ALGEBRA, ANALYTICAL GEOMETRY 3D AND TRIGONOMETRY

SUBJECT CODE: 16SACMM2

UNIT-V

TOPIC: HYPERBOLIC FUNCTIONS

Prepared by

Mrs.C.PRIYA,

HEAD, Assistant Professor of Mathematics,

AIMAN College of Arts and Science for Women,

Trichy-21.

Subject: Algebra, Analytical Geometry (3D) & Trigonometry (1)

UNIT - V

Hyperbolic Functions

Result :1

$\cosh^2 x - \sinh^2 x = 1$

Proof :

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \left[\frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4}\right] - \left[\frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\right]$$

$$= \left[\frac{e^{2x} + 2 + e^{-2x}}{4}\right] - \left[\frac{e^{2x} - 2 + e^{-2x}}{4}\right]$$

$$= \frac{1}{4}\left[\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}\right]$$

$$= \frac{1}{4}[4]$$

$$= 1$$

$$\therefore \boxed{\cosh^2 x - \sinh^2 x = 1}$$

Scanned by CamScanner

Result : 2

$\sinh 2x = 2\sinh x \cosh x$

Proof

$$2\sinh x \cosh x = 2\left[\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\right]$$

$$= 2\left[\frac{(e^x - e^{-x})(e^x + e^{-x})}{4}\right]$$

$$= \frac{2}{4}\left[(e^x - e^{-x})(e^x + e^{-x})\right]$$

$$= \frac{1}{2}\left[(a-b)(a+b) = a^2 - b^2\right]$$

$$= \frac{1}{2}\left[(e^x)^2 - (e^{-x})^2\right]$$

$$= \frac{1}{2}\left[e^{2x} - e^{-2x}\right]$$

$$= \sinh 2x$$

$$\therefore \boxed{2\sinh x \cosh x = \sinh 2x}$$

Scanned by CamScanner

Relation b/w hyperbolic functions & Circular trigonometric functions.

Theorem: (i) $\sin(ix) = i\sinh x$

(ii) $\cos(ix) = \cosh x$

(iii) $\tan(ix) = i\tanh x$

Proof:

Case (i) $\sin(ix) = i\sinh x$

We know $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

$$\sin(ix) = (ix) - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \dots$$

$$= (ix) - \frac{i^3x^3}{3!} + \frac{i^5x^5}{5!} - \dots$$

Put $i^2 = -1$

$i^3 = -i$

$i^5 = i$

$$= ix - \frac{(-i)x^3}{3!} + \frac{ix^5}{5!} - \dots$$

$$= ix + \frac{ix^3}{3!} + \frac{ix^5}{5!} \dots$$

$$= i\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right]$$

Scanned by CamScanner

$\therefore \sin(ix) = i \sinh x$

case (ii) $\cos(ix) = \cosh x$

we know $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$

$$\cos(ix) = 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \ldots$$

$$= 1 - \frac{i^2 x^2}{2!} + \frac{i^4 x^4}{4!} - \ldots$$

Put $i^2 = -1$

$i^4 = 1$

$$= 1 - \frac{(-1)x^2}{2!} + \frac{1(x^4)}{4!} - \ldots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$= \cosh x$$

$\therefore \cos(ix) = \cosh x$

(iii) case (iii) $\tan(ix) = \frac{\sin(ix)}{\cos(ix)}$

$$= \frac{i \sinh x}{\cosh x} \quad \text{(Previous Result)}$$

$\tan(ix) = i \tanh x$

Scanned by CamScanner

Inverse Hyperbolic Functions

Thm

(i) $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right)$

(ii) $\cosh^{-1} x = \log_e\left(x + \sqrt{x^2-1}\right)$

(iii) $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof:

(i) $\sinh^{-1} x = y$

$y = \sinh^{-1} x$

$y = \frac{1}{\sinh} x$

$\sinh y = x$

$$\frac{e^y - e^{-y}}{2} = x$$

$e^y - e^{-y} = 2x$

$e^y - \frac{1}{e^y} = 2x$

$$\frac{e^y \times e^y - 1}{e^y} = 2x$$

$$\frac{e^{2y} - 1}{e^y} = 2x$$

$e^{2y} - 1 = 2x e^y$

$e^{2y} - 2x e^y - 1 = 0$

which is a quadratic equation in e^y

(6)

$$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \quad \begin{matrix} a = 1 \\ b = -2x \\ c = -1 \end{matrix}$$

$$= \frac{-(-2x) \pm \sqrt{(-2x)^2-4(1)(-1)}}{2 \times 1}$$

$$= \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= \frac{\cancel{2}\left[x \pm \sqrt{x^2+1}\right]}{\cancel{2}}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Log on both sides

$$\log(e^y) = \log_e\left(x+\sqrt{x^2+1}\right)$$

$$y = \log_e\left(x+\sqrt{x^2+1}\right) \quad \left[\because y = \sinh^{-1}x\right]$$

$$\therefore \sinh^{-1}x = \log_e\left(x+\sqrt{x^2+1}\right)$$

Scanned by CamScanner

(ii) Let $y = \cosh^{-1} x$

$$y = \frac{1}{\cosh} x$$

$$\cosh y = x$$

$$\frac{e^y + e^{-y}}{2} = x$$

$$e^y + e^{-y} = 2x$$

$$\frac{e^y}{1} + \frac{1}{e^y} = 2x$$

$$\frac{e^y \times e^y + 1}{e^y} = 2x$$

$$\frac{e^{2y} + 1}{e^y} = 2x$$

$$e^{2y} + 1 = 2xe^y$$

$$e^{2y} - 2xe^y + 1 = 0$$

which is a quadratic equation in e^y

Scanned by CamScanner

(8)

$$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a = 1$
$b = -2x$
$c = 1$

$$= \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2 \times 1}$$

$$= \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2-1)}}{2}$$

$$= \left[\frac{2x \pm 2\sqrt{x^2-1}}{2}\right]$$

$$= \cancel{2}\left[\frac{x \pm \sqrt{x^2-1}}{\cancel{2}}\right]$$

$$e^y = x \pm \sqrt{x^2-1}$$

Take log on both sides

$$\log_{\cancel{e}}(\cancel{e}^y) = \log_e\left(x \pm \sqrt{x^2-1}\right)$$

$$y = \log_e\left(x \pm \sqrt{x^2-1}\right)$$

Scanned by CamScanner

$\therefore \cosh^{-1}x = \log_e(x+\sqrt{x^2-1})$ $\quad (y = \cosh^{-1}x)$

(iii) $y = \tanh^{-1}x$

$y = \frac{1}{\tanh} x$

$\tanh y = x$

$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x$

$e^y - e^{-y} = x(e^y + e^{-y})$

$e^y - \frac{1}{e^y} = x(e^y + e^{-y})$

$\frac{e^y \times e^y - 1}{e^y} = x(e^y + e^{-y})$

$\frac{e^{2y} - 1}{e^y} = x(e^y + e^{-y})$

$e^{2y} - 1 = xe^y(e^y + e^{-y})$

Scanned by CamScanner

(10)

$$e^{2y} - 1 = x\,e^{2y} + x\,e^{y}e^{-y}$$

$$e^{2y} - 1 = x\,e^{2y} + x$$

$$e^{2y} - 1 = x\,e^{2y} + x$$

$$e^{2y} - x\,e^{2y} = 1 + x$$

$$e^{2y}(1-x) = 1 + x$$

$$\therefore \boxed{e^{2y} = \frac{1+x}{1-x}}$$

log on both sides

$$\log(e^{2y}) = \log_e\left(\frac{1+x}{1-x}\right)$$

$$2y = \log_e\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

$$\therefore \tanh^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

Scanned by CamScanner

Problem :

If $x+iy = \sin(A+iB)$ P.T $\frac{x^2}{\sin^2 A} - \frac{x^2}{\cos^2 A} = 1$

Solu :

$x+iy = \sin(A+iB)$

$= \sin A \cos iB + \cos A \sin(iB)$

$x+iy = \sin A \cosh B + i\cos A \sinh B$ $\quad \left[\begin{array}{l}\cos iB = \cosh B \\ \sin iB = i\sinh B\end{array}\right.$

$x = \sin A \cosh B, \; y = \cos A \sinh B$

$\therefore \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \frac{(\sin A \cosh B)^2}{\sin^2 A} - \frac{(\cos A \sinh B)^2}{\cos^2 A}$

$= \frac{\cancel{\sin^2 A}\cosh^2 B}{\cancel{\sin^2 A}} - \frac{\cancel{\cos^2 A}\sinh^2 B}{\cancel{\cos^2 A}}$

$= \cosh^2 B - \sinh^2 B$

$= 1$

$\therefore \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

Scanned by CamScanner

(ii) $$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

Already to find

$$x = \sin A \cosh B, \quad y = \cos A \sinh B$$

We have $$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \frac{(\sin A \cosh B)^2}{\cosh^2 B} + \frac{(\cos A \sinh B)^2}{\sinh^2 B}$$

$$= \frac{\sin^2 A \cancel{\cosh^2 B}}{\cancel{\cosh^2 B}} + \frac{\cos^2 A \cancel{\sinh^2 B}}{\cancel{\sinh^2 B}}$$

$$= \sin^2 A + \cos^2 A$$

$$= 1$$

$$\therefore \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

Scanned by CamScanner

Problem : 6

If $\tan A = \tan\alpha \tanh\beta$, $\tan B = \cot\alpha \tanh\beta$

Prove that $\tan(A+B) = \sinh 2\beta \cdot \operatorname{cosec} 2\alpha$

Solu :

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\tan\alpha \tanh\beta + \cot\alpha \tanh\beta}{1 - \tan\alpha \tanh\beta \cot\alpha \tanh\beta}$$

$$= \frac{\tan\alpha \tanh\beta + \cot\alpha \tanh\beta}{1 - \cancel{\tan\alpha} \tanh\beta \frac{1}{\cancel{\tan\alpha}} \tanh\beta}$$

$$= \frac{\tan\alpha \tanh\beta + \cot\alpha \tanh\beta}{1 - \tanh^2\beta}$$

$$= \frac{\tanh\beta \left[\tan\alpha + \cot\alpha\right]}{\operatorname{sech}^2\beta}$$

Note: $[1 - \tanh^2\beta = \operatorname{sech}^2\beta]$

Scanned by CamScanner

$$= \frac{\tanh\beta\left[\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right]}{\operatorname{sech}^2\beta}$$

$$= \frac{\tanh\beta\left[\frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha}\right]}{\operatorname{sech}^2\beta}$$

$$= \frac{\tanh\beta \; \frac{1}{\sin\alpha\cos\alpha}}{\operatorname{sech}^2\beta}$$

$\boxed{\sin^2\alpha + \cos^2\alpha = 1}$

$$= \frac{\tanh\beta \times \frac{1}{\sin\alpha\cos\alpha}}{\operatorname{sech}^2\beta}$$

$$= \frac{\frac{\sinh\beta}{\cosh\beta} \times \frac{1}{\sin\alpha\cos\alpha}}{\frac{1}{\cosh^2\beta}}$$

$$= \frac{\sinh\beta}{\cancel{\cosh\beta}} \times \frac{1}{\sin\alpha\cos\alpha} \times \cosh^{\cancel{2}}\beta$$

$$= \frac{\sinh\beta \cosh\beta}{\sin\alpha \cos\alpha}$$

Multiple & divide by 2

$$= \frac{2 \sinh\beta \cosh\beta}{2 \sin\alpha \cos\alpha}$$

$[\sin 2A = 2\sin A \cos A]$

$$= \frac{\sinh 2\beta}{\sin 2\alpha}$$

$$= \sinh 2\beta \times \frac{1}{\sin 2\alpha}$$

$$\tan(A+B) = \boxed{\sinh 2\beta \ \mathrm{cosec}\, 2\alpha}$$

Problem :

If $\cosh u = \sec\theta$ S.T. $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Solu :-

$\cosh u = \sec\theta$

$u = \cosh^{-1}(\sec\theta)$

$[$put $\cosh^{-1} x = \log_e (x + \sqrt{x^2+1})$

Scanned by CamScanner

$\therefore \boxed{x = \sec\theta}$

Apply the formula, we get

$$u = \cosh^{-1}(\sec\theta)$$

$$= \log_e\left[x + \sqrt{x^2 - 1}\right]$$

$$= \log_e\left[\sec\theta + \sqrt{\sec^2\theta - 1}\right]$$

$$= \log_e\left[\sec\theta + \tan\theta\right]$$

$\left[\sqrt{\sec^2\theta - 1} = \tan\theta\right]$

$$= \log_e\left[\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right]$$

$$= \log_e\left[\frac{1 + \sin\theta}{\cos\theta}\right]$$

$$= \log_e\left[\frac{1 + \frac{2\tan\theta/2}{1 + \tan^2\theta/2}}{\frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2}}\right]$$

$$\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$= \log_e\left[\frac{\frac{1 + \tan^2\theta/2 + 2\tan\theta/2}{\cancel{1 + \tan^2\theta/2}}}{\frac{1 - \tan^2\theta/2}{\cancel{1 + \tan^2\theta/2}}}\right]$$

Scanned by CamScanner

$$= \log_e \left[\frac{1+\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \right]$$

$$= \log_e \left[\frac{\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} + 1}{1-\tan^2\frac{\theta}{2}} \right]$$

$$= \log_e \left[\frac{\left(1+\tan\frac{\theta}{2}\right)^{\cancel{2}}}{\cancel{\left(1+\tan\frac{\theta}{2}\right)} \left(1-\tan\frac{\theta}{2}\right)} \right]$$

$$\left[\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} + 1 = \left(1+\tan\frac{\theta}{2}\right)^2\right]$$

$$\left[1-\tan^2\frac{\theta}{2} = \left(1+\tan\frac{\theta}{2}\right)\left(1-\tan\frac{\theta}{2}\right)\right]$$

$$= \log_e \left[\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} \right]$$

Put $= 1 = \tan \pi/4$

$$= \log_e \left[\frac{\tan \pi/4 + \tan \theta/2}{1-(\tan \pi/4)(\tan \theta/2)} \right]$$

$$\boxed{u = \log_e \left[\tan\left(\pi/4 + \theta/2\right) \right]}$$

Conversely let $u = \log_e \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Take exponential on both sides

$$e^u = e^{\log_e \tan(\pi/4 + \theta/2)}$$

Scanned by CamScanner

$$\therefore e^{u} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$e^{\frac{u}{2}} \times e^{\frac{u}{2}} = \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$\frac{e^{\frac{u}{2}}}{e^{-\frac{u}{2}}} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$\frac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}} = \frac{\cancel{1}+\tan\theta/2 - [\cancel{1}-\tan\theta/2]}{1+\cancel{\tan\theta/2}+[1-\cancel{\tan\theta/2}]}$$

$$\tanh(u/2) = \frac{\tan\theta/2 + \tan\theta/2}{2}$$

$$\tanh(u/2) = \frac{\cancel{2}\tan\theta/2}{\cancel{2}}$$

$$\tanh(u/2) = \tan\theta/2$$

$$\therefore \cosh u = \frac{1+\tanh^2(u/2)}{1-\tanh^2(u/2)} = \frac{1+\tan^2(\theta/2)}{1-\tan^2(\theta/2)}$$

$$\boxed{\cosh u = \sec\theta}$$

Scanned by CamScanner

Problem

If $\cos(x+iy) = \cos\theta + i\sin\theta$ Prove that

$\cos 2x + \cosh 2y = 2$

Solu:

$$\cos\theta + i\sin\theta = \cos(x+iy)$$

$$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$$

$$= \cos x \cosh y - \sin x\,(i\sinh y)$$

$$\cos\theta + i\sin\theta = \cos x\cosh y - i\sin x\sinh y$$

Take Real & Imaginary parts

$\boxed{\cos\theta = \cos x \cosh y}$, $\boxed{\sin\theta = -\sin x \sinh y}$

W.K.T $\sin^2\theta + \cos^2\theta = 1$

$$(-\sin x\sinh y)^2 + (\cos x\cosh y)^2 = 1$$

$$\sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y = 1$$

$$(1-\cos^2 x)\sinh^2 y + \cos^2 x \cosh^2 y = 1$$

Scanned by CamScanner

$$\Rightarrow \sinh^2 y - \cos^2 x \sinh^2 y + \cos^2 x \cosh^2 y = 1$$

$$\Rightarrow \sinh^2 y + \cos^2 x (\cosh^2 y - \sinh^2 y) = 1$$

$$\Rightarrow \sinh^2 y + \cos^2 x (1) = 1$$

$$\Rightarrow \sinh^2 y + \cos^2 x = 1$$

$$\frac{\cosh 2y - 1}{2} + \frac{1 + \cos 2x}{2} = 1$$

$$\frac{\cosh 2y \cancel{-1} + \cancel{1} + \cos 2x}{2} = 1$$

$$\cosh 2y + \cos 2x = 1(2)$$

$$\boxed{\cosh 2y + \cos 2x = 2}$$

Scanned by CamScanner

problem : 3

If $\tan(a+ib) = (x+iy)$ P.T $\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$

Solu:

$$x+iy = \tan(a+ib)$$

$$= \frac{\sin(a+ib)}{\cos(a+ib)}$$

Conjugate Nr & Dr by $\cos(a-ib)$

$$= \frac{\sin(a+ib)}{\cos(a+ib)} \times \frac{\cos(a-ib)}{\cos(a-ib)}$$

$$= \frac{\sin(a+ib)\cos(a-ib)}{\cos(a+ib)\cos(a-ib)}$$

Multiple & divide by 2

$$x+iy = \frac{2\sin(a+ib)\cos(a-ib)}{2\cos(a+ib)\cos(a-ib)}$$

Scanned by CamScanner

(22)

$$\boxed{\begin{aligned} 2\sin A\cos A &= \sin(A+B)+\sin(A-B) \\ 2\cos A\cos B &= \cos(A+B)+\cos(A-B) \\ \text{put } A=a+ib,\ & B=a-ib \end{aligned}}$$ formula

$$\therefore\ x+iy = \frac{2\sin(a+ib)\cos(a-ib)}{2\cos(a+ib)\cos(a-ib)}$$

$$= \frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}$$

$$= \frac{\sin(a+\cancel{ib}+a-\cancel{ib})+\sin(\cancel{a}+ib-\cancel{a}+ib)}{\cos(a+\cancel{ib}+a-\cancel{ib})+\cos[(\cancel{a}+ib)-\cancel{a}+ib]}$$

$$= \frac{\sin(2a)+\sin(2ib)}{\cos(2a)+\cos(2ib)}$$

$$x+iy = \frac{\sin 2a + i\sinh 2b}{\cos 2a+\cosh 2b}$$

$$\boxed{\begin{aligned} &\text{Put } \sin 2ib = i\sinh 2b \\ &\cos 2ib = \cosh 2b \end{aligned}}$$

Scanned by CamScanner

23

$x = \dfrac{\sin 2a}{\cos 2a + \cosh 2b}$, $y = \dfrac{\sinh 2b}{\cos 2a + \cosh 2b}$

$$\therefore \frac{x}{y} = \frac{\dfrac{\sin 2a}{\cos 2a + \cosh 2b}}{\dfrac{\sinh 2b}{\cos 2a + \cosh 2b}}$$

$$= \frac{\sin 2a}{\cancel{\cos 2a + \cosh 2b}} \times \frac{\cancel{\cos 2a + \cosh 2b}}{\sinh 2b}$$

$$\boxed{\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}}$$

Problem:

X. Separate into real and imaginary parts

(i) $\tan^{-1}(x+iy)$

(ii) $\sin^{-1}(\cos\theta + i\sin\theta)$

(iii) $\sinh(\alpha + i\beta)$

(iv) $\tanh(1+i)$

Scanned by CamScanner

Solu :

Case (i) Let $\tan^{-1}(x+iy) = A+iB$

Put $\tan^{-1}(x+iy) = A+iB$

$$\boxed{(x+iy) = \tan(A+iB)} \quad \text{--- (1)}$$

$$\boxed{(x-iy) = \tan(A-iB)} \quad \text{--- (2)}$$

Now

$$\tan 2A = \tan[A+iB + A-iB]$$

$$= \frac{\tan(A+iB) + \tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{(x+\cancel{iy}) + (x-\cancel{iy})}{1-[(x+iy)(x-iy)]}$$

$$= \frac{2x}{1-[x^2+y^2]}$$

$$\tan 2A = \frac{2x}{1-x^2-y^2}$$

Scanned by CamScanner

$$2A = \tan^{-1}\left[\frac{2x}{1-x^2-y^2}\right]$$

Real Part A is

$$A = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-x^2-y^2}\right]$$

Now $\tan 2(iB) = \tan\left((A+iB) - (A-iB)\right)$

$$= \frac{\tan(A+iB) - \tan(A-iB)}{1 + \tan(A+iB)\tan(A-iB)}$$

$$= \frac{(\cancel{x}+iy) - (\cancel{x}-iy)}{1 + (x+iy)(x-iy)}$$

$$= \frac{iy+iy}{1+(x^2+y^2)}$$

$$\tan 2(iB) = \frac{2iy}{1+x^2+y^2}$$

Scanned by CamScanner

(26)

$$\tanh 2B = \frac{2y}{1+x^2+y^2}$$

$$2B = \tanh^{-1}\left[\frac{2y}{1+x^2+y^2}\right]$$

The Imaginary Part B is

$$B = \frac{1}{2}\tanh^{-1}\left[\frac{2y}{1+x^2+y^2}\right]$$

Case (ii) Let $\sin^{-1}(\cos\theta + i\sin\theta) = x+iy$

$$(\cos\theta + i\sin\theta) = \sin(x+iy)$$

$$\cos\theta + i\sin\theta = \sin x \cosh y + i\cos x \sinh y$$

$\therefore \cos\theta = \sin x \cosh y$ — (1), $\sin\theta = \cos x \sinh y$ — (2)

(1) & (2) squaring & adding we get

$$\cos^2\theta + \sin^2\theta = (\sin x \cosh y)^2 + (\cos x \sinh y)^2$$

$$1 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$1 = \sin^2 x (1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y$$

$1 = \sin^2 x + \cancel{\sin^2 x \sinh^2 y} + \sinh^2 y - \cancel{\sin^2 x \sinh^2 y}$

$1 = \sin^2 x + \sinh^2 y$

$1 - \sin^2 x = \sinh^2 y$

$\cos^2 x = \sinh^2 y$

$\sqrt{\cos^2 x} = \sinh y$

$\boxed{\cos x = \sinh y}$ — (3)

From (2) & (3) we get

Take eqn (2) $\rightarrow \sin\theta = \cos x \sinh y$ $\quad [\because \sinh y = \cos x]$

$= \cos x \cos x$

$\sin\theta = \cos^2 x$

$\therefore \cos^2 x = \sin\theta$

$\cos x = \sqrt{\sin\theta}$

Hence the real part is $x = \cos^{-1}\sqrt{\sin\theta}$

Scanned by CamScanner

Take eqn ③ we get

$$\sinh y = \cos x$$

Put $\cos x = \sqrt{\sin\theta}$

$$\sinh y = \sqrt{\sin\theta}$$

Hence the Imaginary part is

$$y = \sinh^{-1}(\sqrt{\sin\theta})$$

Case (iii) Let $x+iy = \sinh(\alpha+i\beta)$

$$\therefore x+iy = \frac{1}{i}\sin[i(\alpha+i\beta)]$$

$$= -i\sin(i\alpha+i^2\beta)]$$

$$= -i\sin(i\alpha-\beta)$$

$$= -i[\sin(i\alpha)\cos\beta - \cos(i\alpha)\sin\beta]$$

$$= -i[i\sinh\alpha\cos\beta - \cosh\alpha\sin\beta]$$

$$= -i^2\sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$$

$$= -(-1)\sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$$

Scanned by CamScanner

$$x+iy = \sinh\alpha \cos\beta + i \cosh\alpha \sin\beta$$ (29)

Real part is

$$x = \sinh\alpha \cos\beta$$

Imaginary part is

$$y = \cosh\alpha \sin\beta$$

Scanned by CamScanner

Case:

(iv) $\tanh(1+i) = x+iy$

$$x+iy = \tanh(1+i)$$

$$= \frac{\sinh(1+i)}{\cosh(1+i)}$$

$$= -i\,\frac{\sin i(1+i)}{\cos i(1+i)}$$

$$= -i\,\frac{2\sin(i-1)\cos(i+1)}{2\cos(i-1)\cos(i+1)}$$

$$= -i\left[\frac{\sin 2i - \sin 2}{\cos 2i + \cos 2}\right]$$

$$= -i\,\frac{(i\sinh 2 - \sin 2)}{\cosh 2 + \cos 2}$$

$$= \frac{-i^2\sinh 2 + i\sin 2}{\cosh 2 + \cos 2}$$

$$= \frac{-(-1)\sinh 2 + i\sin 2}{\cosh 2 + \cos 2}$$

Scanned by CamScanner

$$x+iy = \frac{\sinh 2 + i\sin 2}{\cosh 2 + \cos 2}$$

$$\text{Real Part} = \frac{\sinh 2}{\cosh 2 + \cos 2}$$

$$\text{Imaginary Part} = \frac{\sin 2}{\cosh 2 + \cos 2}$$

Unit - V Completed

Scanned by CamScanner