UNIT : V

Harmonic function

Definition : Harmonic (or) potential function :

(R) A real valued function $u(z)$ (or) $u(x,y)$ defined and single-valued in a region Ω is said to be harmonic on Ω (or) a potential function if it is continuous together with its partial derivatives of the 1st two orders and satisfies Laplace's equation.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The simplest harmonic function are the linear of Laplace functions $ax+by$.

In polar co-ordinates (r,θ) equation.

(polar form)

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0$$

Definition : conjugate harmonic function : $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$.

(R)

$$f\,dz = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right)$$

In this expression the real part is the differential of u.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If u has a conjugate harmonic function v, then the imaginary part can be written as,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Theorem 8 1

(R) If u_1 and u_2 are harmonic in a region Ω, Then $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ for every cycle γ which is homologous to zero in Ω.

Proof :

For $u_1 = 1$, $u_2 = u$ the formula reduces to

(52) $\Rightarrow \int_\gamma * du = \int_\gamma -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = 0$

In the classical Notation.

$\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ would be written as $\Rightarrow \int_\gamma u_1 \frac{\partial v}{\partial n} |dz| - u_2 \frac{\partial u}{\partial n} |dz|$

$$\int_\gamma \left(u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n} \right) |dz| = 0 \quad = \int_\gamma \left(u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u}{\partial n} \right) |dz| = 0$$

The mean value property :

Let us apply above theorem with $u_1 = \log r$ and u_2 equal to a function u harmonic in $|z| \leq \rho$ for Ω we choose the punctured disk $0 < |z| < \rho$ and for γ we take the cycle $C_1 - C_2$ where C_i is a cycle $|z| = r_i < \rho$ discribed in the +ve sense on a circle. $|z| = r$ We have $*du = r\left(\frac{\partial u}{\partial r}\right) d\theta$ and hence

$u_1 = \log r \quad u_2 = u$

$\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ yields $\quad *du = r(\partial u/\partial r)\, d\theta$

$$\log r_1 \int_{C_1} r_1 \frac{\partial u}{\partial r} d\theta - \int_{C_1} u\, d\theta = \log r_2 \int_{C_2} r_2 \frac{\partial u}{\partial r} d\theta - \int_{C_2} u\, d\theta$$

In other words the expansion.

$$\int_{|z|=r} u\, d\theta - \log r \int_{|z|=r} r \frac{\partial u}{\partial r} d\theta \text{ is constant.}$$

and this is true even. If u is only known to be harmonic in an annulus By $\int_\gamma * du = \int_\gamma -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = 0$. We find in the same way that,

$$\int_{|z|=r} r \frac{\partial u}{\partial r} d\theta.$$

In constant and this case of an annulus and zero of u is harmonic in the whole disk combining there results we obtain

Theorem : 2.

The Arithmatic mean of a harmonic function and over concentric circle $|z| = r$. is a linear function of $\log r$.

$\frac{1}{2\pi} \int_{|z|=r} u\, d\theta = \alpha \log r + \beta$ and if u is harmonic in a disk $\alpha = 0$ then the arithmatic mean is constant.

Proof :

$\beta = u(0)$ by continuity and changing to a new origin we find.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \quad —①$$

It is clear that equ- ① could also have been derived from the corresponding formula for analytic function.

It leads directly to the maximum principal for harmonic function.

poisson formula :

The above equ ① determines the value of u at the centers of the disk. But this is all we need for there exists a linear transformation which carries any point to the center to be explicit suppose that $u(z)$ is harmonic in the closed disk $|z| \leq R$. The linear transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}$$

maps $|\zeta| \leq 1$ onto $|z| \leq R$ with $\zeta = 0$ corresponding to $z = a$. The function $u(S(\zeta))$ is harmonic in $|\zeta| \leq 1$ and by ① we obtain,

$$u(a) = \frac{1}{2\pi}\int_{|\zeta|=1} u(S(\zeta))\, d\arg\zeta$$

From,
$$\zeta = \frac{R(z-a)}{R^2 - \bar{a}z}.$$

We compute.

$$d\arg\zeta = -i\frac{d\zeta}{\zeta} = -i\left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right) = \left(\frac{z}{z-a} + \frac{\bar{a}z}{R^2 - \bar{a}z}\right)d\theta$$

on sub $R^2 = z\bar{z}$ the coefficient of $d\theta$ in the last expression can be rewritten as,

linear inasmuch

$$\frac{z}{z-a} + \frac{\bar{a}}{\bar{z}-\bar{a}} = \frac{R^2 - |a|^2}{|z-a|^2}.$$

os, equivalently, as

$$\frac{1}{2}\left(\frac{z+a}{z-a}+\frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}\right)=\operatorname{Re}\frac{z+a}{z-a}$$

We obtain the two forms,

$$u(a)=\frac{1}{2\pi}\int_{|z|=R}\frac{R^2-|a|^2}{|z-a|^2}\,u(z)\,d\theta=\frac{1}{2\pi}\int_{|z|=R}\operatorname{Re}\frac{z+a}{z-a}\,u(z)\,d\theta \quad \hookrightarrow *$$

of poisson's formula (possion formula)

In polar coordinates

$$u\left(re^{i\phi}\right)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}\,u\left(Re^{i\theta}\right)d\theta$$

In the derivation we have assumed that $u(z)$ is Harmonic in the closed disk. however, the result remains true under the weaker condition that $u(z)$ is harmonic in the open disk and ~~conditions~~ continuous in the closed disk, and we obtain.

$$u(ra)=\frac{1}{2\pi}\int_{|z|=R}\frac{R^2-|a|^2}{|z-a|^2}\,u(rz)\,d\theta$$

Now all ~~the~~ we need to do is to let r let r tend to 1 Because $u(z)$ is uniformly condinuous on $|z|\le R$ it is true that $u(rz)\to u(z)$ uniformly for $|z|=R$, and we conclude that (eqn $*$) remains valid.

Hence the proof.

disk $\alpha=0$ the

Theorem : 3

Suppose that $u(z)$ is harmonic for $|z| < R$, Continuous for $|z| \leq R$. Then

(*) $$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta \quad \forall\ |a| < R. \longrightarrow (A)$$

Proof.:

$u(z) = \frac{1}{2\pi i} \int_{|\xi|=R} \mathrm{Re}$

The theorem lends at once to an explicit expression for the conjugate function of u. Indeed formula (*) gives,

$$u(z) = \mathrm{Re}\left[\frac{1}{2\pi i} \int_{|\xi|=R} \frac{\xi + z}{\xi - z} u(\xi) \frac{d\xi}{\xi}\right] \longrightarrow (2)$$

=> **

The bracketed expression is an analytic function of z for $|z| < R$. It follows that $u(z)$ is th real part of

$$f(z) = \frac{1}{2\pi i} \int_{|\xi|=R} \frac{\xi + z}{\xi - z} u(\xi) \frac{d\xi}{\xi} + ic \longrightarrow (3)$$

Where c is an arbitary real constant. This formula known as schwasz's formula.

As a special case (A), $n=1$ yields.

$$\int_{|z|=R} \frac{R^2 - |z|^2}{|z-a|^2} d\theta = 2\pi \quad \forall\ |a| < R \longrightarrow (4)$$

6.4 schwarz's theorem :

Choosing $R=1$, we define, for any Piecewise continuous function $u(\theta)$ in $0 \leq \theta \leq 2\pi$

$$P_u(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z} U(\theta)\, d\theta$$

and call this the poisson integral of U. observe that $P_U(z)$ is not only a function of z, but also a function U, as such it is called a functional. The functional is linear inasmuch as $P_{U+V} = P_U + P_V$

and $P_{cu} = cP_u$ for constant c. More over $u \geq 0$ implies $P_u(z) \geq 0$ because of this property P_U is said to be a positive linear functional.

We deduce from equ (1) that $P_c = P$ from this property together with the linear and +ve character of the functional is follows that any inequality

$m \leq U \leq M \Rightarrow m \leq P_U \leq M$.

Theorem: 23.

The function $P_U(z)$ is harmonic for $|z|<1$ and $\lim_{z \to e^{i\theta}} P_U(z) = U(\theta_0) \to B$ provided that U is cts at θ_0.

Proof:

W.K.T P_U is harmonic

Let c_1 and c_2 be complementary arcs of the unit circle and denote by U_1 the function which coincides with U on c_1 and vanishes on c_2 by U_2 the corresponding function for c_2 clearly $P_U = P_{U_1} + P_{U_2}$.

Since P_{U_1} can be regared as a line integral over c_1 it is by the same reasoning as before harmonic everywhere except on the closed arc c_1.

The expression.

$$Re \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}$$

vanishes on $|z| = 1$ for $z \neq e^{i\theta}$.

It follows that P_{U_1} is zero on the open arc c_2 and since it is cts $P_{U_1}(z) \to 0$ as $z \to e^{i\theta} \in c_2$.

In proving equ (B) we may suppose that $U(\theta_0)=0$ (7)
for if this is not the case we need only replace $U - U(\theta_0)$

Given $\epsilon_1 > 0$ we can find c_1 and $c_2 \ni e^{i\theta}$ is an interior point of c_2 and $|U(\theta)| \le \epsilon_1/2 \; \forall \, e^{i\theta} \in c_2$.

Under this condition $|U_2(\theta)| < \epsilon_1/2 \; \forall \, \theta$ Hence.

$|P_{U_2}(z)| < \epsilon_1/2 \; \forall \, |z|<1$ on the otherhand, since, U_1 is cts and vanishes at $e^{i\theta}$, there exists a $\delta \ni |P_{U_1}(z)| < \epsilon_1/2$ for $|z - e^{i\theta}| < \delta$.

It follows that $|P_U(z)| \le |P_{U_1}| + |P_{U_2}| < \epsilon_1$ as soon as $|z|<1$ and $|z - e^{i\theta}| < \delta$.

Which is precisely what we had to prove there is an interesting geometric interpretation of poisson formula also due to schwarz.

Given a fixed z inside the unit circle we determine for each $e^{i\theta}$ the point $e^{i\theta^*}$ which is $\ni e^{i\theta}, z$ and $e^{i\theta^*}$ are in a straight line.

It is clear geometrically or by simple calculation that,

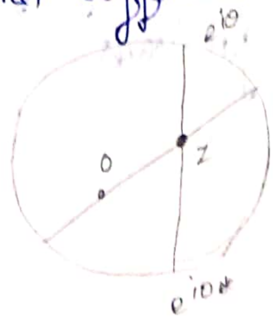
$$|e^{i\theta^*} - z| = \frac{1-|z|^2}{|e^{i\theta} - z|}.$$

$$1-|z|^2 = |e^{i\theta} - z|\,|e^{i\theta^*} - z| \quad \text{— (5)}$$

But the ratio $(e^{i\theta} - z)/(e^{i\theta^*} - z)$ is negative so we must have,

$$1-|z|^2 = -(e^{i\theta} - z)(e^{i\theta^*} - \bar{z})$$

We regard θ^* as a function of θ and different since z is constant we obtain.



$$\frac{e^{i\theta}\, dz}{e^{i\theta} - z} = \frac{e^{i\theta^*} d\theta^*}{e^{-i\theta^*} - \bar{z}}$$

and on taking absolute values,

$$\frac{d\theta^*}{d\theta} = \left| \frac{e^{i\theta^*} - z}{e^{i\theta} - z} \right| \quad \text{—⑥}$$

$$⑤ \Rightarrow \frac{d\theta^*}{d\theta} = \frac{|e^{i\theta^*} - z|}{|e^{i\theta} - z|} = \frac{1-|z|^2}{|e^{i\theta}-z|\,|e^{i\theta}-z|} = \frac{1-|z|^2}{|e^{i\theta}-z|^2}.$$

It follows by ⑤ & ⑥ that,

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2} = \frac{d\theta^*}{d\theta}.$$

and hence,

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} U(\theta)\, d\theta^* = \frac{1}{2\pi}\int_0^{2\pi} U(\theta^*)\, d\theta.$$

In other words, to find $P_U(z)$ replace each value of $U(\theta)$ by the value at the point opposite to z, and take the average over the circle.

The Reflection Principle :

The principle of reflection is based on the observation that if $u(z)$ is a harmonic funct. then $u(\bar{z})$ is likewise harmonic and if $f(z)$ is an analytic funl. Then $\overline{f(\bar{z})}$ is also analytic.

More precisely if $u(z)$ is harmonic and $f(z)$ analytic region then $u(\bar{z})$ is harmonic and $\overline{f(\bar{z})}$ analytic as funl. as funl. of z in the region Ω^* obtained by reflecting Ω in the real axis. That is $z \in \Omega^*$ if and only if $\bar{z} \in \Omega$.

The proof of these statement consists in trivial verifications.

Theorem : 24.

Let Ω^+ be the part in the upper half plane of a symmetric region Ω, and let σ be the part of the real axis in Ω. Suppose that $v(z)$ is cts in $\Omega^+ \cup \sigma$, harmonic in Ω^+ and zero on σ Then v has a harmonic extension to Ω which satisfies the symmetry relation $v(\bar{z}) = -v(z)$ On the same situation if v is the imaginary part of an analytic function $f(z)$ in Ω^+ The $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\bar{z})}$.

Proof :

The function $v(z)$ which is equal to $v(z)$ in Ω^+, 0 on σ and equal to $-v(\bar{z})$ in the mirror image of Ω^+.

We have to p.T v is harmonic on σ for a point $x_0 \in \sigma$ consider a disk with respective to this disk. formed with the boundary values v.

The difference $v - P_v$ is harmonic in the upper half of disk.

It vanishes on the half circle by above theorem and also on the diameters because v tends to zero by defn and P_v vanishes by obvious symmetry.

The maximum and minimum principle implies that $v = P_v$ in the upper half disk and the same proof can be repeated for the lowers half we conclude that v is harmonic in the whole disk and in particular at x_0.

For the remaining part of the theorem let us again consider a disk with center on σ.

We have already extended v to the whole disk and v

a conjugate harmonic function $-u_0$ in the same disk which

We may normalise so that $u_0 = \operatorname{Re} f(z)$ in the upper half.

consider $U_0(z) = U_0(z) - U_0(\bar{z})$

on the real diameter it is clear that $\frac{\partial U_0}{\partial x} = 0$ and also,

$$\frac{\partial U_0}{\partial y} = 2\frac{\partial U_0}{\partial y} = -2\frac{\partial v}{\partial x} = 0$$

It follows that the analytic function $\frac{\partial U_0}{\partial x} - i\frac{\partial V_0}{\partial y}$ vanishes on the real axis and hence identically.

Therefore u_0 is a constant and this constant is evidently zero we have p.t $U_0(z) = U_0(\bar{z})$.

The construction can be repeated for arbitary disk. It is clear that the U_0 coincide in overlapping disk. the defn can be extended to all of Ω, and the theorem follows.

The theorem has obvious generalization the domain Ω can be taken to be symmetric with respect to a circle C rather than w.r.to a straight line, and when $z \to C$ it may be assumed that $f(z)$ approch as another circle C' under such conditions $f(z)$ has an analytic continuation which maps symmetric points w.r.to C onto symmetric points w.r.to C'.

Weierstrass theorem :

The central theorem concerning the convergence of analytic function asserts that the limit of a uniformly convergent sequence of analytic function is an analytic function.

considering the sequence $\{f_n(z)\}$ where each $f_n(z)$ is defined and analytic in a region Ω_n.

The limit function $f(z)$ must also be considered in some region Ω and clearly. if $f(z)$ is to be defined in Ω each points of Ω must along to all Ω_n for n greater than a certain n_0.

In the general case n_0 will not be the same $\forall$ points of Ω and for this reason it would not make sense to required that the convergence be uniformly Ω.

In fact in the most typical case the region Ω_n from an increasing sequence $\Omega_1 \subset \Omega_2 \subset \Omega_3 \ldots \subset \Omega_n \subset \ldots$ and Ω is the union of the Ω_n.

In these circumstances no single function $f_n(z)$ is defined in all of Ω. yet the limit $f(z)$ may exists at all points of Ω, although the convergence cannot be uniform.

Theorem: 1.

Suppose that $f_n(z)$ is analytic in the region Ω_n and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω moreover $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.

Proof:

The analyticity of $f(z)$ follows that most easily by use of morera's Theorem.

Let $|z-a| \leq r$ be a closed disk contained in Ω by assumption implies that this disk lie in Ω_n $\forall$ n greater

Than a certain n_0. If γ is any closed curve contained in $|z-a|<r$ We have.

$$\int_\gamma f_n(z)\,dz = 0 \quad \int_\gamma f(z)\,dz = \int_\gamma f_n(z)\,dz$$

for $n>n_0$ by cauchy's theorem.

Because of the uniform convergence on γ We obtain.

Taking limit on both sides $n\to\infty$

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty}\int_\gamma f_n(z)\,dz = 0.$$

and by morera's theorem it follows that $f(z)$ is analytic in $|z-a|<r$ consequently $f(z)$ is analytic in the whole region Ω.

$$f_n(z) = \frac{1}{2\pi i}\int_c \frac{f_n(\zeta)\,d\zeta}{\zeta - z}. \qquad z = \zeta - a.$$

Where c is the circle $|\zeta-a| = r$ and $|z-a|<r$

letting $n\to\infty$.

We obtain by uniform convergence, $(f_n = f)$

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

and this formula shows that $f(z)$ is analytic in the disk.

starting from the formula,

$$f_n'(z) = \frac{1}{2\pi i}\int_c \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^2}$$

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} = f'(z).$$

and simple estimates S.T the converges is uniformly for $|z-a|\le \rho < r$.

Any compact subset of Ω can be covered by a finite number of such closed disks and $\therefore$ the convergence is uniform on every compact subset.

If a series with analytic terms $f(z) = f_1(z) + \dots + f_n(z)$ converges uniformly on every compact subset of a region Ω Then the sum $f(z)$ is analytic in Ω and the series can be expressed differentiated term by term

Theorem : 2.

bounded, cls

semi Roja If the function $f_n(z)$ are analytic and $\neq 0$ in a region Ω and if $f_n(z)$ converges to $f(z)$ uniformly on every compact bounded, closed subset of Ω, Then $f(z)$ is either identically zero (or) never equal to zero in Ω.

Proof :

Suppose that $f(z)$ is identically zero. The zero's of $f(z)$ are in any case isolated.

For any point $z_0 \in \Omega$ There is Therefore a number $r > 0$ such that $f(z)$ is defined and $\neq 0$ for $0 < |z - z_0| \leq r$.

In particular, $|f(z)|$ has a +ve minimum on the circle $|z - z_0| = r$, which we denote by c.

It follows that $1/f_n(z)$ converges uniformly to $1/f(z)$ on c.

since it is also true that $f_n'(z) \to f'(z)$ uniformly on c.

We may conclude that,

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_c \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} dz.$$

But the integral on the left are all zero for they give the number of roots of the equ. $f_n(z) = 0$ inside of c.

The integral on the right is therefore zero and consequently $f(z_0) \neq 0$ by the same interpretation of the integral

arbitrary

The Taylor series :

Show that every analytic function can be developed in a convergent Taylor series. This is an almost immediate consequence of the finite Taylor development with the corresponding representation of the remainder according to the theorem.

Proof :

If $f(z)$ is analytic in a region Ω containing z_0

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + f_{n+1}(z)(z-z_0)^{n+1}$$

With

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}(\zeta-z)}$$

In this last formula C is any circle $|z-z_0| = \rho$ → conclude closed disk $|z-z_0| \le \rho$ is contained in Ω.

If M denotes the maximum of $|f(z)|$ on C we obtain at once the estimate:

$$\left|f_{n+1}(z)(z-z_0)^{n+1}\right| \le \frac{M|z-z_0|^{n+1}}{\rho^n(\rho-|z-z_0|)}$$

We conclude that the remainder term tends uniformly to zero in every disk $|z-z_0| \le r < \rho$

On the other hand ρ can be choosen arbitarily close to the shortest distance from z_0 to the boundary of Ω.

Hence the proof.

Theorem : 3.

If $f(z)$ is analyt

the representation $f(z$

is valid in the largest

Proof :

The radius of co

atleast equal to the

boundary of Ω.

It may well be

that the series still re

simultaneously in Ω,

We shall devel

$e^z = 1 + z + \frac{z^2}{2!} + \cdots$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} -$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} -$

served as definitions

Every convergent power

gave earlier a direct

differentiated term by

of Weierstrass's theorem

$(1+z)^{\mu}$ (or) $\log(1+z$

Which is respectively

Since this branch

$|z|<1$, the radius of

It is elementary

obtain.

Theorem : 3.

(R) If $f(z)$ is analytic in the region Ω containing z_0, Then the representation $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \cdots + \frac{f^n(z_0)}{n!}(z-z_0)^n + \cdots$ is valid in the largest open disk of center z_0 contained in Ω.

whole region

Proof :

The radius of convergence of the Taylor series is thus atleast equal to the shortest distance from z_0 to the boundary of Ω.

It may well be larger but if it is there is no guarantee that the series still represents $f(z)$ at all points which are simultaneously in Ω, and in the circle of convergence.

We shall developments,

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

served as definitions of the function the represent. Every convergent power series is it own Taylor series We gave earlier a direct proof that power series can be differentiated term by term this is also a direct consequence of Weistrass's theorem.

$(1+z)^\mu$ cos) $\log(1+z)$ about the origin chooting the branch which is respectively equal to 1 cos) 0 at the origin.

Since this branch is single-valued and analytic in $|z|<1$, the radius of convergence is atleast 1.

It is elementary to compute the co-efficients and we obtain.

$$(1+z)^{\mu} = 1+\mu z + \binom{\mu}{2} z^2 + \dots + \binom{\mu}{n} z^n + \dots$$

$$\log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + z^5/5 - \dots$$

The series developments of the cyclometric function arc $\tan z$ and arc $\sin z$ are most easily obtained by consideration of the derived series.

From the expansion,

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

We obtained by integration

$$\int \frac{dz}{1+z^2} = \int dz - \int z^2 dz + \int z^4 dz - \int z^6 dz + \dots$$

$$\text{arc} \tan z = z - z^3/3 + z^5/5 - z^7/7 + \dots$$

Where the branch is uniquely determined as.

$$\text{arc} \tan z = \int_0^z \frac{dz}{1+z^2}.$$ for any path inside that unit circle.

The radius of convergence cannot be greater than that of the derived series and hence it is exactly 1.

If $\sqrt{1-z^2}$ is the branch with a +ve real part we have,

$$\frac{1}{\sqrt{1-z^2}} = 1 + \tfrac{1}{2} z^2 + \tfrac{1}{2} z^4 \cdot \tfrac{3}{4} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} z^6 + \dots$$

for $|z|<1$, and through integration we obtain, arc

$$\sin z = z + \tfrac{1}{2} z^3/3 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{z^5}{5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} z^7/7 + \dots$$

The series represents the principal branch of arc $\sin z$ with a real part between $-\pi/2$ and $\pi/2$.

$[z^n]$ denotes a function which contains the factor z^n can be written as,

$$f(z) = a_0 + a_1 z + \dots + a_n z^n + [z^{n+1}]$$

Where the co-efficients are uniquely determinant and equal to the Taylor co-efficients of $f(z)$.

Thus is order to find the first n co-efficients of the Taylor expansion it is sufficient to determine a polynomial $P_n(z)$ $\ni$ $f(z) - P_n(z)$ has a zero of atleast order $n+1$ at the origin.

The degree of $P_n(z)$ does not matter.

It is true in any case that the co-efficients of z^m, $m \le n$, are the Taylor co-efficients of $f(z)$.

For instance suppose that,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

$$g(z) = b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n + \dots$$

With an abbrevated notation we write,

$$f(z) = P_n(z) + (z^{n+1}), \quad g(z) = Q_n(z) + (z^{n+1})$$

If this then clear that $f(z)\, g(z) = P_n(z)\, Q_n(z) + [z^{n+1}]$ and the co-efficients of the terms of degree $\le n$ in $P_n Q_n$ are the Taylor co-efficients of the product $f(z)\, g(z)$.

To expand the inverse function of an analytic function $w = g(z)$.

Here we may suppose that $g(0) = 0$ and we are looking for the branch of the inverse function $z = g^{-1}(w)$ which is analytic in a neighborhood of the origin and vanishes for $w = 0$

Expansion of $\tan w$ from the series,

$$w = \arctan z = z - z^3/3 + z^5/5 - \dots$$

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n.$$

All we have to show that $f(z)$ can be written as a sum $f_1(z) + f_2(z)$. $f(z) = f_1(z) + f_2(z)$

Where $f_1(z)$ is analytic for $|z-a| < R_2$ & $f_2(z)$ is analytic for $|z-a| > R_1$ with a removable singularity at ∞

under these circumstances $f_1(z)$ can be developed in non-negative powers of $z-a$ and $f_2(z)$ can be developed in non-negative powers of $1/z-a$.]

To find the representation $f(z) = f_1(z) + f_2(z)$ define $f_1(z)$ by,

$$f_1(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

for $|z-a| < r < R_2$ and $f_2(z)$ by,

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\, d\zeta}{\zeta - z}. \quad \text{for } R_1 < r < |z-a|.$$

In both integrals the values of r is irrelevant as long as the inequality is fulfilled for it is an immediate consequence of cauchy's theorem.

That the value of the integral does not change with r provided that the circle does not pass over the point z

For this reason $f_1(z)$ and $f_2(z)$ are uniquely defined and represent analytic function in $|z-a| < R_2$ & $|z-a| > R_1$ respectively.

Moreover by cauchy's integral theorem $f(z) = f_1(z) + f_2(z)$

20 In order to find the development of $f_2(z)$ we perform the transformation $\zeta = a + 1/\zeta'$, $z = a + 1/z'$

This transformation carries $|\zeta - a| = r$ into $|\zeta'| = 1/r$ with negative orientation and by simple calculation we obtain,

$$f_2(a + 1/z') = -\frac{1}{2\pi i} \int_{|\zeta'| = 1/r} \frac{z'}{\zeta'} \frac{f(a + 1/\zeta')\, d\zeta'}{\zeta' - z'} \sum_{n=1}^{\infty} B_n z'^n$$

With $B_n = \frac{1}{2\pi i} \int_{|\zeta'| = 1/r} \frac{f(a + 1/\zeta')\, d\zeta'}{\zeta'^{\,n+1}} = \frac{1}{2\pi i} \int_{|\zeta - a| = r} f(\zeta)(\zeta - a)^{n}$

This formula s.t we can write.

$$f(z) = \sum_{n=\infty}^{\infty} A_n (z-a)^n.$$

Where all the co-efficients A_n are determined by (3)

observe that the $\int$ in (3) is independent of r along as $R_1 < r < R_2$.

If $R_1 = 0$ the point a is an isolated singularity and $A_{-1} = B_1$ is the residue at a, for $f(z) - A_{-1}(z-a)^{-1}$ is the derivative of a single valued function in $0 < |z-a| < R_2$

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

$$f_2(a + 1/z') = -\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(a + 1/\zeta')\, d\zeta}{a + 1/\zeta' - (a + 1/z')}$$

$$= -\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(a + 1/\zeta')\, d\zeta'}{1/\zeta' - 1/z'}$$

$$= -\frac{1}{2\pi i} \int_{|\zeta - a| = r} f(a + 1/\zeta')\, d\zeta' \Big/ \frac{z' - \zeta'}{z'\zeta'}$$

$$\Rightarrow -\frac{1}{2\pi i} \int_{|z|} f(a + 1/\zeta')\, d\zeta'$$