The spectrum:-

Defn:

Let T be an operator on a non-trivial Hilbert space.

We find the spectrum of T to be the set

$$\sigma(T) = \{\lambda : T - \lambda I \text{ is singular}\}$$

consider an element x in our general Banach algebra A.

We define the spectrum of x to be the following subset of the complex plane.

$$\sigma(x) = \{\lambda : x - \lambda \cdot 1 \text{ is singular}\}$$

The spectrum of x depends on A as well as x,

So we use the notation $\sigma_A(x)$.

It is easy to see that $x - \lambda 1$ is a continuous function of λ with values in A and since the set of singular elements in x is closed, it follows that at once that $\sigma(x)$ is closed.

Note :-

1. $\sigma(x)$ is a subset of the closed disc $\{z : |z| \leq \|x\|\}$ for if λ is a complex number such that $|\lambda| > \|x\|$.

Then $\|x/\lambda\| < 1$, $\|1 - (1 - x/\lambda)\| < 1$.

$1 - x/\lambda$ is regular and therefore $x - \lambda I$ is regular.

2. $\sigma(x)$ is always non-empty.

3. The resolvent set of x denoted by $\rho(x)$ is the complement of $\sigma(x)$.

It is clearly an open subset of the complex plane which contains

$$\{z : |z| > \|x\|\}.$$

[The resolvent of x is the function with values in A defined on $\rho(x)$ by]

$$x(\lambda) = (x - \lambda I)^{-1}$$

$x(\lambda)$ is a continuous function of λ.

and the fact that

$x(\lambda) = \lambda^{-1}\left(\frac{x}{\lambda} - 1\right)^{-1}$

since $x(\lambda) = (x-\lambda 1)^{-1} = \lambda^{-1}\left(\frac{x}{\lambda}-1\right)^{-1}$

for $\lambda \neq 0 \Rightarrow x(\lambda) \to 0$ as $\lambda \to \infty$.

If λ & μ are both in $\rho(x)$.

Then $x(\lambda) = x(\lambda)\left[(x-\mu.1)\right]x(\mu)$

$= x(\lambda)\left[x-\lambda.1 + (\lambda-\mu).1\right]x(\mu)$

$= \left[1 + (\lambda-\mu)\, x(\lambda)\right]x(\mu)$

$= x(\mu) + (\lambda-\mu)\, x(\lambda)\, x(\mu)$

so $x(\lambda) - x(\mu) = (\lambda-\mu)\, x(\lambda)\, x(\mu)$

This relation is called the resolvent equation.

Theorem: A :

$\sigma(x)$ is non-empty.

Proof

Let f be a fundamental on A, i.e. An element of the conjugate space A^*.

Define $f(\lambda)$ by $f(\lambda) = f(r(\lambda))$.

It is clear that $f(\lambda)$ is a complex function is defined and continuous on the resolvent set $\rho(x)$.

The resolvent equation s.t

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = f\left(r(\lambda)\, r(\mu)\right)$$

and it follows from this that

$$\lim_{\lambda \to \mu} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = f\left(r(\mu)\right)^2$$

$f(\lambda)$ has a derivative μ each point of $\rho(x)$.

Further,

$$|f(\lambda)| \leq \|f\| \, \|r(\lambda)\|.$$

So $f(\lambda) \to 0$ as $\lambda \to \infty$.

We now assume that $\sigma(x)$ is empty. so that $\rho(x)$ is the entire complex plane Liouville's then from complex analysis conclude that $f(\lambda) = 0 \;\forall\; \lambda$.

since f is arbitrary functional on x,

thm-2 (Hahn Banach sec)

$\Rightarrow x(\lambda) = 0 \ \forall \lambda.$

This is impossible for no inverse can equal 0, and therefore it cannot be true that $\sigma(x)$ is empty.

[since $x(\lambda) = (x - \lambda 1)^{-1}$]

Hence $\sigma(x)$ is non-empty.

Note :-

1. w.k.t $\sigma(x)$ is non-empty also it is a complete subspace of the complex plane.

The number $r(x)$ is defined by

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

is called the spectral radius of x.

It is clear that $0 \leq r(x) \leq \|x\|$.

2. A division algebra is an algebra with identity in which each non-zero

Theorem B: ⑦

If A is a division algebra, then it equals the set of all scalar multiples of the identity.

proof:

We must S.T if x is an element of A. Then x equals $\lambda 1$. for some scalar λ.

Suppose that $x \neq \lambda 1$ for every λ.

Then $x - \lambda \cdot 1 \neq 0$ for every λ.

$x - \lambda \cdot 1$ is regular for every λ, and therefore $\sigma(x)$ is empty.

This contradicts thm - A and complete the proof.

$\therefore$ $\sigma(x)$ is non-empty.

Hence proved.

Note:

The mapping $\lambda 1 \rightarrow \lambda$ is clearly an

... the set of all

scalar multiples of the identity and the Banach algebra $\mathbb{C}$ of all complex numbers.

2. Thm - B says that any Banach algebra which is a division algebra equals $\mathbb{C}$.

3. It is obviously that $\mathbb{C}$ itself which is the simplest of all Banach algebras, is a division algebra. So thm - B characterises $\mathbb{C}$ as the only Banach algebra with this property.

4. Since 0 is a divisor of zero, it is a topological divisor of zero in every Banach algebra. In the Banach algebra $\mathbb{C}$, 0 is plainly the only topological divisor of zero.

Theorem - C

If zero is the only topological divisor of zero in A, then $A = \mathbb{C}$.

Proof:

Let x be an element of A. Its spectrum $\sigma(x)$ is non-empty. So it has a boundary point λ and $x - \lambda 1$ is easily seen to be a boundary point of the set S of all singular elements.

By thm - B (Topological divisor of zero).

$x - \lambda 1$ is a topological divisor of zero.

So it follows from our hypotheses that $x - \lambda \cdot 1 = 0$ or $x = \lambda \cdot 1$.

Note :-

The basic link between multiplication in A and the norm is given by the inequality $\|xy\| \leq \|x\| \|y\|$. and when $A = C$ this inequality can be reversed.

Theorem D :-

If the norm in n satisfies the inequality $\|ny\| \geq k \|n\| \|y\|$ for some the constant k, Then $A = C$.

proof:

By given $\|ny\| \geq k \|n\| \|y\|$, zero is only top - divisor of zero.

$\|zz_n\| \geq \|z\| \|z_n\| \neq 0$

Theorem E :-

If A is a Banach subalgebra of a Banach algebra A'. then the spectra of an element n in A w.r.t A & A' are related as follows

1. $\sigma_{A'}(n) \leq \sigma_A(n)$

2. each boundary point of $\sigma_A(n)$ is also a boundary point of $\sigma_{A'}(n)$.

proof:

If $n - \lambda 1$ is singular in A'. Then it is contained ...

So (1) is clear.

T.P (2).

we let λ be a boundary point of $\sigma_A(x)$.

It is easy to see that $x-\lambda 1$ is a boundary point of the set of singular elements in A.

So by thm - B (Topological divisor of zero) it is a topological divisor of zero in A.

It is therefore a topological divisor of zero in A' as well.

So it is singular in A', and λ is an $\sigma_{A'}(x)$.

The fact that λ is actually a boundary point of $\sigma_{A'}(x)$ is immediate from (1).

So the proof of (2) is complete.

$\therefore$ λ is boundary point of $\sigma_A(x)$

Note:

The spectrum of an element shrinks when its containing Banach algebra is enlarged.

Ex:

The disc algebra A of all complex function which are defined and continuous on

$$D = \{ z : |z| \leq 1 \}$$

and analytic in the interior of this set.

If f is a function in A, then the maximum modulus thm from complex analysis implies that

$$\|f\| = \sup \{ |f(z)| ; |z| \leq 1 \}$$

$$= \sup \{ |f(z)| : |z| = 1 \}$$

This allows us to identity A with the Banach algebra of all the restrictions of its functions to the boundary of D, which is a Banach subalgebra of

$$A' = C\{z : |z| = 1\}.$$

If we now consider the element f in A, defined by $f(z) = z$,

Then it is easy to see that $\sigma_A(f)$ equals D and that $\sigma_{A'}(f)$ equals the boundary of D.

The formula for the spectral Radius:

Defn:

Let x be an element of Banach algebra A, and consider its spectral radius $r(x)$ which is defined.

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}.$$

Now let A' be the Banach subalgebra of A regenerated by x

i.e) The closure of the set of all polynomials in x.

Then E. S.T $r(x)$ has the same value if it is computed w.r.t A',

$r(x) = \sup\{ |\lambda| : \lambda \in \sigma_{A'}(x) \}$

clearly $r(x)$ depends only on the sequence powers of x.

Lemma :-

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$$\sigma(x^n) = \sigma(x)^n$$

proof:

Let λ be a non zero complex number and $\lambda_1 \dots \lambda_n$ its distinct n^{th} roots, so that,

$$x^n - \lambda_1 = (x - \lambda_1')(x - \lambda_2')\dots(x - \lambda_n')$$

The statement of the Lemma follows easily from the fact that

$x^n - \lambda_1$ is singular,

$$\sigma(x) = \{ \lambda : x - \lambda_1 \text{ is singular} \}$$

$\Leftrightarrow$ $x - \lambda_i 1$, is singular for atleast one i.

$$\lambda \in \sigma(x^n) \Leftrightarrow \lambda_i \in \sigma(x^n)$$

$$\Leftrightarrow \lambda \in \sigma(x^n)$$

Hence proved.

Theorem :- A

$$r(x) = \lim \|x^n\|^{1/n}$$

proof:-

$$\sigma(x^n) = \sigma(x)^n$$

w.k.T $r(x^n) = r(x)^n$ and

since $r(x^n) \leq \|x^n\|$.

we have $r(x)^n \leq \|x^n\|$ (or)

$$r(x) \leq \|x^n\|^{1/n} \quad \forall\, n. \quad \text{—— (1)}$$

T.S.T if a is any real number such that $r(x) < a$ Then

$$\|x^n\|^{1/n} \leq a$$

for all but a finite number of n's.

It follows from Thm - A. (Regular & singular elements). and our work in above see that is ~~$\|x\lambda\|$~~ $\|x\| < |\lambda|$

Then $x(\lambda) = (x - \lambda 1)^{-1} = \lambda^{-1}\left(\frac{x}{\lambda} - 1\right)^{-1}$

$$= -\lambda^{-1}\left(1 - \frac{x}{\lambda}\right)^{-1}$$

$$= -\lambda^{-1}\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}\right] \text{—— (2)}$$

If f is any functional on A, then

② yields

$$f(x(\lambda)) = -\lambda^{-1}\left[f(1) + \sum_{n=1}^{\infty} f\left(\frac{x^n}{\lambda^n}\right)\right]$$

$$= -\lambda^{-1}\left[f(1) + \sum_{n=1}^{\infty} f(x^n)\,\lambda^{-n}\right] \quad \text{③}$$

for all $|\lambda| > \|x\|$.

Thm - A. (The spectrum) that

w.k.T $f(x(\lambda))$ is an analytic function in the region $|\lambda| > r(x)$, and

series (3) is its Laurent expansion

for $|\lambda| > \|x\|$.

w.k.T from complex analysis that this expansion is valid for $|\lambda| > r(x)$

Let α be any real number such that $r(x) < \alpha < a$.

Then it ~~follow~~ follows from the series $\sum_{n=1}^{\infty} f(x^n/\alpha^n)$ converges, so its transforms a bounded sequence.

Since this is true for every f in A^*, T.S.T The elements x^n/α^n form a bounded sequence in A. Thus

$$\| x^n/\alpha^n \| < k.$$

$$\Rightarrow \| x^n \| \le k \| \alpha^n \|$$

$$\Rightarrow \| x^n \|^{1/n} \le k^{1/n} \| \alpha \|$$

for some +ve constant k and every n.

Since $k^{1/n} \alpha \le a$.

for every sufficiently large n, we have

$\| x^n \|^{1/n} \le a$ for all but a finite number of n's and replacing a by $r(x)$.

$$\| x^n \|^{1/n} \le a \ge r(x)$$

$$\| x^n \|^{1/n} \le r(x) \quad \text{—— (4)}$$

(1) & (4) $\therefore r(x) = \| x^n \|^{1/n}$

The Radical and semi - simplicity :

Defn:-

An ideal in A is a subset I with the following three properties.

1. I is a linear subspace of A.

3. $i \in I \Rightarrow in \in I$ for every element $n \in A$

If I is assumed only to satisfy conditions ① & ② [or conditions ① & ②]. it is called a left ideal (or a right ideal).

One which satisfies all three of these conditions is often called a two-sided ideal.

The properties of the ideals in A are closely related to the properties of its regular and singular elements.

The statement that an element n in A is ~~singular~~ regular has meant that there exists an element y such that

$$ny = yn = 1.$$

n is left regular if $\exists$ an element $y \ni yn = 1$ & if n is not left regular, it is called left singular.

The terms right regular & right singular are defined similarly.

If x is both left regular and right regular.

So that there exists elements y & z $\ni$ $yx = 1$ & $xz = 1$, Then the relation $y = y.1 = y(xz) = (yx)z = 1.z$
$= z$

S.T x is regular in the ordinary sense and that $x^{-1} = y = z$.

We define a maximal left ideal in A is defined to lyed in any other proper left ideal.

We define the radical R of A to be the intersection of all its maximal left ideals.

It will be convenient to abbreviated this definition by writting $R = \cap MLI$.

R is clearly a proper left ideal R is also the intersection of all the maximal right ideals in A. That is that

Lemma : 1

If r is an element of R, then $1-r$ is left regular.

proof:

We assume that $1-r$ is left singular so that
$$L = A(1-r)$$
$$= \{ x - xr : x \in A \}$$
is a proper that left ideal which contains $1-r$,

We next embed L in a maximal left ideal M, which of course also contains $1-r$.

Since r is in R, it is also in M and therefore
$$1 = (1-r) + r \text{ is in } M.$$
This $\Rightarrow M = A$.

which is contradiction.

M is proper left ideal.

$\therefore$ our assumption is wrong.

$1-r$ is left regular.

Lemma 2 :.

If r is an element of R, then $1-r$ is regular.

proof:

By the lemma just proved, $\exists$ an element s such that

$$s(1-r) = 1.$$

So it is right regular and

$$s = 1-(1-s)r$$

The fact that R is a left ideal implies that $(-s)r$ is in R along with r, and another application of the preceding Lemma s.t

$$1-(-s)r = s \text{ is left regular.}$$

Since s is both left regular and right regular. it is regular with inverse $1-r$.

So $1-r$ is also regular.

Lemma :

If r is an element of A with the property that $1-nr$ is regular for every n, then r is in R.

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Proof) We assume that r is not in R,
So that r is not in some maximal left ideal M.

It is easy to see that the set

$$M + Ar = \{ m + nr : m \in M \text{ and } n \in A \}$$

is a left ideal which contains both M and r, So $M + Ar = A$ and

$$m + nr = 1$$

for some m & n.

It now follows that $1 - nr = m$ is a regular element in M and this is impossible, for no proper ideal can contain any regular element.

$\therefore$ r is in R.

Hence proved.

Remark :-

The effect of these lemmas is to establish the quantity of two sets.

Now $1 - nr$ is regular for

precisely the same arguments, when applied to maximal right ideals. S.T

$nmRI = \{r : 1 - rn$ is regular for every $n\}$ — ②

We now P.T all four of these sets are the same by S.T the two sets on the right of ① & ② are equal to one another.

Lemma:-

If $1 - nr$ is regular, then $1 - rn$ is also regular.

Proof

We assume that $1 - nr$ is regular with inverse.

$$S = (1 - nr)^{-1}$$

This means of course that $\forall r$

$$(1 - nr)S = S(1 - nr) = 1$$

$$S = (1 - nr)^{-1} = 1 + nr + (nr)^2 + \dots$$

$1 - rn$ ... $rn + (rn)^2 + (rn)^3 + \dots$

$= 1 + rn\ (1 + nr + nrnr + \cdots)\ n$

$(1 - rn)^{-1} = 1 + r s n$.

Also $(1 - rn)(1 + r s n) = (1 + r s n)(1 - rn)$

$= 1$.

So that $1 - rn$ is regular with inverse $1 + r s n$.

Algebras of operators :-

Theorem A :-

The radical of R of A equals each of the four sets in (1) & (2) and is therefore a proper two sided ideal.

Defn :-

A is said to be semi - simple if its radical equals the zero ideal $\{0\}$ that is, if each non - zero element of A is outside of some maximal left ideal.

Theorem - B

Every maximal left ideal in A is closed.

proof If any maximal left ideal L is not closed, the L is, a proper subset of the proper left ideals, L and this cannot happen, for it contradicts the maximality of L.

Theorem C ::

The radical R of A is a proper closed two sided ideal.

proof

By thm - A. Radical R is a proper two-sided ideal.

By thm - B. Every - maximal left ideal is closed

since $R = \cap M L I$,

R is closed.

(intersecting of closed set is closed.).

R is a proper closed two - sided ideal.