spacetime components of the wavefunction.

The (partial) list of such conservation laws that have never been shown to be violated (– till now –), i.e., exact laws:

* electric charge
* color charge: a property of quarks & gluons ⇒ coupling w/ strong interaction
* Weak isospin: is a complement of weak hypercharge, which unifies weak interactions with EM interactions
* CPT symmetry: simultaneous inversion of charge conjugation, parity, time

C → particle-antiparticle conjugation

P → inversion of the coord axes: $\vec{r} \to -\vec{r}$

T → time reversal: $t \to -t$

There are also approximate conservation laws which are true only for certain interactions, but violated by others. In others words, the underlying symmetries are broken by some interactions

* Baryon Number, $B = \frac{1}{3}\left(n_q - n_{\bar{q}}\right)$

quarks # antiquarks

This is conserved in nearly all interactions of the Standard Model. (i.e., B.# of all incoming particles is the same as sum of B # resulting). Proton decay would be an example of its violation (but not observed). The only known violation is chiral anomaly.

* Lepton Number, $L = n_\ell - n_{\bar{\ell}}$

$$\begin{cases} \text{leptons have} & +1 \\ \text{anti leptons} \;\; ,, & -1 \\ \text{non-leptons :} & 0 \end{cases}$$

The neutrino oscillations is an example of violation of this conservation

* Flavor

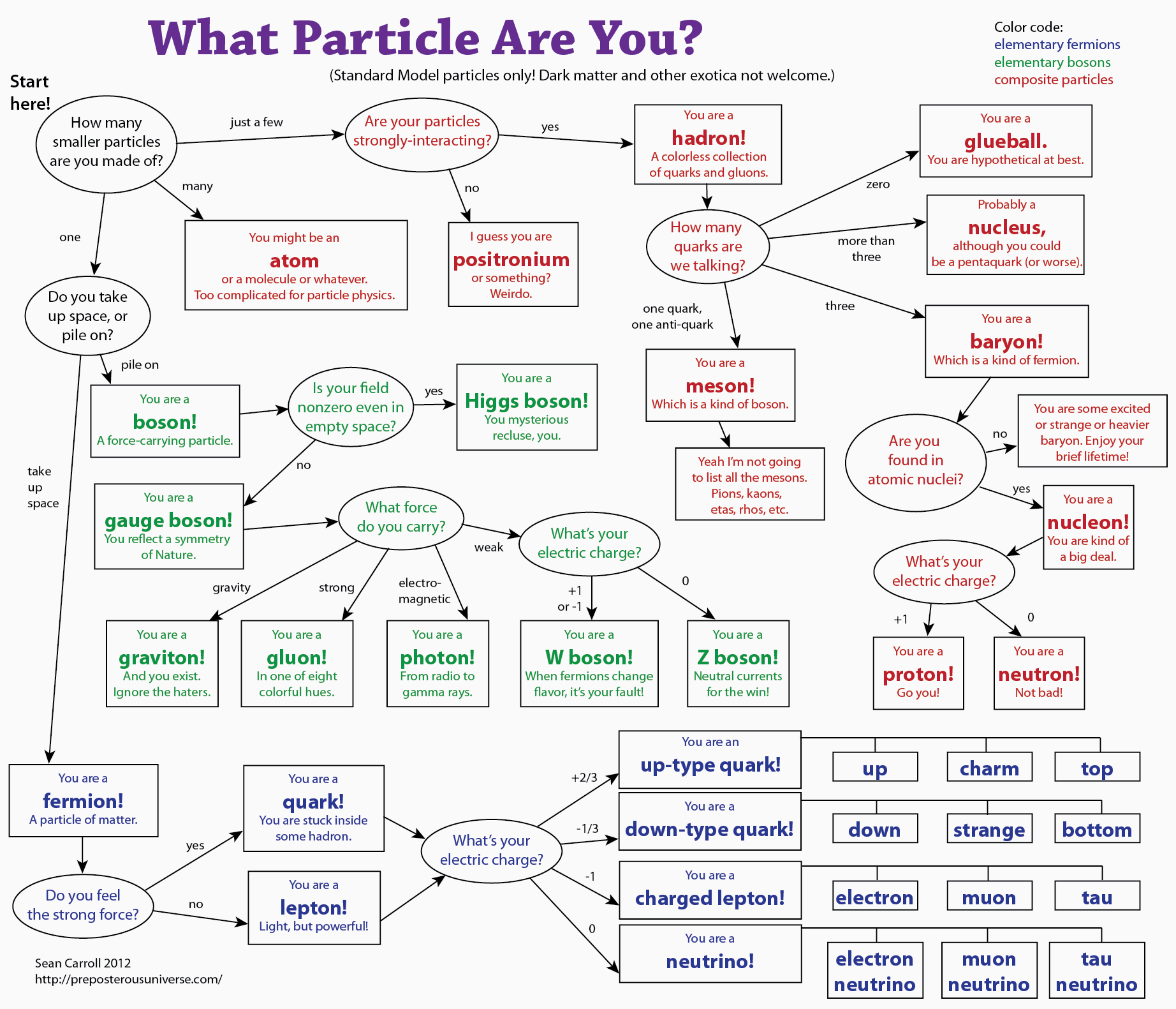
Strong interactions conserve all flavors, but weak interactions don't. (such as neutrino oscillations)

* Parity $\vec{r} \to -\vec{r}$

Conserved by EM, strong interaction and gravity but violated in weak interactions. Lee & Yang proposed its violation in 50's and shown by Wu.

* Charge conjugation
* Time reversal

} again violated by only the weak interaction


What Particle Are You?
(Standard Model particles only! Dark matter and other exotica not welcome.)
Color code:
elementary fermions
elementary bosons
composite particles
Start here!
How many smaller particles are you made of?
just a few
many
one
Are your particles strongly-interacting?
yes
no
You are a hadron! A colorless collection of quarks and gluons.
You might be an atom or a molecule or whatever. Too complicated for particle physics.
I guess you are positronium or something? Weirdo.
How many quarks are we talking?
zero
more than three
three
one quark, one anti-quark
You are a glueball. You are hypothetical at best.
Probably a nucleus, although you could be a pentaquark (or worse).
You are a baryon! Which is a kind of fermion.
You are a meson! Which is a kind of boson.
Yeah I'm not going to list all the mesons. Pions, kaons, etas, rhos, etc.
Are you found in atomic nuclei?
no
yes
You are some excited or strange or heavier baryon. Enjoy your brief lifetime!
You are a nucleon! You are kind of a big deal.
What's your electric charge?
+1
0
You are a proton! Go you!
You are a neutron! Not bad!
Do you take up space, or pile on?
pile on
take up space
You are a boson! A force-carrying particle.
Is your field nonzero even in empty space?
yes
no
You are a Higgs boson! You mysterious recluse, you.
You are a gauge boson! You reflect a symmetry of Nature.
What force do you carry?
gravity
strong
electro-magnetic
weak
What's your electric charge?
+1 or -1
0
You are a graviton! And you exist. Ignore the haters.
You are a gluon! In one of eight colorful hues.
You are a photon! From radio to gamma rays.
You are a W boson! When fermions change flavor, it's your fault!
You are a Z boson! Neutral currents for the win!
You are a fermion! A particle of matter.
Do you feel the strong force?
yes
no
You are a quark! You are stuck inside some hadron.
You are a lepton! Light, but powerful!
What's your electric charge?
+2/3
-1/3
-1
0
You are an up-type quark!
up
charm
top
You are a down-type quark!
down
strange
bottom
You are a charged lepton!
electron
muon
tau
You are a neutrino!
electron neutrino
muon neutrino
tau neutrino
Sean Carroll 2012
http://preposterousuniverse.com/

Standard Model of

FUNDAMENTAL PARTICLES AND INTERACTIONS

The Standard Model summarizes the current knowledge in Particle Physics. It is the quantum theory that includes the theory of strong interactions (quantum chromodynamics or QCD) and the unified theory of weak and electromagnetic interactions (electroweak). Gravity is included on this chart because it is one of the fundamental interactions even though not part of the "Standard Model."

FERMIONS

matter constituents
spin = 1/2, 3/2, 5/2, ...

Leptons spin = 1/2

Flavor	Mass GeV/c^2	Electric charge
ν_e electron neutrino	$<1\times10^{-8}$	0
e electron	0.000511	–1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	–1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	–1

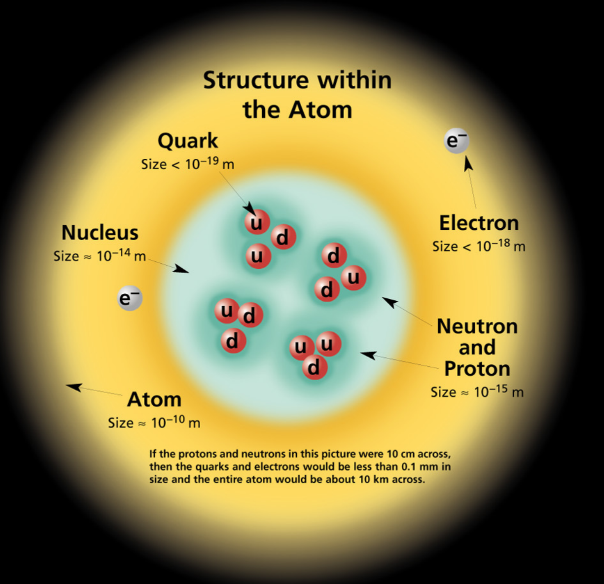
Quarks spin = 1/2

Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.003	2/3
d down	0.006	–1/3
c charm	1.3	2/3
s strange	0.1	–1/3
t top	175	2/3
b bottom	4.3	–1/3

Spin is the intrinsic angular momentum of particles. Spin is given in units of $\hbar$, which is the quantum unit of angular momentum, where $\hbar = h/2\pi = 6.58\times10^{-25}$ GeV s $= 1.05\times10^{-34}$ J s.

Electric charges are given in units of the proton's charge. In SI units the electric charge of the proton is 1.60×10^{-19} coulombs.

The **energy** unit of particle physics is the electronvolt (eV), the energy gained by one electron in crossing a potential difference of one volt. **Masses** are given in GeV/c^2 (remember $E = mc^2$), where 1 GeV $= 10^9$ eV $= 1.60\times10^{-10}$ joule. The mass of the proton is 0.938 GeV/c^2 $= 1.67\times10^{-27}$ kg.



BOSONS

force carriers
spin = 0, 1, 2, ...

Unified Electroweak spin = 1

Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.4	–1
W^+	80.4	+1
Z^0	91.187	0

Strong (color) spin = 1

Name	Mass GeV/c^2	Electric charge
g gluon	0	0

Color Charge

Each quark carries one of three types of "strong charge," also called "color charge." These charges have nothing to do with the colors of visible light. There are eight possible types of color charge for gluons. Just as electrically-charged particles interact by exchanging photons, in strong interactions color-charged particles interact by exchanging gluons. Leptons, photons, and ***W*** and ***Z*** bosons have no strong interactions and hence no color charge.

Quarks Confined in Mesons and Baryons

One cannot isolate quarks and gluons; they are confined in color-neutral particles called **hadrons**. This confinement (binding) results from multiple exchanges of gluons among the color-charged constituents. As color-charged particles (quarks and gluons) move apart, the energy in the color-force field between them increases. This energy eventually is converted into additional quark-antiquark pairs (see figure below). The quarks and antiquarks then combine into hadrons; these are the particles seen to emerge. Two types of hadrons have been observed in nature: **mesons** $q\bar{q}$ and **baryons** qqq.

Residual Strong Interaction

The strong binding of color-neutral protons and neutrons to form nuclei is due to residual strong interactions between their color-charged constituents. It is similar to the residual electrical interaction that binds electrically neutral atoms to form molecules. It can also be viewed as the exchange of mesons between the hadrons.

PROPERTIES OF THE INTERACTIONS

Property \ Interaction	Gravitational	Weak (Electroweak)	Electromagnetic (Electroweak)	Strong: Fundamental	Strong: Residual
Acts on:	Mass – Energy	Flavor	Electric Charge	Color Charge	See Residual Strong Interaction Note
Particles experiencing:	All	Quarks, Leptons	Electrically charged	Quarks, Gluons	Hadrons
Particles mediating:	Graviton (not yet observed)	W^+ W^- Z^0	γ	Gluons	Mesons
Strength relative to electromag for two u quarks at: 10^{-18} m	10^{-41}	0.8	1	25	Not applicable to quarks
Strength relative to electromag for two u quarks at: 3×10^{-17} m	10^{-41}	10^{-4}	1	60	
for two protons in nucleus	10^{-36}	10^{-7}	1	Not applicable to hadrons	20

Baryons qqq and Antibaryons $\bar{q}\bar{q}\bar{q}$

Baryons are fermionic hadrons.
There are about 120 types of baryons.

Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
p	proton	uud	1	0.938	1/2
$\bar{p}$	anti-proton	$\bar{u}\bar{u}\bar{d}$	–1	0.938	1/2
n	neutron	udd	0	0.940	1/2
Λ	lambda	uds	0	1.116	1/2
Ω^-	omega	sss	–1	1.672	3/2

Mesons $q\bar{q}$

Mesons are bosonic hadrons.
There are about 140 types of mesons.

Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
π^+	pion	$u\bar{d}$	+1	0.140	0
K^-	kaon	$s\bar{u}$	–1	0.494	0
ρ^+	rho	$u\bar{d}$	+1	0.770	1
B^0	B-zero	$d\bar{b}$	0	5.279	0
η_c	eta-c	$c\bar{c}$	0	2 .980	0

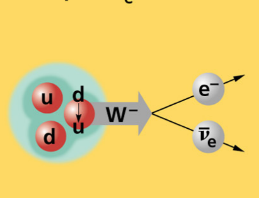
Matter and Antimatter

For every particle type there is a corresponding antiparticle type, denoted by a bar over the particle symbol (unless + or – charge is shown). Particle and antiparticle have identical mass and spin but opposite charges. Some electrically neutral bosons (e.g., Z^0, γ, and $\eta_c = c\bar{c}$, but not $K^0 = d\bar{s}$) are their own antiparticles.

Figures

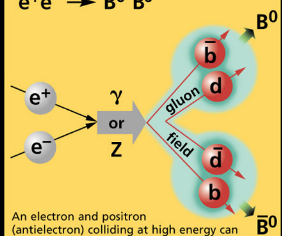
These diagrams are an artist's conception of physical processes. They are *not* exact and have *no* meaningful scale. Green shaded areas represent the cloud of gluons or the gluon field, and red lines the quark paths.

$n \rightarrow p\, e^-\, \bar{\nu}_e$



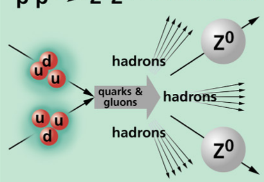
A neutron decays to a proton, an electron, and an antineutrino via a virtual (mediating) W boson. This is neutron β decay.

$e^+e^- \rightarrow B^0\, \bar{B}^0$



An electron and positron (antielectron) colliding at high energy can annihilate to produce B^0 and $\bar{B}^0$ mesons via a virtual Z boson or a virtual photon.

$p\, p \rightarrow Z^0 Z^0$ + assorted hadrons



Two protons colliding at high energy can produce various hadrons plus very high mass particles such as Z bosons. Events such as this one are rare but can yield vital clues to the structure of matter.

The Particle Adventure

Visit the award-winning web feature *The Particle Adventure* at
http://ParticleAdventure.org

This chart has been made possible by the generous support of:
U.S. Department of Energy
U.S. National Science Foundation
Lawrence Berkeley National Laboratory
Stanford Linear Accelerator Center
American Physical Society, Division of Particles and Fields
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The Expanding Universe

There are remarkably close ties bet. physics on the smallest scale (particle physics) and physics on the largest scale (cosmology).

Until early in the 20th century it was assumed that the universe was static. But if everything is initially sitting still in the universe, why doesn't gravity just pull it all together into one big clump? [Can we all delegate it to the elliptical orbits of the celestial objects?] Newton himself recognized the seriousness of this troubling question.

The motions of galaxies relative to Earth can be observed by observing the shifts in the wavelengths of their spectra. For receding objects λ redshifts

$$\underset{\text{velocity}}{v} = \frac{(\lambda_0/\lambda_s)^2 - 1}{(\lambda_0/\lambda_s)^2 + 1}\, c$$

λ_0: measured while receding

λ_s: measured in rest frame of obj.

This expression above is the Doppler shift (i.e. special relativistic effect). The redshift from distant galaxies is actually caused by an effect explained by general relativity and is not a Doppler shift. But for $\frac{v}{c} \ll 1$ those expressions approach to that of the Doppler shift expression.

The analysis of redshifts from many distant galaxies yielded

$$v = H_0 r \quad \ldots \quad \text{Hubble Law}$$

H_0: Hubble constant

more distant galaxies are receding with even faster speeds!

Another aspect is that this holds in <u>all directions</u>. At any given time,
'the universe looks more or less the same, no matter where in the universe we are.'
This is the cosmological principle; the laws of physics are the same everywhere.

Critical Density

We need to look at the role of gravity in an expanding universe. For strong enough attractions, the universe should expand more and more slowly, eventually stop, and then begin to contract to perhaps a Big Crunch.
The situation is analogous to the problem of escape speed of a projectile launched from Earth.

$$E = \frac{1}{2} m v^2 - \frac{GmM}{R}$$

If the total energy, E for our galaxy (universe) has enough energy to escape from the gravitational attraction of the mass M inside the sphere; in this case universe should keep expanding for ever.

Use $v = H_0 R$ (Hubble Law)

$$\frac{1}{2} \cancel{m} (H_0 R)^2 = \frac{G\cancel{m}}{R} \left(\frac{4}{3} \pi R^3 \rho_c \right)$$

$$\Rightarrow \quad \rho_c = \frac{3 H_0^2}{8 \pi G} \quad \ldots \text{critical density} = 9.5 \times 10^{-27} \, kg/m^3 = \frac{6 \text{ H atoms}}{m^3}$$

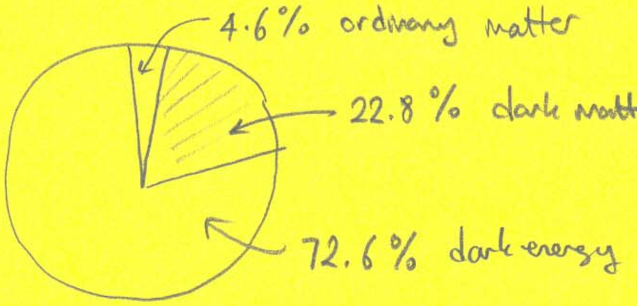
Dark Matter, Dark Energy

Astronomers have made extensive studies of the average density of matter in the universe. According to average density of luminous matter (that is, matter that emits EM radiation) is only 4.6 % of the critical density ρ_c, whereas that of all matter, i.e., including dark matter that does not radiate is 27.4 %. We know about dark matter by their gravitational effect as measured by redshifts. So, what is the nature of this dark matter? They cannot be any matter we know of: Cosmological models that are convincingly corroborated by the observed abundances of light elements do not allow for anywhere near enough baryons to account for dark matter. One of the prime aims of LHC is to get some clues for new particles to solve the dark matter mystery.

According to the above numbers, as the avg. density of matter in the universe is less than ρ_c, the universe will continue to expand indefinitely, and that gravitational attraction bet. matter in different parts of the universe should slow the expansion down (albeit not enough to stop it).

<u>But</u>, according to measurements since 1998 on distant galaxies, the expansion is speeding up (rather than slowing down). The explanation generally accepted is that space is stuffed with a kind of energy that

has no gravitational effect and emits no EM radiation, but rather acts as a kind of "antigravity" that produces a universal repulsion, called dark energy. Observations show that its energy density is 72.6% of the critical density times c^2. Because the energy density of dark energy is nearly x3 greater than that of matter, the expansion of the universe will continue to accelerate.



So, we can say that the physics we know only accounts for the 4.6% of the universe. The nature of dark matter and dark energy are fields of active research.

Temperature Scale

In the following discussions of the chronology of the universe, we shall use absolute temperature, as a means to quantify the energy through the relation $E \to k_B T$, $k_B : 8.617 . 10^{-5}\ eV/K$

For instance, ionization energy of H, 13.6 eV $\to 10^5$ K,

rest energy of $\bar{e}$, 0.511 MeV $\to 10^{10}$ K,

" " " p , 938 MeV $\to 10^{13}$ K

In the interior of Sun, temperatures in excess of 10^5 K are found, so most of H there is ionized. However, 10^{10} K or 10^{13} K are not found anywhere in the solar system; such high T's were available in the very early universe.

Cooling & Uncoupling of Interactions

Under the expansion of the universe, the total gravitational potential energy increases, so that the average K.E. hence the temperature of the particles **decreases**. As this happened, the fundamental forces of nature became progressively uncoupled. For instance, γ, $\underbrace{W^{\pm}, Z^{0}}$ mediators of the unified electroweak interaction have masses 0 and 100 GeV/c^2, resp.

At energies much less than 100 GeV, the EM and weak interactions seem quite different, but at energies much greater than 100 GeV they become part of a single interaction.

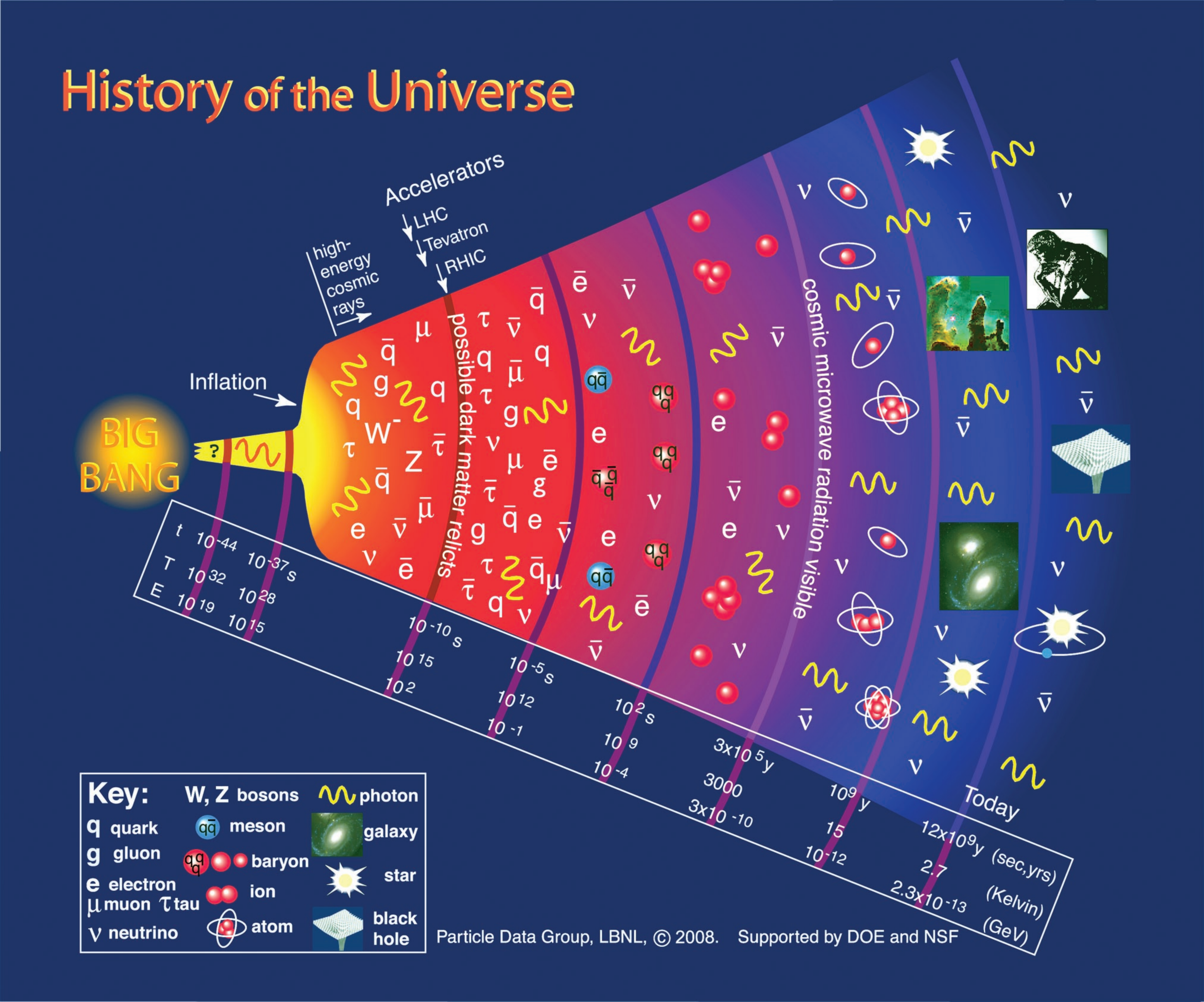
The grand unified theories (GUTs) provide similar behavior for the strong force. It becomes unified at energies of the order of 10^{14} GeV, but at lower energies the two appear quite distinct. Since current accelarator can only generate ~ 10 TeV they are 10^{10} factor away from verifying this unification, therefore such GUTs are still speculative.

Planck Scale

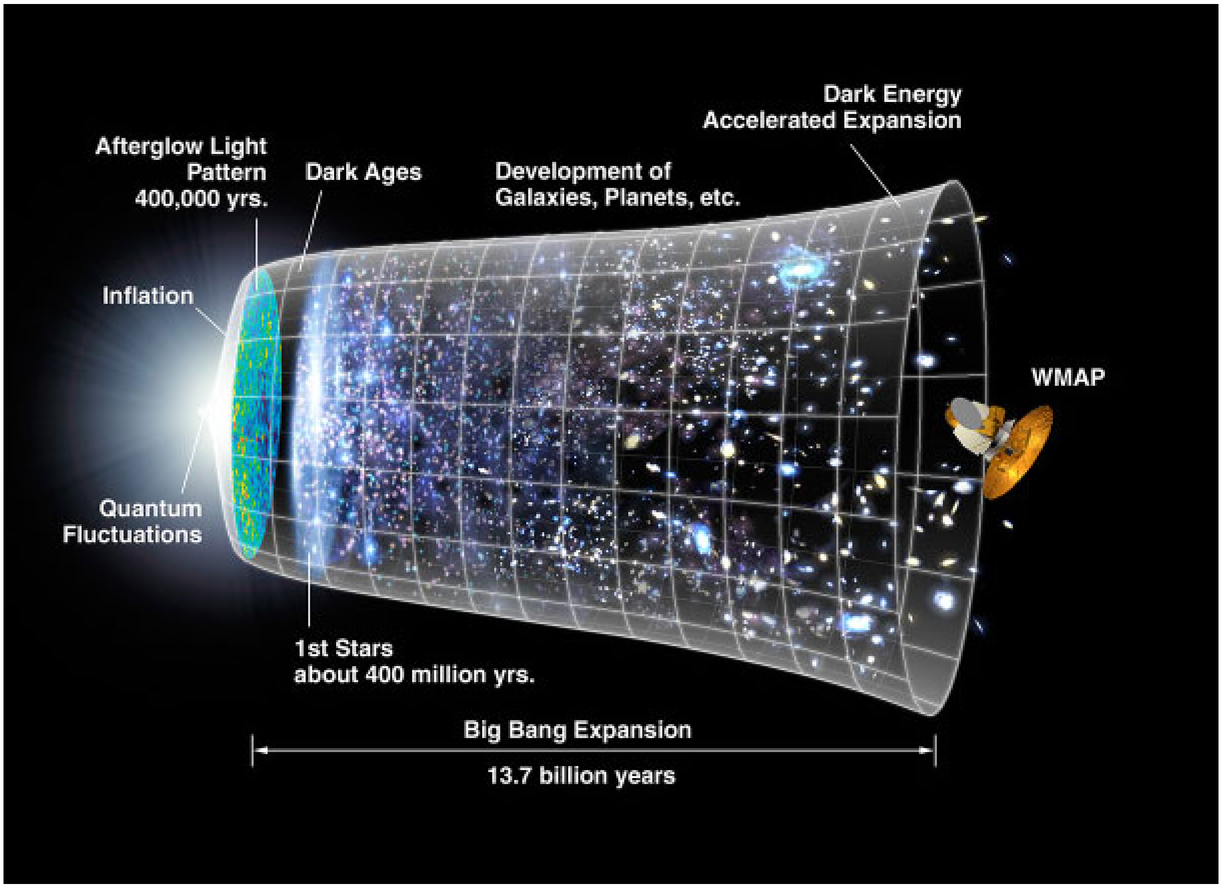
Using fundamental constants, we can introduce to so-called Planck $\left\{ \begin{array}{l} \text{length} \\ \text{time} \end{array} \right.$

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \cdot 10^{-35}\, m \,, \quad t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} = 0.539 \cdot 10^{-43}\, s$$

This is the scale beyond which our current theories breakdown. We have no way of predicting what might have happened at times earlier than 10^{-43} s or when its size was less than 10^{-35} m. A theory on quantum gravity is much needed to speculate on the Planck scale physics.


History of the Universe
BIG BANG
?
Inflation
high-energy cosmic rays
Accelerators
LHC
Tevatron
RHIC
possible dark matter relicts
cosmic microwave radiation visible
t 10^{-44} 10^{-37} s
T 10^{32} 10^{28}
E 10^{19} 10^{15}
10^{-10} s
10^{15}
10^{2}
10^{-5} s
10^{12}
10^{-1}
10^{2} s
10^{9}
10^{-4}
$3x10^{5}$ y
3000
$3x10^{-10}$
10^{9} y
15
10^{-12}
Today
$12x10^{9}$ y (sec,yrs)
2.7 (Kelvin)
$2.3x10^{-13}$ (GeV)
Key:
q quark
g gluon
e electron
μ muon τ tau
ν neutrino
W, Z bosons
meson
baryon
ion
atom
photon
galaxy
star
black hole
Particle Data Group, LBNL, © 2008. Supported by DOE and NSF

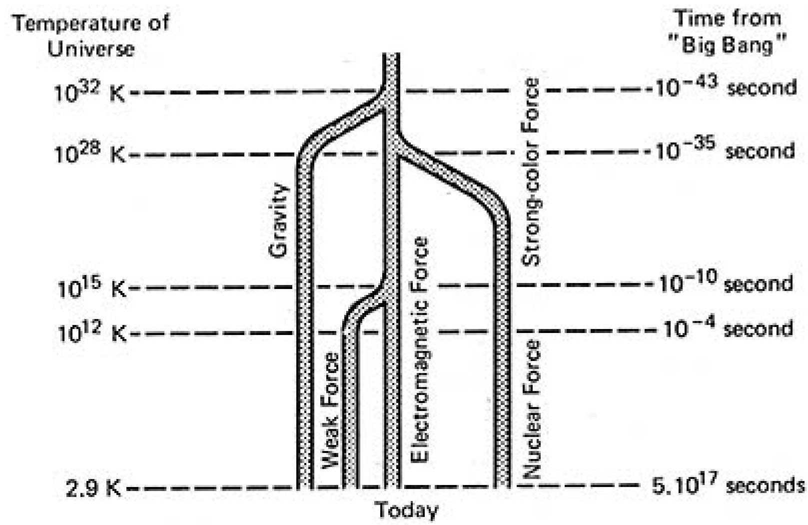
Big Bang (Encyclopedia of Science http://www.daviddarling.info)



Time line of the Universe. The expansion of the universe over most of its history has been relatively gradual. The notion that a rapid period of "inflation" preceded the Big Bang expansion was first put forth 25 years ago. Recent observations, including those by NASA's WMAP orbiting observatory favor specific inflation scenarios over other long held ideas. Credit: NASA

The event in which, according to standard modern cosmology, the Universe came into existence some 13.7 billion years ago. The Big Bang is sometimes described as an "explosion;" however, it is wrong to suppose that matter and energy erupted into a pre-existing space. Modern Big Bang theory holds that space and time came into being simultaneously with matter and energy. The possible overall forms that space and time could take – closed, open, or flat – are described by three different cosmological models.

Creation to inflation



According to current theory, the first physically distinct period in the Universe lasted from "time zero" (the Big Bang itself) to 10^{-43} second later, when the universe was about 100 million trillion times smaller than a proton and had a temperature of 10^{34} K. During this so-called Planck era, quantum gravitational effects dominated and there was no distinction between (what would later be) the four fundamental forces of nature – gravity, electromagnetism, the strong force, and the weak force. Gravity was the first to split away, at the end of the Planck era, which marks the earliest point at which present science has any real understanding. Physicists have successfully developed a theory that unifies the strong, weak, and electromagnetic forces, called the Grand Unified Theory (GUT). The GUT era lasted until about 10^{-38} second after the Big Bang, at which point the strong force broke away from the others, releasing, in the process, a vast amount of energy that, it is believed, caused the Universe to expand at an extraordinary rate. In the brief ensuing interval of so-called inflation, the Universe grew by a factor of 10^{35} (100 billion trillion trillion) in 10^{-32} seconds, from being unimaginably smaller than a subatomic particle to about the size of a grapefruit.

Postulating this burst of exponential growth helps remove two major problems in cosmology: the horizon problem and the flatness problem. The horizon problem is to explain how the cosmic microwave background – a kind of residual glow of the Big Bang from all parts of the sky – is very nearly isotropic despite the fact that the observable universe isn't yet old enough for light, or any other kind of signal, to have traveled from one side of it to the other. The flatness problem is to explain why space, on a cosmic scale, seems to be almost exactly flat, leaving the universe effectively teetering on a knife-edge between eternal expansion and eventual collapse. Both near-isotropy and near-flatness follow directly from the inflationary scenario.

Electroweak era (10^{-38} to 10^{-10} second)

At the end of inflationary epoch, the so-called vacuum energy of space underwent a phase transition (similar to when water vapor in the atmosphere condenses as water droplets in a cloud) suddenly giving rise to a seething soup of elementary particles, including photons, gluons, and quarks. At the same time, the expansion of the universe dramatically slowed to the "normal" rate governed by the Hubble law. At about 10^{-10} seconds, the electroweak force separated into the electromagnetic and weak forces, establishing a universe in which the physical laws and the four distinct forces of nature were as we now experience them.

Particle era (10^{-10} to 1 second)

The biggest chunks of matter, as the Universe ended its first trillionth of a second or so, were individual quarks and their antiparticles, antiquarks – the underlying particles out of which future atoms, asteroids, and astronomers would be made. As time went on, quarks and anti-quarks annihilated each other. However, either because of a slight asymmetry in the behavior of the particles or a slight initial excess of particles over antiparticles, the mutual destruction ended with a surplus of quarks. Only because of this (relatively minor) discrepancy do stars, planets, and human beings exist today.

Between 10^{-6} and 10^{-5} second after the beginning of the Universe, when the ambient cosmic temperature had fallen to a balmy 10^{15} K, quarks began to combine to form a variety of hadrons. All of the short-lived hadrons quickly decayed leaving only the familiar protons and neutrons of which the nuclei of atoms-to-come would be made. This hadron era was followed by the lepton era, during which most of the matter in the Universe consisted of leptons and their antiparticles. The lepton era drew to a close when the majority of leptons and antileptons annihilated one another, leaving, again, a comparatively small surplus to populate the future universe.

One to 100 seconds

Up to this stage, neutrons and protons had been rapidly changing into each other through the emission and absorption of neutrinos. But, by the age of one second, the Universe was cool enough for neutron-proton transformations to slow dramatically. A ratio of about seven protons for every neutron ensued. Since to make a hydrogen nucleus, only one proton is needed, whereas helium requires two protons and two neutrons, a 7:1 excess of protons over neutrons would lead to a similar excess of hydrogen over helium – which is what is observed today. At about the 100-second mark, with the temperature at a mere billion K, neutrons and protons were able to stick together. The majority of neutrons in the Universe wound up in combinations of two protons and two neutrons as helium nuclei. A small proportion of neutrons contributed to making lithium, with three protons and three neutrons, and the leftovers ended up in – an isotope of hydrogen with one proton and one neutron.

The first 10,000 years

Most of the action, at the level of particle physics, was compressed into the first couple of minutes after the Big Bang. Thereafter, the universe settled down to a much lengthier period of cooling and expansion in which change was less frenetic. Gradually, more and more matter was created from the high energy radiation that bathed the cosmos. The expansion of the Universe, in other words, caused matter to lose less energy than did the radiation, so that an increasing proportion of the cosmic energy density came to be invested in nuclei rather than in massless, or nearly massless, particles (mainly photons). From a situation in which the energy invested in radiation dominated the expansion of spacetime, the Universe evolved to the point at which matter became the determining factor. Around 10,000 years after the Big Bang, the radiation era drew to a close and the matter era began.

When the Universe became transparent

About 300,000 years after the Big Bang, when the cosmic temperature had dropped to just 3,000 K, the first atoms formed. It was then cool enough to allow protons to capture one electron each and form neutral atoms of hydrogen. While free, the electrons had interacted strongly with light and other forms of electromagnetic radiation, making the Universe effectively opaque. But bound up inside atoms, the electrons lost this capacity, matter and energy became decoupled, and, for the first time, light could travel freely across space. This, then, marks the earliest point in time to which we can see back. The cosmic microwave background is the greatly redshifted first burst of light to reach us from the early Universe and provides an imprint of what the Universe looked like about a third of a million years after the Big Bang. Fluctuations in the nearly-uniform density of the infant Universe show up as tiny temperature differences in the microwave background from point to point in the sky. These fluctuations are believed to be the seeds from which future galaxies and clusters of galaxies arose.

Relativistic Kinematics

Special Theory of Relativity (STR) is not hard to understand. It is hard to believe! As a consequence of this disbelief, the intuition one uses for solving problems cannot be trusted, and STR gets harder to apply.

In STR the gravity is neglected so that one works in a Euclidean (i.e., flat spacetime) as opposed to Riemanian (curved) spacetime under gravitational warpings.

So in STR, we limit our considerations to so-called Inertial frames of reference, where Newton's law of inertia hold: Bodies in motion continue indefinitely in a straight line unless some force acts.

→ How to check if you are in an inertial frame?

Throw stones in three orthogonal dir's, and if you observe any one bending, it means you are not in an inertial frame.

Einstein based STR on two simple postulates:

1) The speed of light is the same in all inertial frames.

2) The laws of physics should have the same form when they are determined in any inertial frame of reference.

Laws that follow this 2nd postulate are said to be covariant, which means that all of their variables change in coordination (covary) in just the right way to keep their functional forms unchanged under a frame transformation (even though the numerical values of the variables do change).

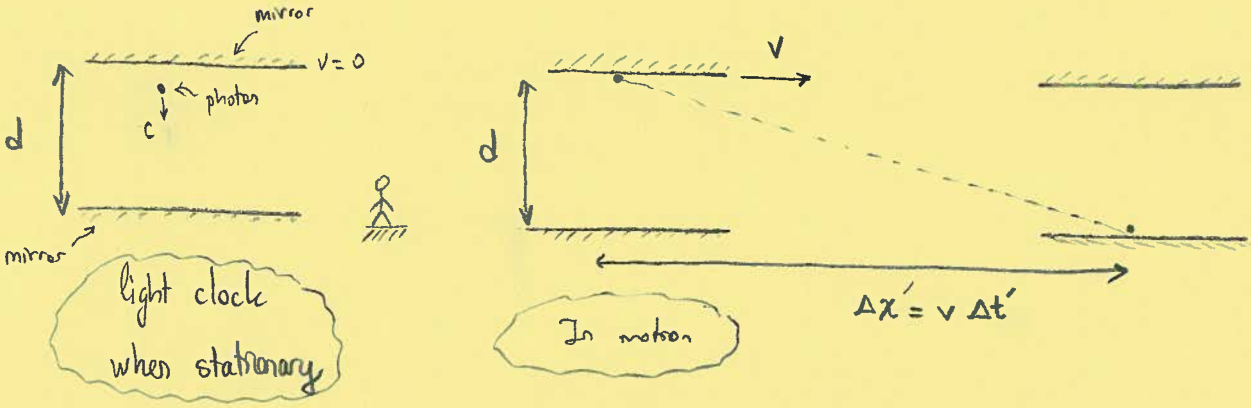
$$\left[\frac{\hbar^2 \nabla^2}{2m} + V(\vec{r})\right]\Psi(\vec{r},t) = i\hbar \frac{\partial}{\partial t}\Psi(\vec{r},t) \quad \Big\}\ \begin{array}{l}\text{1st order in time,} \\ \text{2nd order in space} :-(\end{array}$$

For instance, Schrödinger eq. is not covariant, meaning it is not relativistically correct, whereas the Dirac eqn. is. Maxwell's eq's are covariant even though they were developed before STR (1905). This is because Maxwell's eq's were based on experimental observation (other than the displacement current).

All of STR follows from those two postulates! The constancy of the speed of light in all inertial frames (quite a counter intuitive idea from our daily experiences) results in the mixing of time and space. Basically a stationary observer sees the time on a moving frame running slow and its dimension along the dir. of motion contracted.

We shall demonstrate these using the simplest possible clock one can imagine: a light clock, made up of two mirrors with a single photon bouncing back and forth (tick-tack).

Time Dilation



$$\Delta t = \frac{d}{c}$$

ticking interval

$$\Delta t' = \frac{\sqrt{d^2 + (v\,\Delta t')^2}}{c}$$

$$c^2 \Delta t'^2 = d^2 + v^2 \Delta t'^2$$

$$\Delta t' \left(c^2 - v^2\right)^{1/2} = d$$

$$\Delta t' = \frac{d}{\sqrt{c^2 - v^2}} = \underbrace{\frac{d}{c}}_{\Delta t} \; \underbrace{\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}_{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$$

$$\beta \equiv \frac{v}{c}$$

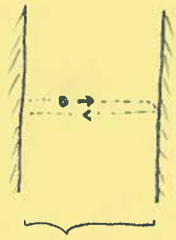
ticking interval dilated wrt Δt $\longrightarrow$ $\Delta t' = \gamma\, \Delta t$

Note that $\gamma \geq 1$, so the time on the moving clock is seen from the stationary frame to be running at a slower rate.

The fact that two observers in relative motion wrt each other do not have an <u>absolute time</u> means, two events happening simultaneously in one frame will not be agreed on the other frame. Farewell to simultaneity & Newton's 3rd law for non-contact forces.

Space Contraction (Lorentz Contraction)

Now rotate the light clock so that the light travels in the dir. of clock motion.

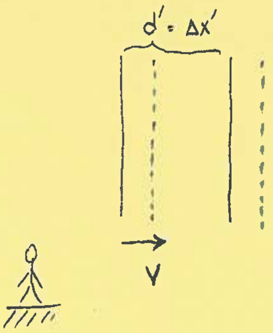


$d = \Delta x$

Mirror Frame

The roundtrip time for an observer on the frame of mirrors is:

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{2d}{c} = 2\,\frac{\Delta x}{c}$$



Stationary Frame

As seen by the ground observer (with the light clock moving with v), the roundtrip time becomes

$d' \longrightarrow \Delta x'$

$$\Delta t' = \Delta t_1' + \Delta t_2' = \frac{\Delta x' + v\,\Delta t_1'}{c} + \frac{\Delta x' - v\,\Delta t_2'}{c}$$

$$\Rightarrow \quad \Delta t_1' = \frac{\Delta x'}{c-v} \quad , \quad \Delta t_2' = \frac{\Delta x'}{c+v}$$

$$\Rightarrow \quad \Delta t' = 2\,\frac{\Delta x'}{c}\,\underbrace{\frac{1}{1-v^2/c^2}}_{\gamma^2} = 2\,\frac{\Delta x'}{c}\,\gamma^2$$

Inserting the time dilation expression from previous consideration: $\Delta t' = \gamma\,\Delta t$

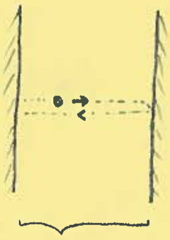
$$\Rightarrow \quad \cancel{\gamma}\,\Delta t = \frac{2}{c}\,\Delta x'\,\gamma^{\cancel{2}} \quad , \quad \Delta x' = \overbrace{\frac{\Delta t\, c}{2}}^{\Delta x}\,\frac{1}{\gamma} = \frac{\Delta x}{\gamma}$$

So, the ground observer sees the moving object contracted by a factor of γ along the dir. of motion.

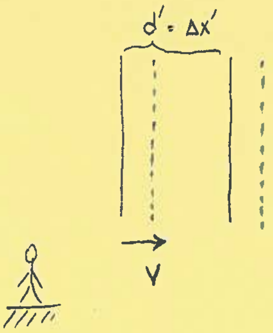
NB: The dimensions $\perp$ to $\vec{v}$ are not affected!

Space Contraction (Lorentz Contraction)

Now rotate the light clock so that the light travels in the dir. of clock motion.



$d = \Delta x$



Mirror Frame

The roundtrip time for an observer on the frame of mirrors is:

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{2d}{c} = 2\,\frac{\Delta x}{c}$$

Stationary Frame

As seen by the ground observer (with the light clock moving with v), the roundtrip time becomes

$d' \longrightarrow \Delta x'$

$$\Delta t' = \Delta t_1' + \Delta t_2' = \frac{\Delta x' + v\,\Delta t_1'}{c} + \frac{\Delta x' - v\,\Delta t_2'}{c}$$

$$\Rightarrow \quad \Delta t_1' = \frac{\Delta x'}{c-v} \quad , \quad \Delta t_2' = \frac{\Delta x'}{c+v}$$

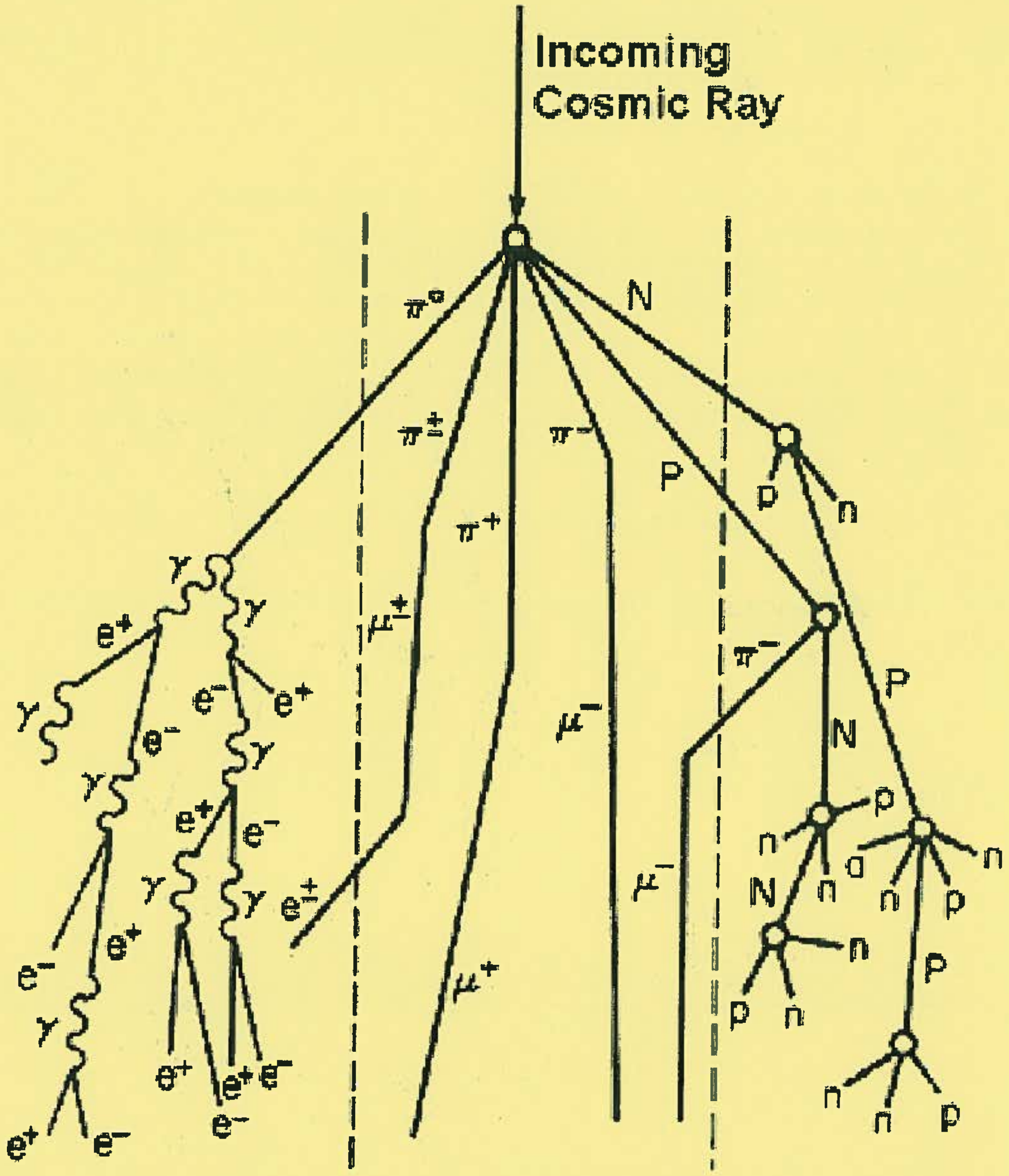
$$\Rightarrow \quad \Delta t' = 2\,\frac{\Delta x'}{c}\underbrace{\frac{1}{1-v^2/c^2}}_{\gamma^2} = 2\,\frac{\Delta x'}{c}\,\gamma^2$$

Inserting the time dilation expression from previous consideration: $\Delta t' = \gamma\,\Delta t$

$$\Rightarrow \quad \cancel{\gamma}\,\Delta t = \frac{2}{c}\,\Delta x'\,\gamma^{\cancel{2}} \quad , \quad \Delta x' = \overbrace{\frac{\Delta t\, c}{2}}^{\Delta x}\,\frac{1}{\gamma} = \frac{\Delta x}{\gamma}$$

So, the ground observer sees the moving object contracted by a factor of γ along the dir. of motion.

NB: The dimensions $\perp$ to $\vec{v}$ are not affected!

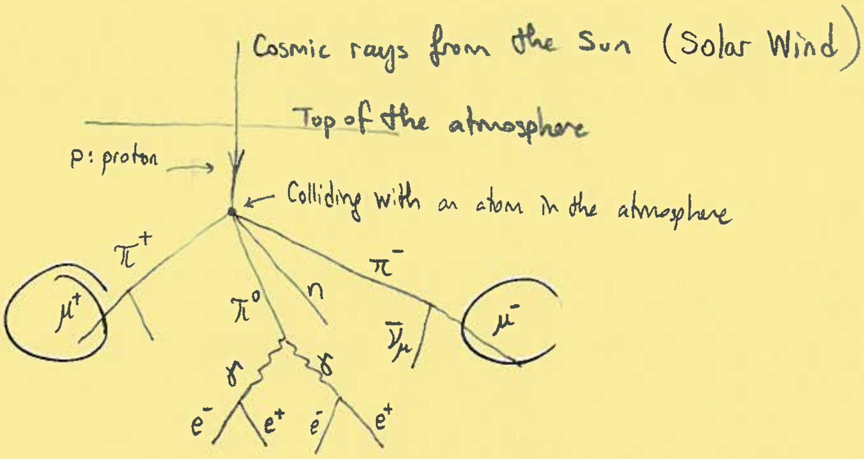


KEY

P	**Proton**	e	**Electron**
n	**Neutron**	μ	**Muon**
π	**Pion**	γ	**Photon**

Source: neutronm.bartol.udel.edu/catch/cr2b.gif

An Example on Cosmic Rays (Griffiths 3.4)



Cosmic ray muons are produced high in the atmosphere (at 8 km, say) and travel toward the Earth at very nearly the speed of light (0.998, say).

The lifetime of $\mu^\pm$ (in its rest frame) is $2.2 \mu s$.

Let's calculate how far it will travel before it decays (into $e^\pm$)

<u>Classical Consideration:</u>

$$d = 2.2 \cdot 10^{-6} \cdot 0.998 \cdot 3 \cdot 10^{8} = 660 \text{ m} \ll 8000 \text{ m}$$

So, a $\mu^\pm$ would <u>not</u> reach the surface of Earth if classical mech. were true!

<u>Relativistic Consideration:</u>

$$v = 0.998 c \Rightarrow \gamma = 15.8$$

i) According to an <u>observer on the ground</u>: the $\mu^\pm$ lifetime will be dilated by γ times.

the distance travelled would be:

$$d = 15.8 \cdot 2.2 \cdot 10^{-6} \cdot 0.998 \cdot 3 \cdot 10^{8} = 10{,}400\,m > 8000\,m$$

so on the average a lot of μ's will reach the ground.

ii) According to <u>muon's frame</u> of reference: this time the distance will be contracted by the same factor γ. So even though in its lifetime it travels 660 m, the distance to Earth will be contracted to $\frac{8000}{15.8} = 506\,m < 660\,m$, hence again we reach to the same conclusion that muons will reach the ground.

to be introduced shortly.

NB: We can calculate the energy of muons from $E_\mu = \gamma m_\mu c^2 \sim 1.7$ GeV

This is not particularly energetic, even so, such "low energy" particles travel close to the speed of light.

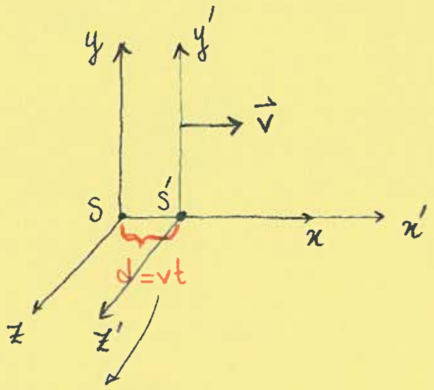
→ What about the pions $\pi^\pm$. Their lifetimes are much shorter $2.6 \cdot 10^{-8}$ s. If we assume that they also travel at 0.998 c, can they reach the Earth?

No, they can only travel (on the average):

$$10{,}400 \, \frac{2.6 \cdot 10^{-8}}{2.2 \cdot 10^{-6}} = 123\,m \ll 8000\,m$$

Lorentz Transformations

We shall combine time dilation and length contraction into overall frame transformation expressions, known as Lorentz transformations. Consider two inertial frames S and S', with S' moving at uniform velocity $\vec{v}$ wrt S along a common x/x' axis. Let the two frames coincide at $t=0$.



distance bet. origins
(no Lorentz contraction for this part, as <u>point</u> S' has no extention)

Recalling Lorentz contraction bet. frames

$$x - vt = \frac{x'}{\gamma}$$

$\gamma=1$ yields Galilean xf.

or from the point of view of S' $(v \to -v)$

$$\frac{x}{\gamma} = x' + vt'$$

Eliminate x'

$$\Rightarrow \quad x = \gamma\left[\gamma(x-vt)+vt'\right]$$

Solve for t' as

$$t' = \frac{x(1-\gamma^2)}{\gamma v} + \gamma t$$

$$= \gamma\left[t + \underbrace{\frac{x}{v}\left(\frac{1}{\gamma^2}-1\right)}_{\frac{1}{v}\left[1-\left(\frac{v}{c}\right)^2-1\right]}\right]$$

$$\Rightarrow \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Since directions $\perp$ to $\vec{v}$ are not affected by these transformations, we have the following form for the Lorentz xf:

$$x' = \gamma(x - vt) \qquad\qquad x = \gamma(x' + vt')$$

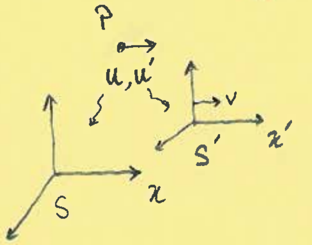
$$y' = y \qquad \xleftrightarrow{v \to -v} \qquad y = y'$$

$$z' = z \qquad\qquad z = z'$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad\qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

" Originally in 1887, German physicist Woldemar Voigt has formulated these xf's. However, Voigt himself declared that xf was aimed for a specific problem and did not carry with it the ideas of relativity." — Wikipedia

Einstein Velocity Addition Rule



Suppose a particle P moves at a velocity u, wrt S (and u' wrt S')
moves with v wrt S

$$dx' = \gamma\,(dx - v\,dt)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}\,dx\right)$$

$$\Rightarrow u' = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{\overbrace{dx/dt}^{u} - v}{1 - \frac{v}{c^2}\underbrace{\frac{dx}{dt}}_{u}} = \frac{u - v}{1 - \frac{uv}{c^2}}$$

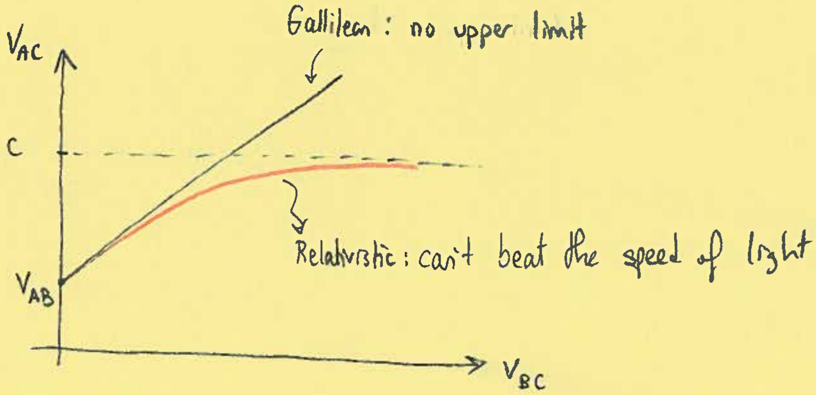
This expression becomes more transparent if we change our notation as:

$V_{AB} \equiv u \,, \quad V_{AC} \equiv u' \,, \quad \Rightarrow \quad V = V_{CB} = -V_{BC}$

$$\Rightarrow \quad V_{AC} = \frac{V_{AB} + V_{BC}}{1 + \frac{V_{AB} V_{BC}}{c^2}}$$

$\xrightarrow{V_{AB}, V_{BC} \ll c}$ $V_{AC} \simeq V_{AB} + V_{BC}$ ... Gallieon Rule

$\xrightarrow{V_{AB} \equiv c}$ $V_{AC} \equiv c$ ... speed of light is the same for all ref. frames!



Observe that velocity addition gets highly <u>nonlinear</u> as any of the two gets comparable to c.

Four-Vectors

As stated by R. Feynman, much of mathematics (or theoretical physics) is about introducing a good notation. The Lorentz xf. and relativistic algebra becomes much more simplified and transparent by the so-called four-vectors, metrics and covariant/contravariant notations.

To begin with, an ordinary vector of space coord's will be designated as a 3-vector, $\vec{r}$, $\vec{p}$ etc.

Since in STR we get space and time "mixed", we define the position-time four-vector x^μ, $\mu = \overset{\text{time-like component}}{0}, \underbrace{1,2,3}_{\text{space-like components}}$ as

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

superscripts, not powers

The Lorentz xf. are written in the following matrix form

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{\Lambda}} \,:\, \text{Lorentz xf matrix}} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

For two frames in relative motion along a common x, x' axes.

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu \quad ; \quad \Lambda^{\mu \leftarrow \text{labels rows}}{}_{\nu \leftarrow \text{labels columns}}$$

Einstein summation convention: repeated (Greek) indices one as subscript (covariant) and one as superscript (contravariant) are assumed to be summed from 0 to 3

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

Note that the entries of $\underline{\underline{\Lambda}}$ will be more complicated when the relative motion dir. is along an arbitrary direction, but the form of general expressions are immune to this internal matter, that is, $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ still holds

Scalar Product

Given two four-vectors, a^{μ} and b^{μ}, their scalar product is defined as

$$a \cdot b \equiv a^0 b^0 - \vec{a} \cdot \vec{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

This definition makes the result invariant (under frame xf.), also called Lorentz scalar.

To get rid of different signs in front of the components, we introduce the metric, $g_{\mu\nu}$

$$\underline{\underline{g}} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \quad ; \quad \underline{\text{NB}}: \quad \underline{\underline{g}}^{-1} = \underline{\underline{g}}$$

Using this, we can express the dot product as

$$a \cdot b = g_{\mu\nu}\, a^{\mu} b^{\nu}$$

Covariant Four-Vectors

Another alternative is to introduce covariant 4-vectors as: $a_{\mu} \equiv g_{\mu\nu}\, a^{\nu}$

Covariant 4-vector

Contravariant 4-vector

Since $\underline{\underline{g}}^{-1} = \underline{\underline{g}}$, we also have $a^{\mu} = g^{\mu\nu} a_{\nu}$

has the same entries as $g_{\mu\nu}$

So, $a_0 = a^0, \quad a_1 = -a^1, \quad a_2 = -a^2, \quad a_3 = -a^3$

Hence, by means of covariant-contravariant vectors we can express dot product w/o explicitly using metric $g_{\mu\nu}$ as:

$$a \cdot b = a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

$$a^2 \equiv a \cdot a = \underbrace{(a^0)^2 - \vec{a} \cdot \vec{a}}_{\text{need } \underline{\text{not}} \text{ be +ve}}$$

If $a^2 > 0$, a^μ is timelike

$a^2 < 0$, a^μ is spacelike

$a^2 = 0$, a^μ is lightlike

Tensors: 2^{nd}-rank and higher-rank tensors

$$\left.\begin{aligned} s'^{\mu\nu} &= \Lambda^\mu_\kappa \Lambda^\nu_\sigma s^{\kappa\sigma} \\ t'^{\mu\nu\lambda} &= \Lambda^\mu_\kappa \Lambda^\nu_\sigma \Lambda^\lambda_\tau t^{\kappa\sigma\tau} \end{aligned}\right\} \text{Each index requires one } \underline{\underline{\Lambda}} \text{ matrix to xf.}$$

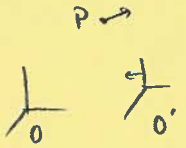
We can construct mixed tensors using the metric:

$$s^\mu_{\ \nu} = g_{\nu\lambda} s^{\mu\lambda}, \qquad s_{\mu\nu} = g_{\mu\kappa} g_{\nu\lambda} s^{\kappa\lambda}$$

NB: $a^\mu b^\nu$ is a 2^{nd}-rank tensor, $a^\mu t^{\mu\lambda\sigma} \to 4^{th}$ rank, $s^\mu_{\ \mu} \to$ scalar
summation conv.

Energy and Momentum

Consider an object in motion relative to a number of inertial frames.



The time as seen by the particle's frame is the so-called **proper time**, τ

self (in french)

Surely, according to any other frame, it is running slow by $dt = \overbrace{\gamma\, d\tau}^{\text{tick intervals appear dilated}}$.

The nice thing about proper time is that all observers (by calculation) will obtain the same τ, in other words proper time is (trivially) invariant.

proper velocity: $\vec{\eta} \equiv \dfrac{d\vec{x}}{d\tau}$ ← proper time

The advantage of proper velocity, as opposed to 'ordinary vel. $\vec{v} = \dfrac{d\vec{x}}{dt}$' is that the latter has cumbersome xf rules as both numerator & denom undergoes xf whereas in $\vec{\eta}$ only the numerator xf.

If we extend to a 4-vector as $\eta^\mu = \dfrac{dx^\mu}{d\tau}$

with $\eta^0 = \dfrac{dx^0}{d\tau} = \dfrac{d(ct)}{(1/\gamma)dt} = \gamma c$, $\vec{\eta} = \dfrac{d\vec{x}}{(dt)/\gamma} = \gamma \vec{v}$

$$\therefore \quad \eta^\mu = \gamma\,(c, v_x, v_y, v_z)$$

NB: $\eta_\mu \eta^\mu = \gamma^2 (c^2 - v_x^2 - v_y^2 - v_z^2) = \gamma^2 c^2 \left(1 - \dfrac{v^2}{c^2}\right) = c^2$ ← invariant, as it should

Momentum

In relativity, the momentum is defined through the proper velocity:

$$\vec{p} \equiv m\vec{\eta}$$

and extended to a 4-vector as $p^\mu = m\,\eta^\mu$

"temporal" $\swarrow$ $\searrow$ spatial

$$p^0 = \gamma m c \qquad \vec{p} = \gamma m \vec{v} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

Relativistic Energy

$$E \equiv \gamma m c^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} \quad , \quad \text{then} \quad p^0 = E/c$$

Thus, energy and 3-vector momentum altogether make up the energy-momentum 4-vector: $p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$, also called momentum 4-vector.

Its scalar product with itself $p_\mu p^\mu = \frac{E^2}{c^2} - |\vec{p}|^2 = \frac{m^2}{1-\frac{v^2}{c^2}} - \frac{m^2 v^2}{1-\frac{v^2}{c^2}} = m^2c^2$

which is manifestly invariant (as it should).

For $v \ll c$ we can expand $E = \gamma m c^2$ into Taylor series:

$$E = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) = mc^2 + \underbrace{\frac{1}{2}mv^2}_{\text{classical K.E.}} + \frac{3}{8}m\frac{v^4}{c^2} + \cdots$$

Rest energy

$$T \equiv mc^2(\gamma - 1) = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots$$ Relativistic Kinetic Energy

WARNING: The term "relativistic mass" is superfluous. One can use $m_{rel} \equiv \gamma m$, but this is just E/c^2 (so no need for it.)

Throughout this course when we use mass, it means the rest mass.

→ What happens for massless particles like photons?

These particles travel at the speed of light, $v = c$

Their energy-momentum relation is given by $E = |\vec{p}|c$

For the energy of a photon, we resort to QM: $E = h\nu$

From these two relations, we can obtain the wavelength associated to a particle with momentum $|\vec{p}|$ as:

$$E = h\nu = h\frac{c}{\lambda} = |\vec{p}|c \quad \Rightarrow \quad \lambda = \frac{h}{|\vec{p}|}$$ ... well-known de Broglie (matter) wavelength

Note that for massless particles

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2c^2 \equiv 0$$ ← lightlike

Collisions

$$A + B \to C + D + \cdots$$

By its nature, a collision is something that happens so fast that no external force, such as gravity, or friction has an appreciable influence.

Classical Collisions

1. Total mass is conserved: $m_A + m_B = m_C + m_D$
2. Total momentum is cons'd: $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$

$\to$ What about the total kinetic energy?

a) Sticky : $T_A + T_B > T_C + T_D$ ... K.E. decreases

b) Explosive : $T_A + T_B < T_C + T_D$ ... K.E. increases

c) Elastic : $T_A + T_B = T_C + T_D$ ... K.E. conserved

Limiting Cases: $A + B \to C$ extreme case of (a)

$A \to C + D$ ... A decays into C & D.

Relativistic Collisions

$$\left.\begin{array}{ll} 1.\ \text{Total Energy is conserved:} & E_A + E_B = E_C + E_D \\ 2.\ \text{" Momentum is conserved:} & \vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D \end{array}\right\} p^\mu_A + p^\mu_B = p^\mu_C + p^\mu_D$$

The kinetic energy may or may not be conserved.

a) Sticky (k.e. decreases): rest energy and mass increase

b) Explosive (k.e. increases): rest energy and mass decrease

c) Elastic (k.e. is conserved): rest energy and mass are conserved

An Example on "Sticky" Collisions

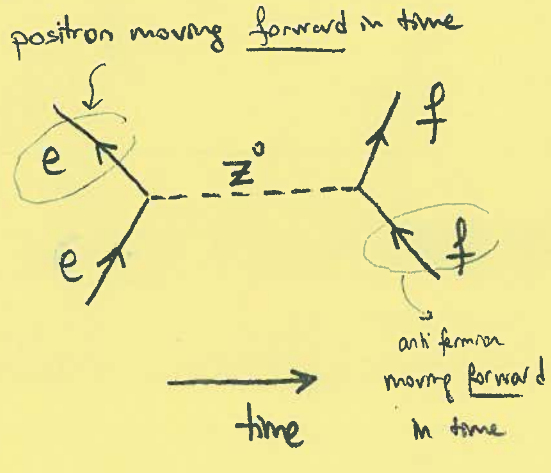
In the electron-positron collision, when they have sufficiently high K.E. the Z^0 gauge boson can be formed (for a short time)

$$e^+ \longrightarrow Z^0 \longleftarrow e^-$$

In the center-of-~~mass~~ momentum (CM) frame, the Z^0 is produced at rest. $M_{Z^0} \sim 91.2\ \mathrm{GeV}/c^2$. Since $M_{e^{-/+}} \sim 0.511\ \mathrm{MeV}/c^2$ we need 45.6 GeV energy e^-/e^+, that is with mostly K.E. of each beam contributing to production of Z^0.

The Z^0 almost immediately decays into any fermion/anti-fermion pair with $m_f < M_{Z^0}/2$

$$\bar{f} \longleftarrow Z^0 \longrightarrow f$$



Examples from Griffiths

The physics involved is minimal (conservation of energy-momentum 4-vector), but the algebra can become formidable, if not done with 4-vector machinary and not with the suitable frame of reference.

Ex. 3.1 (Sticky)

$M = ?$

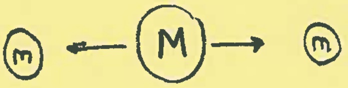
Conservation of momentum is trivially sat'd.

Cons. of energy: $2E_m = Mc^2$

$$2\frac{mc^2}{\sqrt{1-(3/5)^2}} = Mc^2 \Rightarrow M = \frac{5}{2}m > 2m$$ as it must be!

Ex. 3.2 (Explosive)

What's the speed of each m?

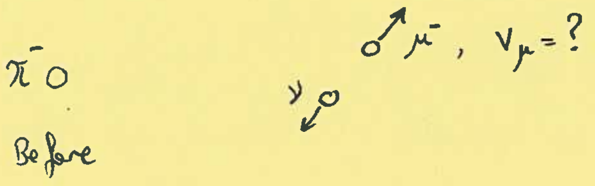


Again, only cons. of energy is nontrivial.

$$Mc^2 = 2\gamma mc^2 \Rightarrow M = 2\gamma m = \frac{2m}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow v = c\sqrt{1-\left(\frac{2m}{M}\right)^2}$$ (only possible for $M > 2m$)

Ex. 3.3 A pion decays into muon plus neutrino.



This example can be solved in a number of ways (see Griffiths).

For those sub-optimal strategies Griffiths gives two suggestions:

1) To get energy of a particle, when you know its momentum (or vice versa), use the invariant: $E^2 - |\vec{p}|^2 c^2 = m^2 c^4$

2) If you know both the energy and momentum of a particle, to determine its velocity, use $\dfrac{\vec{p} = \gamma m \vec{v}}{E = \gamma m c^2} \Rightarrow \vec{v} = \dfrac{\vec{p}}{E} c^2$

(This avoids extracting out v factors in γ terms)

The best way is to use 4-vector momentum:

Notation: p's without vector sign ($\vec{p}$) denote 4-vectors

Initial State: $p_i = p_\pi = \left(\dfrac{E_\pi}{c}, \vec{0} \right) = (m_\pi c, \vec{0})$

Final State: $p_f = p_\nu + p_\mu = \left(\dfrac{E_\nu}{c}, \vec{p} \right) + \left(\dfrac{E_\mu}{c}, -\vec{p} \right)$

$m_\pi c \cdot \dfrac{E_\mu}{c} = m_\pi E_\mu$

$p_i = p_\pi = p_f = p_\nu + p_\mu$

$p_\nu = p_\pi - p_\mu \Rightarrow \underbrace{p_\nu \cdot p_\nu}_{0} = \underbrace{p_\pi^2}_{m_\pi^2 c^2} + \underbrace{p_\mu^2}_{m_\mu^2 c^2} - 2 p_\pi \cdot p_\mu$

massless particles (lightlike)

$$\therefore \quad 0 = m_\pi^2 c^2 + m_\mu^2 c^2 - 2 m_\pi E_\mu$$

$$\Rightarrow \quad E_\mu = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi} c^2$$

To find out $|\vec{v}_\mu|$, we also need $|\vec{p}_\mu|$ (see 2nd suggestion in prev. page)

Simply use $\quad \frac{E_\mu^2}{c^2} - |\vec{p}_\mu|^2 = m_\mu^2 c^2$

$$\Rightarrow \quad |\vec{p}_\mu|^2 = c^2 \frac{\left(m_\pi^2 + m_\mu^2\right)^2}{4 m_\pi^2} - m_\mu^2 c^2$$

$$\Rightarrow \quad |\vec{p}_\mu| = c \, \frac{m_\pi^2 - m_\mu^2}{2 m_\pi}$$

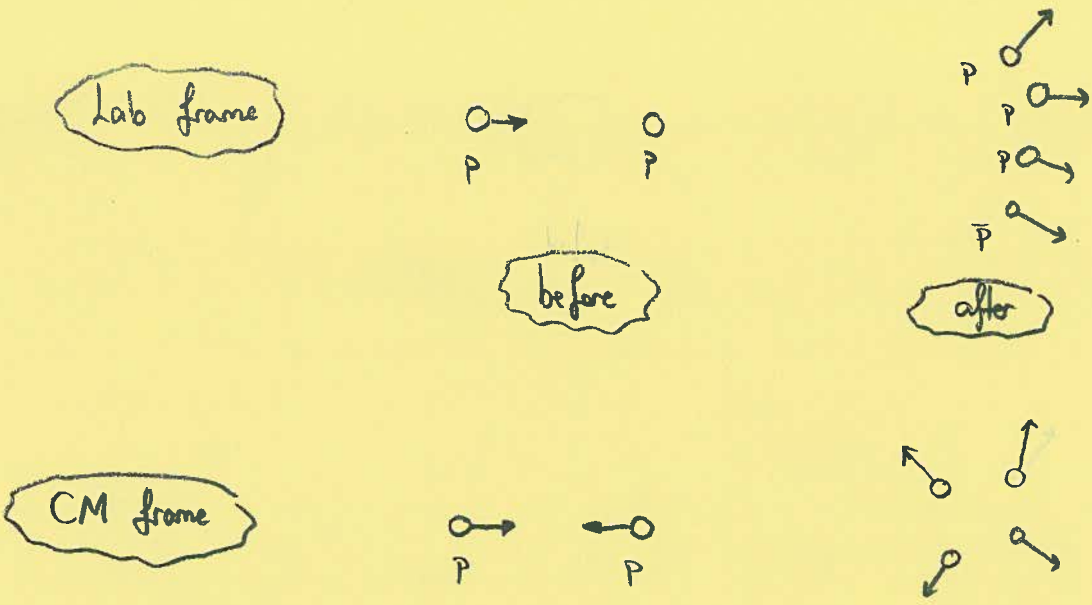
From E_μ and $|\vec{p}_\mu|$ we get

$$v_\mu = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} c = 0.271 \, c$$

billions of eV synchrotron (GeV)

Ex. 3.4 The Bevatron at Berkeley was built in 1955 with the idea of producing antiprotons, and it did leading to 1959 Nobel prizes for E. Segré and O. Camberlain. The reaction is $\quad p + p \rightarrow p + p + p + \bar{p}$

A high energy proton strikes a proton at rest, creating additionally a proton-antiproton pair. What is the minimum energy of the incident proton for this reaction to occur?



In the Lab frame, it is hard to figure out the threshold energy, but in the CM it is simple: all four final particles must be at rest so that nothing is 'wasted' in the form of K.E.

energy of the acc'd p to be det'd

momentum " "

In the Lab frame: $p^{\mu}_{TOT} = \left(\frac{E + mc^2}{c} , \vec{p} \right)$

(We use the case before the collision, but since p_{TOT}^{μ} is cons'd it could be after as well)

The same quantity in CM: $p'^{\mu}_{TOT} = \left(4mc^2, \vec{0} \right)$

here we use after the coll.

Since any scalar product is (inertial frame) invariant:

$$\underbrace{p^{\mu}_{TOT} \cdot p_{\mu,TOT}}_{} = p'^{\mu}_{TOT} \cdot p'_{\mu,TOT}$$

$$\left(\frac{E}{c} + mc \right)^2 - |\vec{p}|^2 = (4mc)^2 \quad \Rightarrow \quad E = 7mc^2$$

$E^2 - p^2c^2 = m^2c^4$

As expected, the first antiprotons were discovered when Bevatron reached about $6\,\text{GeV} = 6m_pc^2$ $(+1m_p^2c$ from the rest energy of the proton$)$

Note that to create an additional $p/\bar{p}$ pair, that is $2mc^2$ of rest energy, it takes an incident K.E. of $6mc^2$. This illustrates the inefficiency of scattering off a <u>stationary</u> target; conservation of momentum forces us to waste a lot of K.E. in the final state. Much more economical would be the <u>collider</u> scheme

$$\underset{p}{\circ}\!\longrightarrow \qquad \longleftarrow\!\underset{p}{\circ}$$

where each proton will have a K.E. of mc^2 (1/6 of what the stationary-target exp. requires). This realization led, in early 70's to switch to collider-beam machines.

Eg: Large Hadron Collider (LHC)

Started in 2008, uses two counter-propagating proton beams up to 7 TeV per nucleon (in the lab frame) or Pb nuclei at an energy of 574 TeV per nucleus (2.76 TeV/nucleon).

Angular Momentum

The main purpose of this section is to offer some practical tools for how to <u>add</u> angular momenta (especially spins).

Before we get on to that task, a few refreshments on angular momentum will be in order.

Consider some vector, say $\vec{J}$. If there is the following commutation rule among its Cartesian components:

Levi-Civita

$$[J_i, J_j] = i\hbar\, \varepsilon_{ijk}\, J_k, \qquad \varepsilon_{ijk} = \begin{cases} +1 \,; & ijk \text{ all distinct and cyclic} \\ -1 \,; & \text{" " " not cyclic} \\ 0 \,; & \text{o.w.} \end{cases}$$

then we say that $\vec{J}$ corresponds to some angular momentum (orbital or intrinsic).

Because of this commutation relation, we cannot simultaneously measure any two Cartesian components. But since $\vec{J}^2 \equiv \vec{J}\cdot\vec{J}$ commutes with J_i, i.e., $[\vec{J}^2, J_i] = 0$ (easy to show), we can simultaneously measure $\vec{J}^2$ and say J_z

Orbital Angular Momentum

$$\vec{J} \to \vec{L} = \vec{r} \times \vec{p} = -i\hbar\, \vec{r} \times \vec{\nabla}$$

According to QM measurements yield only certain discrete values for L^2 and L_z

L^2 measurements : $l(l+1)\hbar^2$, where $l = \overbrace{0,1,2,\ldots}^{\text{non negative integers}}$

L_z measurements : $m_l \hbar$, where $m_l = \underbrace{-l, -l+1, \ldots, l-1, l}_{2l+1 \text{ possibilities}}$

Note that even though the magnitude of $\vec{J}$ is $|\vec{J}| = \hbar\sqrt{l(l+1)}$

the z-component cannot reach this value $J_z < |\vec{J}|$, otherwise

J_x and J_y would become zero, violating the uncertainty constraint.

Spin Angular Momentum

For 'fundamental' particles, spin is an intrinsic property.

* Fundamental fermions are all spin-1/2 (quarks, leptons)
* Force carriers (gauge bosons) are all spin 1
* Higgs boson is spin 0, graviton is believed to be spin 2

Spin cannot be explained by a spinning only in 3D space around an axis.

To refute such a picture, suppose we pick say an $\bar{e}$ and interpret it literally as a classical solid sphere of radius r, mass m, spinning with angular momentum $\frac{\hbar}{2}$.

Experimentally it is known that $r < 10^{-16}$ cm. If we take it equal to this value and proceed with classical physics

rot. inertia of a solid sphere

$$I = \frac{2}{5} m r^2, \quad \omega = \frac{v}{r} \quad \Rightarrow \quad \overset{\hbar/2}{L} = I\omega \quad \Rightarrow \quad v = \frac{5}{4}\frac{\hbar}{\underset{9\cdot 10^{-31}\,\text{kg}}{m}\,r}$$

So, for $r < 10^{-18}$ m, $v > 10^{14}$ m/s $= 10^6 \times c$!

That means, in this classical picture of a rotation solely in our 3D space can only make up one millionth of its measured intrinsic angular momentum: $\hbar/2$

Denoting the spin angular momentum vector with $\vec{S}$:

S^2 measurements: $s(s+1)\hbar^2$, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$

So note that unlike orbital angular momentum, the spin quantum number s can also become a <u>half</u>-integer (fermions) as well as an integer (bosons).

S_z measurements: $m_s \hbar$, where $m_s = \overbrace{-s, -s+1, \dots, s-1, s}^{2s+1 \text{ possibilities}}$

When we say spin-s particle, we refer to this s.

NB: A given particle can be given any <u>orbital</u> angular momentum l, but for each type of particle, the value of s is <u>fixed</u>.

Spin - 1/2

Since all quarks and all leptons have spin 1/2, it deserves special attention. [Also note that proton and neutron which are baryons made up of three quarks are also spin-1/2.]

$$|s \; m_s\rangle \xrightarrow{\text{spin } 1/2} \left|\tfrac{1}{2} \; \pm\tfrac{1}{2}\right\rangle$$

with alternative terminology: $m_s = +\frac{1}{2}$: ↑ (spin-up)
$m_s = -\frac{1}{2}$: ↓ (down)

The '↑', '↓' notation is OK for quick reference, but in doing math we should work with spinors (column vectors with $2s+1$ entries):

$$\left|\tfrac{1}{2} \; \tfrac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \left|\tfrac{1}{2} \; -\tfrac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The most general spin-1/2 spinor would be:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ and since $|\alpha|^2$ and $|\beta|^2$ are the probabilities that a measurement of S_z would yield the values $+\hbar/2$ and $-\hbar/2$ respectively, we need the normalization: $|\alpha|^2 + |\beta|^2 = 1$.

Pauli Spin Matrices

Since observables are represented by (Hermitian) operators, which can in turn be put into a matrix form, for the 2-dimensional spin-1/2 space, the form of these operators take 2x2 matrices.

If we introduce the Pauli spin matrices, with Cartesian components:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then, we can express the spin operator as: $\hat{\vec{S}} = \frac{\hbar}{2} \hat{\vec{\sigma}}$.

Let's work out the eigenkets and values of each Cartesian component.

$$S_i |\chi_i\rangle = \lambda_i |\chi_i\rangle \quad , \quad |\chi_i\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \text{ with } |a|^2 + |b|^2 = 1$$

<u>x-component</u>:

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\det\left(\lambda_x \bar{\bar{I}} - \bar{\bar{S}}_x\right) = 0 \Rightarrow \lambda_x^2 - \frac{\hbar^2}{4} = 0 \quad , \quad \lambda_x = \pm \frac{\hbar}{2}$$

For the eigenvectors, insert each λ_x into eigenvalue eqn.

<u>$\lambda_{x,+} = + \hbar/2$</u>

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \left.\begin{matrix} a = b \\ + \\ |a|^2 + |b|^2 = 1 \end{matrix}\right\} |\chi_{x+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>$\lambda_{x,-} = - \hbar/2$</u>

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \left.\begin{matrix} a = -b \\ + \\ \text{Norm.} \end{matrix}\right\} |\chi_{x-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

y-component:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda_y \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\det\left(\lambda_y \bar{\bar{I}} - \bar{\bar{S}}_y\right) = 0 \quad \Rightarrow \quad \lambda_y^2 - \frac{\hbar^2}{4} = 0 \,, \qquad \lambda_y = \pm \frac{\hbar}{2}$$

$\lambda_{y,+} = +\hbar/2$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \left.\begin{matrix} a = -ib \\ + \\ \text{Norm.} \end{matrix}\right\} \; |\chi_{y+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\lambda_{y,-} = -\hbar/2$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \left.\begin{matrix} a = ib \\ + \\ \text{Norm.} \end{matrix}\right\} \; |\chi_{y-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

z-component:

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda_z \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\det\left(\lambda_z \bar{\bar{I}} - \bar{\bar{S}}_z\right) = 0 \quad \Rightarrow \quad \lambda_z^2 - \frac{\hbar^2}{4} = 0 \,, \qquad \lambda_z = \pm \frac{\hbar}{2}$$

$\lambda_{z,+} = +\hbar/2$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \left.\begin{matrix} b = 0 \\ + \\ \text{Norm.} \end{matrix}\right\} \; |\chi_{z+}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda_{z,-} = -\hbar/2$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \left.\begin{matrix} a = 0 \\ + \\ \text{Norm.} \end{matrix}\right\} \; |\chi_{z-}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let's see how we use these matrices and the associated eigenkets etc.

Exercise (Problem 4.18 from Griffiths)

Suppose an $\bar{e}$ is in the state $|\psi\rangle = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ → note that it is normalized

a) What's the expectation value of S_x?

What values you might get for S_x measurement and with what probablities?

b) Redo part (a) for S_y and S_z.

Solution:

a) $\langle S_x \rangle = \langle \psi | \hat{S}_x | \psi \rangle$; NB: if $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, then $\langle \psi | = (\alpha^* \;\; \beta^*)$

$$\langle S_x \rangle = \frac{\hbar}{2} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$= \frac{\hbar}{2} \cdot \frac{4}{5}$ (That means quite closer to $+\frac{\hbar}{2}$ value)

Let's work out the individual probabilities for $+\frac{\hbar}{2}$ & $-\frac{\hbar}{2}$ occurances for S_x

$$P\left(S_x = +\frac{\hbar}{2}\right) = \left|\langle \chi_{x+} | \psi \rangle\right|^2 = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right|^2 = \left(\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{10}} \right)^2 = \frac{9}{10}$$

$$P\left(S_x = -\frac{\hbar}{2}\right) = \left|\langle \chi_{x-} | \psi \rangle\right|^2 = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right|^2 = \left(\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} \right)^2 = \frac{1}{10}$$

This verifies the expectation value: $\langle S_x \rangle = \frac{\hbar}{2} \cdot \frac{9}{10} + \left(\frac{-\hbar}{2}\right) \cdot \frac{1}{10} = \frac{4\hbar}{10}$ ✓

b) $$\langle S_y \rangle = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} 1/\sqrt5 \\ 2/\sqrt5\end{pmatrix} = \frac{\hbar}{2}\left(\frac{-2i}{5} + \frac{2i}{5}\right) = 0$$

$$P\left(S_y = +\frac{\hbar}{2}\right) = \left|\langle \chi_{y+} | \psi \rangle\right|^2 = \left| \begin{pmatrix} 1/\sqrt2 & -i/\sqrt2 \end{pmatrix}\begin{pmatrix} 1/\sqrt5 \\ 2/\sqrt5\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$P\left(S_y = -\frac{\hbar}{2}\right) = \left|\langle \chi_{y-} | \psi \rangle\right|^2 = \left| \begin{pmatrix} 1/\sqrt2 & +i/\sqrt2 \end{pmatrix}\begin{pmatrix} 1/\sqrt5 \\ 2/\sqrt5\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$\langle S_z \rangle = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 1/\sqrt5 \\ 2/\sqrt5\end{pmatrix} = \frac{\hbar}{2}\left(\frac{1}{5} - \frac{4}{5}\right) = -\frac{\hbar}{2}\cdot\frac{3}{5}$$

$$\left.\begin{aligned} P\left(S_z = +\frac{\hbar}{2}\right) &= \frac{1}{5} \\ P\left(S_z = -\frac{\hbar}{2}\right) &= \frac{4}{5} \end{aligned}\right\}$$

we can immediately write these as $\hat{S}_z$ is diagonal in this (conventional) representation.

Some Properties of Pauli Spin Matrices (Proofs: HW-2)

* $\bar{\bar{\sigma}}_i \bar{\bar{\sigma}}_j = \delta_{ij} \bar{\bar{I}} + i\, \varepsilon_{ijk} \bar{\bar{\sigma}}_k$

2x2 identity matrix

* $[\bar{\bar{\sigma}}_i, \bar{\bar{\sigma}}_j] = 2i\, \varepsilon_{ijk} \bar{\bar{\sigma}}_k$

* $\{\bar{\bar{\sigma}}_i, \bar{\bar{\sigma}}_j\} = 2\delta_{ij} \bar{\bar{I}}$

anticommutator: $\{A, B\} \equiv AB + BA$

* For any two vectors $\vec{a}$ and $\vec{b}$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\bar{\bar{I}} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

Rotation of Spinors

The most remarkable property of spinors is reflected by the way they rotate in space. A rotation by an angle θ around the z-axis is executed by multiply the spinor with the matrix $U_z(\theta) = e^{-i\theta\sigma_z/2}$.

A general rotation by an angle θ around an axis $\hat{n}$ becomes:

$$U_{\hat{n}}(\theta) = e^{-i\vec{\sigma}\cdot\hat{n}\,\theta/2}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - i n_z \sin\frac{\theta}{2} & \text{c.c.} \\ (-in_x + n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + i n_z \sin\frac{\theta}{2} \end{pmatrix}$$

So $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U(\theta)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

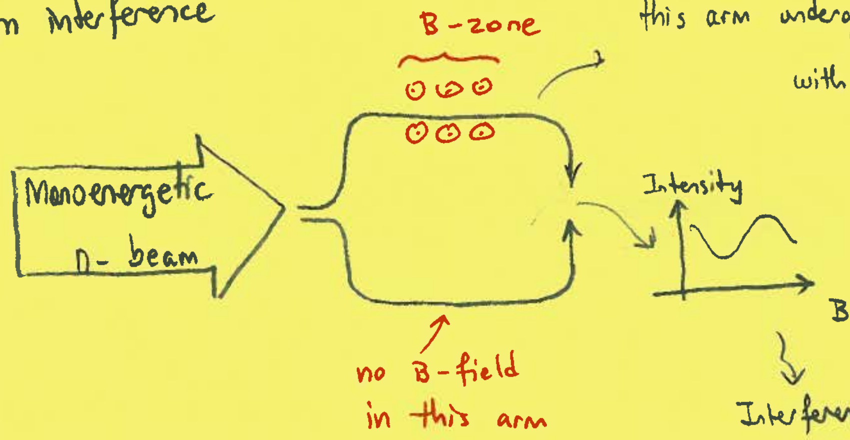
$U(\theta)$ is a unitary matrix of determinant 1; the set of all such rotation matrices constitutes the group $SU(2)$: special unitary group of degree 2. The spin-½ particles transform under rotations according to the 2D representation of $SU(2)$; spin-1 particles (vectors) transform as 3D rep. of $SU(2)$; spin-3/2 (4 component obj.) xform as 4D rep. of $SU(2)$.

From the practical point of view, if we consider $U_z(2\pi) = -1$,

hence spin-$1/2$ spinors recover their direction only after 4π rotation!

This is real! It has been expermentally demonstrated by neutron $\rightarrow$ spin-$1/2$

beam interference



this arm undergoes spin Larmor precession

with $\omega = \dfrac{g_n e B}{m_p}$

Interference as a fn. B

(verifies that only after 4π precession constructive interference is recovered)

Addition of Angular Momenta

In nuclear, particle, atomic and solid-state physics, we repeatedly end up needing to add two or more angular momenta. Once we learn how to add two of them, the addition of more is just adding them cummulatively, with the order not being important.

Take two angular momenta $\vec{J}_1, \vec{J}_2$ with either one corresponding to orbital or intrinsic degree of freedom. What is $\vec{J} = \vec{J}_1 + \vec{J}_2$?

If they were classical variables, we would just add the components, $J_i = J_{1i} + J_{2i}$, and the result would be a single (definite) answer.

But in quantum mechanics we do not have access to all three components; we need to live with uncertainity and work with one component (J_z) and the magnitude (J), which inherently yields not a single summation but a set of possible sums, each with their certain probability weights.

Assume that we are adding a spin-j_1 with a spin-j_2 having z-projections m_1 and m_2, resp.

$$|j_1 m_1\rangle \;\&\; |j_2 m_2\rangle \longrightarrow |j, m\rangle \; ; \; j = ? \, , \, m = ?$$

The z components simply add, i.e., $m = m_1 + m_2$

Note that, for a given j_1 and j_2, the associated subspace is spanned by either the individual z-projections or the total momenta, in mathematical terms:

$$\sum_{m_1, m_2} |m_1\, m_2\rangle\langle m_1\, m_2| = I$$

over their all allowed values

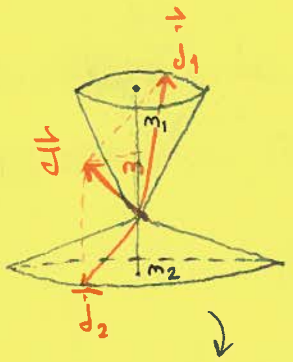
$$\sum_{j, m} |j\, m\rangle\langle j\, m| = I$$

Completeness relations

With this property, we can express any ket in one set with those of the other set. This is usually required when we want to find out what specific $|j\, m\rangle$ states with what weights are needed to express any two angular momenta, that is

$$\underbrace{|j_1\, m_1\rangle\, |j_2\, m_2\rangle}_{\substack{\text{shorthand:} \\ |m_1\, m_2\rangle}} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \underbrace{\langle j\, m | m_1\, m_2\rangle}_{C^{j\, j_1\, j_2}_{m\, m_1\, m_2}} |j\, m\rangle$$

$C^{j\, j_1\, j_2}_{m\, m_1\, m_2}$: Clebsch - Gordan Coefficients



Assume $m_2 < 0$

Geometric Illustration of possible $\vec{J}, m$

So, C-G coefficients tell us the probability of each $|j\, m\rangle$ contribution among the overall possible total spin states.

What about j ?

Depending on the relative orientation of $\vec{j}_1$ and $\vec{j}_2$, (as it turns out) we get every j from (j_1+j_2) down to $|j_1-j_2|$.

We can convince ourselves for this by checking the number of states before and after the addition: (Assume $j_1 \geqslant j_2$)

$$\sum_{\substack{j=j_1-j_2 \\ (j_1 \geqslant j_2)}}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1) \quad ✓$$

Ex. 4.1 (Griffiths)

<u>Meson</u> : $q\bar{q}$ each with zero orbital angular momentum ($l=0$).

What are the possible values of the meson's spin?

$$\frac{1}{2}+\frac{1}{2}=1 \quad \text{or} \quad \frac{1}{2}-\frac{1}{2}=0$$

$\frac{1}{2}+\frac{1}{2}=1$ → eg. (ρ's, K^*'s, ϕ, ω)

$\frac{1}{2}-\frac{1}{2}=0$ → eg. (π's, K's, η, η')

Ex. 4.2 (Griffiths)

<u>Baryon</u> : qqq each with $l=0$. What are possible baryon spins?

$q+q$: ↗ $\frac{1}{2}+\frac{1}{2}=1$ $+q$ ↗ $1+\frac{1}{2}=3/2$

 ↘ $1-\frac{1}{2}=1/2$ (in two different routes)

 ↘ $\frac{1}{2}-\frac{1}{2}=0$ $+q$ ↗ $1/2$

If we permit quarks to have $l>0$, possibilities will increase but sum = half integer

always!

Ex. 4.3 (Griffiths)

For an $\bar{e}$ in a H-atom having the orbital state $|2\ -1\rangle$ and spin state $|\frac{1}{2}\ \frac{1}{2}\rangle$, if we measure J^2, what values do we get, and with what probabilities for each?

$l+s = 2+\frac{1}{2} = \frac{5}{2}$ and $l-s = 2-\frac{1}{2} = \frac{3}{2}$

Since the z-components directly add: $m = -1+\frac{1}{2} = -\frac{1}{2}$

With this info, refer to C-G table (next page) for the section $\overset{j_1}{2} \times \overset{j_2}{1/2}$ to the horizontal row $\underset{m_1}{-1}, \underset{m_2}{\frac{1}{2}}$. Read the two entries there

$$\Rightarrow |2\ -1\rangle\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}\,|\tfrac{5}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$$

$\frac{2}{5}$: prob. of getting $j = \frac{5}{2}$; $\frac{3}{5}$: prob. of getting $j = \frac{3}{2}$

Ex. 4.4 (Griffiths)

Find C-G decomposition for the addition of two spin-1/2. Using the tables:

$$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |1\ 1\rangle$$

$$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}|1\ 0\rangle + \tfrac{1}{\sqrt{2}}|0\ 0\rangle$$

$$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}|1\ 0\rangle - \tfrac{1}{\sqrt{2}}|0\ 0\rangle$$

$$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = |1\ -1\rangle$$

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	1, +1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2×1/2

m_1	m_2	5/2, +5/2
+2	+1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2	7/2, +7/2
+2	+3/2	1

m_1	m_2	7/2, +5/2	5/2, +5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2, −5/2	5/2, −5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2, −7/2
−2	−3/2	1

3/2×3/2

m_1	m_2	3, +3
+3/2	+3/2	1

m_1	m_2	3, +2	2, +2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3, +1	2, +1	1, +1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3, 0	2, 0	1, 0	0, 0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3, −1	2, −1	1, −1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3, −2	2, −2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3, −3
−3/2	−3/2	1

2×2

m_1	m_2	4, +4
+2	+2	1

m_1	m_2	4, +3	3, +3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4, +2	3, +2	2, +2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4, +1	3, +1	2, +1	1, +1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4, −1	3, −1	2, −1	1, −1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4, −2	3, −2	2, −2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4, −3	3, −3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4, −4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

From these 4 eq's we can solve for the three spin-1 states:

$$\left.\begin{aligned} |1\,1\rangle &= |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |1\,0\rangle &= \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right] \\ |1\,-1\rangle &= |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \end{aligned}\right\}$$

triplet: symmetric under interchange of particles ($1 \leftrightarrow 2$)

and the spin-0 state as

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$$

singlet: antisymmetric

NB: The above relations for $|j\,m\rangle$ could be written also from the C-G table (this time read down the columns), since the C-G are the same in either direction:

$$|j\,m\rangle = \sum_{m_1, m_2} C^{j\,j_1\,j_2}_{m\,m_1\,m_2}\,|j_1\,m_1\rangle|j_2\,m_2\rangle$$

The Feynman Calculus

A great deal of high energy experiments are about decays and scatterings of the elementary particles, as they give insight into how they interact internally (decays) and among themselves (scatterings). One needs a relativistic framework to analyze such events.

Decay Rates

In practical terms the most important parameter of a metastable particle is its lifetime, τ which is quoted in its own rest frame (i.e., proper lifetime). This is an average value of a random process, in other words an expectation value for a quantum process to happen.

We shall obtain an expression for the (mean) lifetime of a collection of particles, say muons. Let there be N_0 of them at $t=0$. We shall denote by Γ the decay rate, i.e.,

Γ : Probability per unit time that any given particle will decay.

So, if we have at hand $N(t)$ muons remaining at t, then

$N(t)\,\Gamma\,dt$: total # muons to decay in the interval, $[t, t+dt]$

Hence the change in muon number $dN = N(t+dt) - N(t) = \overset{\text{decrease}}{-}N\Gamma\,dt$

Solving the simple d.e. $\frac{dN}{dt} = -N(t)\,\Gamma$ subject to $N(t=0) = N_0$

yields $N(t) = N_0\, e^{-\Gamma t}$

From this, let's obtain the probability that an individual particle selected at random from the initial sample will decay between t and $t+dt$:

$N(t) = N_0\, e^{-\Gamma t}$ ... number of particles at t (already far less than N_0)

$$N(t+dt) = N_0\, e^{-\Gamma(t+dt)} = N_0\, e^{-\Gamma t} \underbrace{e^{-\Gamma dt}}_{1 - \Gamma dt + \cdots}$$

$\simeq N_0\, e^{-\Gamma t}(1 - \Gamma dt)$ ... number of particles remaining at $t+dt$

The fraction of particles decayed bet. t, $t+dt$: $\frac{N(t) - N(t+dt)}{N_0} = \Gamma e^{-\Gamma t} dt$

wrt initial ensemble

Hence $p(t)\, dt = \Gamma\, e^{-\Gamma t}\, dt$

Note the distinction bet. these two:

$\Gamma\, dt$: probability that a given particle will decay in the next instant

$\Gamma e^{-\Gamma t} dt$: " " an individual from the initial ensemble will decay bet. t and $t+dt$. This is more unlikely as we demand that the chosen particle must have survived till then. (This is actually a conditional probability)

Using this probability $p(t)$, we can calculate the mean lifetime as, the expectation value of t:

$$\tau = \int_0^\infty t\, p(t)\, dt = \Gamma \int_0^\infty t\, e^{-\Gamma t}\, dt = \frac{1}{\Gamma}$$

Note that the probability $p(t) = \Gamma e^{-\Gamma t}$ is the special form of Poisson distribution $f(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$ where k: # occurance of an event

In the (radioactive) decay we have $k=1$. ← you die only once! The Poisson probability shows up commonly in processes which counts the number of events and the time these events occur in a given time interval.

Generalization: Several available decay modes

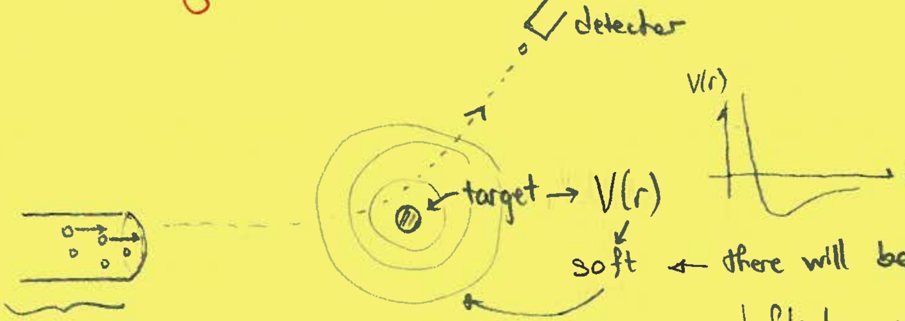
$$\pi^+ \xrightarrow{\text{usually}} \mu^+ + \nu_\mu$$
$$\pi^+ \xrightarrow{\text{sometimes}} e^+ + \nu_e$$
$$\pi^+ \xrightarrow{\text{occasionally}} \mu^+ + \nu_\mu + \gamma$$
$$\pi^+ \xrightarrow{\text{rarely}} e^+ + \nu_e + \pi^0$$

$$\Gamma_{tot} = \sum_i \Gamma_i$$

$\frac{\Gamma_i}{\Gamma_{tot}}$: Branching ratio of the i'th decay mode

The lifetime in such a case, $\tau = \frac{1}{\Gamma_{tot}}$

Scattering Cross Section



← there will be still some deflection even we have an offset in "hitting" the target (described by impact parameter, b)

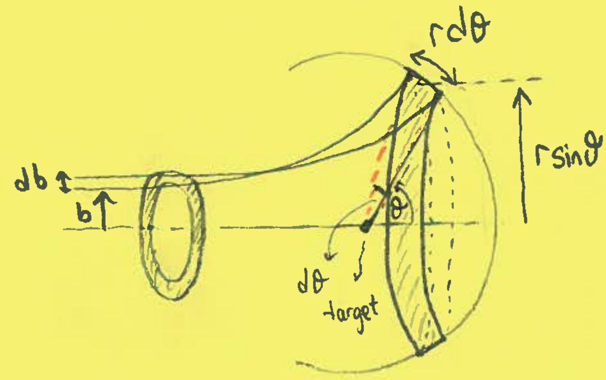
Size of beam ≫ target size

The cross section, σ is the measure of interaction bet. the particles in the beam and the target. It defines the effective area for collision. SI unit is m^2, but the standard unit is barn.

$$1 \text{ barn} = 10^{-24} \text{ cm}^{-2} = 10^{-28} \text{ m}^2$$

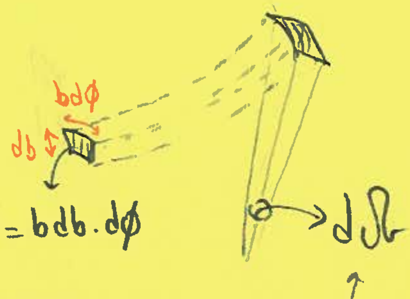
Geometrical Considerations

by how much u "miss" the target { impact parameter [Length]



Assuming a spherically symmetric, $V(\vec{r})$

no azimuthal ∡ dependence : ϕ - independent



$$d\sigma = b\,db \cdot d\phi$$

An incident differential spot of the beam will scatter into this solid ∡ $d\Omega$

The larger we make $d\sigma$, the larger $d\Omega$ will be.

The proportionality factor is called differential scattering cross section, $\frac{d\sigma}{d\Omega}$ ← Griffiths uses $D(\theta)$ for this

From the previous drawing: $d\sigma = b\,d\phi\,db$, $d\Omega = \sin\theta\,d\theta\,d\phi$

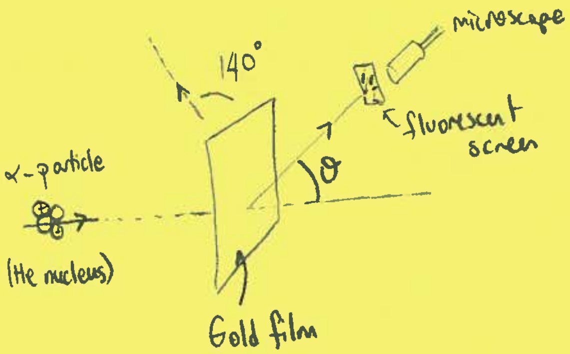
$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta}$$

The total cross section is the integral of $d\sigma$ over all solid ∡.

$$\sigma = \int d\sigma = \int \frac{d\sigma}{d\Omega}\cdot d\Omega$$

Best example to illustrate these concepts would be the pioneering scattering experiment of modern physics, Rutherford scattering.

Rutherford Scattering (1911)



backscattering!

Rutherford observed 140° deflection, which would be impossible if +ve charge were uniformly distributed as in the Thompson model.

⇒ There must be a nuclear core!

Let's keep the central potential for the time-being as general.

$$\frac{1}{2}mv^2 + V(r) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r) = E \leftarrow \text{will be conserved}$$

$$\vec{L} = \vec{r} \times m\vec{v} = mr\hat{r} \times (\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) = mr^2\dot{\phi}\,\underbrace{(\hat{r}\times\hat{\phi})}_{\hat{z}}$$

$$= \underbrace{mr^2\dot{\phi}}_{L\text{ : will be conserved}}\hat{z}$$

The values for these well before interaction:

$$E = \frac{1}{2}mv_0^2, \qquad L = \overset{r\sin\alpha}{b}\,m v_0 = bm\sqrt{\frac{2E}{m}}$$

From cons. of ang. mom.: $\dot{\phi} = \dfrac{L}{mr^2} = \dfrac{bm\sqrt{2E/m}}{mr^2} = \dfrac{b}{r^2}\sqrt{\dfrac{2E}{m}}$

From cons. of energy: $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{b^2}{r^4}\frac{2E}{m} + V(r)$

$$\Rightarrow \dot{r}^2 = \frac{2}{m}\left[E - \frac{Eb^2}{r^2} - V(r)\right]$$

For the trajectory, we need, not $r(t)$ but $r(\phi)$:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{b}{r^2}\sqrt{\frac{2E}{m}}$$

Let $u \equiv \dfrac{1}{r}$, then $\dfrac{dr}{d\phi} = \dfrac{dr}{du}\dfrac{du}{d\phi} = -\dfrac{1}{u^2}\dfrac{du}{d\phi}$

Combining these: $\dot{r} = -r^2\dfrac{du}{d\phi}\dfrac{b}{r^2}\sqrt{\dfrac{2E}{m}} = -b\sqrt{\dfrac{2E}{m}}\dfrac{du}{d\phi}$

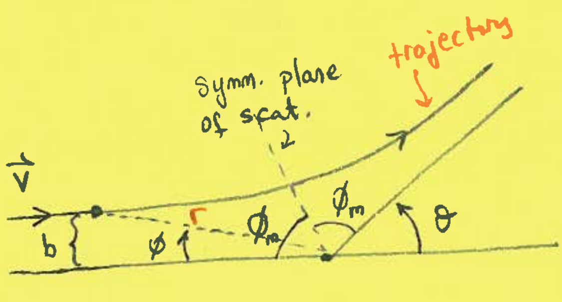
Derivation with Classical Mechanics

As it turns out, nonrelativistic QM result exactly matches that of the classical treatment. So we present the classical derivation.

Assumptions:

* Atom contains a nucleus with +ve charge with almost the entire mass
* " " Z e^-'s moving around the nucleus, but we discard any interaction with these e^-'s.
* Target nucleus (here, Au) is much more heavy than incident particles (He nucleus), so no recoil of the nucleus $\Rightarrow$ force is central
* CM is valid (The relativistic case is called Mott scattering)
* Interaction is $V(r) \sim \frac{1}{r}$ ... Coulomb ; both target and projectile are point-like charges
* Scattering is elastic (consider only mechanical form of energy)

NB: central force, $V(|\vec r|)$ + conservation of ang. mom $\Rightarrow$ trajectory is planar (azimuthal symm.)



Use polar coord's (r, ϕ)

$$\vec v = \dot r \hat r + r \dot \phi \hat \phi$$

incident particle velocity

From the trajectory, we extract $b(\theta)$ relation, which is what we use in scattering

Thus $$b^2 \frac{2E}{m}\left(\frac{du}{d\phi}\right)^2 = \frac{2E}{m}\left(1-\frac{b^2}{r^2}-\frac{V}{E}\right)$$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} - \underbrace{\frac{1}{r^2}}_{u^2} - \frac{V}{b^2 E}$$

$$\frac{du}{d\phi} = \frac{1}{b}\sqrt{1-b^2u^2-\frac{V}{E}} \quad ; \qquad d\phi = \frac{b\,du}{\sqrt{1-b^2u^2-V/E}}$$

Integrate this from distant past: $\phi=0$, $r=\infty \Rightarrow u=0$; to the point of closest approach: $\phi=\phi_m$, $r=r_{min} \Rightarrow u=u_{max}$, the point at which $\frac{du}{d\phi}=0$ ← see the trajectory

$$\Rightarrow \phi_m = b\int_0^{u_m} \frac{du}{\sqrt{1-b^2u^2-\frac{V(u)}{E}}} \Bigg\} \; b(\phi_m) \text{ relation } \xrightarrow[\underbrace{2\phi_m=\pi-\theta}_{\text{see scattering illustration}}]{\text{convert to}} b(\theta) \text{ relation}$$

Up to here $V(r)$ was kept general, now insert

$$V(r) = \frac{Z_1 e\, Z_2 e}{4\pi\varepsilon_0}\underbrace{\frac{1}{r}}_{u}$$

The integrand is very similar to Kepler problem ($V_{Gravitational} \propto \frac{1}{r}$)

Use the indefinite integral $$\int \frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2}} = \frac{1}{\sqrt{-\gamma}}\cos^{-1}\left(-\frac{\beta+2\gamma x}{\sqrt{q}}\right)$$

where $q \equiv \beta^2 - 4\alpha\gamma$

Working out the integral

$$\Rightarrow b(\theta) = \frac{Z_1 e\, Z_2 e}{4\pi\varepsilon_0}\frac{1}{4E}\cot\frac{\theta}{2}$$

or using $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\vartheta}\frac{db(\vartheta)}{d\vartheta} = \left(\frac{Z_1 e \cdot Z_2 e}{4\pi\varepsilon_0}\right)^2 \frac{1}{16E^2}\frac{1}{\sin^4\frac{\vartheta}{2}}$

So note that the differential cross section, i.e., probability for scattering in a certain direction:

* does not depend on ϕ (trajectory is planar) as expected
* proportional to the strength of the interaction squared
* decreases drastically for large scattering angles ϑ
* high energy particles scatter less

If we now calculate the total cross section

$$\sigma = 2\pi\left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{16E^2}\int_0^{\pi}\frac{\sin\vartheta\, d\vartheta}{\sin^2\left(\frac{\theta}{2}\right)} = \infty \text{ !!}$$

This is an outcome of the fact that the Coulomb potential has infinite range.

Note that typically subatomic forces go like $\frac{e^{-\frac{r}{a}}}{r^2}$, where a is the 'range'; this is called Yukawa force.

So, as Coulomb force $\frac{1}{r^2}$ therefore has infinite range, whereas for strong force $a \sim 1$ fm.

* Rutherford backscattering spectroscopy (RBS) is routinely used to detect heavy elements in a lower atomic number matrix (heavy metal impurities in semiconductors).

Luminosity and Event Rate

Some more terminology for the particle physics experiments.

$\mathcal{L}$: luminosity – # particles flowing down the beam per unit time, per unit area (i.e., flux of particles)

If we assume a uniform luminosity, then $dN = \mathcal{L}\, d\sigma$ will be the # particles per unit time passing through $d\sigma$, and hence also # per unit time scattered into solid ∡ $d\Omega$:

$$\underbrace{dN}_{\text{also called event rate}} = \mathcal{L}\, d\sigma = \underbrace{\mathcal{L}}_{\text{controlled by the accelerator}} \underbrace{\frac{d\sigma}{d\Omega}}_{\text{scatterer}} \underbrace{d\Omega}_{\text{controlled by the detection part}}$$

With these parameters established, the differential cross section can be measured simply by counting particles entering the detector.

$$\frac{d\sigma}{d\Omega} = \frac{dN}{\mathcal{L}\, d\Omega}$$

If the detector completely surrounds the target, then $N = \sigma\, \mathcal{L}$

i.e., event rate is the cross section times the luminosity.

Fermi's Golden Rule

Rate of transitions among the stationary states of a system due to an additional perturbation (unaccounted in the original case) is calculated by FGR. Its general structure is:

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar} \underbrace{|\langle f|\mathcal{H}'|i\rangle|^2}_{|\mathcal{M}|^2} \underbrace{\rho_f}_{\text{Density of final states}}$$

So, processes like decay or scattering cross sections have two ingredients:

1) $\mathcal{M}$: Matrix element – amplitude for the process – Feynman Rules
   This contains dynamics of the process

2) ρ_f: Density of final states – available phase space (room to maneuver)
   Depends on the kinematics (masses, energies, momenta of the constituents)

Note that we need the relativistic version of FGR. We shall apply it to decay and scatterings separately.

Golden Rule for Decays

Within the reference frame of a particle 1, which decays into: $2+3+\cdots+n$

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \underbrace{\delta^{(4)}(p_1 - p_2 - \cdots - p_n)}_{\text{energy-momentum conservation}} \prod_{j=2}^{n} 2\pi \underbrace{\delta(p_j^2 - m_j^2 c^2)}_{\text{each outgoing particle lies on its "mass shell" } p_j^2 = m_j^2 c^2} \underbrace{\theta(p_j^0)}_{\text{outgoing energy} > 0} \frac{d^4 p_j}{(2\pi)^4}$$

S: Correction for double counting when there are identical particles in the final state: for each group of s particles $\rightarrow \frac{1}{s!}$

e.g. $a \longrightarrow \underbrace{b+b}_{\frac{1}{2!}} + \underbrace{c+c+c}_{\frac{1}{3!}}$, $\quad \frac{1}{2!}\,\frac{1}{3!} = \frac{1}{12}$

A rule for 2π factors: every $\delta(\cdot)$ gets (2π); every d gets $\frac{1}{2\pi}$

$$d^4 p_j = dp_j^0 \, d^3\vec{p}_j \qquad \text{(drop } j \text{ index from now on)}$$

For $\delta(p^2 - m^2c^2) \equiv \delta\left[(p^0)^2 - \vec{p}^{\,2} - m^2c^2\right]$

Use $\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right]$, $a > 0$

Only one "hits": due to $\theta(p^0)$

$$\theta(p^0)\,\delta\left[(p^0)^2 - \vec{p}^{\,2} - m^2c^2\right] = \frac{1}{2\sqrt{\vec{p}^{\,2} + m^2c^2}}\,\underbrace{\delta\left(p^0 - \sqrt{\vec{p}^{\,2} + m^2c^2}\right)}$$

Using this Dirac delta we trivially do $\int dp_j^0$ integrals

$$\Rightarrow \Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - \cdots - p_n) \prod_{j=2}^{n} \frac{1}{2\sqrt{\vec{p}_j^{\,2} + m_j^2 c^2}} \frac{d^3\vec{p}_j}{(2\pi)^3}$$

p_j^0 terms appearing here should be taken as $p_j^0 = \sqrt{\vec{p}_j^{\,2} + m_j^2 c^2}$ as a result of Dirac delta $\int dp_j^0$

Now, to make more (and easy) progress we apply this to two-particle decay

Two-Particle Decay $\quad 1 \longrightarrow 2+3$

This case is a lot easy since we can carry out all the integrals w/o ever knowing the functional form of $\mathcal{M}$. The reason is that, physically, two-particle decays are kinematically determined: particles have to come out back-to-back with opposite momenta.

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \frac{\delta^{(4)}(p_1 - p_2 - p_3)}{\sqrt{\vec{p}_2^{\,2} + m_2^2 c^2}\sqrt{\vec{p}_3^{\,2} + m_3^2 c^2}} d^3\vec{p}_2\, d^3 p_3$$

$$\delta(p_1^0 - p_2^0 - p_3^0)\,\delta(\cancel{\vec{p}_1}^{\,0} - \vec{p}_2 - \vec{p}_3)$$

$\sqrt{\vec{p}_j^{\,2} + m_j^2 c^2}$ $\quad m_1 c$ $\quad$ rest frame

$$\Rightarrow \Gamma = \frac{S}{32\pi^2\hbar m_1}\int |M|^2 \frac{\delta\left(m_1c - \sqrt{\vec{p}_2^{\,2}+m_2^2c^2} - \sqrt{\vec{p}_3^{\,2}+m_3^2c^2}\right)}{\sqrt{\vec{p}_2^{\,2}+m_2^2c^2}\ \sqrt{\vec{p}_3^{\,2}+m_3^2c^2}}\ \delta^{(3)}(\vec{p}_2+\vec{p}_3)\, d^3\vec{p}_2\, d^3\vec{p}_3$$

$\vec{p}_3 \to -\vec{p}_2$

$$\Gamma = \frac{S}{32\pi^2\hbar m_1}\int |M|^2 \frac{\delta\left(m_1c - \sqrt{\vec{p}_2^{\,2}+m_2^2c^2} - \sqrt{\vec{p}_2^{\,2}+m_3^2c^2}\right)}{\sqrt{\vec{p}_2^{\,2}+m_2^2c^2}\ \sqrt{\vec{p}_2^{\,2}+m_3^2c^2}}\, d^3\vec{p}_2$$

Switch to spherical coord's: $d^3\vec{p}_2 \longrightarrow r^2 \sin\theta\, d\theta\, d\phi$

$|\vec{p}_2|$

Since the amplitudes M should be scalars, i.e., only depend on r not θ and ϕ, this leaves angular $\int$'s trivial $\int \sin\theta\, d\theta\, d\phi = 4\pi$

$$\Rightarrow \Gamma = \frac{S}{8\pi\hbar m_1}\int_0^\infty |M(r)|^2 \frac{\delta\left[m_1c - \left(\sqrt{r^2+m_2^2c^2} + \sqrt{r^2+m_3^2c^2}\right)\right]}{\sqrt{r^2+m_2^2c^2}\ \sqrt{r^2+m_3^2c^2}}\, r^2\, dr$$

u

Use $\delta(f(r)) = \sum_i \frac{\delta(r-r_i)}{|f'(r_i)|}$

roots of $f(r)=0$

with $\frac{du}{dr} = \frac{ur}{\sqrt{r^2+m_2^2c^2}\ \sqrt{r^2+m_3^2c^2}}$

$$\Gamma = \frac{S}{8\pi\hbar m_1}\int_{(m_2+m_3)c}^{\infty} |M(r)|^2\, \delta(m_1c-u)\,\frac{r}{u}\, du$$

if $m_1 < m_2+m_3$ the Delta fn does not hit, i.e., a particle cannot decay into heavier secondaries

Hence, carrying out final integration yields $|\vec{p}_2| = r_0$ as

$$|\vec{p}_2| = r_0 = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

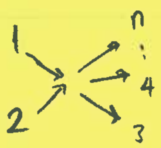
with the decay rate

magnitude of either outgoing momentum, $|\vec{p}_2| = |\vec{p}_3| = r_0$

$$\Gamma = \frac{S\,|\vec{p}|}{8\pi\hbar m_1^2 c}\,|\mathcal{M}|^2$$

evaluated at the momenta dictated by the conservation laws, $\mathcal{M}(r_0)$

Golden Rule for Scattering



$$1 + 2 \rightarrow 3 + 4 + \cdots + n$$

The scattering cross section (essentially FGR) becomes

same statistical factor (over end products)

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \overbrace{\delta^4(p_1 + p_2 - p_3 - \cdots - p_n)}^{\text{energy-momentum cons.}}$$

$$\times \prod_{j=3}^{n} 2\pi \underbrace{\delta\left(p_j^2 - m_j^2 c^2\right)}_{\text{outgoing momenta on their mass shell}} \underbrace{\theta\left(p_j^0\right)}_{\text{outgoing energy +ve}} \frac{d^4 p_j}{(2\pi)^4}$$

integrate over all outgoing momenta

Performing p_j^0 integrals within $d^4 p_j$, exploiting $\delta(p_j^2 - m_j^2 c^2)\,\theta(p_j^0)$ yields

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n) \prod_{j=3}^{n} \frac{d^3\vec{p}_j/(2\pi)^3}{2\sqrt{\vec{p}_j^{\,2} + m_j^2 c^2}}$$

with $\quad p_j^0 = \sqrt{\vec{p}_j^{\,2} + m_j^2 c^2}$

Two-body scattering (in the CM frame)

For further (easy) progress we specialize to

$$1 + 2 \rightarrow 3 + 4$$

In the CM frame:

Since $\vec{p}_1 = -\vec{p}_2$, let's simplify the term in the denominator: $(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2$

$$p_1 \cdot p_2 = \frac{E_1}{c}\frac{E_2}{c} - \vec{p}_1 \cdot (-\vec{p}_1) = \frac{E_1 E_2}{c^2} + \vec{p}_1^{\,2}$$

$$(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2 = \left(\frac{E_1 E_2}{c^2} + \vec{p}_1^{\,2}\right)^2 - (m_1 m_2 c^2)^2$$

$$= \frac{E_1^2 E_2^2}{c^4} + 2\frac{E_1 E_2}{c^2}\vec{p}_1^{\,2} + \vec{p}_1^{\,4} - m_1^2 m_2^2 c^4$$

Using $\; m_1^2 c^2 = \frac{E_1^2}{c^2} - \vec{p}_1^{\,2}$, and $\; m_2^2 c^2 = \frac{E_2^2}{c^2} - \vec{p}_2^{\,2} = \frac{E_2^2}{c^2} - \vec{p}_1^{\,2}$

$$\Rightarrow \left(p_1 \cdot p_2\right)^2 - \left(m_1 m_2 c^2\right)^2 = \cancel{\frac{E_1^2 E_2^2}{c^4}} + 2\frac{E_1 E_2}{c^2}\vec{p}_1^{\,2} + \cancel{\vec{p}_1^{\,4}} - \underbrace{\left(\frac{E_1^2}{c^2} - \vec{p}_1^{\,2}\right)\left(\frac{E_2^2}{c^2} - \vec{p}_1^{\,2}\right)}_{\cancel{\frac{E_1^2 E_2^2}{c^4}} - p_1^2 \frac{(E_1^2 + E_2^2)}{c^2} + \cancel{\vec{p}_1^{\,4}}}$$

$$= \frac{\vec{p}_1^{\,2}}{c^2} \underbrace{\left(E_1^2 + E_2^2 + 2E_1E_2\right)}_{(E_1+E_2)^2}$$

$$\therefore \sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = \frac{1}{c} |\vec{p}_1| \, (E_1 + E_2)$$

$$\Rightarrow \sigma = \frac{S\hbar^2 c}{64\pi^2 (E_1+E_2)|\vec{p}_1|} \int |M|^2 \frac{\delta^4(p_1+p_2-p_3-p_4)}{\sqrt{\vec{p}_3^{\,2} + m_3^2c^2}\sqrt{\vec{p}_4^{\,2}+m_4^2c^2}} \, d^3\vec{p}_3 \, d^3\vec{p}_4$$

$$\delta\left(\frac{E_1+E_2}{c} - p_3^0 - p_4^0\right) \delta^3(\vec{p}_3 + \vec{p}_4)$$

$$p_j^0 = \sqrt{\vec{p}_j^{\,2} + m_j^2 c^2}$$

Carry out $\int d^3\vec{p}_4$ using $\delta(\cdot)$ fn yields $\vec{p}_4 \to -\vec{p}_3$

$$\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{cS}{(E_1+E_2)|\vec{p}_1|} \int |M|^2 \frac{\delta\left[\frac{E_1+E_2}{c} - \sqrt{\vec{p}_3^{\,2}+m_3^2c^2} - \sqrt{\vec{p}_3^{\,2}+m_4^2c^2}\right]}{\sqrt{\vec{p}_3^{\,2}+m_3^2c^2}\sqrt{\vec{p}_3^{\,2}+m_4^2c^2}} \, d^3\vec{p}_3$$

Switching to spherical coord's $d^3\vec{p}_3 = r^2 dr \sin\theta \, d\theta \, d\phi$, $|\vec{p}_3| \equiv r$

Since $|\mathcal{M}|^2$ also depends on θ, we cannot perform the angular integration. However, since $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$, we can extract the differential xsection as

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{p}_1|} \int_0^\infty |\mathcal{M}|^2 \frac{\delta\left[\frac{E_1+E_2}{c} - \sqrt{r^2+m_3^2c^2} - \sqrt{r^2+m_4^2c^2}\right]}{\sqrt{r^2+m_3^2c^2}\,\sqrt{r^2+m_4^2c^2}} r^2 dr$$

This becomes identical to two-particle decay under $m_2 \to m_4$, $m_1 \to \frac{E_1+E_2}{c^2}$

So, we can directly write

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}, \text{ where } \begin{cases} |\vec{p}_f| = |\vec{p}_3| = |\vec{p}_4| \\ |\vec{p}_i| = |\vec{p}_1| = |\vec{p}_2| \end{cases}$$

Note that two-particle final state has the nice feature that we are able to carry out the calculation till the end w/o knowing explicit functional form of $\mathcal{M}$, as it was the case in two-particle decay.

Dimension of $\mathcal{M}$: If there are n external lines (incoming + outgoing), then dim. of $\mathcal{M}$ becomes $(\text{momentum})^{4-n}$

e.g.: $A \to B+C$, $n=3$ $[\mathcal{M}]$ = momentum

$A+B \to C+D$, $n=4$ $[\mathcal{M}]$ = dimensionless

Feynman Diagrams (for QED)

"Julian Schwinger once said rather bitterly that "Feynman brought quantum field theory to the masses," by which he meant that any dullard could memorize a few "Feynman rules", call himself or herself a field theorist, and build a credible career. Generations learned Feynman diagrams w/o understanding field theory. Heavens to Betsy, there are still university professors like that walking around!" A. Zee

First, some remarks on the meaning of Feynman diagrams

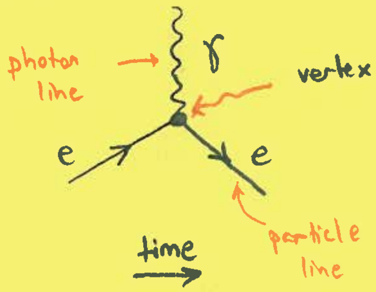
* Feynman diagrams are purely symbolic; they do not represent particle trajectories (like in a bubble chamber).

* Horizontal dimension is (usually) time, but the vertical spacing does not correspond to physical separation.

* Quantitatively, each Feynman diagram stands for a particular number, which can be calculated using the so-called Feynman rules. To analyze a certain process (say, Møller scattering): first, you draw all the diagrams that have the appropriate external lines, then evaluate the contribution of each diagram using the Feynman rules, and add it up. The sum total of all Feynman diagrams with the given external lines represents the actual physical process.

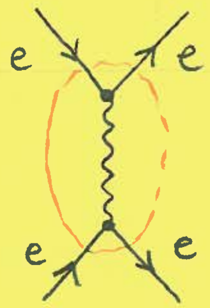
* Even though, there are infinitely many Feynman diagrams for any particular problem, since each **vertex** within a diagram introduces a factor of $\alpha = \frac{e^2}{\hbar c} \simeq 1/137$, the fine structure constant, only a few leading diagrams suffice — in QED a calculation seldom uses more than 4 vertices.

primitive vertex (QED)



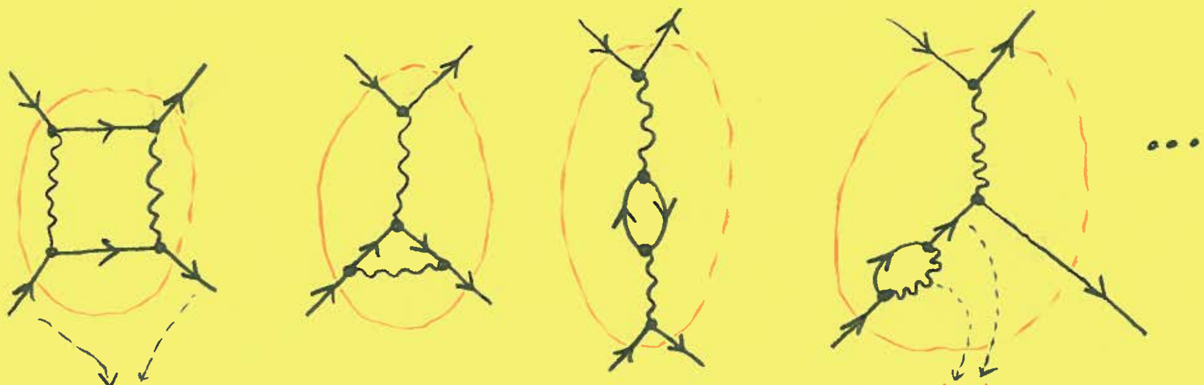
A charged particle, e, enters, emits (or absorbs) a photon, γ, and exits.

We build higher-order diagrams using this vertex:



Møller Scattering: describes the interaction bet. two $\bar{e}$'s (the classical Coulomb repulsion of like charges); the interaction is mediated by the exchange of a (virtual) photon.

There are higher-order diagrams which also contribute to Møller scattering:



External lines tell what physical process is occurring, internal lines describe the **mechanism** involved.

Virtual Particles: Internal lines (those which begin and end within the diagram) represent particles that are not observed. Although energy-momentum must be conserved at each vertex, a virtual particle do not necessarily lie on their mass shell, i.e., $p_\mu p^\mu = E^2 - \vec{p}^2 c^2$, but can take on any value (whereas a real particle must be on its mass shell, $E^2 - \vec{p}^2 c^2 = m^2 c^4$)

As a matter of fact any particle will be eventually vanish (absorbed etc.) for detection purposes etc. A photon from a star will end up in our eyes etc. However, the farther a virtual particle is from its mass shell the shorter it lives, so a photon from a distant star would have to be extremely close to its "correct" mass — it would have to be almost 'real'. So, we might say that a real particle is a virtual particle that lasts long enough that we don't care to inquire how it was produced, or how it is eventually absorbed.

Antiparticles: You are allowed to twist 'Feynman diagrams' around into any topological configuration you like. A particle line running 'backward in time' (arrow pointing leftwards) is to be interpreted as the corresponding antiparticle going forward in time. (Since photon is its own antiparticle, we do not need to put an arrow on the photon line.)

Crossing Symmetry: Suppose that a reaction of the form

$$A + B \rightarrow C + D$$

is known to occur, then any of the particles can be 'crossed' over the other side of the equation, provided it is turned into its antiparticle.

Such a crossed (or reversed) reaction will be dynamically permissible (but it may or may not be kinematically allowed – a kinetic energy threshold may exist for the input particles).

So, for above example:

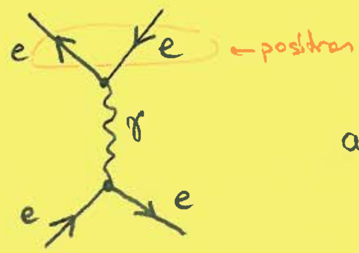
$$A \rightarrow \bar{B} + C + D$$
$$A + \bar{C} \rightarrow \bar{B} + D$$
$$\bar{C} + \bar{D} \rightarrow \bar{A} + \bar{B}$$

The crossing symmetry also tells us that Compton scattering is really the same process as pair annihilation:

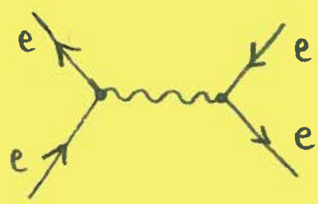
$$\gamma + e^- \rightarrow \gamma + e^- \qquad \text{vs.} \qquad e^- + e^+ \rightarrow \gamma + \gamma$$

In terms of Feynman diagrams, crossing symmetry corresponds to twisting or rotating the figure. Under crossing symmetry, Møller scattering goes into so-called Bhabha scattering – which is the name for electron-positron scattering:

named after the Indian physicist, Homi J. Bhabha



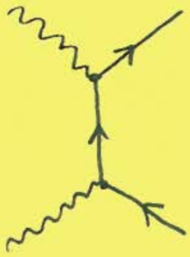
and



Other possibilities using only two vertices (all crossing-symmetry products):

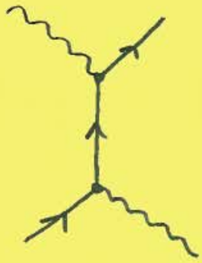
$e^- + e^+ \longrightarrow \gamma + \gamma$

(pair annihilation)



$\gamma + \gamma \to e^- + e^+$

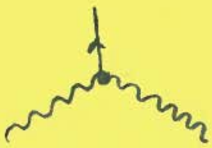
(pair production)

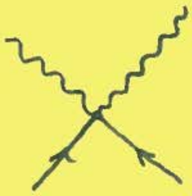


$e^- + \gamma \to e^- + \gamma$

(Compton scattering)

Impossible vertices:





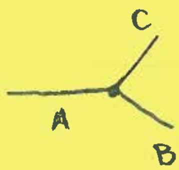
Any of these is impossible within QED.

Feynman Rules for a Toy Theory

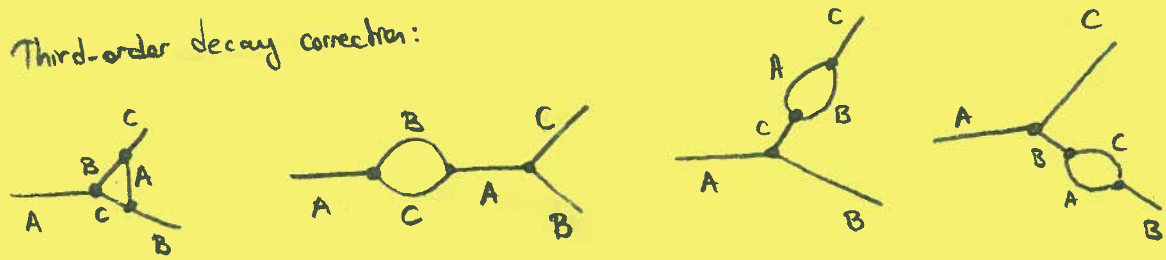
Now it comes to state the rules that goes into the calculation of a matrix element M using Feynman diagrams. First, to get the basic notion, we remove the complications brought by spin, and consider a toy theory for spin-0 particles. [We shall get back to Feynman rules for QED after we learn about the Dirac Equation]

Toy Model: Only three kinds of particles A, B, C, all with spin-0 and each is its own antiparticle (so we don't need arrows on the lines). We assume $m_A \geqslant m_B + m_C$ so that $A \to B + C$ is kinematically possible.

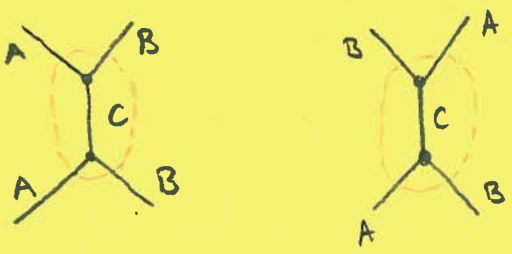
Lowest-order decay:



Third-order decay correction:

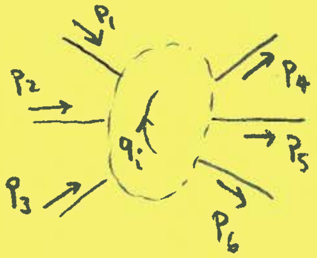


Lowest-order $A + A \to B + B$ and $A + B \to A + B$ scattering



To find the amplitude $\mathcal{M}$ associated with a given Feynman diagram, we apply the following rules:

1. **Labeling:** Label the incoming and outgoing four-momenta, i.e., <u>external lines</u> with $p_1, \dots, p_n$ (use forward in time arrows)
Label <u>internal</u> momenta with $q_1, q_2, \dots$ (use arbitrary dir.)



coupling constant

2. **Vertex Factors:** For each vertex, include a $-ig$ term
In 'real-world' theories, g is always dimensionless (like $\alpha = \frac{e^2}{\hbar c}$)
whereas in this toy model g has the dimensions of momentum.

3. **Propagators:** For each <u>internal</u> line include a $\dfrac{i}{q_j^2 - m_j^2 c^2}$
with q_j

NB: $q_j^2 \neq m_j^2 c^2$ ... virtual particle

4. **Conservation of Energy & Momentum**
For each vertex include a $(2\pi)^4 \, \delta^4(k_1 + k_2 + k_3)$
where k_i's are 4-momenta <u>into</u> the vertex — like Kirchhoff's J.L.

5. Integration For each internal line, include $\int \frac{1}{(2\pi)^4} d^4 q_j$

NB: Every δ gets a 2π, and $d \rightarrow 1/(2\pi)$

6. Cancel the delta fn: The result will contain a delta fn.

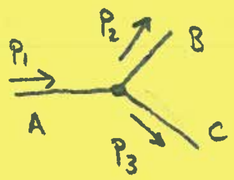
$(2\pi)^4 \delta^4(p_1 + p_2 + \cdots - p_n)$ reflecting overall conservation of energy-mom.

Replace this factor with i.

This yields the contribution of this diagram to $\mathcal{M}$.

Now we consider some applications of these rules.

Decay of A (lowest order)



Rule-5 $\rightarrow i$

$$-ig \, (2\pi)^4 \delta^4(p_1 - p_2 - p_3)$$

$$\Rightarrow \mathcal{M} = g$$

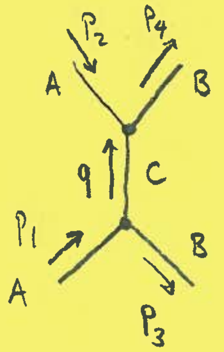
Inserting into $\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m_A^2 c} |\mathcal{M}|^2$ for the decay rate

($S \rightarrow 1$, $\mathcal{M} \rightarrow g$)

where $|\vec{p}| = |\vec{p}_B| = |\vec{p}_C| = \frac{c}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$

Lifetime of A is then $\tau = \frac{1}{\Gamma} = \frac{8\pi\hbar m_A^2 c}{g^2 |\vec{p}|}$

A + A → B + B



Applying Rules 1-5:

$$\int \frac{d^4q}{(2\pi)^4}\,(ig)^2\,\frac{i}{q^2-m_c^2c^2}\,(2\pi)^4\,\delta^4(p_1-p_3-q)\,(2\pi)^4\,\delta^4(p_2+q-p_4)$$

$q \to p_4 - p_2$

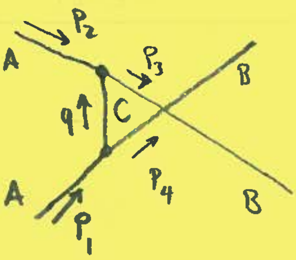
$$= -ig^2\,\frac{1}{(p_4-p_2)^2-m_c^2c^2}\,(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)$$

Rule - 6

i

$$\Rightarrow \quad \mathcal{M} = \frac{g^2}{(p_4-p_2)^2-m_c^2c^2}$$

There is yet another 2^{nd}-order diagram that contributes to $\mathcal{M}$



which can be directly written from prev. case by

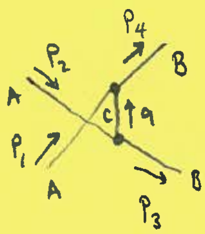
$p_3 \leftrightarrow p_4$

So that to order g^2 we have:

$$\mathcal{M} = \frac{g^2}{(p_4-p_2)^2-m_c^2c^2} + \frac{g^2}{(p_3-p_2)^2-m_c^2c^2}$$

... observe that it is Lorentz-invariant i.e., a scalar (built into Feynman rules)

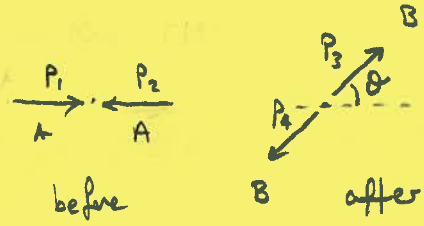
NB:



is not a new diagram, but exact replica of the second diagram

Let's connect this $\mathcal{M}$ to a measurable quantity, $\frac{d\sigma}{d\Omega}$ in CM frame.

Assume, say $m_A = m_B = m$, $m_C = 0$



CM frame

$$\Rightarrow \quad (p_4 - p_2)^2 - \cancel{m_c^2} c^2 = p_4^2 + p_2^2 - 2 p_2 \cdot p_4$$

$$= -2\,\vec{p}^{\,2}(1-\cos\theta)$$

$\vec{p} \equiv \vec{p}_1$

$$(p_3 - p_2)^2 - \cancel{m_c^2} c^2 = p_3^2 + p_2^2 - 2 p_3 \cdot p_2 = -2\,\vec{p}^{\,2}(1+\cos\theta)$$

$$\Rightarrow \quad \mathcal{M} = -\frac{g^2}{\vec{p}^{\,2}\sin^2\theta}$$

(now more transparent that it is Lorentz-inv.)

From our previous expression for $\frac{d\sigma}{d\Omega}$, we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{\hbar c g^2}{16\pi E \vec{p}^{\,2}\sin^2\theta}\right)^2$$

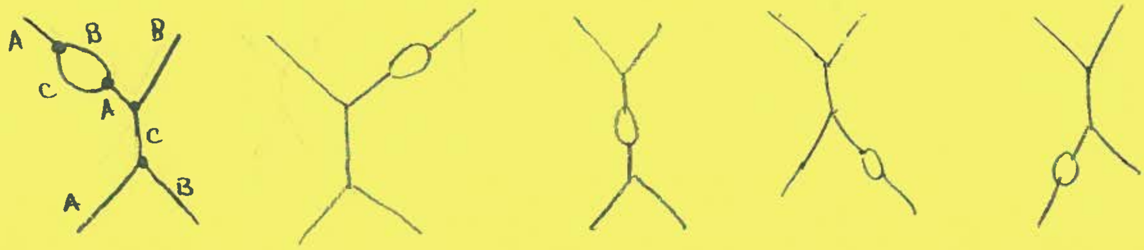
from $S = \frac{1}{2!}$ $(\rightarrow B + B)$

NB: As in the case of Rutherford scattering the <u>total</u> xsection $\rightarrow \infty$.

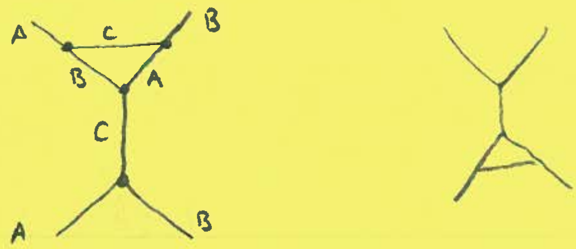
Higher-order Diagrams for $A + A \to B + B$

In addition to lowest order (tree)  which is 2^{nd}-order,

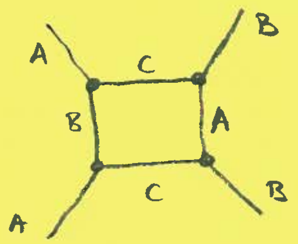
we have <u>eight</u> more 4^{th}-order diagrams:



5 "self-energy"  diagrams

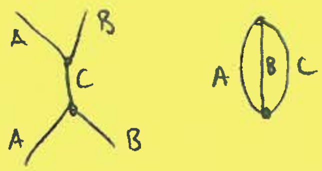


2 "vertex correction"



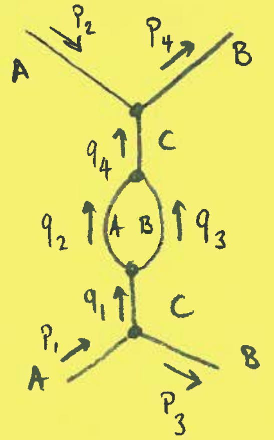
← 1 'box' diagram

<u>NB</u>: Disconnected diagrams do not count (they are cancelled by vacuum polarization)

e.g., 

There is an unpleasant surprise hidden in these diagrams.

Consider, for instance:



From Rules 1-5:

$$g^4 \int \frac{\delta^4(p_1 - q_1 - p_3)\,\delta^4(q_1 - q_2 - q_3)\,\delta^4(q_2 + q_3 - q_4)\,\delta^4(q_4 + p_2 - p_4)}{(q_1^2 - m_C^2 c^2)(q_2^2 - m_A^2 c^2)(q_3^2 - m_B^2 c^2)(q_4^2 - m_C^2 c^2)}\, d^4q_1\, d^4q_2\, d^4q_3\, d^4q_4$$

$\int dq_1 : \quad q_1 \to p_1 - p_3 \quad , \quad \int dq_4 : \quad q_4 \to p_4 - p_2$

$$\frac{g^4}{[(p_1 - p_3)^2 - m_C^2 c^2][(p_4 - p_2)^2 - m_C^2 c^2]} \int \frac{\delta^4(p_1 - p_3 - q_2 - q_3)\,\delta^4(q_2 + q_3 - p_4 + p_2)}{(q_2^2 - m_A^2 c^2)(q_3^2 - m_B^2 c^2)}\, d^4q_2\, d^4q_3$$

$\int d^4q_2 : \quad q_2 \to p_1 - p_3 - q_3$ yielding $\delta^4(p_1 + p_2 - p_3 - p_4) \xrightarrow{\text{Rule 6}} \frac{i}{(2\pi)^4}$

$q_3 \to q$

$$\Rightarrow \mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \frac{1}{[(p_1 - p_3)^2 - m_C^2 c^2]^2} \int \frac{d^4q}{[(p_1 - p_3 - q)^2 - m_A^2 c^2]\,(q^2 - m_B^2 c^2)}$$

At large q

$$\int^{\infty} \frac{1}{q^4}\, q^3\, dq = \ln q \Big|^{\infty} \to \infty$$

This was the main stumbling point in QED and such divergences halted the progress of QED for more than a decade. The resolution was to realize that such divergences arouse because of using bare e^- mass and charges as opposed to physically measured values. The technical procedure to get rid of these divergences is to regularize such integrals by using a suitable cutoff procedure that renders it finite w/o spoiling its other desirable features (such as Lorentz invariance). We introduce

$$\frac{-M^2c^2}{q^2 - M^2c^2} \quad ; \quad M: \text{cutoff mass}$$

under the integral; [at the end of the calculation M will be taken to infinity]

the integral is then carried out, yielding two parts:

$$\underbrace{\mathrm{I}}_{\substack{\text{finite} \\ (\text{independent of } M)}} + \underbrace{\mathrm{II}}_{\sim \log M}$$

$\downarrow$ miracle

all divergent, M-dependent terms appear in the form of additions to the masses and the coupling constant

The point is that physical masses and couplings are not m's and g's that appeared in the original Feynman rules, but rather the "renormalized" ones, containing these extra factors:

$$\underbrace{m_{\text{physical}}}_{\substack{\text{finite} \\ \text{(measured in the lab)}}} = \underbrace{m + \delta m}_{\text{contain compensating infinities}}; \qquad g_{\text{physical}} = g + \delta g$$

So, in the so-called renormalization procedure, we take account of infinities by using the physical values of m and g in the Feynman rules, and then systematically ignoring the divergent contributions from higher-order diagrams. That means we discard term II.

How about term I?

This term gives rise to further modifications in m & g which are fn's of 4-momentum (like $P_1 - P_3$ in the above example). This is meaningful; indeed masses and coupling constants can depend on the energies of the particles involved. They have measurable consequences, in the form of Lamb shift (in QED) and asymptotic freedom (in QCD).

If all the infinities arising from higher-order diagrams can be accommodated in this way, we say the theory is renormalizable.

ABC theory, QED, and all gauge theories (QCD, electroweak) are renormalizable.

Non-renormalizable theory yields answers that are cutoff-dependent — nonsense!

Dirac Equation

Schrödinger equation is not covariant, hence cannot be relativistically correct. Spinors, whenever needed, has to be introduced by hand. The restoration of the respect for the spacetime by Dirac resulted in his famous equation.

:-) When Dirac met the young Richard Feynman at a conference, he said after a long silence "I have an equation. Do you have one too?"

With the Dirac equation, we not only automatically get spinors, but also the very concept of antiparticles Note that Dirac eqn. is specifically for spin-1/2 particles. The relativistic QM for spin-0 are governed by Klein-Gordon equation and spin-1 by the Proca equation.

Since Schrödinger eqn heuristically follows from the classical energy-momentum relation $\frac{\vec{p}^2}{2m} + V = E$ by overloading with operators: $\vec{p} \to -i\hbar\vec{\nabla}$, $E \to i\hbar\frac{\partial}{\partial t}$

acting on w.f. $\Psi(\vec{r},t)$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\Psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)$$

whatever that means!

so, if we start from the relativistic energy-momentum relation (setting $V \equiv 0$, free particles) we may hope to get a better eqn.

$$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m^2c^2$$

where we should substitute $p_\mu \to i\hbar\partial_\mu$

$\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$, so it is a <u>covariant</u> four-vector.

This last remark requires a little (technical) elaboration:

Contravariant Lorentz xf are:

$$(1)\begin{cases} x'^0 = \gamma(x^0 - \beta x^1) \\ x'^1 = \gamma(x^1 - \beta x^0) \\ x'^2 = x^2 \\ x'^3 = x^3 \end{cases} \quad \longleftrightarrow \quad \left.\begin{matrix} x^0 = \gamma(x'^0 + \beta x'^1) \\ x^1 = \gamma(x'^1 + \beta x'^0) \\ x^2 = x'^2 \\ x^3 = x'^3 \end{matrix}\right\} (3)$$

Covariant xf are:

$$(2)\begin{cases} x'_0 = \gamma(x_0 + \beta x_1) \\ -x'_1 = \gamma(-x_1 - \beta x_0) \Rightarrow x'_1 = \gamma(x_1 + \beta x_0) \\ x'_2 = x_2 \\ x'_3 = x_3 \end{cases}$$

So, we have to check whether $\dfrac{\partial \phi}{\partial x^\mu}$ transforms as set $\underset{\text{contravariant}}{(1)}$ or $\underset{\text{covariant}}{(2)}$:

$$\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial \phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{\text{from set (3)}} \partial_\nu \phi$$

$$\frac{\partial x^0}{\partial x'^0} = \gamma, \quad \frac{\partial x^0}{\partial x'^1} = \gamma\beta, \quad \frac{\partial x^1}{\partial x'^0} = \gamma\beta, \quad \frac{\partial x^1}{\partial x'^1} = \gamma, \quad \frac{\partial x^2}{\partial x'^2} = 1, \quad \frac{\partial x^3}{\partial x'^3} = 1$$

and all the rest are zero.

So, we have

$$\left.\begin{aligned}
\frac{\partial \phi}{\partial x'^0} &= \gamma\left[(\partial_0 \phi) + \beta(\partial_1 \phi)\right] \\
\frac{\partial \phi}{\partial x'^1} &= \gamma\left[(\partial_1 \phi) + \beta(\partial_0 \phi)\right] \\
\frac{\partial \phi}{\partial x'^2} &= \partial_2 \phi, \qquad \frac{\partial \phi}{\partial x'^3} = \partial_3 \phi
\end{aligned}\right\} \text{ same xf. as in set (2)}$$

This observation proves that $\frac{\partial}{\partial x^\mu}$ transforms as a covariant 4-vector, so we should attribute the shorthand notation ∂_μ.

Likewise, $\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$ is a contravariant 4-vector

$\therefore$ $\vec{p} \to -i\hbar\vec{\nabla}$, $E \to i\hbar\frac{\partial}{\partial t}$ will be achieved by setting $p_\mu \to i\hbar\partial_\mu$ $\begin{cases} \frac{E}{c} = i\hbar\frac{\partial}{c\partial t} \\ -\vec{p} = i\hbar\vec{\nabla} \end{cases}$

Similarly, $p^\mu \to i\hbar\partial^\mu$ $\begin{cases} \frac{E}{c} = i\hbar\frac{\partial}{c\partial t} \\ \vec{p} = -i\hbar\vec{\nabla} \end{cases}$

$p^\mu p_\mu - m^2c^2 = 0$

$$\Rightarrow -\hbar^2 \partial^\mu \partial_\mu \psi - m^2c^2\psi = 0$$

$$\Rightarrow -\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2 \psi \quad \ldots \text{ Klein-Gordon Equation}$$

In the first years of QM, KGE was disgraced as it's wf normalization was time-dependent. However, in 1934 Pauli and Weisskopf showed that the statistical interpretation must be reformulated in relativistic quantum theory; the reason is that such a theory has to account for pair production and annihilation and hence the ~~#~~ particles can no longer be a conserved quantity.

KGE applies to particles with spin 0. For particles of spin 1/2, Dirac's strategy was to start from the relativistic energy-momentum relation and factor it as:

as yet, unknown

$$(p^\mu p_\mu - m^2c^2) = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$$

Recall Einstein S.C.: There are summations over μ, κ and λ

$$= \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc\underbrace{(\beta^\kappa p_\kappa - \gamma^\lambda p_\lambda)}_{(\beta^\kappa - \gamma^\kappa)p_\kappa} - m^2c^2$$

change dummy var. $\lambda \to \kappa$

no linear term on the LHS so, must vanish

$$\Rightarrow \beta^\kappa = \gamma^\kappa$$

$$\Rightarrow p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$$

$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (\gamma^0)^2(p^0)^2 + (\gamma^1)^2(p^1)^2 + (\gamma^2)^2(p^2)^2 + (\gamma^3)^2(p^3)^2$$

$$+ (\gamma^0\gamma^1 + \gamma^1\gamma^0)p_0p_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0)p_0p_2$$

$$+ (\gamma^0\gamma^3 + \gamma^3\gamma^0)p_0p_3 + (\gamma^1\gamma^2 + \gamma^2\gamma^1)p_1p_2$$

$$+ (\gamma^1\gamma^3 + \gamma^3\gamma^1)p_1p_3 + (\gamma^2\gamma^3 + \gamma^3\gamma^2)p_2p_3$$

1

0

Only if γ's become matrices, the prev. eqn. can be sat'd ⇒

← with that anticipation, we wrote γ as a 4-component vector

$$(\gamma^0)^2 = \underset{\substack{\text{identity}\\\text{matrix}}}{I_{4\times4}}, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -I_{4\times4}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \underset{\text{zero matrix}}{0_{4\times4}} \quad \text{for } \mu \neq \nu$$

or, equivalently $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

anticommutator

Minkowski metric $\begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$

The solution for 'gamma matrices' yields

Pauli matrix

$$\gamma^0 = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -I_{2\times2} \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{bmatrix} \qquad i = 1, 2, 3$$

NB: γ's are traceless.

In summary, the relativistic energy-momentum relation does factor when carried over to a 4×4 matrix

$$p^\mu p_\mu - m^2c^2 = (\gamma^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

either one can be used to form Dirac eqn.

using $p_\mu \to i\hbar\partial_\mu$

and acting on a wf ψ

we go with $(\gamma^\lambda p_\lambda - mc) = 0$ piece

This gives us the

thanks to factorization

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$ Dirac Eqn. (a first-order in spacetime 4×4 matrix d.e.)

where $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ ... bi-spinor or Dirac spinor

that's why it is not labelled as 0, 1, 2, 3 but 1,2,3,4

NB: Even though ψ has 4 components, it is not a 4-vector. It undergoes a different inertial frame xf than the Lorentz xf.

Solving the Dirac Equation

We shall consider some typical cases, going from simple to more difficult.

* Zero-momentum Solutions

If ψ is independent of position: $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0$

i.e., a particle at rest ($\vec{p} = 0$),

then the Dirac eqn. becomes

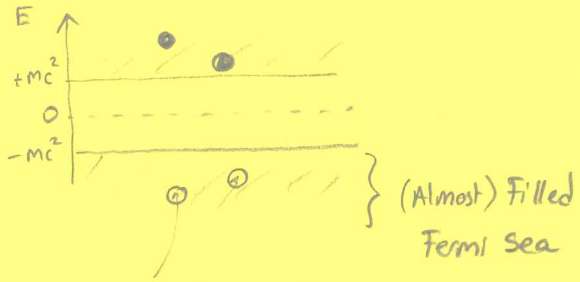
$$\frac{i\hbar}{c}\gamma^0\frac{\partial\psi}{\partial t} - mc\psi = 0 \quad \text{or} \quad \left\{ \begin{array}{l} \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_A}{\partial t}\\ \frac{\partial\psi_B}{\partial t}\end{pmatrix} = -i\frac{mc^2}{\hbar}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} \\ \\ \text{where} \quad \psi_A = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \psi_B = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} \end{array} \right.$$

Aside: Dirac vs. Feynman Interpretations

* Dirac (1930) : Particle-hole picture

$E > 0$: particles

$E < 0$: hole states



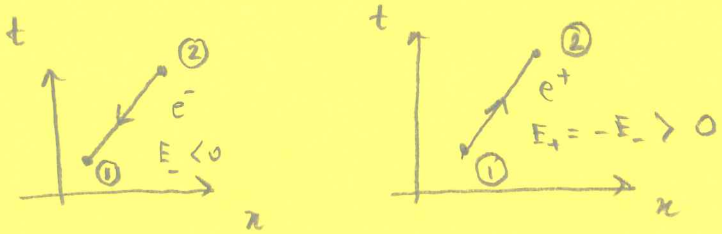
vacancy of $\bar{e}$'s (act like +ve charges with opposite spin of $\bar{e}$ and same mass)

For the infinite -ve charge of the filled Fermi sea, Dirac commented that only charge differences would be observable.

* Feynman Interpretation :

$E < 0$, $e < 0$ $\rightarrow$ $\underbrace{E > 0,\ e > 0}$

antiparticle travelling forward in time



$\Rightarrow$ Emission of an antiparticle with 4-momentum p_μ is equivalent to the absorption of a particle with 4-momentum $-p_\mu$ (and vice versa)

So, according to modern QFT, the quantization of a classical field (which has only +ve solutions) gives rise to both particle and anti-particle excitations.

Thus, $\dot{\psi}_A = -i \frac{mc^2}{\hbar} \psi_A$ and $-\dot{\psi}_B = -i \frac{mc^2}{\hbar} \psi_B$

$$\Rightarrow \quad \psi_A(t) = e^{-i\frac{mc^2}{\hbar}t} \psi_A(0) \,, \qquad \psi_B(t) = e^{+i\frac{mc^2}{\hbar}t} \psi_B(0)$$

$E = mc^2$ (for particle at rest), then $E = -mc^2$?

ψ_A solutions are perfectly reasonable, but what do we make of these negative energy solutions coming from ψ_B component ?

NB: To get rid of them by setting $\psi_B(0) = 0$ is <u>not</u> a good idea, to span the space (completeness) we need these solutions as well.

There are two alternatives to deal with negative energy states

1) Unseen infinite sea of negative-energy particles which <u>fill</u> up these states — Dirac's original proposal [Abandoned in modern treatments, as this argument does not work for charge <u>boson</u> case — no Pauli Exc. Principle]

2) These correspond to positive-energy antiparticles having anomalous time dependence wrt particles.

So, up to a normalization factor, there are 4 independent $\vec{p} = 0$ solutions:

$$\underbrace{\psi^{(1)} = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{\bar{e} \text{ with spin } \uparrow}, \quad \underbrace{\psi^{(2)} = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\bar{e} \text{ w spin } \downarrow}, \quad \underbrace{\psi^{(3)} = e^{+i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{e^+ \text{ w spin } \downarrow}, \quad \underbrace{\psi^{(4)} = e^{i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{e^+ \text{ w spin } \uparrow}$$

* Plane-wave Solutions

This time we look for solutions of the form $\psi(x) = a\, e^{-ik\cdot x}\, u(k)$

for normalization — a bispinor to be det'd

where $k\cdot x = k^0 ct - \vec{k}\cdot\vec{r}$, with $k^0 \to \frac{\omega}{c}$

Before inserting into Dirac Eqn. recall

$$\partial_0 e^{-ik\cdot x} = \frac{1}{c}\frac{\partial}{\partial t} e^{-ik^0 ct + i\vec{k}\cdot\vec{r}} = -ik^0 e^{-ik\cdot x}$$

$$\partial_1 e^{-ik\cdot x} = \frac{\partial}{\partial x}\left[e^{-ik^0 ct + ik_x x + ik_y y + ik_z z}\right] = ik_x e^{-ik\cdot x} = -i(\underbrace{-k^1}_{k_1}) e^{-ik\cdot x}$$

$$\therefore \quad \partial_\mu \psi = -ik_\mu \psi$$

So, inserting into Dirac Eqn. $i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0$ yields

$$\hbar\gamma^\mu k_\mu e^{-ik\cdot x} u - mc\, e^{-ik\cdot x} u = 0$$

$$\Rightarrow \left(\hbar \gamma^\mu k_\mu - mc\,\mathbb{I}\right) u = 0 \quad \ldots \text{ purely algebraic eqn.}$$

$$\gamma^0 k^0 - \vec{\gamma}\cdot\vec{k} = k^0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \vec{k}\cdot\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} k^0 & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -k^0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \hbar k^0 - mc & -\hbar \vec{k}\cdot\vec{\sigma} \\ \hbar \vec{k}\cdot\vec{\sigma} & -\hbar k^0 - mc \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$\Rightarrow u_A = \frac{1}{k^0 - mc/\hbar} (\vec{k}\cdot\vec{\sigma})\, u_B \quad , \quad u_B = \frac{1}{k^0 + mc/\hbar} (\vec{k}\cdot\vec{\sigma})\, u_A$$

$$\Rightarrow u_A = \frac{1}{(k^0)^2 - \left(\frac{mc}{\hbar}\right)^2} (\vec{k}\cdot\vec{\sigma})^2\, u_A$$

where $\vec{k}\cdot\vec{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$

$$\text{So, } (\vec{k}\cdot\vec{\sigma})^2 = \begin{pmatrix} k_z^2 + (k_x - ik_y)(k_x + ik_y) & k_z \cancel{(k_x - ik_y)} - \cancel{(k_x - ik_y)} k_z \\ k_z \cancel{(k_x + ik_y)} - k_z \cancel{(k_x + ik_y)} & (k_x + ik_y)(k_x - ik_y) + k_z^2 \end{pmatrix}$$

$$= |\vec{k}|^2 \, I_{2\times 2}$$

$$\therefore \quad u_A = \frac{|\vec{k}|^2}{(k^0)^2 - \left(\frac{mc}{\hbar}\right)^2} u_A$$

$$\Rightarrow (k^0)^2 - \left(\frac{mc}{\hbar}\right)^2 = |\vec{k}|^2 \quad \text{or} \quad k^\mu k_\mu = \left(\frac{mc}{\hbar}\right)^2$$

But we already know a 4-vector that has this $\}$ $k^\mu = \pm p^\mu / \hbar$

$e^{-iEt/\hbar}$ particle states

antiparticle states $e^{+iEt/\hbar}$

So, we can write down the four canonical solutions for u

under the normalization condition $u^{\dagger} u = \frac{2E}{c}$

so that $N \equiv \sqrt{\frac{E+mc^2}{c}}$

$$u^{(1)} = N \begin{bmatrix} 1 \\ 0 \\ \frac{c p_z}{E+mc^2} \\ \frac{c(p_x + i p_y)}{E+mc^2} \end{bmatrix}, \qquad u^{(2)} = N \begin{bmatrix} 0 \\ 1 \\ \frac{c(p_x - i p_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \end{bmatrix}$$

$$v^{(1)} = N \begin{bmatrix} \frac{c(p_x - i p_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \\ 0 \\ 1 \end{bmatrix}, \qquad v^{(2)} = -N \begin{bmatrix} \frac{c(p_z)}{E+mc^2} \\ \frac{c(p_x + i p_y)}{E+mc^2} \\ 1 \\ 0 \end{bmatrix}$$

$$\psi = a\, e^{-ip\cdot x/\hbar}\, u \;\; \ldots \text{ particles } \xrightarrow[\text{momentum space}]{\text{satisfy in}} \left(\gamma^{\mu} p_{\mu} - mc\right) u = 0$$

$$\psi = a\, e^{+ip\cdot x/\hbar}\, v \;\; \ldots \text{ antiparticles } \xrightarrow{\;\;''\;\;} \left(\gamma^{\mu} p_{\mu} + mc\right) v = 0$$

NB: We do not have clean entries for each bispinor (unlike $p=0$ solutions); this is b/c $\vec{L}$ and $\vec{S}$ are not conserved quantities in Dirac eqn. but $\vec{J} = \vec{L} + \vec{S}$ is. (See HW-4)

Bilinear Covariants

First we introduce some more definitions & matrices:

$$\bar{\psi} \equiv \psi^{+}\gamma^{0} = \left(\psi_1^{*} \;\; \psi_2^{*} \;\; -\psi_3^{*} \;\; -\psi_4^{*}\right)$$

$$\gamma^{5} \equiv i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}\right)$$

Using $\psi_i^{*}\psi_j$ we can form 16 products as $i = 1, \ldots, 4$ $j = 1, \ldots, 4$.

These 16 products can be assembled into various <u>linear combinations</u> to construct quantities with distinct transformation behavior, as:

$$\left.\begin{array}{l} \bar{\psi}\psi = \text{scalar} \quad (\text{one component}) \xrightarrow[\vec{r} \to -\vec{r}]{\text{under parity}} \text{preserve sign} \\ \bar{\psi}\gamma^{5}\psi = \text{pseudoscalar} \; (\text{one component}) \xrightarrow[\vec{r} \to -\vec{r}]{\text{under parity}} \text{reverse sign} \end{array}\right\} \begin{array}{l}\text{Both} \\ \text{Lorentz} \\ \text{invariant}\end{array}$$

$\bar{\psi}\gamma^{\mu}\psi$ = vector (four components)

$\bar{\psi}\gamma^{\mu}\gamma^{5}\psi$ = pseudovector (four components)

$\bar{\psi}\sigma^{\mu\nu}\psi$ = antisymmetric tensor (four comp's)

$+$

---

16 components

These are the only classes available out of combining two Dirac spinors. Observe that from the <u>look</u> of each, one can guess how it will behave. γ^{5} makes pseudo, γ^{μ} makes vector, $\sigma^{\mu\nu}$ makes tensor etc.

Finally, let's state how a Dirac spinor transfroms bet. inertial frames:

$\psi \rightarrow \psi' = S\psi$ ... not the usual Lorentz xf. matrix!

$$S = a_+ + a_- \gamma^0 \gamma^1 = \begin{pmatrix} a_+ I_{2\times2} & a_- \sigma_1 \\ a_- \sigma_1 & a_+ I_{2\times2} \end{pmatrix} = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}$$

with $a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$; $\beta = \frac{v}{c}$ ← For a frame moving along x with v.

Electrodynamics in 4-vector Notation (Gaussian cgs Units)

In the discussion of relativity we only dealt with kinematics which was used in collision calculations. Now, as we come to the discussion of Feynman rules for QED, it is time to refresh classical electrodynamics (which is a covariant theory) in 4-vector notation.

Maxwell's Equations in 3-vector notation for sources ρ, $\vec{J}$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \,, \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \,, \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

in 4-vector notation becomes

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

where $J^\nu = (c\rho, \vec{J})$, $F^{\mu\nu} = \begin{bmatrix} 0 & & & \\ E_x & 0 & & \\ E_y & B_z & 0 & \\ E_z & -B_y & B_x & 0 \end{bmatrix}$ anti-symm.

As $F^{\mu\nu} = -F^{\nu\mu}$, we have $\partial_\mu J^\mu = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ } Continuity Eqn. (Local conservation of charge)

Fields can be expressed using potentials $A^\mu = (V, \vec{A})$ as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \xrightarrow{\text{3-vector}} \begin{cases} \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{E} = -\vec{\nabla}V - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \end{cases}$$

So, the inhomogeneous Maxwell Eq's become:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu$$

As we know, potentials are not uniquely det'd, and they can be "tuned" as we like w/o affecting the actual fields $F^{\mu\nu}$.

Any transformation such as $A'_\mu = A_\mu + \underbrace{\partial_\mu \lambda}_{\text{a gauge function}}$ is known as gauge xf.

One possible choice is to set $\partial_\mu A^\mu = 0$ ... Lorentz gauge

which yields $\Box A^\mu = \frac{4\pi}{c} J^\mu$

$$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \quad \ldots \text{ d'Alembertian}$$

At this level (Lorentz Gauge) we have the nice manifestly covariant form, but there is also further degrees of freedom that can be exploited at the expense of spoiling this aesthetical manifest covariance.

* In source-free region of space, $J^\mu = 0$ it is usually convenient to opt for switching to Coulomb gauge: $A^0 = 0 \Rightarrow \vec{\nabla} \cdot \vec{A} = 0$

* In source-free region of space

$\Box A^\mu = 0$ ... KGE for a massless particle

Seeking for plane-wave type solutions:

$$A^\mu(x) = a\, e^{-\frac{i}{\hbar} p \cdot x} \underbrace{\epsilon^\mu(p)}_{\text{polarization vector}}$$

Normalization factor

(spin of the photon)

Inserting this form into KGE yields: $p^\mu p_\mu = 0$ or $E = |\vec{p}|\, c$ which is what we expect for a massless particle, photon.

In the Lorentz gauge, $\partial_\mu A^\mu = 0$, the plane-wave form leads to

$p^\mu \epsilon_\mu = 0$ ... so, only 3 components out of 4 are independent

NB: Out of the 3 indep. pol's, the <u>longitudinal</u> one does not interact with anything!

In the Coulomb gauge, $A^0 = 0$, $\vec{\nabla} \cdot \vec{A} = 0$, the plane-wave form leads to $\epsilon^0 = 0$, $\underbrace{\vec{\epsilon} \cdot \vec{p} = 0}$

free photons are transversely polarized

So, essentially both in Lorentz and Coulomb gauges, there are two linearly independent 3-vectors which are $\perp$ $\vec{p}$

If $\vec{p} \parallel \hat{z}$, $\epsilon^{(1)} = \hat{x}$, $\epsilon^{(2)} = \hat{y}$

Usually, right- and left-circular polarization superpositions are preferred:

$$\hat{\epsilon}_{\pm} = \mp \frac{\left(\epsilon^{(1)} \pm i\, \epsilon^{(2)}\right)}{\sqrt{2}} \quad \text{with } m_s = \pm 1$$

helicity

<u>NB</u>: Even though a massive ($W^\pm, Z^0$) particle of spin s admits $2s+1$ different spin orientations, a massless (γ) particle of spin s has only 2, regardless of its spin (except for $s=0$, which has 1) scalar fields

Along its dir. of motion ($\vec{p}$) it can have $m_s = \pm s$

Feynman Rules for QED (The proof for these requires QFT)

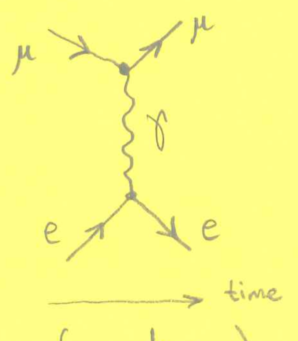
1. **Notation**: label incoming and outgoing 4-momenta & corresponding spins; label the internal 4-momenta; assign arrows to lines as follows:

NB: Arrows **on** fermion lines indicate whether they represent particles or antiparticles (a backwards in time arrow represents an antiparticle).

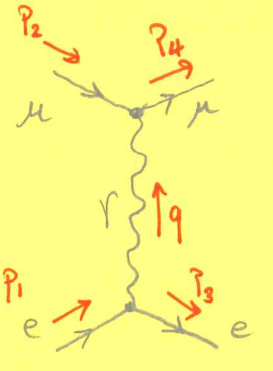
* To each **external** line associate a momentum $p_1, \ldots, p_n$ and draw an arrow **next to the line**, along the **positive** direction (forward in time). There is no relation bet. the dir. of the arrow <u>next</u> to line and <u>on</u> the line.

* To each **internal** line associate a momentum $q_1, q_2, \ldots$ put next to line indicating the "positive" direction (arbitrarily assigned).

<u>Example</u>: $e^- \mu^- \to e^- \mu^-$



(raw diagram)



(with arrows added)

2. **External Lines**: Contribute factors to matrix element to account for <u>spinors</u>:

• fermion (e^-'s): $\begin{cases} \text{incoming}: u \\ \text{outgoing}: \bar{u} \equiv u^\dagger \gamma^0 \end{cases}$

• boson (γ): $\begin{cases} \text{incoming}: \epsilon^\mu \\ \text{outgoing}: \epsilon^{\mu *} \end{cases}$

• anti-fermion (e^+'s): $\begin{cases} \text{incoming}: v \\ \text{outgoing}: \bar{v} \equiv v^\dagger \gamma^0 \end{cases}$

3. **Vertex factors:** Each vertex of the diagram contributes a factor $iq_e\gamma^\mu$ to $\mathcal{M}$ expression

$$q_e = e\sqrt{\frac{4\pi}{\hbar c}} = \sqrt{4\pi\alpha} \quad \ldots \text{ dimensionless coupling const.}$$

4. **Propagators:** Each internal line with q^μ contributes a factor to $\mathcal{M}$ as:

- e^-'s, e^+'s: $$\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2} = \frac{i(\not{q} + mc)}{q^2 - m^2c^2}$$

- γ's: $$\frac{-ig_{\mu\nu}}{q^2}$$ metric tensor

Feynman Slash Notation:

$$\gamma^\mu q_\mu = q^\mu \gamma_\mu \equiv \not{q}$$

Eg. For Dirac Eqn.:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

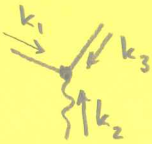
$$\Rightarrow i\hbar\not{\partial}\psi - mc\psi = 0$$

NB: In the case of weak interaction where force is mediated by massive particles ($W^\pm$, Z^0), the boson propagator gets modified as:

$$\frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2c^2}\right)}{q^2 - M^2c^2}, \quad \text{where } M = M_W \text{ or } M_Z$$

But throughout this course, we shall remain in QED and ignore weak interactions.

5. **Conservation of energy and momentum:** For each vertex of the diagram, write a delta fn. of the form



$$(2\pi)^4\,\delta(\underbrace{k_1 + k_2 + k_3})$$

use a consistent sign, say incoming momenta +ve, outgoing −ve.

6. Integrate over Internal Momenta : For each internal momentum q, add the operator : $\int \frac{d^4 q}{(2\pi)^4}$

7. Cancel the delta fn: The result (after carrying out the integrations) will include a factor $(2\pi)^4 \, \delta(\underbrace{p_1 + p_2 + \cdots}_{\text{incoming}} \underbrace{- p_n}_{\text{outgoing}})$ corresponding to overall energy-momentum conservation. Cancel this factor and multiply by i. At this stage we shall have the contribution of the chosen Feynman diagram to the $\mathcal{M}$.

Caveat : Unlike scalar ABC theory that we considered as a toy model, in QED we work with matrices, so the correct ordering of the terms is critical. Safest procedure is to track each fermion line backward through the diagram. More on this with examples...

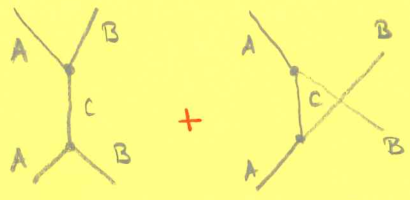
For a chosen order of calculation, all diagrams contributing to that process must be added into the amplitude $\mathcal{M}$. Here, the critical thing is to watch for anti-symmetrization.

8. Antisymmetrization : Include a minus sign bet. diagrams that differ only in the interchange of two incoming (or outgoing) e^-'s (or e^+'s), or of an incoming e^- with an outgoing e^+ or vice versa. So, only the external fermion lines are under consideration here. Two such diagrams can be referred as "twisted" or "exchange" diagrams. ← not in common use

NB: One final Feynman rule will be stated later on.

Antisymmetrization of QED Diagrams

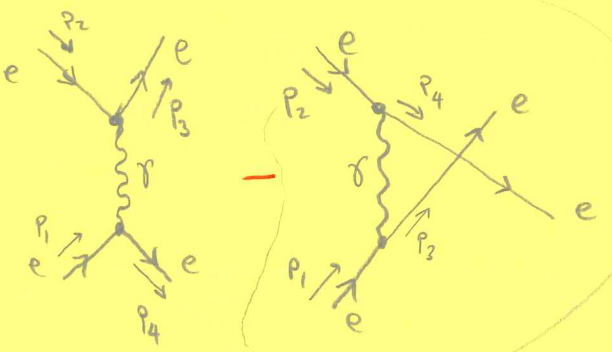
In the ABC toy model, for the process $AA \to BB$, we had two diagrams that contributed in the lowest order:



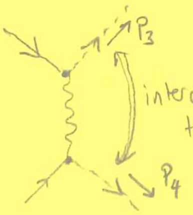
As ABC was a scalar theory, these lines were **bosons** lines, so we add the two diagrams with a relative **positive** sign to get the total amplitude.

For **fermions** the relative sign bet. such diagrams is **negative**.

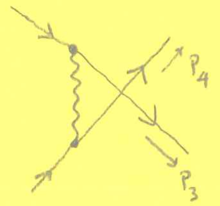
* $e^- e^- \to e^- e^-$



take this

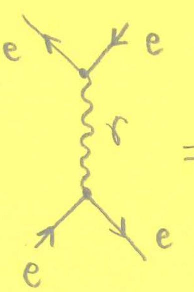


interchange vertices to which these lines are connected to

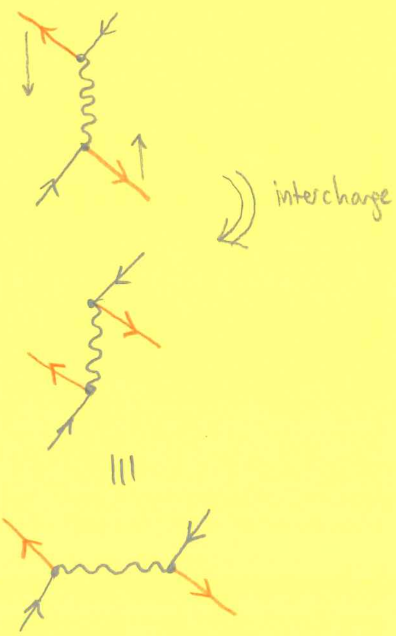


get this

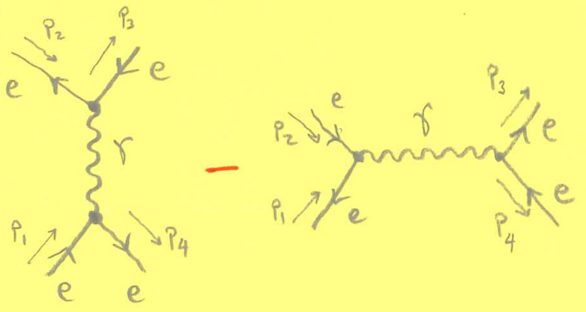
* $e^+ e^- \to e^+ e^-$ (Bhabha scattering)



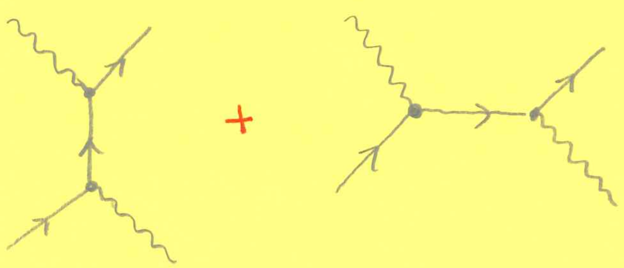
$\Rightarrow$ Now, we shall transform this diagram into another by exchanging incoming positron and outgoing e^-



So, for these two diagrams we have to include sign difference (does not matter which one picks up the -ve sign as we need $|\mathcal{M}|^2$ in FGR)



Note that the two following diagrams in Compton scattering are not like this



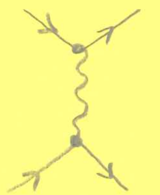
We cannot obtain one from the other by swapping two incoming e^- (outgoing e^+) or one incoming e^- and outgoing e^+ (vice versa) lines.

The root of this -ve sign lies on the indistinguishability of two like quantum particles. For fermions exchange of two fermion labels should result in a -ve sign so as to respect Pauli Exc. Pr.

Some Examples on Executing Feynman Rules

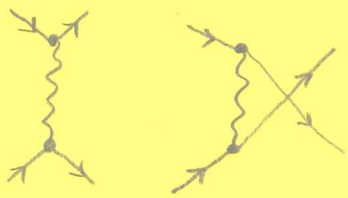
After basic foundations of QED were established the following historical calculations were performed (thanks to renormalization program that removed the divergence problems - more on this later.)

* 2^{nd}- order processes - elastic



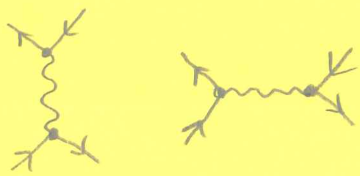
$e + \mu \rightarrow e + \mu$

Mott scattering ($M \gg m$, no recoil), Rutherford scatt. (non-rel $v \ll c$)



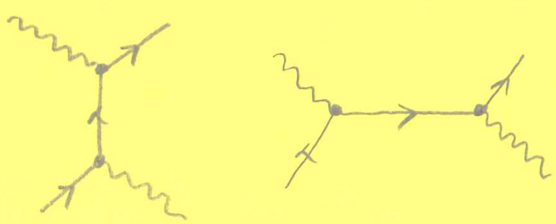
$e^- + e^- \rightarrow e^- + e^-$

Møller scattering



$e^- + e^+ \rightarrow e^- + e^+$

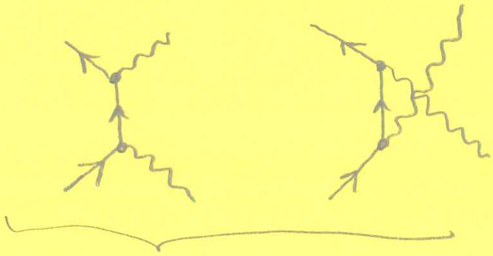
Bhabha scattering



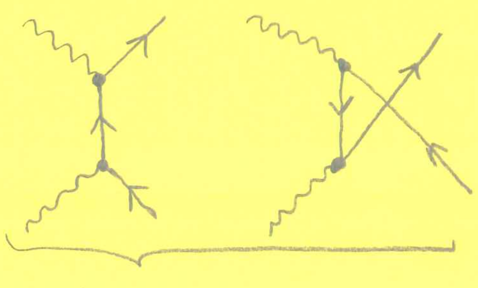
$\gamma + e^- \rightarrow \gamma + e^-$

Compton Scattering

* 2^{nd}-order - inelastic

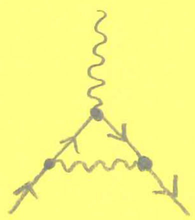


$e^- + e^+ \rightarrow \gamma + \gamma$ pair annihilation



$\gamma + \gamma \rightarrow e^- + e^-$ pair production

* 3^{rd}-order process



This is the fundamental diagram that is used in calculating the **anomalous magnetic dipole moment** (a) of an $\bar{e}$.

The Dirac equation states that $\bar{e}$ is a spinor and it has a magnetic moment (in dimensionless form, known as g-factor); the Dirac equation predicts $g = 2$. But the experimental value differs by a small amount, denoted by $a = \frac{g-2}{2}$.

One of the triumphs of QED is to calculate this!

The one-loop correction (or vertex correction) diagram shown above accounts for most of it: $a_{\text{one-loop}} = \frac{\alpha}{2\pi} \simeq 0.0011614$ (as first found by J. S. Schwinger in 1948)

The current experimental value is $a_{exp} = 0.00115965218073(28)$

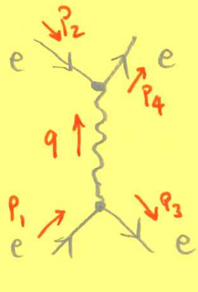
The latest QED value agrees with experiment to more than 10 significant figures!

The anomalous magnetic moment of the **muon** is calculated in a similar way which provides a precision test for the **standard model** as it contains not only QED, but weak and strong contributions.

$$a^{\mu}_{exp} = 0.00116592089(54)$$

Next we consider some of the 2nd-order diagrams to illustrate how we execute the Feynman rules. We shall leave the $e^- - \mu$ scattering for the full calculation.

* $e^- - e^-$ scattering



$\mathcal{M}_1$

track fermion lines backward from output to input

We shall use the following shorthand notation:

$$\underset{\substack{\uparrow\\ \text{4-momentum}\\ \text{of the line}}}{u^{(s_i)}(p_i)} \longrightarrow u(i) \qquad (s_i) \leftarrow \text{spinor index}$$

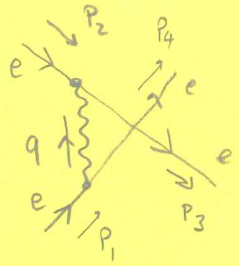
$$\int \frac{d^4q}{(2\pi)^4} \underbrace{\underbrace{\bar{u}(3)}_{1\times4}\, \underbrace{ig_e\gamma^\mu}_{4\times4}\, \underbrace{u(1)}_{4\times1}}_{\substack{\text{scalar}\\(1\times1)}} \frac{-ig_{\mu\nu}}{q^2} \underbrace{\bar{u}(4)\, ig_e \gamma^\nu\, u(2)}_{\substack{\text{scalar}\\(1\times1)}} (2\pi)^4\delta(p_1-q-p_3)\,(2\pi)^4\delta(p_2+q-p_4)$$

Performing q integral $q \to p_4 - p_2$; take into account $g_{\mu\nu}$ by changing $\gamma^\nu \to \gamma_\mu$

$$\underbrace{(2\pi)^4\delta(p_1+p_2-p_3-p_4)}_{i}\; \bar{u}(3)\, ig_e\gamma^\mu u(1)\; \bar{u}(4)\, ig_e\gamma_\mu u(2)\, \frac{-i}{(p_4-p_2)^2}$$

$$\Rightarrow \mathcal{M}_1 = -\frac{g_e^2}{(p_4-p_2)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]$$

The contribution of the 'twisted' diagram

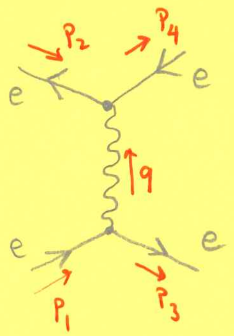


is trivially obtained from the previous expression by $3 \leftrightarrow 4$ swapping. Hence the 2^{nd}-order <u>total</u> matrix element becomes:

with (-) sign incorporated!

$$\mathcal{M} = -\frac{g_e^2}{(p_4-p_2)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right] + \frac{g_e^2}{(p_3-p_2)^2}\left[\bar{u}(4)\gamma^\mu u(1)\right]\left[\bar{u}(3)\gamma_\mu u(2)\right]$$

* $e^- - e^+$ scattering



anti-particle spinors!

$$(2\pi)^4 \int \bar{u}(3)\, i g_e \gamma^\mu u(1) \frac{-i g_{\mu\nu}}{q^2} \bar{v}(4)\, i g_e \gamma^\nu v(2)\, \delta^{(4)}(p_1-p_3-q)\, \delta^{(4)}(p_2+q-p_4)\, d^4 q$$

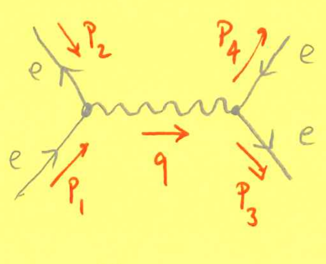
integrate & cancel $\delta(\cdot)$

$\mathcal{M}_1$

$$\Rightarrow \mathcal{M}_1 = -\frac{g_e^2}{(p_1-p_3)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{v}(2)\gamma_\mu v(4)\right]$$

The exchange diagram is:

skipping details, we get



$$\mathcal{M}_2 = -\frac{g_e^2}{(p_1+p_2)^2}\left[\bar{u}(3)\gamma^\mu v(4)\right]\left[\bar{v}(2)\gamma_\mu u(1)\right]$$

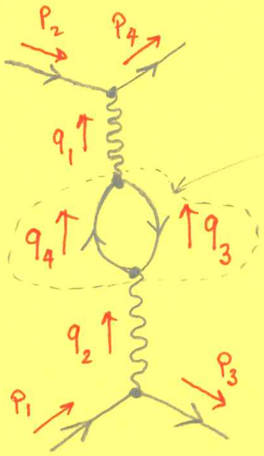
$\mathcal{M}_2$

minus sign required for exchange diagram

Total matrix element is: $\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$

Last Feynman Rule: For a closed fermion loop include a factor -1 and take the trace.

Vacuum Polarization Diagram



This Feynman diagram contains a closed fermion loop, so we shall also make use of the last Feynman rule. Let's start with loop part:

$$\underbrace{-\mathrm{Tr}\left[(ig_e\gamma^\lambda)\frac{i(\not q_4+mc)}{q_4^2-m^2c^2}(ig_e\gamma^\kappa)\frac{i(\not q_3+mc)}{q_3^2-m^2c^2}\right]}$$

Putting the rest as well

$$\int \left[\bar u(3)(ig_e\gamma^\mu u(1)\right]\frac{-ig_{\mu\lambda}}{q_2^2}\left[\text{loop}\right]\frac{-ig_{\kappa\nu}}{q_1^2}\left[\bar u(4)(ig_e\gamma^\nu)u(2)\right]$$

$$\times\ \cancel{(2\pi)^4}\,\delta^4\big(\underbrace{p_1-p_3-q_2}_{q_2\to p_1-p_3\equiv q}\big)\ (2\pi)^4\delta^4(q_2-q_3-q_4)\ \cancel{(2\pi)^4}\,\delta(q_3+q_4-q_1)$$

$$\times\ (2\pi)^4\,\underbrace{\delta^4(q_1+p_2-p_4)}_{q_1\to p_4-p_2}\ \frac{d^4q_1}{(2\pi)^4}\ \frac{d^4q_2}{\cancel{(2\pi)^4}}\ \frac{d^4q_3}{\cancel{(2\pi)^4}}\ \frac{d^4q_4}{(2\pi)^4}$$

Working out delta fn. integrations leads to a final delta fn. (as expected) $\underbrace{(2\pi)^4\delta^4(p_1-q_3+p_2-p_4)}$

i

with a new labeling $p_1-p_3\equiv q$ and $q_4\to k$, we are left with a final integral over k

$$\mathcal{M} = -\frac{i g_e^4}{q^4}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left\{\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma_\mu(\not{k}+mc)\gamma_\nu(\not{q}-\not{k}+mc)\right]}{\left[k^2-m^2c^2\right]\left[(q-k)^2-m^2c^2\right]}\right\}$$

$$\times\left[\bar{u}(4)\gamma^\nu u(2)\right]$$

Note that this integral is divergent and the divergence is remedied by a renormalization procedure.

Spin-Averaged Amplitudes – Casimir's Trick

There is a great simplification if one does not care about the spin degrees of freedom. Some of the experiments utilize beams of particles with random spins, and simply count the number of particles scattered in a given direction. In this case the relevant cross section is the average over all initial spin configurations, s_i and the sum over all final spin configurations, s_f.

$\langle |\mathcal{M}|^2 \rangle \equiv$ average over all initial spins, sum over final spins

The good news is that $\langle |\mathcal{M}|^2 \rangle$ can be calculated directly, w/o ever evaluating the individual amplitudes, using so-called Casimir's trick.

The key observation is that $|\mathcal{M}|^2$ expression contains spinor products typically of the form:

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1-p_3)^4} \underbrace{[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)]}_{\mathcal{M}} \underbrace{[\bar{u}(3)\gamma^\nu u(1)]^* [\bar{u}(4)\gamma_\nu u(2)]^*}_{\mathcal{M}^*}$$

If we regroup it, it will contain the following generic quantities

$$G \equiv [\bar{u}(a)\,\Gamma_1\, u(b)][\bar{u}(a)\,\Gamma_2\, u(b)]^*$$

Some 4×4 matrices

Such terms can be easily worked out as (see Griffiths sec. 7.7)

$$\sum_{\text{all spins}} \left[\bar{u}(a)\,\Gamma_1\,u(b)\right]\left[\bar{u}(a)\,\Gamma_2\,u(b)\right]^* = \underset{\text{trace}}{\mathrm{Tr}}\left[\Gamma_1\,(\not{p}_b + m_b c)\,\bar{\Gamma}_2\,(\not{p}_a + m_a c)\right]$$

$$\bar{\Gamma}_2 \equiv \gamma^0\,\Gamma_2^\dagger\,\gamma^0$$

Casimir's trick reduces the major task to a calculation of the trace of some complicated product of γ matrices. So, we shall list some identities that are useful

Trace Properties:

If A, B, C any three same size square matrices, α some scalar

* $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$
* $\mathrm{Tr}(\alpha A) = \alpha\,\mathrm{Tr}(A)$
* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
* $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$

Before moving to γ matrices, we state one identity on metric tensor

$$g_{\mu\nu}\,\underbrace{g^{\mu\nu}}_{\substack{\text{inverse of}\\ \text{metric tensor}}} = \mathrm{Tr}\left(g\,g^{-1}\right) = 4$$

Some Properties of γ matrix Products — Clifford Algebra

* $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\{\not{a}, \not{b}\} = 2a\cdot b$

anti-commutator

* $\gamma_\mu \gamma^\mu = 4$, $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$, $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

* $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$, $\gamma_\mu \not{a} \not{b} \gamma^\mu = 4(a\cdot b)$

* $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu$, $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$

Some trace theorems:

* The trace of the product of an odd number of gamma matrices is zero.

* $\mathrm{Tr}(\underset{I_{4\times4}}{1}) = 4$, $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$, $\mathrm{Tr}(\not{a}\not{b}) = 4(a\cdot b)$

* $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda})$

$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4(a\cdot b\, c\cdot d - a\cdot c\, b\cdot d + a\cdot d\, b\cdot c)$

Some traces involving γ^5 matrix.

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

From the above rules we have $Tr(\gamma^5\gamma^\mu) = Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$

all involving odd # γ matrices

* $Tr(\gamma^5) = 0$, $Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$, $Tr(\gamma^5 \not{a}\not{b}) = 0$

* $Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\,\epsilon^{\mu\nu\lambda\sigma}$

$$Tr(\gamma^5 \not{a}\not{b}\not{c}\not{d}) = 4i\,\epsilon^{\mu\nu\lambda\sigma} a_\mu b_\nu c_\lambda d_\sigma$$

where $\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} -1, & \text{if } \mu\nu\lambda\sigma \text{ is an } \underline{\text{even}} \text{ permutation of } 0123 \\ +1, & \text{'' '' '' '' } \underline{\text{odd}} \text{ '' '' ''} \\ 0, & \text{if any two indices are the same} \end{cases}$

NB: There will be one question on this in the HW set

Caveat: High-energy and nuclear physics using accelerators has reached a point where a very large fraction of experiments require polarized-spin beams. For instance, RHIC - relativistic heavy ion collider at Brookhaven National Lab. See for instance: Mane et al. Rep. Prog. Phys. 68, 1997 (2005).

Example: (7.6 From Griffiths)

We shall work out the following trace that comes out in e-μ scatt:

$$\mathrm{Tr}\left[\gamma^{\mu}(\not{p}_1+mc)\gamma^{\nu}(\not{p}_3+mc)\right]$$

expand & take out scalars

$$= \underbrace{\mathrm{Tr}\left(\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3\right)}_{\text{use rule for } \mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma})} + mc\left[\mathrm{Tr}\left(\gamma^{\mu}\not{p}_1\gamma^{\nu}\right) + \underbrace{\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\not{p}_3\right)}_{0}\right] + (mc)^2\underbrace{\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\right)}_{4g^{\mu\nu}}$$

$$\not{p}_1 = \sum_{\lambda}\gamma^{\lambda}p_{\lambda}$$

$$\sum_{\lambda} p_{\lambda}\underbrace{\mathrm{Tr}\left(\gamma^{\lambda}\gamma^{\nu}\gamma^{\mu}\right)}_{\mathrm{Tr}(\text{odd }\#\ \gamma\text{'s}) \equiv 0}$$

similarly

$$\underbrace{\sum_{\lambda,\sigma}(p_1)_{\lambda}(p_3)_{\sigma}\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\lambda}\gamma^{\nu}\gamma^{\sigma}\right)}$$

$$\underbrace{(p_1)_{\lambda}(p_3)_{\sigma}\,4\left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\lambda\nu}\right)}$$

$$4\left[p_1^{\mu}p_3^{\nu} - g^{\mu\nu}(p_1\cdot p_3) + p_3^{\mu}p_1^{\nu}\right]$$

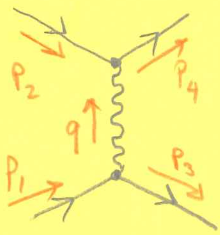
$$\Rightarrow \mathrm{Tr}\left[\gamma^{\mu}(\not{p}_1+mc)\gamma^{\nu}(\not{p}_3+mc)\right] = 4\left\{p_1^{\mu}p_3^{\nu} + p_3^{\mu}p_1^{\nu} + g^{\mu\nu}\left[(mc)^2 - (p_1\cdot p_3)\right]\right\}$$

There are people addicted to this kind of γ matrix gymnastics (i.e., Clifford Algebra). Well, sadly (for them) there are now codes built-in which do these simplifications, in Mathematica, Maple and elsewhere...

Renormalization

Now we shall demonstrate (this time mathematically) an important technique, the so-called renormalization, that rescued QED (and also QCD).

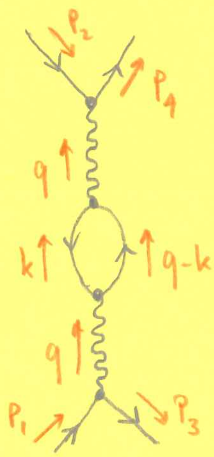
For this purpose, consider first the e-μ scattering in the lowest order diagram:



$$\mathcal{M} = -g_e^2 \left[\bar{u}(3)\gamma^\mu u(1)\right] \frac{g_{\mu\nu}}{q^2} \left[\bar{u}(4)\gamma^\nu u(2)\right]$$

with $q = p_1 - p_3$

Among the 4^{th}-order corrections, the most interesting is the vacuum polarization:



The amplitude of this diagram is:

$$\mathcal{M} = \frac{-i g_e^4}{q^4} \left[\bar{u}(3)\gamma^\mu u(1)\right] \times \left\{ \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma_\mu(\not{k}+mc)\gamma_\nu(\not{k}-\not{q}+mc)\right]}{(k^2-m^2c^2)\left[(k-q)^2-m^2c^2\right]} \right\}$$

$$\times \left[\bar{u}(4)\gamma^\nu u(2)\right]$$

If we compare the two expressions above, the inclusion of vacuum pol. diagram amounts to modifying the photon propagator to:

$$\frac{g_{\mu\nu}}{q^2} \rightarrow \frac{g_{\mu\nu}}{q^2} - \frac{i}{q^4} I_{\mu\nu} \quad , \text{where } I_{\mu\nu} = -g_e^2 \times$$

This term $I_{\mu\nu}$ is logarithmically divergent. The form of $I_{\mu\nu}$ after integration is:

$$I_{\mu\nu} = -i g_{\mu\nu} q^2 I(q^2) + \underbrace{q_\mu q_\nu J(q^2)}$$

this term does not contribute when we insert it back to $\mathcal{M}$ expression due to $\bar{u}(3) \not{q} u(1) = 0$

$$I(q^2) = \frac{g_e^2}{12\pi^2} \left\{ \int_{m^2}^{\infty} \frac{dz}{z} - 6 \underbrace{\int_0^1 z(1-z) \ln\left[1 - \frac{q^2}{m^2c^2} z(1-z)\right] dz}_{} \right\}$$

cut off

$$\int_{m^2}^{\infty} \frac{dz}{z} \xrightarrow{\infty \to M^2} \int_{m^2}^{M^2} \frac{dz}{z} = \ln\left(\frac{M^2}{m^2}\right)$$

$$f(x) \equiv -\frac{5}{3} + \frac{4}{x} + \frac{2(x-2)}{x} \sqrt{\frac{x+4}{x}} \tanh^{-1} \sqrt{\frac{x}{x+4}}$$

$$f(x) \simeq \begin{cases} \frac{x}{5} & x \ll 1 \text{ ... non-relativistic scat.} \\ \ln x & x \gg 1 \text{ ... ultrarelativistic scat.} \end{cases}$$

With this cutoff fix the matrix element including the vacuum pol. becomes:

$$\mathcal{M} = -g_e^2 \left[\bar{u}(3) \gamma^\mu u(1)\right] \frac{g_{\mu\nu}}{q^2} \left\{ 1 - \frac{g_e^2}{12\pi^2} \left[\ln\left(\frac{M^2}{m^2}\right) - f\left(\frac{-q^2}{m^2c^2}\right) \right] \right\} \left[\bar{u}(4) \gamma^\nu u(2)\right]$$

Now, introduce the renormalized coupling constant as:

$$g_R \equiv g_e \sqrt{1 - \frac{g_e^2}{12\pi^2} \ln\left(\frac{M^2}{m^2}\right)}$$

$$\Rightarrow \mathcal{M} = -g_R^2 \left[\bar{u}(3)\gamma^\mu u(1)\right] \frac{g_{\mu\nu}}{q^2} \Big\{ \underbrace{1 + \frac{g_R^2}{12\pi^2} f\left(\frac{-q^2}{m^2c^2}\right)}_{} \Big\} \left[\bar{u}(4)\gamma^\nu u(2)\right]$$

We note that :

* So, we interpret that the bare coupling g_e was actually logarithically divergent as well, and g_e was only a theoretical construct. Whereas if we use the experimentally measured (i.e., dressed) value, g_R the value is well-behaved.

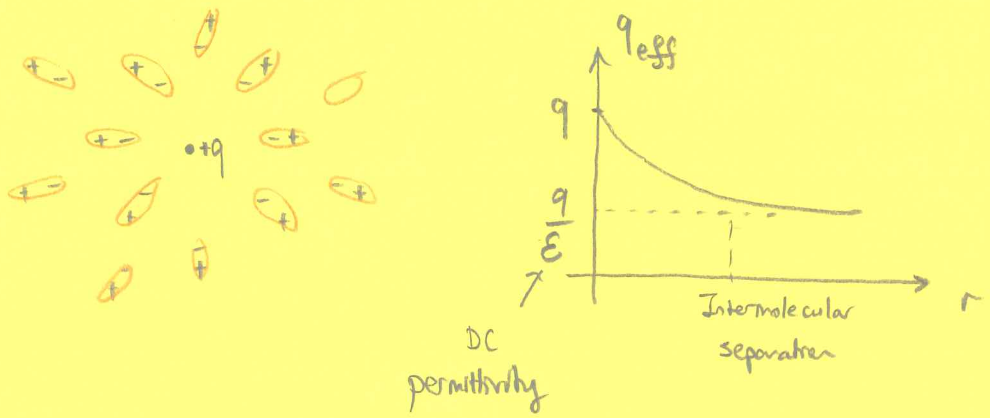
* Still there remains, this time finite, correction term that depends on the momentum xfer $q = p_1 - p_3$. So, we can absorb this into g_R by making it fn. of q^2, i.e., a running coupling constant:

$$g_R(q^2) = g_R(0)\sqrt{1 + \frac{g_R(0)^2}{12\pi^2} f\left(\frac{-q^2}{m^2c^2}\right)}$$

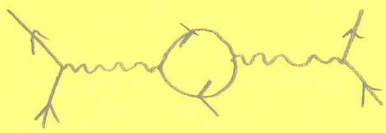
or, since $g_e = \sqrt{4\pi\alpha}$

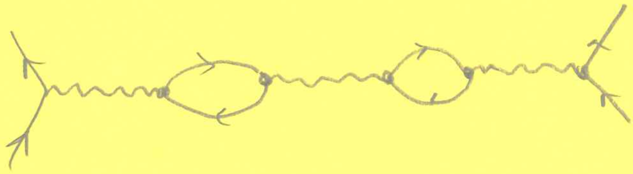
$$\alpha(q^2) = \alpha(0)\left\{1 + \frac{\alpha(0)}{3\pi} f\left(\frac{-q}{m^2c^2}\right)\right\} \quad \ldots \text{ a running fine-str. constant}$$

Hence, the effective charge of the $\bar{e}$ (and the μ), then depends on the momentum xfer in the collision, i.e., depends on how far apart each charge are. This is a consequence of the vacuum polarization, which 'screens' each charge. This is reminiscent of the screening of an external charge injected to a dielectric (polarizable) medium.



In the case of QED, the vacuum is not static, as an quantum object it has zero point energy, so has vacuum fluctuations. This is in the form of creation of e^+-e^- pairs and subsequent annihilations such as in the vacuum pol. diagrams:





etc.

The resulting vacuum polarization partially screens the charge and reduces its field. If, however, one gets too close to q, screening disappears.
The analog of 'intermolecular spacing' in this case is Compton wavelength, $\lambda_c = \frac{h}{mc} = 2.43 \cdot 10^{-12}$ m

At 'ordinary' separations of chemistry, atomic/molecular physics, an $\bar{e}$ is at its fully screened value. Even in a head-on collision at $c/10$, the correction term is about $6 \cdot 10^{-6}$. So, for most purposes $\alpha(0) = 1/137$ works just fine.

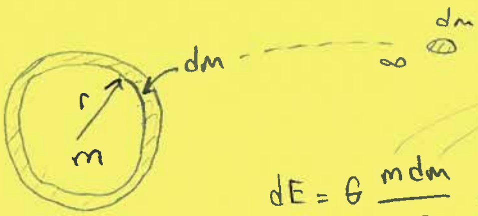
Neutrino Oscillations

This subject illustrates one of the most interesting puzzles that kept physics community busy, until 2002 or so. It is about the solar neutrino problem. To state it, we shall have to get into the following question:

Where does Sun's power (radiation) come from?

In 1850's Lord Rayleigh dealt with this problem, stating that the source was gravity - energy accumulated when all matter 'fell down' from ∞, is liberated in time in the form of radiation. We can easily reproduce this calculation.

As a differential mass dm falls in (from ∞), that amounts to a potential energy:



$$dE = G\frac{m\,dm}{r} = \frac{G}{r}\left(\rho\frac{4}{3}\pi r^3\right)\left(\rho\,4\pi r^2 dr\right) = \frac{G}{3}(4\pi\rho)^2 r^4 dr$$

$$\therefore\ E = \frac{GR^5}{15}(4\pi\rho)^2 = \frac{GR^5}{15}\left(\frac{3M}{R^3}\right)^2 = \frac{3}{5}\frac{GM^2}{R} \simeq 2.28 \cdot 10^{41}\,J$$

The solar luminosity $P_{rad} = 3.85 \times 10^{26}\,W$, so the lifetime (assuming the radiation rate to be constant) yields for the sun's lifetime $\frac{E}{P_{rad}} = 18.7$ million years.

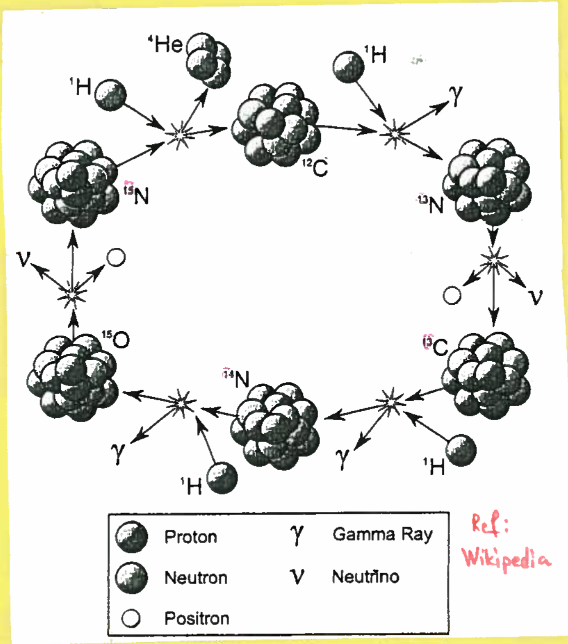
But this value is far below from the earth's age (modern value calculated by radiometric age dating of meteorite mat'l is 4.54 ± 0.05 billion years)
↳ age of the sun should be of this order

So, this 'gravitational potential energy being converted gradually to radiation' as the ultimate source for Sun's radiation must be fatally wrong.

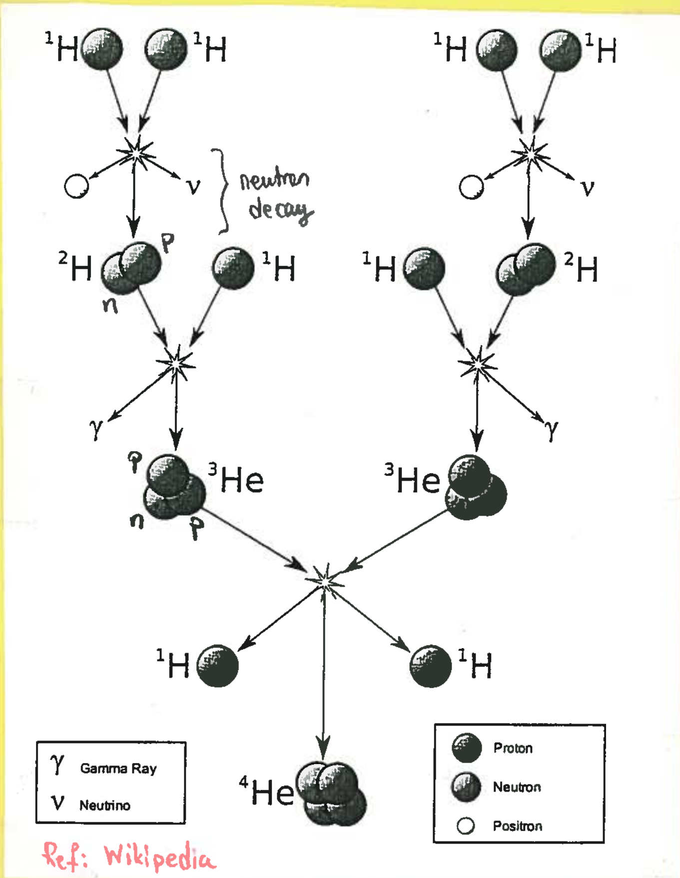
In 1896, after Bequerel's discovery of radioactivity, nuclear fission for the source of Sun's energy came into discussion. However, the problem here is that Sun did not appear to be out of radiactive elements like uranium, radium, but rather made up of H (and small amount of light elements).

In 1920 Eddington suggested nuclear fusion for the powering of the sun. Details started to be worked out by H. Bethe in 1938 who identified CNO cycle

This is one of the two mechanisms by which stars convert $H \rightarrow \alpha$ (He nucleus). It is the dominant source of energy in stars more massive than $1.3\ M_\odot$ (sun mass). The other mechanism is the proton-proton chain.



Ref: Wikipedia



Ref: Wikipedia

pp-chain is more important in stars the mass of Sun or less. The difference stems from temperature dependency differences bet. the two reactions; pp-chain reactions start occuring @ $4 \cdot 10^6$ K, making it the dominant energy source in smaller stars. The fact that Sun is still shining is due to the slow nature of this reaction; otherwise it would have exhausted its H long ago.

Leaving aside the details, pp-chain essentially produces

$$H \to \alpha + e^-\text{'s}, e^+\text{'s}, \underbrace{\gamma\text{'s}}_{\text{takes 1000 years to reach to the Sun's surface}}, \nu\text{'s}$$

almost non-interacting
perfect for studying Sun's interior

There's a huge neutrino flux from the Sun. An expert in the field, John Bahcall, said '100 billion neutrinos pass through your thumnail every second, and yet you can look forward to only one or two neutrino-induced reactions in your body during your entire lifetime.'

Solar Neutrino Problem:

in South Dakota-US — also called Davis Experiment — cleaning fluid

The Homestake mine experiment in 1968 using a huge tank of Chlorine

$$\nu_e + {}^{37}Cl \to {}^{37}Ar + e \qquad (\text{essentially } \nu_e + n \to p + e)$$

This experiment collected Ar atoms for several months (expectation: one atom every 2 days). The surprise was that it was about 1/3 of what Bahcall predicted. So, what's going on?

Neutrino Oscillations

An explanation came immediately in 1968 by Bruno Pontecorvo who proposed that e^- neutrinos produced by the Sun are transformed in flight into different species (muon neutrinos, say, or antineutrinos) to which Davis' experiment was insensitive.

To see this mathematically, consider just two neutrino types (for simplicity), say ν_e and ν_μ. If one can spontaneously convert into the other, it means that neither is an eigenfn. of the Hamiltonian. The <u>true</u> stationary states must be some orthogonal linear combinations. Let's use sine/cosine expansion coef's in the anticipation of normalization constraints, so that

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} -\sin\vartheta & \cos\vartheta \\ \cos\vartheta & \sin\vartheta \end{bmatrix} \begin{bmatrix} \nu_e \\ \nu_\mu \end{bmatrix}$$

; ϑ : mixing angle (usually det'd by by experiment, $\theta_{solar} \simeq \pi/6$).

With their time evolution as governed by the $\underbrace{\text{stationary}}_{\text{also called, mass eigenstates}}$ state energies $E_{1,2}$,

i.e., $\nu_1(t) = e^{-iE_1 t/\hbar}\,\nu_1(0)$, $\nu_2(t) = e^{-iE_2 t/\hbar}\,\nu_2(0)$

Assuming that the particle starts out as $\nu_e(0) = 1$, $\nu_\mu(0) = 0$, so that

$\nu_1(0) = -\sin\vartheta$, $\nu_2(0) = \cos\vartheta$

$$\Rightarrow \nu_\mu(t) = \cos\vartheta \underbrace{\nu_1(t)}_{-\sin\vartheta\, e^{-iE_1 t/\hbar}} + \sin\vartheta \underbrace{\nu_2(t)}_{\cos\vartheta\, e^{-iE_2 t/\hbar}} = \sin\vartheta\cos\vartheta\left(-e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar}\right)$$

After some algebraic simplification:

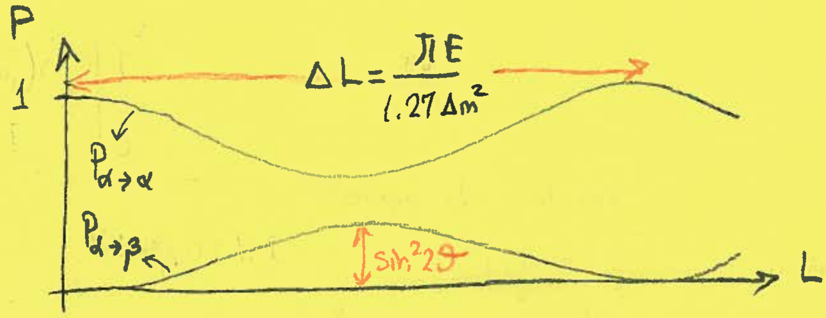
$$|\nu_\mu(t)|^2 = P_{\nu_e\to\nu_\mu} = \left[\sin 2\vartheta \sin\left(\frac{E_2 - E_1}{2\hbar} t\right)\right]^2 = \frac{\sin^2 2\vartheta}{2}\left[1 - \cos\left(\frac{E_2 - E_1}{\hbar}\right)t\right]$$

Using $\Delta E = E_2 - E_1 = \Delta m^2/2p$, $E \simeq pc$, $L \simeq tc$ $\Rightarrow \frac{t}{p} \simeq \frac{L}{E}$

We can convert oscillations from $t \to L$: (in convenient units)

$$P_{\nu_e\to\nu_\mu}(L) = \sin^2 2\vartheta \sin^2\left(\frac{1.267\ \Delta m^2[eV^2]\ L[km]}{E[GeV]}\right)$$

So, in the two-flavor approximation, the oscillation bet. flavors is of the form: $P_{\alpha \to \beta,\, \alpha \neq \beta}(L) = \sin^2 2\theta \, \sin^2\left(1.27 \frac{\Delta m^2 L}{E}\right)$



Osc's Depend on:

Exp. Parameters: $\left\{\begin{array}{l} L - \text{dist. from source to detector} \\ E - \text{Energy of } \nu\text{'s} \end{array}\right.$

Fundamental Par's: $\{ \Delta m^2, \ \sin^2 2\theta$

NB: For oscillations, we need $\theta \neq 0$ and $m_1 \neq m_2$; $\theta_{solar} \simeq \pi/6$, $\Delta(m^2)_{solar} \simeq 8.10^{-5} (eV/c^2)^2$

Experimental Confirmation

* The Super-Kamiokande collaboration in Japan (2001) used water as the detector. The process is elastic neutrino-$\bar{e}$ scatt: $\nu + e \to \nu + e$; the outgoing $\bar{e}$ is detected by the Cerenkov radiation it emits in water. → This exp. counted only electron neutrin[os]

* Sudbury Neutrino Observatory (SNO) used rather heavy water (D_2O), with the advantage over H_2O is that the neutrons (present in D) admit two other reactions (in addition to elastic scattering off $\bar{e}$'s).

  The experiments done in 2001 and 2002, together with Super K overall confirmed the neutrino oscillation proposal. That is, the Sun is indeed producing electron neutrinos at the rate predicted by theory, but 2/3 of these ν_e's are xformed to ν_μ or ν_τ during their flight from Sun's core → Earth.

NB: The ideal test for ν osc. would involve a fixed source (reactor/accelerator) and a movable detector.

↓ usually huge

osc. lengths ~ 100's of km's ← not practical!

The Mixing Matrix

So far we discussed oscillations bet. <u>two</u> neutrino species (ν_e & ν_μ). But, as we know there are actually three (ν_τ). As it turns out, if one of the three neutrino masses is substantially different from the two (and there is strong evidence for this), then quasi-two neutrino oscillation (as discussed above) remains an excellent approximation.

Nevertheless, we can mention how it is generalized to three coupled neutrinos. If we denote the mass eigenstates by ν_1, ν_2, and ν_3, then the 'MNS matrix' that describes the neutrino mixing is given by

$$\underbrace{\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}}_{\text{flavor states}} = \underbrace{\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}}_{\text{mass eigenstates}}$$

or in terms of angles θ_{12}, θ_{23}, θ_{13} and one phase factor δ

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$

$\theta_{12} \approx \theta_{solar} = 34 \pm 2^\circ$, $\theta_{23} \simeq \theta_{atm} = 45 \pm 8^\circ$, $\theta_{13} < 10^\circ$

Two 2012 Experiments:

(China) Daya Bay Collaboration: $\theta_{13} \simeq 8.83^\circ$

(Korea) RENO Experiment: $\theta_{13} \simeq 9.82^\circ$

The unitary matrix $\underline{\underline{U}}$ can be easily inverted to solve for expressing the mass eigenstates in terms of flavor states

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \underline{\underline{U}}^{-1} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

which yields $\begin{cases} \nu_3 \text{ as an almost perfect 50-50 blend of } \nu_\mu \text{ \& } \nu_\tau \\ \nu_2 \text{ as } 1/3 \text{ of each flavors} \\ \nu_1 \text{ as mostly } \nu_e \end{cases}$

Note that the observation of neutrino oscillations have proven that neutrinos should have mass (and each with different values). The Standard Model does not rule out this finite mass, even though for most considerations it is easier to set them to zero. This is unlike the case for photons which should have <u>zero mass</u> according to the Standard Model.

Quoting from D. Wark, Nature Physics, 8, 859 (2012).

"The oscillations arise b/c ν's can show two different faces, depending on how you look at them. If ν's are interacting with other particles via weak interaction, they appear as three flavors of ν — the e^-, μ & τ ν's (so called b/c they produce e^-'s, μ's or τ's when interacting). However, when ν's travel, they show another face, appearing as three ν's having well-defined masses, and unimaginatively called ν's one, two, and three. The interference bet. these leads to ν oscillations, where a beam of one flavor of ν will disappear and another appear, back and forth."