NUMERICAL ANALYSIS AND STATISTICS

Subject Code: 16SACMA2

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NUMERICAL DIFFERENTIATION AND INTEGRATION.

Newton's forward difference formula:

If $u = \dfrac{x - x_0}{h}$

then

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right]$$

At $x = x_0$, $u = 0$, then

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

Newton's Backward difference formula: If $v = \dfrac{x - x_n}{h}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2!}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{2v^3+9v^2+11v+3}{12}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{(6v^2+18v+11)}{12}\nabla^4 y_n + \cdots\right)$$

At $x = x_n$ $v = 0$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Problems:

1) Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 51$ from the following table

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Soln: Difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	19.96				
		16.69			
60	36.65		5.47		
		22.16		-9.23	
70	58.81		-3.76		11.99
		18.4		2.76	
80	77.21		-1		
		17.4			
90	94.61				

$$u = \frac{x - x_0}{h} = \frac{51 - 50}{10} = 0.1$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right.$$

$$= \frac{1}{10}\left[16.69 + \frac{2(0.1)-1}{2} \times 5.47 + \frac{3(0.1)^2 - 6(0.1)+2}{6} \times -9.23 + \frac{4(0.1)^3 - 18(0.1)^2 + 22(0.1) - 6}{24} \times 11.99\right]$$

$$= \frac{1}{10}\left[16.69 - 2.188 - 2.1998 - 1.9863\right]$$

$$= \frac{10.3159}{10} = 1.0316$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2 - 18u + 1}{12}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{100}\left[5.47 + (0.1-1) \times -9.23 + \frac{6(0.1)^2 - 18(0.1)+1}{12} \times 11.99\right]$$

$$= \frac{1}{100}\left[5.47 + 8.307 - 0.7394\right]$$

$$= 0.1304$$

NUMERICAL INTEGRATION:

TRAPEZOIDAL RULE:

$$y(x) = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

Simpson's 1/3 Rule:

$$y(x) = \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + y_6 + \cdots) + 4(y_1 + y_3 + \cdots)\right].$$

This rule is applicable when the no. of subinterval is even.

Simpson's 3/8 rule:

$$y(x) = \frac{3h}{8}\left[y_0 + y_n + 2(y_3 + y_6 + y_9 + \cdots) + 3(y_1 + y_2 + y_4 + y_5 + \cdots)\right]$$

This rule is applicable when the no. of subinterval is multiple of 3.

2) Dividing the range into 10 equal parts, find the value of $\int_0^{\pi/2} \sin x \, dx$ by (i) Trapezoidal rule (ii) Simpson's rule

Soln:

Here $f(x) = \sin x$ $x_0 = 0$ $x_n = \frac{\pi}{2}$ $n = 10$.

$$h = \frac{x_n - x_0}{n} = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$$

x	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$
$f(x)=\sin x$	0	0.1564	0.3090	0.454	0.5878	0.7071

x	$6\pi/20$	$7\pi/20$	$8\pi/20$	$9\pi/20$	$\frac{10\pi}{20}$
$f(x)=\sin x$	0.809	0.8091	0.9511	0.9877	1

Trapezoidal rule:

$$I = \frac{h}{2}\left[y_0 + y_{10} + 2(y_1 + y_2 + \cdots + y_9)\right]$$

$$= \frac{\frac{\pi}{20}}{2}\left[0 + 1 + 2(0.1564 + 0.309 + 0.454 + 0.5878 + 0.7071 + 0.809 + 0.8091 + 0.9511 + 0.9877)\right]$$

$$= \frac{\pi}{40} \times 12.5424 = 0.9851.$$

(ii) By simpson's $\frac{1}{3}$ rule

$$I = \frac{h}{3}\left[y_0 + y_{10} + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)\right]$$

$$= \frac{\pi/20}{3}\left[0 + 1 + 2(0.309 + 0.5878 + 0.809 + 0.9511) + 4(0.1564 + 0.454 + 0.7071 + 0.8091 + 0.9877)\right]$$

$$= \frac{\pi}{60}\left[1 + 2(2.6569) + 4(3.1143)\right]$$

$$= \frac{\pi}{60} \times 18.771 \qquad = 0.9828$$

Solutions of System of linear equations:

Consider a system of 'm' linear equations in 'n' unknown variables is

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} - \text{I}$$

It can be written in matrix form $AX = B$,

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Solution of I can be obtained by

(i) Direct method (ii) Indirect method

Direct method:

Gauss elimination & Gauss Jordon method are direct methods.

Gauss elimination method:

Procedure:

(i) To write the eqns in to matrix form $AX = B$

(ii) To write the augmented matrix (A/B).

(iii) To change the matrix (A/B) in to upper Δ^{le} matrix

(iv) To find the values of variables by back subtitution method solve the equations $2x+y+4z=12$, $8x-3y+2z=20$, $4x+11y-z=33$

Soln: G.T
$$2x+y+4z=12$$
$$8x-3y+2z=20$$
$$4x+11y-z=33$$

It can be written as $AX=B$,

Where $A=\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}$ $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B=\begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$

$$(A|B) = \begin{bmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{bmatrix} \quad R_3 \to 7R_3 - 9R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & 189 & 189 \end{bmatrix}$$

$\Rightarrow$ $2x+y+4z=12 \Rightarrow 2x+2+4=12 \Rightarrow 2x=12-6 \Rightarrow 2x=6$
$x=3$

$-7y-14z=-28 \Rightarrow -7y-14=-28 \Rightarrow -7y=-14 \Rightarrow y=2$

$189z=189 \Rightarrow$
$z=1$

$\boxed{x=3 \quad y=2 \quad z=1}$

Indirect method or Iterative method:

Diagonally dominant:

A matrix is said to be deagonally dominant if the numerical value of the leading diagonal ~~value~~ elt in each row is greater than or equal to the sum of the numerical values of the other elts in that row.

Eg: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is said to diagonally

dominant if $|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{32}|$.

Gauss - Seidal Method:

consider the system of m eqns in n unkno

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$.

Procedure:

(i) To check whether the co.effi. matrix is deagonally dominant.

(ii) If not, change it to diagonally dominant.

(iii) Let us assume the initial values $x_1 = x_2 = \cdots = x_n = 0$.

(iv) To find $x_1^* = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$

$x_2^* = \frac{1}{a_{22}}[b_2 - a_{21}x_1^* - a_{23}x_3 - \cdots - a_{2n}x_n)$

$\vdots$

$x_n^* = \frac{1}{a_{mn}}[b_m - a_{m1}x_1^* - a_{m2}x_2^* - \cdots a_{m(n-1)}x_{n-1}^*)$

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(iv) ~~To~~ To continue this procedure till we get ~~two~~ same answers.

Problems:

Solve the following eqns by Gauss seidal method:

$4x+2y+z=14$, $x+5y-z=10$ $x+y+8z=20$.

Soln: G.T $4x+2y+z=14$

$x+5y-z=10$

$x+y+8z=20$

The Co.effi. matrix is diagonally dominant.

Let $x_1=0$. $y=0$ $z=0$.

$x^*=\frac{1}{4}[14-2y-z]$

$y^*=\frac{1}{5}[10-x^*+z]$

$z^*=\frac{1}{8}[20-x^*-y^*]$

I iteration:

$x^*=\frac{1}{4}[14-0-0]=3.5$.

$y^*=\frac{1}{5}[10-3.5]=1.3$

$z^*=\frac{1}{8}[20-3.5-1.3]=\frac{15.2}{8}.=1.9$.

IInd iteration:

$x=\frac{1}{4}[14-2\times1.3-1.9]=\frac{1}{4}[14-2.6-1.9]=\frac{9.5}{4}=2.375$

$y=\frac{1}{5}[10-2.375+1.9]=\frac{1}{5}[9.525]=1.905$

$z=\frac{1}{8}[20-2.375-1.905]=\frac{15.72}{8}=1.965$

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IIIrd iteration:

$$x = \frac{1}{4}[14 - 2\times1.905 - 1.965] = \frac{1}{4}[14 - 3.81 - 1.965] = \frac{8.225}{4} = 2.0563$$

$$y = \frac{1}{5}[10 - 2.0563 + 1.965] = \frac{9.9087}{5} = 1.9817$$

$$z = \frac{1}{8}[20 - 2.0563 - 1.9817] = \frac{15.962}{8} = 1.9953$$

IVth iteration:

$$x = \frac{1}{4}[14 - 2\times1.9817 - 1.9953] = \frac{1}{4}[14 - 3.9634 - 1.9953] = \frac{8.0413}{4}$$
$$= 2.0103$$

$$y = \frac{1}{5}[10 - 2.0103 + 1.9953] = \frac{9.985}{5} = 1.997$$

$$z = \frac{1}{8}[20 - 2.0103 - 1.997] = \frac{15.9927}{8} = 1.9991$$

Vth iteration:

$$x = \frac{1}{4}[14 - 2\times\overset{1.997}{\cancel{2.0103}} - 1.9991] = \frac{1}{4}[14 - 3.994 - 1.9991] = \frac{8.0069}{4}$$
$$= 2.0017$$

$$y = \frac{1}{5}[10 - 2.0017 + 1.9991] = \frac{9.9974}{5} = 1.9995$$

$$z = \frac{1}{8}[20 - 2.0017 - 1.9995] = \frac{15.9988}{8} = 1.9999$$

VIth iteration

$$x = \frac{1}{4}[14 - 2\times1.9995 - 1.9999] = \frac{8.0011}{4} = 2.0003$$

$$y = \frac{1}{5}[10 - 2.0003 + 1.9999] = \frac{9.9996}{5} = 1.9999$$

$$z = \frac{1}{8}[20 - 2.0003 - 1.9999] = \frac{15.9998}{8} = 2$$

$\overline{\text{VI}}^{\text{th}}$ iteration

$$x = \frac{1}{4}[14 - 2\times 1.9999 - 2] = \frac{8.0002}{4} = 2.0001$$

$$y = \frac{1}{5}[10 - 2.0001 + 1.9999] = \frac{9.9999}{5} = 2.0000$$

$$z = \frac{1}{8}[20 - 2.0001 - 2.0000] = \frac{15.9999}{8} = 2.$$

$\overline{\text{VII}}^{\text{th}}$ iteration:

$$x = \frac{1}{4}[14 - 2\times 2 - 2] = \frac{8}{4} = 2.$$

$$y = \frac{1}{5}[10 - 2 + 2] = \frac{10}{5} = 2.$$

$$z = \frac{1}{8}[20 - 2 - 2] = \frac{16}{8} = 2.$$

$\overline{\text{VIII}}^{\text{th}}$ iteration:

$$x = \frac{1}{4}[14 - 4 - 2] = 2.$$

$$y = \frac{1}{5}[10 - 2 + 2] = 2$$

$$z = \frac{1}{8}[20 - 2 - 2] = 2.$$

$\therefore$ the soln is $x = y = z = 2$

Gauss-Jacobi Method:-

Procedure:-

1. Check whether the co-efficient matrix is diagonally dominant. If not rearrange the rows such that the matrix is diagonally dominant.

2. Let initially $x^{(0)} = y^{(0)} = z^{(0)} = 0$

3. $$x^{n+1} = \frac{1}{a_1}\left(d_1 - b_1 y^{(n)} - c_1 z^{(n)}\right)$$

$$y^{n+1} = \frac{1}{b_2}\left(d_2 - a_2 x^{(n+1)} - c_2 z^{(n)}\right)$$

$$z^{n+1} = \frac{1}{c_3}\left(d_3 - a_3 x^{(n+1)} - b_3 y^{(n+1)}\right)$$

4. Continue this process till we get a same solution in successive iteration

Problems:-

solve the system of equations $4x+2y+z=14$, $x+5y-z=10$, $x+y+8z=20$ using Gauss-Jacobi

Soln:-

G.T $4x+2y+z=14$

$x+5y-z=10$

$x+y+8z=20$

Here the coefficient matrix is diagonally domi

Let $x^{n+1} = \frac{1}{4}(14-2y^{(n)} - z^{(n)})$

$y^{n+1} = \frac{1}{5}(10 - x^{(n)} + z^{(n)})$

$z^{n+1} = \frac{1}{8}(20 - x^{(n)} - y^{(n)})$.

$x^{(0)} = y^{(0)} = z^{(0)} = 0$.

Ist iteration:-

$x^{(1)} = \frac{1}{4}(14-0-0) = 3.5$

$y^{(1)} = \frac{1}{5}(10-0-0) = 2$

$z^{(1)} = \frac{1}{8}(20-0-0) = 2.5$

Second iteration:-

$x^{(1)} = 3.5 \quad y^{(1)} = 2 \quad z^{(1)} = 2.5$

$x^{(2)} = \frac{1}{4}(14 - 2\times 2 - 2.5) = 1.875$

$y^{(2)} = \frac{1}{5}(10 - 3.5 + 2.5) = 1.8$

$z^{(2)} = \frac{1}{8}(20 - 3.5 - 2) = 1.8125$

iteration	x	y	z
0	0	0	0
1	3.5	2	2.5
2	1.875	1.8	1.8125
3	2.1469	1.9875	2.0406
4	1.9164	1.9787	2.0016
5	2.0414	2.0170	2.1277
6	1.9474	2.0173	1.9927
7	2.0281	2.0091	2.0044
8	1.9943	1.9953	1.9954
9	2.0035	2.0002	2.0013
10	1.9996	1.9996	1.9994
11	2.0004	2	2.0001
12	2	1.9999	1.9999
13	2.0001	2	2
14	2	2	2
15	2	2	2

$\therefore$ the solution is

$x = 2 \quad y = 2 \quad z = 2.$