M.Sc. II yr (Maths)

Advanced Numerical Analysis

Unit V

Singlestep Method.

A general single-step method can be written as $u_{j+1} = u_j + h\phi(t_{j+1}, u_{j+1}, u_j, h)$ where ϕ is a fn of the arguments $t_j, t_{j+1}, u_j, u_{j+1}, h$ and also depends of f. We often write it as $\phi(t, u, h)$. This function ϕ is called the increment function.

The truncation error of the method is given by $T_{j+1} = \frac{1}{(P+1)!} h^{P+1} u^{P+1}(t_j + \theta h)$

This method is called Taylor Series Method.

Ex: Given the intial value pb $u' = t^2 + u^2$, $u(0) = 0$ determine the first 3 non zero terms in the Taylor's series $u(t)$ & hence obtain the value for $u(1)$. Also determine t when the error $u(t)$ obtain from the first two non zero terms is to be less then 10^{-6} after rounding.

We have $u(0) = 0$, $u'(0) = 0$

gn. $u' = t^2 + u^2$

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$u'' = 2t + 2\underbrace{uu'}_{uv}$ $\quad | \because uv = uv' + vu'$

$u''(0) = 0.$

$u''' = 2 + 2\underbrace{(uu'' + (u')^2)}_{0 \; (\text{sub})}$

$u'''(0) = 2.$

$u^4 = 2(uu''' + 3u'u'')$

$u^4(0) = 0.$

$u^5 = 2[uu^4 + 4u'u''' + 3u''^2]$

$u^5(0) = 0$

$u^6 = 2(uu^5 + 5u'u^4 + 10u''u''']$

$u^6(0) = 0.$

$u^7 = 2[uu^6 + 6u'u^5 + 15u''u^4 + 10u'''^2]$

$u^7(0) = 80 \quad$ i.e., $2[0+0+0+10(2)^2] \;\because u'''(0) = 2.$

$u^8(0) = u^9(0) = u^{10}(0) = 0.$

Taylor Series of $u(t)$ becomes.

$u(t) = u(t_j) \dots + \frac{1}{3!} u'''(t_j) \dots \frac{1}{7!} u^7(t_j) \dots$

$u(t) = \frac{1}{3!} 2t^3 + \frac{1}{7!} t^7 + \cdot \frac{2}{11!} t^{11}.$

$= \frac{1}{3} t^3 + \frac{1}{63} t^7 + \frac{2}{2079} t^{11}$

If only the first two terms are used then the values of t is obtained from

$\left| \frac{2}{2079} t^{11} \right| < 0.5 \times 10^{-7}.$

Runge - Kutta Methods.

Integrating the differential equation $u' = f(t,u)$ on the interval $[t_j, t_{j+1}]$ we get.

$$\int_{t_j}^{t_{j+1}} \frac{du}{dt}\,dt = \int_{t_j}^{t_{j+1}} f(t,u)\,dt.$$

By Mean Value Thm of integral calculus.

$$u(t_{j+1}) = u(t_j) + h f(t_j + \theta h, u(t_j + \theta h)), \quad 0<\theta<1.$$

any value of $\theta \in [0,1]$

Case (i) if $\theta = 0$

$$u_{j+1} = u_j + h f(t_j, u)$$

Which is Euler Method.

Case (ii) if $\theta = 1$

$$u(t_{j+1}) = u(t_j) + h f(t_j + u(t_{j+1}))$$

We have Numerical method as

$$u_{j+1} = u_j + h f_{j+1}$$

Which is backward Euler Method.

W.K.T $\quad u_{j+1} = u_j + h f(t_{j+1}, u_j + h f_j)$

If we set $\quad k_1 = h f_j$

$$k_2 = h f(t_{j+1}, u_j + k_1)$$

we get the Method as

$$u_{j+1} = u_j + k_2$$

Case (iii) if $\theta = 1/2$

$$u(t_{j+1}) = u(t_j) + h f(t_j + h/2, u(t_j + h/2)) \quad \text{—} (1)$$

However $t_j + h/2$ is not a nodal points.

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If we approximate $u(t_j + h/2)$ of above eqn. by Euler method with spacing $h/2$ we get

$$u(t_j + h/2) = u_j + \frac{h}{2} f_j$$

Then we approx.

$$u_{j+1} = u_j + h f(t_j + h/2, u_j + \frac{h}{2} f_j) \quad \text{--- (2)}$$

If we set $K_1 = h f_j$

$$K_2 = h f(t_j + h/2, u_j + \tfrac{1}{2} K_1)$$

(2) Can be written as

$$u_{j+1} = u_j + K_2$$

Thus (1) may be approximated by

$$u_{j+1} = u_j + \frac{h}{2}\left[f(t_j, u_j) + f(t_{j+1}, u_j + h f_j) \right] \quad \text{--- (3)}$$

If we set $K_1 = h f_j$

$$K_2 = h f(t_{j+1}, u_j + K_1)$$

(3) Can be written as

$$u_{j+1} = u_j + \frac{h}{2}[K_1 + K_2]$$

This Method is also called Euler - Cauchy Method

Minimization of Local Truncation Error

Third order Method

Now use three evaluations of f & define the methods as

$$u_{j+1} = u_j + W_1 K_1 + W_2 K_2 + W_3 K_3 \quad \text{--- (1)}$$

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Where $k_1 = hf(t_j, u_j)$ —— (2)

$k_2 = hf(t_j + c_2 h, u_j + a_{21} k_1)$ — (3)

$k_3 = hf(t_j + c_3 h, u_j + a_{31} k_1 + a_{32} k_2)$ — (4)

We obtain the following six eqn. in 8 parameter $W_1, W_2 W_3, c_2 c_3, a_{21}, a_{31}$, & a_{32}

∴ The Methods contain two arbitrary parameters.

The eqn are.

$$a_{21} = c_2 \quad \text{—— (5)}$$

$$a_{31} + a_{32} = c_3 \quad \text{—— (6)}$$

$$W_1 + W_2 + W_3 = 1 \quad \text{—— (7)}$$

$$c_2 W_2 + c_3 W_3 = 1/2 \quad \text{—— (8)}$$

$$c_2^2 W_2 + c_3^2 W_3 = 1/3 \quad \text{—— (9)}$$

$$c_2 a_{32} W_3 = 1/6 \quad \text{—— (10)}$$

Eq (8) Multiplying $c_2 a_{32}$

$$c_2^2 a_{32} W_2 + c_2 c_3 W_3 a_{32} = \frac{c_2 a_{32}}{2}$$

$$c_2^2 a_{32} W_2 + c_3 \cdot 1/6 = \frac{c_2 a_{32}}{2}$$

using eqn. (10)

$$6 c_2^2 a_{32} W_2 + c_3 = 3 c_2 a_{32} \quad \text{—— (11)}$$

Eqn (9) Multiplying $c_2 a_{32}$.

$$c_2^3 a_{32} W_2 + c_3^2 \cdot c_2 a_{32} W_3 = 1/3 \; c_2 a_{32}$$

$$c_2^3 a_{32} W_2 + c_3^2 \frac{1}{6} = \frac{c_2 a_{32}}{3}$$

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$$6c_2^3 a_{32} w_2 + c_3^2 = 2c_2 a_{32}$$

$$6c_2^3 a_{32} w_2 = 2c_2 a_{32} - c_3^2 \quad \text{--- (12)}$$

(11) $\Rightarrow$

$$6c_2^2 a_{32} w_2 = 3c_2 a_{32} - c_3 \quad \text{--- (13)}$$

Eliminating w_2 from eq (12) & (13)

$$6c_2^3 a_{32} \cancel{w_2} = 2c_2 a_{32} - c_3^2$$

$$- 6c_2^3 a_{32} w_2 = \underline{3} c_2^2 a_{32} \mp c_2 c_3$$

$$0 = 2c_2 a_{32} - c_3^2$$

$$\quad + c_2 c_3 - 3c_2^2 a_{32}$$

$$0 = a_{32}(2c_2 - 3c_2^2) - c_3^2 + c_2 c_3$$

$$a_{32}(2c_2 - 3c_2^2) = c_3^2 + c_2 c_3$$

$$a_{32} = \frac{c_3^2 + c_2 c_3}{2c_2 - 3c_2^2}$$

$$= \frac{c_3(c_3 - c_2)}{c_2(2 - 3c_2)} \quad \text{--- (14)}$$

Usually c_2, c_3 are arbitrarily chosen. & a_{32} is determine from (14).

If $c_2 = c_3$ then eqn (8) & (9) we get

$c_3 w_2 + c_3 w_3 = 1/2$

$c_3^2 w_2 + c_3^2 w_3 = 1/3.$

ii, c_3 Mutiplay on both side by first eqn.

$c_3^2 \cancel{w_2} + c_3^2 \cancel{w_3} = c_3/2$

$- \; \cancel{c_3^2} w_2 + \cancel{c_3^2} w_3 = 1/3.$

$c_3/2 - 1/3 = 0$

$$\frac{3c_3 - 2}{6} = 0 \Rightarrow 3c_3 - 2 = 0$$

$3c_3 = 2$

$c_3 = 2/3$

If $c_2 = c_3$ Then $c_2 = 2/3$

IIIly we can find $a_{21} = 2/3$ $a_{31} = 0$

$a_{32} = 2/3$, $w_1 = 2/8$, $w_2 = 3/8$, $w_3 = 3/8$

(using eqn 5 to 10)

The Runge-Kutta Method is obtained as

$$u_{j+1} = u_j + 1/8 \, (2k_1 + 3k_2 + 3k_3)$$

where $k_1 = h f(t_j, u_j)$

$k_2 = h f(t_j + 2h/3, \, u_j + 2/3 k_1)$

$k_3 = h f(t_j + 2h/3, \, u_j + 2/3 k_2)$

~~~ x ~~~

Ex Given the initial value pb $u' = -2tu^2$, $u(0) = 1$ estimate $u(0.4)$ using modified Euler-Cauchy Methods with $h = 0.2$ & compare Heun Methods.
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Sol

$u_{j+1} = u_j + k_2$ ——— (1)

Where $k_1 = h f(t_j, u_j)$

$= 0.2 [-2 t_j u_j^2]$

$k_1 = 0.4 t_j u_j^2$ ——— (2)

$k_2 = h f(t_j + h/2, u_j + 1/2 k_1)$

$= -0.4 (t_j + \frac{0.2}{2})(u_j + \frac{k_1}{2})^2$ ——— (3)

For $j=0$ we have $t_0 = 0$, $u_0 = 1$

(2) $\Rightarrow$ $k_1 = 0$

(3) $k_2 = -0.4(0.1)(1)^2 = -0.04$.

(1) $\rightarrow$

$u(0.2) = u_1 = u_0 + k_2$

$= 1 - 0.04 = 0.96$

For $j=1$

$t_1 = 0.2$, $u_1 = 0.96$ (above Answer)

$k_1 = -0.4(0.2)(0.96)^2 = -0.073728$

$k_2 = -0.4(0.2+0.1)(0.96 - 0.036864)^2$

$= -0.102262$

$u(0.4) = u_2 = u_1 + k_2$

$= 0.96 - 0.102262$.

$= 0.857738$.

(ii) Heun Method is gn by

$u_{j+1} = u_j + 1/2 (k_1 + k_2)$ ——— (1)

$k_1 = h f(t_j, u_j) = -0.4 t_j u_j^2$

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$k_2 = hf(t_j + h, u_j + k_1)$

$= 0.4(t_j + 0.2)(u_j + k_1)^2$

For $j = 0$.

$t_0 = 0, \ u_0 = 1 \quad k_1 = 0,$

$k_2 = -0.4(0.2)(1)^2 = -0.08$

$u(0.2) = u_1 = u_0 + 1/2\,(k_1 + k_2)$

$= 1 + 1/2\,(-0.08) = 0.96$

For $j = 1$

$t_1 = 0.2, \quad u_1 = 0.96, \quad k_1 = -0.4(0.2)(0.96)^2$

$= -0.073728$

|||ly

$k_2 = -0.125676.$

$u(0.4) = u_2 = u_1 + 1/2\,(k_1 + k_2)$

$= 0.860298.$

Taylor series Method

Ex compute an approximate to $u(1), u'(1), u''(1)$ with Taylor series Method of second order & step length $h = 1$ for the initial value problems,

$u''' + 2u'' + u' - u = \cos t \quad , \ 0 \le t \le 1$

$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 2$

after reducing it to a system of first order equation.

Set $u = v_1$

$v_1' = v_2$

$v_2' = v_3$

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i.e, $V_2 = V_1' = u'$

$V_3 = V_2' = u''$

$V_3' = V_2'' = u'''$

The System of eqn. (above)

$v_1' = v_2$ i.e, u'

$v_2' = v_3$ i.e, u''

$v_3' = \cos t - 2u'' - u' - u$ (given)

i.e, $v_3' = \cos t - 2v_3 - v_2 + v_1$

$v_1(0) = 0$ i.e, u(0)

$v_2(0) = 1$

$v_3(0) = 2$

Matrix form of above eqn.

$$v' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}' = \begin{bmatrix} v_2 \\ v_3 \\ \cos t - 2v_3 - v_2 + v_1 \end{bmatrix}$$

$$v(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{(above answer)}$$

Taylorseries Method of order two is

$$v(t_0+h) = v_0 + h v_0' + \frac{h^2}{2} v''(0).$$

put $h=1$

$$= v_0 + v_0' + \tfrac{1}{2} v_0''.$$

we have

$$v_0' = \begin{bmatrix} v_2(0) \\ v_3(0) \\ \underset{\cos 0}{1} - 2v_3(0) - v_2(0) + v_1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$\because = 1 - 2v_3(0) - v_2(0) + v_1(0)$

$= \cancel{1} - 2(2) - \cancel{1} + 0$

$= -4$

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$$v'' = \begin{bmatrix} v_2' \\ v_3' \\ -\sin t - 2v_3' - v_2' + v_1' \end{bmatrix}$$

$\because$ one time diff

$$\therefore v_0'' = \begin{bmatrix} v_3(0) \\ \begin{bmatrix} u''' \quad u, \cos t - 2v_3 - v_2 - v_1 \\ 1 - 2v_3(0) - v_2(0) + v_1(0) \end{bmatrix} \\ -\sin t - 2(\text{above}) - v_3(0) + v_2(0) \end{bmatrix}$$

$$v_0'' = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

Hence $v(1) = v_0 + v_0' + \frac{1}{2} v_0''$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix}$$

$\therefore u(1) = 2, \quad u'(1) = 1, \quad u''(1) = 3/2.$

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