(2) LAURENT's SERIES :

~~ST~~ :

A series in the form $b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n} + \dots$ — (1). is an ordinary power series in the variable $\frac{1}{z}$ it will therefore converge outside of some circle $|z| = R$ except in $R = \infty$. The convergence is uniformly in every region $|z| \geq \rho > R$ and hence of represent as Analytic function in $|z| > R$.

$\sum_{n=-\infty}^{\infty} a_n z^n$ — (2) will be convergent only if the parts consisting of non-negative powers and negative powers are separately convergent.

Since the first part converges in a disk $|z| < R_2$ and the second in a region $|z| > R_1$ there is a common region of convergence only if $R_1 < R_2$ and (2) represents an analytic function in the annulus $R_1 < |z| < R_2$

conversely,

we may start from analytic function $f(z)$ whose region of definition, contains an annulus $R_1 < |z| < R_2$ or more generally an annulus $R_1 < |z-a| < R_2$ we show that such a function can always be developed in a general power series of the form

$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n$. Now we have

Thm Hadmard's three circle theorem :-

If $f(z)$ is analytic in the annulus $0 < r_1 < |z| < r_2$ and continuous in the closed annulus and $M(r) = \max_{|z|=r} |f(z)|$

Then $M(r) = N(r_1)^{\alpha} M(r_2)^{1-\alpha_2}$ where

$\alpha = \log\left(\frac{r_1}{r}\right) / \log(r_2/r)$.

Proof :-

consider $z^{\alpha} f(z)$

Now $|z^{\alpha} f(z)|$ attains its maximum on the boundary $|z| = r_1$ or $|z| = r_2$

If $r_1 < r < r_2$ then

$r^{\alpha} M(r) = \max\{r_1^{\alpha} M(r_1), r_2^{\alpha} M(r_2)\}$

Taking $r_1^{\alpha} M(r_1) = r_2^{\alpha} M(r_2) \rightarrow ①$.

So $r^{\alpha} M(r_1) \leq r_2^{\alpha} M(r_2)$

$\log(r^{\alpha} M(r_1)) \leq \log(r_2^{\alpha} M(r_2))$

$\alpha \log r + \log M(r_1) \leq \alpha \log r_2 + \log M(r_2)$ - ②

From ①

$$\frac{M(r_1)}{M(r_2)} = \left(\frac{r_2}{r_1}\right)^{\alpha}$$

Power series.

Definition:

A power series is of the form

$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$ where the co officiend ~~an~~ a_n and the variable z are complex.

Also $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ This is a power series w.r.to the centre z_0

eg. $1 + z + z^2 + \cdots + z^n + \cdots$

The partial sum of this

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}$$

$z^n \to 0$ for $|z| < 1$ and $|z^n| \geq 1 \ \forall \ |z| \geq 1$

$\therefore$ This Geometric series converges to $\frac{1}{1-z} \ \forall$ $|z| < 1$ and diverges for $|z| \geq 1$

If an analytic function is cgt

$\therefore$ The limit of the uniformly cgt seq. of analytic function. $\{f_n(z)\}$ where each $f_n(z)$ is defined and analytic in Ω_n

The limit function $f(z)$ must also be considered in some region Ω; and clearly if $f(z)$ is to be defined in Ω. if each point of Ω must belong to all Ω_n for n greater than certain n_0

In general case n_0 will not be the same for all pts of Ω and for this reason it.

Because of the uniform convergence on γ we get.

$$\int_{\gamma} f(z)dz = \lim_{n\to\infty} \int_{\gamma} f_n(z)dz = 0$$ and by.

morera's Theorem it follows That $f(z)$ is analytic in $|z-a|<r$ consequently $f(z)$ is analytic in The whole region Ω

$$f_n(z) = \frac{1}{2\pi i}\int_c \frac{f_n(\xi)d\xi}{\xi - z}$$ where c is The circle $|\xi - a| = r$ & $|z-a|<r$

Letting $n\to\infty$ we get by uniform converges

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\xi)d\xi}{\xi - z}$$ & This formula shows That $f(z)$ is analytic in The dist starting from The formula.

By higher derivative

$$f_n'(z) = \frac{1}{2\pi i}\int_c \frac{f_n(\xi)d\xi}{(\xi - z)^2}$$

The same reasoning gives $\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_c \frac{f(\xi)}{(\xi - z)}$

$$= f'(z)$$

and a simple estimate shows That The convergence is uniform for $|z-a| \le \rho < r$

Any compact subset of Ω such can be covered by a finite number of such closed disks and therefore The convergence is uniform in every compact subsets equation $f_n(z)=0$ inside of C. The integral on the R.H.S is therefore zero and consequently $f(z_0)\neq 0$ ($\because f_n(z_0)\neq 0$)

Since z_0 was arbitrary, the theorem follows.

Theorem:

If $f(z)$ is analytic in the region Ω containing z_0 then the representation

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{f^n(z_0)}{n!}(z-z_0)^n+\cdots$$

is valid in the largest open disk of centre z_0 contained in Ω.

Proof:

If $f(z)$ is analytic in a region Ω containing z_0 we can write

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{f^n(z_0)}{n!}(z-z_0)^n+f_{n+1}(z_0)(z-z_0)^{n+1}$$

where $f_{n+1}(z)=\frac{1}{2\pi i}\int_C \frac{f(\xi)\,d\xi}{(\xi-z_0)^{n+1}(\xi-z)}$ and

C is any circle $|z-z_0|=\rho$ such that the closed disk $|z-z_0|\leq\rho$ is. If M denotes the maximum of $|f(z)|$ on C. we get the estimate

$$= \frac{z(\bar{z}-\bar{a}) + \bar{a}(z-a)}{(z-a)(\bar{z}-\bar{a})}$$

$$= \frac{z\bar{z} - z\bar{a} + \bar{a}z - a\bar{a}}{(z-a)^2}$$

$$= \left(\frac{|z|^2 - |a|^2}{(z-a)^2}\right) d\theta$$

$$d(\arg w) = \frac{R^2 - |a|^2 d\theta}{|z-a|^2} \quad \text{--- (6)}$$

(4) becomes,

$$u(a) = \frac{1}{2\pi} \int_{|w|=1} u(z) \cdot \frac{R^2 - |a|^2}{|z-a|^2} d\theta \quad [\text{from } (6)$$

$$= \frac{1}{2\pi} \int_{|z|=R} u(z) \frac{R^2 - |a|^2}{|z-a|^2} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |a|^2}{|z-a|^2} u(z) d\theta \quad \text{--- (7)}$$

Also $\frac{R^2 - |a|^2}{|z-a|^2} = \frac{|z|^2 - |a|^2}{|z-a|^2}$

$$= \frac{z\bar{z} - a\bar{a}}{(z-a)(\bar{z}-\bar{a})}$$

$$= \frac{1}{2}\left[\frac{z+a}{z-a} + \frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}\right]$$

$$= \operatorname{Re}\left(\frac{z+a}{z-a}\right) \quad \left[\because \operatorname{Re} z = \frac{z+\bar{z}}{2}\right]$$

$$u(a) = \text{Re}\ \frac{1}{2\pi} \int_{|z|=R} \frac{z+a}{z-a}\ u(z)\ \frac{dz}{iz}$$

Putting $a = z$ and $z = \omega$.

$$u(z) = \text{Re}\left[\frac{1}{2\pi i} \int_{|\omega|=R} \frac{\omega+a}{\omega-z}\ u(\omega)\frac{d\omega}{\omega}\right]$$

The bracketed expression is an analytic function of z for $|z| < R$.

$\therefore$ $u(z)$ is the real part of $f(z)$

$$f(z) = \frac{1}{2\pi i} \int_{|\omega|=R} \frac{\omega+z}{\omega-z}\ u(\omega)\ \frac{d\omega}{\omega} + ic$$

where c is a constant

This formula is called schwartz formula.

Remark :-

In particular, if $u(z) = 1$ then $u(a) = 1$

hence poisson formula (7) becomes

$$1 = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\ d\theta.$$

$$\therefore \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\ d\theta = 2\pi$$

DEFN :- Poisson integral of a function choosing $R = 1$ for any piecewise continuous function $u(\theta)$ in or $0 \le \theta \le 2\pi$

$u_1 = \begin{cases} u & \text{on } C_1 \\ 0 & \text{on } C_2 \end{cases}$ and $u_2 = \begin{cases} 0 & \text{on } C_1 \\ u & \text{on } C_2 \end{cases}$

So $u = u_1 + u_2$

$P_U = P_{U_1} + P_{U_2}$

$\because P_{U_1}$ can be recorded as a line integral over C_1

P_{U_1} is harmonic every where except on the closed arc C_1.

If C_1 is $\widehat{ABC}$ and C_2 is $\widehat{CDA}$ [The expression

$$Re\, \frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$ vanishes on $|z|=1$ for

$z \neq e$. It follows that P_{U_1} is zero on the open arc C_2 and since it is continuous $P_{U_1}(z) \to 0$ as $z \to e^{i\theta} \in C_2$.]

Clearly $P_{U_1}(z) = 0$ for z on C_2

Claim:

$P_U(z) - U(\theta_0)$ as $z \to e^{i\theta_0}$

Suppose $U(\theta_0) = 0$, given $\varepsilon > 0$

There exist an arc C_2 such that $|U(\theta)| < \frac{\varepsilon}{2}$ for all $e^{i\theta} \in C_2$.

[$\because U(\theta)$ is continuous at θ_0 and $U(\theta_0) = 0$]

Now $|U_2(\theta)| < \frac{\varepsilon}{2}$ $\forall e^{i\theta} \in C_2$

$\because U_2(\theta) = 0$ if $e^{i\theta} \in C_1$

f(z) has analytic extend to Ω with the relation $f(z) = \overline{f(\bar{z})}$

Proof :-

Construct a function $U(z)$ as

$$U(z) = \begin{cases} u(z) & \text{in } \Omega^+ \\ 0 & \text{on } \sigma \\ -u(\bar{z}) & \text{in } \Omega^- \text{ (image of } \Omega^+) \end{cases}$$

we shall P.T $V(z)$ is required harmonic extend of $u(z)$.

consider a point $z_0 \in \sigma$

Consider a disc Δ with centre z_0. Let P_V be the poisson integral w.r.to this disc formed with the boundary value V.

Consider $V \to P_V$ By schwartz theorem $P_V \to V$.

$\therefore P_V - V = 0$ on σ

So applying maximum modulus Principle $V - P_V \leq 0$ in that semi-disc and $P_V - V \leq 0$ in the same disc by the same principle

$\therefore P_V = V$ on the whole of disc Now

$$P_V = \frac{1}{2\pi} \int_0^{2\pi} Re\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] V(\theta)\, d\theta.$$

$$= \frac{1}{2\pi} \int_{|w|=1} Re\left(\frac{w+z}{w-z}\right) V(w)\, d(\arg w)$$

Proof :

[Note :-

The laplace equation in polar co-ordinates is

$$r\frac{d}{dr}\left(r\frac{du}{dr}\right)+\frac{d^2u}{d\theta^2}=0 \,]$$

Take Ω as $0<|z|<\rho$ and $u_1=\log r$

$u_2=u$.

It is easy to verify that $\log r$ is normal take $\gamma=C_1-C_2$ where C_1 is $|z|=r_1$ and C_2 is $|z|=r_2\,(r_1<r_2)$

By above theorem

$$\int_\gamma u_1\,{}^*du_2-u_2\,{}^*du_1=0 \qquad \because u_1=\log r,\ u_2=u.$$

$$\int_{C_1-C_2}\log r\,{}^*du-u\,{}^*d(\log r)=0 \longrightarrow (1)$$

on a circle $\;{}^*du=r\frac{du}{dr}d\theta \longrightarrow (2)$

$${}^*d(\log r)=r\frac{d}{dr}(\log r)d\theta$$

$$=r\frac{1}{r}d\theta=d\theta \longrightarrow (3)$$

From (1) & (2) and (3)

$$\int_{C_1-C_2}(\log r)\,r\frac{du}{dr}d\theta-u\,d\theta=0.$$

$$\int_{C_1}\log r_1\cdot r_1\frac{du}{dr}d\theta-u\,d\theta\,]-\int_{C_2}[\log r_2\cdot r_2\frac{du}{dr}d\theta$$

$\gamma : |z| = r$

Then $\gamma = 0 \pmod{\Omega}$

Now, $\alpha' = \int_{|z|=r} r \frac{\partial u}{\partial r} d\theta = \int_{|z|=r} {}^*du = \int_{|z|=r} dv = 0$.

From (4) $\int_{|z|=r} u\,d\theta = \beta'$

$\therefore \frac{1}{2\pi} \int_{|z|=r} u\,d\theta = \frac{\beta'}{2\pi} = \beta$.

ie) arithmetic mean $= \beta$.

which proves the second part.

Theorem: Y.Y.I Maximum principle for harmonic function :-

Statement :

A non constant harmonic function has neither a maximum nor a minimum in continuous region of definition. consequently maximum (and minimum) of a harmonic function on a compact set is attained on the boundary.

Proof:

Let u be harmonic in $|z| < \rho < \Omega$

Then for $|z| = r < \rho$ by theorem 2.

$\frac{1}{2\pi} \int_{|z|=r} u\,d\theta = \beta$.

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if $I_0 = 0$, $I = I_0 + re^{i\theta}$

$I = 0 + re^{i\theta}$.

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(z)\,d\theta \quad \text{——} ①$$

consider the Linear Transformation

$$\omega = \frac{R(z-a)}{R^2 - z\bar{a}} \quad \text{——} ②.$$

This maps $|z| \le R$ onto $|\omega| \le 1$.

Further from ②

$\omega(R^2 - z\bar{a}) = R(z-a)$

$\omega R^2 - \omega z\bar{a} = Rz - Ra$.

$\omega R^2 + Ra = Rz + \omega z\bar{a}$

$z(R + \omega\bar{a}) = \omega R^2 + aR$

$$z = \frac{R(\omega R + a)}{R + \omega\bar{a}} = s(\omega) \text{ (say)} \to ③$$

This inverse transformation maps $|\omega| \le 1$ onto $|z| \le R$, if $\omega = 0$.

Then ③ ⇒ $z = \frac{R(0+a)}{R+0} = \frac{Ra}{R}$

$z = a$

Let $u(\omega) = u(s(\omega))$ clearly $u(\omega)$ is from harmonic in $|\omega - 0| \le 1$ The from ①.

$$u(0) = \frac{1}{2\pi}\int_{|\omega|=1} u(e^{it})\,dt \qquad |\omega| = 1, \ \omega = e^{it}$$

$$= \frac{1}{2\pi}\int_{|\omega|=1} u(\omega)\,d(\arg\omega) \qquad = \cos t + i\sin t \ \downarrow \ \arg\omega.$$

$$|f_{n+1}(z)(z-z_0)^{n+1}| = \frac{1}{|2\pi i|}\left|\int_C \frac{f(\xi)\,d\xi}{(\xi-z_0)^{n+1}(\xi-z)}\right|$$

we have to find $|\xi - z|$

$$|z-z_0| = |z-\xi+\xi-z_0|$$

$$= |(\xi-z_0)-(\xi-z)|$$

$$\geq |\xi-z_0| - |\xi-z|$$

$$|z-z_0| - |\xi-z_0| \geq -|\xi-z|$$

$$|z-z_0| - \rho \geq -|\xi-z|$$

$$\rho - |z-z_0| \leq |\xi-z|$$

$$\frac{1}{\rho-|z-z_0|} \geq \frac{1}{|\xi-z|}$$

$$|f_{n+1}(z)(z-z_0)^{n+1}| \leq \frac{1}{2\pi}\cdot\frac{M|z-z_0|^{n+1}}{\rho^{n+1}(\rho-|z-z_0|)}\,2\pi\rho$$

$$\leq \frac{M|z-z_0|^{n+1}}{\rho^n(\rho-|z-z_0|)}$$

Hence we conclude that the remainder tends uniformly to zero in every disk $|z-z_0| \leq r < \rho$

on the other hand ρ can be choosen orbilarily closed to the shortest distance from z_0 to the boundary of Ω.

to shows that ...

form $f_1(z)+f_2(z)$ where $f_1(z)$ is analytic for $|z-a|<R_2$ & $f_2(z)$ is analytic for $|z-a|>R$ with removable singularity at ∞.

Under these circumstances $f_1(z)$ can be developed in non-negative powers of $z-a$ and $f_2(z)$ in the non-negative power $\frac{1}{z-a}$.

Now to find the representation. $f(z)=f_1(z)+f_2(z)$

define $f_1(z)$ by $f_1(z)=\frac{1}{2\pi i}\int_{|\xi-a|=r}\frac{f(\xi)\,d\xi}{\xi-z}$ for $|z-a|<r<R_2$.

and $f_2(z)=-\frac{1}{2\pi i}\int_{|\xi-a|=r}\frac{f(\xi)\,d\xi}{\xi-z}$ for $R_1<r<|z-a|$

$\therefore$ In both integrals the value of r is irrelevant as long as the inequality is fulfilled for it is a consequence of cauchy's theorem that the value of the integral does not change with r provided that the circle does not pass through the point z.

For this reason $[f_1(z)$ and $f_2(z)$ are uniquely defined and they represent analytic function in $|z-a|<R_2$ and $|z-a|>R_1$ respectively.

By cauchy's Integral theorem the taylor's development of $f(z)=f_1(z)+f_2(z)$

$$f_1(z)=\sum_0^\infty A_n(z-a)^n \text{ with } A_n=\frac{1}{2\pi i}\int_{|\xi-a|=r}\frac{f(\xi)\,d\xi}{(\xi-a)^{n+1}} \longrightarrow (1)$$

Now to find $f_2(z)$ put $\xi=a+\frac{1}{\xi}$ and $z=a+\frac{1}{z}$

This transformation carries $|\xi-a|=r$ into $|\xi|=\frac{1}{r}$

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Also
$$\int_{\partial R} d(u_2 v_1) = 0$$

$$\therefore \int_{\partial R} u_1 {}^*du_2 - u_2 {}^*du_1 = 0$$

Theorem: 2 Mean value property of Harmonic function].

Statement:

The arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of $\log r$. ie) $\frac{1}{2\pi}\int_{|z|=r} u\,d\theta = \alpha \log r + \beta$

where α, β are constant. If u is harmonic in a disc, then the arithmetic mean is constant

$= Re\left[\frac{1}{2\pi i}\int_{|w|=1}\left(\frac{w+z}{w-z}\right)v(w)\frac{dw}{w}\right]$

$= Re[f(z)]$

lly we can P.T v is harmonic in the cover half of Δ.

Thus v is harmonic a disc on σ By the first part v is extended to the disc. Now v has a harmonic conjugate $-u_0$ in the disc such that $u_0 = Re(f(z))$

consider $U_0(z) = u_0(z) - u_0(\bar{z})$

$\therefore \frac{dU_0}{dx} = \frac{\partial u_0}{\partial x} - \frac{\partial u_0}{dx} = 0$

$\frac{dU_0}{dy} = \frac{\partial u_0}{dy} - \left(-\frac{\partial u_0}{dy}\right) = \frac{2 d u_0}{dy}$

$\frac{dU_0}{dy} = 2\frac{\partial u_0}{\partial y} = -2\frac{dv}{\partial x} = 0$ [CR equ]

Hence $\frac{dU_0}{\partial x} - \frac{i\,\partial U_0}{\partial y} = 0$ on the real axis

and hence identically $= 0$

$\Rightarrow U_0$ is a constant

$\Rightarrow U_0(z) = U_0(z) - U_0(\bar{z}) = 0$

$\Rightarrow U_0(z) = u_0(\bar{z})$

and hence the result is true for Ω

$\Rightarrow \alpha \log r_2 - \alpha \log r_1 = \log(M(r_1)) - \log M(r_2)$

$$\alpha \log\left(\frac{r_2}{r_1}\right) = \log M(r_1) - \log M(r_2)$$

$$\alpha = \frac{\log M(r_1) - \log M(r_2)}{\log(r_2/r_1)}$$

using this in ②

$$\frac{\log(M(r_1)) - \log(M(r_2))}{\log\left(\frac{r_2}{r_1}\right)}(\log r) + \log M(r_2)$$

$$\leq \frac{\log(M(r_1)) - \log(M(r_2))}{\log\left(\frac{r_2}{r_1}\right)}(\log r_2) + \log(M(r_2))$$

Since $r_1 < r_2$

$$\log\left(\frac{r_2}{r_1}\right) > 0.$$

$$\Rightarrow \log\left(\frac{r_2}{r_1}\right) \log M(r) \leq \log M(r_1)\left[\log r_2 - \log r\right]$$

$$+ \log M(r_2)\left[\log\left(\frac{r_2}{r_1}\right) - \log r_2 + \log r\right]$$

$$\leq \log M(r_1) \log\left(\frac{r_2}{r}\right) + \log M(r_2) \log\left(\frac{r}{r_1}\right)$$

$$\Rightarrow \log(M(r)) \log\left(\frac{r_2}{r_1}\right) \leq \log(M(r_1)) \log\left(\frac{r_2}{r_1}\right) + \log(M(r_2)) \log\left(\frac{r}{r_1}\right)$$

$$\Rightarrow M(r) \leq M(r_1) \cdot \frac{\log\left(\frac{r_2}{r}\right)}{\log\left(\frac{r_2}{r_1}\right)} \cdot \frac{\log\left(\frac{r}{r_1}\right)}{M(r_2) \log\left(\frac{r_2}{r_1}\right)}$$

$$\Rightarrow M(r) \leq M(r_1)^{\alpha} M(r_2)^{1-\alpha}$$

where $\alpha = \frac{\log\left(\frac{r_1}{r}\right)}{\log\left(\frac{r_2}{r}\right)}$

would not make sense to require that the convergence be uniform in Ω

In case. in the most typical case the region Ω_n form an increasing sequence $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_n \subset$ ... of Ω is the union of the Ω_n. In these circumstances no single function $f_n(z)$ is defined in all of Ω yet. The limit $f(z)$ may exist at all pts of Ω although the convergence cannot be uniform

※ Weierstrass Theorem:

Suppose that $f_n(z)$ is analytic in the region Ω_n and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω moreover $f_n'(z)$ converges uniform to $f'(z)$ on every compact subset only.

Proof: $f(z)$ is analytic (It $f_n(z) \to f'(z)$ The use of Morera's Theorem) (state

Let $|z-a| \le r$ be a closed disk contained in Ω this implies that this disk lies in $\Omega_n \forall n$ greater than a certain number

If γ is any closed curve contained in $|z-a| < r$ we have $\int_\gamma f_n(z)\,dz = 0 \;\forall\, n >$ by Cauchy's Theorem.

Theorem Hurwitz Theorem:

If the function $f_n(z)$ are analytic and $\neq 0$ in a region Ω and if $f_n(z)$ cgs to $f(z)$ uniformly on every cpt subset of Ω. Then $f(z)$ is either identically zero or never equal to zero in Ω.

proof:

Suppose that $f(z)$ is not identically zero. The zeros of $f(z)$ are in any case isolated for any point $z_0 \in \Omega$ there is therefore a number $r > 0$ s.t $f(z)$ is defined and $\neq 0$ for $0 < |z - z_0| \leq r$

In particular. $|f(z)|$ has a positive minimum on the circle $|z - z_0| = r$, which we denote by c (since f_n converges uniformly to $f(z)$.

It follows that $\frac{1}{f_n(z)}$ converges uniformly to $\frac{1}{f(z)}$ on c.

Since it is also true that $f_n'(z) \to f'(z)$ uniformly on c.

$\therefore$ we conclude that $\lim\limits_{n \to \infty} \frac{1}{2\pi i} \int_c \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)}$

[$\because$ Argument principle number of zeros.

The integrals on the LHS are all zero for they give the number of roots of the with negative orientation and by simple calculation we get $f_2\left(a+\frac{1}{z'}\right) = \frac{1}{2\pi i}\int\limits_{|\xi'|=\frac{1}{r}} \frac{z'}{\xi'} \frac{f\left(a+\frac{1}{\xi'}\right)}{\xi'-z'} d\xi'$

$$= \sum_{n=1}^{\infty} B_n z'^n \quad \text{with } B_n = \frac{1}{2\pi i}\int\limits_{|\xi|=\frac{1}{r}} \frac{f\left(a+\frac{1}{\xi}\right)}{\xi^{n+1}}$$

$$B_n = \frac{1}{2\pi i}\int\limits_{|\xi-a|=r} f(\xi)\,(\xi-a)^{n-1}\,d\xi.$$

which shows that we can write $f(z) = \sum_{n=-\infty}^{\infty} A_n$ where all the co-efficient of A_n are determined by ③.

Note that the integral in ③ in independent of r as long as $R_1 < r < R_2$.

If $R_1 = 0$ the point a is isolation singularity and $A_{-1} = B$ ~~point a isolation singularity~~ ~~and~~ is the residue at a, for $f(z) = A_{-1}(z-a)^{-1}$ is the derivative of the single valued function on $0 < |z-a| < R$.

——— × ———

$S(0) = a$.

But $u(0) = u(S(0)) = u(a)$

$$u(a) = \frac{1}{2\pi} \int_{|w|=1} u(S(w))\, d(\arg w) \longrightarrow (4).$$

$w = 1.e^{it} \Rightarrow dw = e^{it}.\, i\,dt$.

$\Rightarrow dw = wi\,dt$.

$$\frac{dw}{iw} = d(\arg w) \rightarrow (5)$$

$$w = \frac{R(z-a)}{R^2 - z\bar{a}}$$

$$\log w = \log R + \log(z-a) - \log(z^2 - z\bar{a}).$$

Diff $$\frac{dw}{w} = \frac{dz}{z-a} - \frac{(-\bar{a})\,dz}{R^2 - z\bar{a}}$$

$$\frac{dw}{iw} = \frac{1}{i}\left[\frac{dz}{z-a} + \frac{\bar{a}\,dz}{R^2 - z\bar{a}}\right] \longrightarrow (a).$$

$|z| = R \Rightarrow z = Re^{i\theta}$.

sub (b) in (a)

$$\therefore \frac{dw}{iw} = dz = Re^{i\theta}\, i\,d\theta = zi\,d\theta \quad \text{— (B)}$$

(5) becomes $$\frac{dw}{iw} = \frac{1}{i}\left[\frac{zi\,d\theta}{z-a} + \frac{\bar{a}\,zi\,d\theta}{R^2 - z\bar{a}}\right] \text{— (c)}.$$

becomes

from (c) & (5) we get

$$d(\arg w) = \frac{1}{i}\left[\frac{zi\,d\theta}{z-a} + \frac{\bar{a}\,zi\,d\theta}{R^2 - z\bar{a}}\right]$$

$$= \left[\frac{z}{z-a} + \frac{\bar{a}z}{z\bar{z} - z\bar{a}}\right] d\theta$$

$$= \left[\frac{z}{z-a} + \frac{\bar{a}}{\bar{z} - \bar{a}}\right] d\theta.$$

say
$\rightarrow (3)$

from (a)

$t + i\sin t$
$\downarrow$

(7) becomes $u(a) = \frac{1}{2\pi} \int_{|z|=R} Re\left(\frac{z+a}{z-a}\right) u(z) d\theta$

corr - 1.

[poisson formula for polar co-ordinates

from poisson formula

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |a|^2}{|z-a|^2} u(z) d\theta \rightarrow (8)$$

Put $a = re^{i\phi}$, $z = Re^{i\theta}$

$|a| = r$

$|z-a|^2 = (z-a)\overline{(z-a)}$

$= (z-a)(\bar{z} - \bar{a})$

$= (Re^{i\theta} - re^{i\phi})(Re^{-i\theta} - re^{-i\phi})$

$= R^2 + r^2 - Rr e^{i(\theta-\phi)}$ $R^2 e^{i\theta - i\theta} - Rre^{i\theta - i\phi}$

$= R^2 + r^2 - Rr e^{i(\theta-\phi)} - Rre^{i\phi - i\theta} + r^2 e^{i\phi - i\phi}$

$- rRe^{i(\phi-\theta)}$

$= R^2 + r^2 - Rr[e^{i(\theta-\phi)} + e^{i(\phi-\theta)}]$

$- Rr[e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}]$

$= R^2 + r^2 - Rr[2\cos(\theta-\phi)]$

(8) becomes,

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)\, u(Re^{i\theta})\, d\theta}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

corr - 2.

christofel's or schwartz formula.

W.K.T

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} Re\left(\frac{z+a}{z-a}\right) u(z) d\theta$$

$$P_v(Z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+Z}{e^{i\theta}-Z}\right) u(\theta)\, d\theta$$ is

called the poisson integral of u propertie,

1. $P_{u+v}(Z) = P_u + P_v$

2. $P_{cv}(Z) = c\, P_v$

3. P_v is linear.

4. If $v \geq 0$ then $P_v \geq 0$

5) If $m \leq u \leq M$ then $m \leq P_v \leq M$

Theorem 5 :-

The function $P_v(Z)$ is harmonic in $|Z| < 1$ and $\lim_{Z \to e^{i\theta}} P_v(Z) = u(\theta_0)$ provided v is continuous at θ_0

Proof :-

W.K.T $$P_v(Z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+Z}{e^{i\theta}-Z}\right) u(\theta)\, d\theta$$

$$= Re\left[\frac{1}{2\pi}\int_{|w|=1} \frac{w+Z}{w-Z}\, v(w) \frac{dw}{w}\right.$$

The function inside the bracket is analytic and so real part is also analytic.

Hence $P_u(Z)$ is analytic which proves the first part

Next let C_1 and C_2 be two complementary arcs of the unit circle.

Define functions U_1 and U_2 as

$\therefore |U_2(\theta)| < \frac{\epsilon}{2} \ \forall \theta$.

$|PU_2(z)| < \epsilon/2$ (by property 5).

U_1 is continuous at $e^{i\theta_0}$ and $U_1(e^{i\theta_0}) = 0$

$\exists\ \delta$ such that $|z - e^{i\theta_0}| < \delta \Rightarrow PU_1(e^{i\theta_0}) = 0$

$\therefore PU_1(z) \leq \frac{\epsilon}{2}$.

Thus $P_U = PU_1 + PU_2$.

$\Rightarrow |P_U(z)| \leq |PU_1(z)| + |PU_2(z)|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

when $|z - e^{i\theta_0}| < \delta$, As $z \to e^{i\theta_0}$, $P_U(z) \to 0$

ie) $P_U(z) \to U(\theta_0)$

Hence the theorem.

Reflection PRINCIPLE :

Theorem (Schwartz reflection Principle)

Let Ω^+ be part of the upper half plane of a symmetric region Ω. Let σ be the part of real axis with Ω suppose that $V(z)$ is continuous in $\Omega^+ \cup \sigma$ harmonic in Ω^+ and vanishes on σ. Then V has a harmonic extend to Ω which satisfies the symmetric relation $V(\bar{z}) = -V(z)$

Further of V is the imaginary part of an analytic function $f(z) = u + iV$ in Ω^+, then

$$\Rightarrow (\log r_1)\int_{C_1} r_1 \frac{du}{dr}\, d\theta - \int_{C_1} u\, d\theta = (\log r_2)\int_{C_2} r_2 \frac{du}{dr}\, d\theta - \int_{C_2} u\, d\theta.$$

ii) If u is harmonic in $r_1 < |z| < r_2$ then

$$\int_{|z|=r} u\, d\theta - \log r \int_{|z|=r} r\frac{du}{dr}\, d\theta = \text{constant}$$

[By Property

But $\int_\gamma {}^*du = 0 \quad \forall$ cycle $\gamma \equiv 0 \pmod{\Omega}$

$$\therefore \int_{C_1 - C_2} r\frac{du}{dr}\, d\theta = 0$$

$$\Rightarrow \int_{C_1} r_1 \frac{du}{dr}\, d\theta = \int_{C_2} r_2 \frac{du}{dr}\, d\theta.$$

$$\Rightarrow \int_{|z|=r} r\frac{du}{dr}\, d\theta = \text{constant} = \alpha' \text{ (say)} \quad (5)$$

(4) becomes. Put (5) in (4)

$$\int_{|z|=r} u\, d\theta - \alpha' \log r = \beta'$$

$$\therefore \int_{|z|=r} u\, d\theta = \alpha' \log r + \beta'$$

$$\Rightarrow \frac{1}{2\pi}\int_{|z|=r} u\, d\theta = \frac{\alpha'}{2\pi}\log r + \frac{\beta'}{2\pi}$$

$$= \alpha \log r + \beta.$$

which Prove the 1st part.

If u is harmonic throughout the disc

where $u(0)$ denotes the value of u at the centre 0 of $|z|=r$.

If the circle is $|z-z_0|=r$. then $z=z_0+re^{i\theta}$ is on the circle and z_0 is the centre

① becomes

$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta.$$

∴ The maximum and minimum attains on the boundary.

By similar lines we can prove the maximum principle for analytic function.

Theorem 4 : poisson formula.

S.T

Suppose $u(z)$ is continuous on $|z|\le R$ and is harmonic in $|z|<R$ then

$$u(a)=\frac{1}{2\pi}\int_{|z|=R}\frac{R^2-|a|^2}{|z-a|^2}\,u(z)\,d\theta.$$

$$=\frac{1}{2\pi}\int_{|z|=R}\mathrm{Re}\left(\frac{z+a}{z-a}\right)u(z)\,d\theta.\ \forall |z|<R.$$

Proof :

If u is harmonic in $|z-z_0|=r$ then by theorem 3

$$u(z_0)=\frac{1}{2\pi}\int_{|z|=R}u(z_0+re^{i\theta})\,d\theta.$$