UNIT: 5

Notes :

Definition : Let V be a vector space and T a linear operator on V. If W is a subspace of V, we say that W is invariant under T if for each vector α in W the vector $T\alpha \in W$

(ie) if $T(W)$ is contained in W.

Lemma: Let W be an invariant subspace for T. The charecteristic polynomial for the restriction operator T_W divides the charecteristic polynomial for T. The minimal polynomial for T_W divides the minimal polynomial for T.

Proof:

we have $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$

where $A = [T]_{\mathcal{B}}$, $B = [T_W]_{\mathcal{B}'}$

The block form of the matrix

$$\det(xI - A) = \det(xI - B)\det(xI - D)$$

here I is Identity matrix.

k^{th} power of the matrix A has the block form

$A^k = \begin{bmatrix} B^k & C_k \\ 0 & D^k \end{bmatrix}$ where C_k is some $r \times (n-r)$ matrix.

$\therefore$ Any polynomial which annihilates A also B and D

hence the minimal polynomial for B divides the minimal polynomial fo A.

Definition: Let W be an invariant subspace for T and let α be a vector in V. The T-conductor of α into W is the set $S_T(\alpha; W)$, which consists of all polynomials g such that $g(T)\alpha$ is in W.

Lemma: If W is an invariant subspace for T, then W is invariant under every polynomial in T. Thus for each α in V, the conductor $S(\alpha; W)$ is an ideal in the polynomial algebra $F[x]$.

Proof:

If $\beta \in W$ then $T\beta \in W$

Consequently $T(T\beta) = T^2\beta$, $T^2\beta \in W$

By induction

$T^k\beta \in W$ for each k. Take a linear combination to see that $f(T)\beta \in W$ $\forall$ polynomial f.

The definition of $S(\alpha; W)$ makes sense if W is any subset of V. If W is a subspace then $S(\alpha; W)$ is a subspace of $F[x]$.

$$(cf + g)(T) = cf(T) + g(T).$$

If W is also invariant under T, let g be a polynomial in $S(\alpha; W)$.

i.e) let $g(T)\alpha \in W$.

If any polynomial then $f(T)[g(T)\alpha] \in W$

$\because$ $(fg)(T) = f(T)\,g(T)$, $fg \in S(\alpha; W)$

Thus the conductor absorbs multiplication by any polynomial.

Lemma: Let V be a finite-dimensional vector space over the field F. Let T be a linear operator on V such that the minimal polynomial for T is a product of linear factors $p = (x-c_1)^{r_1} \cdots (x-c_k)^{r_k}$, $c_i \in F$. Let W be a proper ($W \neq V$) subspace of V which is invariant under T. There excits a vector α in V such that

a) α is not in W

b) $(T - cI)\alpha$ is in W, for some characteristic value c of the operator T.

Proof:

Here a) and b) says that the T-conductor of α into W is a linear polyonomial.

Let β be any vector in V which is not in W. Let g be the T-conductor of β into W. Then g divides p, the minimal polynomial for T.

Since β is not in W, the polynomial g is not constant

$$\therefore \quad g = (x-c_1)^{e_1} \cdots (x-c_k)^{e_k}$$

where atleast one of the integers e_i is positive

choose j so that $e_j > 0$. then $(x - c_j)$ divides g

$$g = (x - c_j)h$$

by the definition of g, the vector $\alpha = h(T)\beta$ Cannot be in W. But

$$(T - c_jI)\alpha = (T - c_jI)h(T)\beta$$
$$= g(T)\beta$$

hence $g(T)\beta \in W$.

Theorem: Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \cdots (x - c_k)$ where $c_1, \ldots, c_k$ are distinct elements of F.

Proof: We have noted that if T is diagonalizable, it's minimal polyonmial is a product of distinct linear factors

Converse

Let W be the subspace spanned by all the charecteristic vectors of T and suppose $W \neq V$.

If there is a vector α not in W and a charectoristic value c_j of T such that the vector $\beta = (T - c_j I)\alpha$ lies in W.

$\because \beta \in W$, $\beta = \beta_1 + \cdots + \beta_k$, $T\beta_i = c_i \beta_i$, $1 \leq i \leq k$

$\therefore$ the vector $h(T)\beta = h(c_1)\beta_1 + \cdots + h(c_k)\beta_k$ is in W for every polynomial h.

Now $p = (x - c_j) q$ for some polynomial q.
also $q - q(c_j) = (x - c_j) h$.
we have $q(T)\alpha - q(c_j)\alpha = h(T)(T - c_j I)\alpha$

$$= h(T)\beta$$

But $h(T)\beta \in W$ and since

$0 = p(T)\alpha = T(c_j I) q(T)\alpha$, $q(T)\alpha \in W$

$\therefore$ $q(c_j)\alpha \in W$. Since $\alpha \notin W$ we have $q(c_j) = 0$ that condradicts

Thus p has distinct roots.

Definition: Let $W_1, \ldots, W_k$ be subspa... ... vector space V. We say that $W_1, \ldots, W_k$ are independent if $\alpha_1 + \cdots + \alpha_k = 0$, $\alpha_i \in W_i$ implies that each α_i is 0.

Definition If V is a vector space, a projection of V is a linear operator E on V such that $E^2 = E$.

Theorem: If $V = W_1 \oplus \cdots \oplus W_k$ then there exits k linear operators $E_1, \ldots, E_k$ on V such that

i) each E_i is a projection $(E_i^2 = E_i)$

ii) $E_i E_j = 0$, if $i \neq j$

iii) $I = E_1 + \cdots + E_k$

iv) the range of E_i is W

Conversely if $E_1, \ldots, E_k$ are k linear operators on V which satisfy conditions i), ii) and iii), and if we let W_i be the range of E_i, then $V = W_1 \oplus \cdots \oplus W_k$.

Proof:

Converse part

Suppose $E_1, \ldots, E_k$ are linear operators on V which satisfy the first three condition, and let W_i be the range of E_i.

Then $V = W_1 + \cdots + W_k$

by condition iii) we have

$\alpha = E_1\alpha + \cdots + E_k\alpha$ for each α in V

and $E_i\alpha$ is in W_i

The expression for α is unique because if $\alpha = \alpha_1 + \cdots + \alpha_k$

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with α_i in W_i, $\alpha_i = E_i \beta_i$ then using i) & ii) (6)
we have

$$E_j \alpha = E_j \alpha_1 + \cdots + E_j \alpha_i$$
$$= \sum_{i=1}^{k} E_j \alpha_i$$
$$= \sum_{i=1}^{k} E_j E_i \beta_i$$
$$= E_j E_j \beta_j \qquad \because i = j$$
$$= E_j^2 \beta_j$$
$$= E_j \beta_j \qquad \because E_j^2 = E_j$$
$$= \alpha_j .$$

Hence this shows that V is the direct sum of the W_i.

Theorem: (X) ~~statement~~ primary Decomposition theorem

Statement: Let T be a linear operator on the finite-dimensional vector space V over the field F. Let p be the minimal polynomial for T, $p = p_1^{r_1} \cdots p_k^{r_k}$ where the p_i are distinct irreducible monic polynomials over F and the r_i are positive integers.

Let W_i be the null space of $p_i(T)^{r_i}$, $i = 1, \ldots, k$. Then

i) $V = W_1 \oplus \cdots \oplus W_k$

ii) Each W_i is invariant under T

iii) if T_i is the operator induced on W_i by T, then the minimal polynomial for T_i is $p_i^{r_i}$.

Proof:

If the direct-sum decomposition (i) is valid, The projection E_i will be the identity on W_i and zero on the other W_j.

we shall find a polynomial h_i such that $h_i(T)$ is the identity on W_i and is zero on the other W_j, and so that

$$h_1(T) + \cdots + h_k(T) = I, \ \ldots \text{ for each } i$$

let $$f_i = \frac{p}{p_i^{r_i}} = \prod_{j \neq i} p_j^{r_j}$$

Since $p_1, p_2, \cdots, p_k$ are distinct prime polynomials, the polynomials $f_1, \cdots, f_k$ are relatively prime.

Thus there are polynomials $g_1, \cdots, g_k$ such that $\sum_{i=1}^{n} f_i g_i = 1$

Note that if $i \neq j$, then $f_i f_j$ is divisible by the polynomial p because $f_i f_j$ contains each $p_m^{r_m}$ as a factor.

we shall show that the polynomials

$h_i = f_i g_i$, to

Let $E_i = h_i(T)$
$= f_i(T)\, g_i(T)$

Since $h_1 + \cdots + h_k = 1$ and p divides $f_i f_j$ for $i \neq j$ we have

$$E_1 + \cdots + E_k = I$$

$$E_i E_j = 0, \text{ if } i \neq j$$

Thus the E_i are projection which correspond to some direct-sum decomposition of the space V.

We wish to show that the range of E_i is exactly the subspace W_i. It is clear that each vector in the range of $E_i \in W_i$ for if α is in the range of E_i, then $\alpha = E_i\alpha$

$$p_i(T)^{r_i}\alpha = p_i(T)^{r_i} E_i\alpha$$

$$= p_i(T)^{r_i} f_i(T) g_i(T)\alpha$$

$$= 0$$

because $p^{r_i} f_i g_i$ is divisible by the minimal polynomial p.

Conversely, suppose that α is in the null space of $p_i(T)^{r_i}$

If $j \neq i$, then $f_j g_j$ is divisible by $p_i^{r_i}$ and so $f_j(T) g_j(T)\alpha = 0$

i.e) $E_j\alpha = 0$ for $j \neq i$ immediate that

~~Ep (or~~ $E_i\alpha = \alpha$

i.e) that α is in the range of E_i

This completes the proof of statement(i)

It is clear that the subspace W_i are invariant under T. If T_i is the operator induced on W_i by T, then evidently $p_i(T_i)^{r_i} = 0$, by the definition $p_i(T)^{r_i}$ is 0 on the subspace W_i

This shows that the minimal

Polynomial for T_i divides p_i.

Conversely, let g be any polynomial such that

$g(T_i)=0$. Then $g(T) f_i(T)=0$.

Thus $g f_i$ is divisible by the minimal polynomial p of T

ie) $p_i^{r_i} f_i$ divides $g f_i$ and $p_i^{r_i}$ divides g

Hence the minimal polynomial for T_i is $p_i^{r_i}$.

Definition Let N be a linear operator on the vector space V. We say that N is nilpotent if there is some positive integer r such that $N^r=0$.

Definition If $A=[T]_{\mathcal{B}}$ and $A_i=[T_i]_{\mathcal{B}_i}$, then A has the block form

$$A=\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \rightarrow (1)$$

in (1) A_i is a $d_i \times d_i$ matrix $(d_i = \dim W_i)$ and the 0's are symbols for rectangular blocks of scalar 0's of varies sizes.

It also seems appropriate to describe (1) by saying that A is the direct sum of the matrices $A_1, \ldots, A_k$.

Theorem: Let T be a linear operator on the space V, and let $W_1, \ldots, W_k$ and $E_1, \ldots, E_k$ be linear operator on V. Then a necessary and sufficient condition that each subspace

W_i be invariant under T is that T commute with each of the projection E_i. i.e)

$TE_i = E_i T$, $i = 1, \dots, k$.

Proof: Suppose T commutes with each E_i.

let $\alpha \in W$ then $E_j \alpha = \alpha$

$$T\alpha = T(E_j \alpha)$$
$$= E_j (T\alpha)$$

which shows that $T\alpha$ is in the range of E, i.e) W_j is invariant under T.

Assume each W_i is invariant under T. we shall show that $TE_j = E_j T$.

let α be any vector in V. Then

$$\alpha = E_1\alpha + \dots + E_k\alpha$$

$$T\alpha = TE_1\alpha + \dots + TE_k\alpha \quad , \quad E_i\alpha \in W_i$$

which is invariant under T. We must have $T(E_i\alpha) = E_i\beta_i$ for some vector β_i

$$E_j T E_i \alpha = E_j E_i \beta_i$$
$$= \begin{cases} 0 , & \text{if } i \neq j \\ E_j \beta_j , & \text{if } i = j \end{cases}$$

Thus $E_j T\alpha = E_j TE_1\alpha + \dots + E_j TE_k\alpha$

$$\cancel{E_j} T\alpha = \cancel{E_j}(TE_1\alpha + \dots + TE_k\alpha)$$
$$T\alpha = (TE_1\alpha + \dots + TE_k\alpha)$$
$$T(E_j\alpha) = E_j\beta_j$$
$$= TE_j\alpha$$

$E_j T \cancel{\alpha} = TE_j \cancel{\alpha}$ This holds for each α in V so $E_j T = TE_j$.