## ALGEBRA, ANALYTICAL GEOMETRY 3D AND TRIGONOMETRY

SUBJECT CODE: 16SACMM2

## **UNIT-V**

## **TOPIC: HYPERBOLIC FUNCTIONS**

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Subject: Algebra, Analytical Geometry (39) &  
UNIT-Y  
Hyperbolic Functions  
Result:)  
Cosh<sup>2</sup>x - sinh<sup>2</sup>x = 1  
Proof:  
Cosh<sup>2</sup>x - sinh<sup>2</sup>x = 
$$\left(\frac{e^{x} + e^{-x}}{2}\right)^{2} \left(\frac{e^{x} - e^{-x}}{2}\right)^{2}$$
  
 $= \left(\frac{e^{2x} + \partial e^{x} e^{-x} + e^{-2x}}{4}\right) \left(\frac{e^{2x} - \partial e^{-x} e^{-x}}{4}\right)^{2}$   
 $= \left(\frac{e^{2x} + \partial e^{x} e^{-x} + e^{-2x}}{4}\right) \left(\frac{e^{2x} - \partial e^{-x} e^{-x}}{4}\right)^{2}$   
 $= \left(\frac{e^{2x} + \partial e^{-2x}}{4}\right) - \left(\frac{e^{2x} - \partial e^{-x} e^{-x}}{4}\right)^{2}$   
 $= \frac{1}{4} \left(\frac{e^{2x} + \partial e^{-2x}}{4}\right) - \left(\frac{e^{2x} - \partial e^{-2x} e^{-2x}}{4}\right)^{2}$   
 $= \frac{1}{4} \left(\frac{e^{2x} + \partial e^{-2x}}{4}\right) - \left(\frac{e^{2x} - \partial e^{-2x} e^{-2x}}{4}\right)^{2}$   
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Proof  
ginh 
$$dx = dsin hx coshx$$
  
proof  
 $gsinhx coshx = d\left[\left(\frac{e^{x} \cdot e^{x}}{2}\right)\left(\frac{e^{x} + e^{x}}{2}\right)\right]$   
 $= d\left[\left(\frac{e^{x} - e^{-x}\right)\left(e^{x} + e^{-x}\right)}{4}\right]$   
 $= \frac{d}{h}\left[\left(e^{x} - e^{-x}\right)\left(e^{x} + e^{-x}\right)\right]$   
 $= \frac{1}{d}\left[\left(e^{x}\right)^{2} - \left(e^{-x}\right)^{2}\right]$   
 $= \frac{1}{d}\left[\left(e^{x}\right)^{2} - \left(e^{-x}\right)^{2}\right]$   
 $= \frac{1}{d}\left[e^{2x} - e^{-2x}\right]$   
 $= sinh2x$   
 $dsin hx coshx = sinh dx$ 

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Relation blu hyperbolic functions & Circular trigonometric functions. Theorem ! (i) Sin (in) = isinha (ii) coscine) = coshn (iii) tan (ix) = itanhx G Proof '. case (i) sin(ix) = isinhx We know  $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$  $Sin(ix) = (ix) - (ix)^3 + (ix)^5$ 31 51  $=(ix) - \frac{i^3 x^3}{3!} + \frac{i^5 x^5}{5!} - \cdots$ Puf c = -1ig=-i  $2ix - \frac{(-i)x^3}{31} + \frac{ix^5}{51}$ .5 0  $= ix + \frac{ix^{3}}{3!} + \frac{ix^{5}}{5!}$  $= i \left[ x + \frac{x^3}{21} + \frac{x^5}{51} + \cdots \right]$ 

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(i)  
Sin (ix) = i sin hx  
case (ii) 
$$\cos(ix) = \cosh x$$
  
We know  $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}$   
 $\cos(ix) = 1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!}$   
 $= 1 - \frac{i^2 x^2}{2!} + \frac{i^4 x^4}{4!}$   
 $= 1 - \frac{i^2 x^2}{2!} + \frac{i^4 x^4}{4!}$   
 $= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$   
 $= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$   
 $= \cosh x$   
 $\cos(ix) = \cosh x$   
(iii)  $\tan(ix) = \frac{\sin(ix)}{\cosh x}$   
 $= i \frac{\sinh x}{\cosh x}$  (previous  
 $\frac{\cos hx}{\cosh x}$ 

(i) 
$$\sinh^{2}x = \log e \left( x + \sqrt{x^{2} + 1} \right)$$
  
(ii)  $\sinh^{2}x = \log e \left( x + \sqrt{x^{2} + 1} \right)$   
(iii)  $\cosh^{2}x = \log e \left( x + \sqrt{x^{2} - 1} \right)$   
(iii)  $\tanh^{2}x = \frac{1}{2} \log \left( \frac{1 + x}{1 - x} \right)$   
Proof:  
(i)  $\sinh^{2}x = \frac{1}{2} \log \left( \frac{1 + x}{1 - x} \right)$   
Proof:  
(i)  $\sinh^{2}x = \frac{1}{2} \log \left( \frac{1 + x}{1 - x} \right)$   
 $y = \sinh^{2}x$   
 $y = \frac{1}{\sin h} x$   
 $\sin hy = x$   
 $e^{y} - e^{-y} = x$   
 $e^{y} - e^{-y} = 2x$   
 $e^{y} - \frac{1}{2} = 2x$   
 $e^{y} - 1 = 2x e^{y}$   
 $e^{y} - 1 = 2x e^{y}$   
 $e^{y} - 1 = 0$   
which is a quadratic equation in ey

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$$e^{y} = -b \pm \sqrt{b^{2} - hqc}$$

$$a = 1$$

$$b = -2x$$

$$a = 1$$

$$= -(-2x) \pm \sqrt{(-2x)^{2} - H(1)(-1)}$$

$$= \frac{2x}{2x} \pm \sqrt{Ax^{2} + A}$$

$$= \frac{2x}{2} \pm \sqrt{Ax^{2} + A}$$

$$= \frac{2x}{2} \pm \sqrt{x^{2} + 1}$$

$$= \frac{2}{2} \left[ x \pm \sqrt{x^{2} + 1} \right]$$

$$e^{y} = x \pm \sqrt{x^{2} + 1}$$

$$bg \text{ on both Sides}$$

$$bg(e^{x}) = bge(x + \sqrt{x^{2} + 1})$$

$$y = bge(x + \sqrt{x^{2} + 1})$$

$$f = bge(x + \sqrt{x^{2} + 1})$$

$$f = bge(x + \sqrt{x^{2} + 1})$$

(ii) Let g=costila  $y = \frac{1}{\cosh} \alpha$ Coshy = x ey te-y = x sca / t x c 2  $e^{y}+e^{-y}=2x$  $e^{y} + \frac{1}{e^{y}} = ax$ eyxey tit x = 2% 04 e<sup>2y</sup> +1 -= 2x x / 1 x - 16 04 e<sup>2</sup>y +1 = 2xe<sup>y</sup> e<sup>2</sup>y - 2xey +1 =0 which is a guadratic equation in et

ey = -6 ± 162-49C 8 a=1 21a b=-27 C  $= -(-2\pi) \pm \sqrt{(-2\pi)^2 - 4(1)(1)}$ Jx1 = 21x + 1 4x2-4 2  $= 2x \pm \sqrt{4(x^2 - 1)}$  $= 22 \pm 21 = 2^{2} - 1$  $= \cancel{2} \underbrace{2 \pm \sqrt{2^2 - 1}}_{\cancel{2}}$  $e^{\gamma} = \chi \pm \sqrt{\chi^2 - 1}$ Take log on both sides  $lg(q^{4}) = log_{e}(x \pm \sqrt{x^{2}-1})$  $y = log_{e}(x \pm \sqrt{x^{2}-1})$ 

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·· Costin = Loge(n+ $\sqrt{n^2-1}$ ) (y=costin) (9) (iii) y=tanhtx  $y = \frac{1}{tanh} \chi$ H\_SCH/ + xB = tounhy = x eJ\_0-4 (1- s) H / t sis = 2 ey+p-y  $e^{y} - e^{-y} = \chi(e^{y} + e^{-y})$  $e^{y} - \frac{1}{e^{y}} = \chi \left( e^{y} + e^{-y} \right)$ exey-1  $= \chi (e^{y} + e^{-y})$ y - 2x 1 1 x = 1 3c - 1 24  $= \chi \left( e^{y} + e^{-y} \right) \circ \chi$  and oy  $= ke^{y} \left( e^{y} + e^{-y} \right)$ 

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x e<sup>2y</sup> + xe<sup>y</sup>e<sup>y</sup>  $e^{2y} - 1 = \chi e^{2y} + \chi$ = 2 2 + 2 24 -24 - xey = 1+x  $e^{29}(1-3e)^{-1} = 1+3e^{-1}(0)^{-1}$  $\frac{1}{e} = \frac{1+x}{1-x}$ (di) 20) by on both sides  $log (2y) = log \left(\frac{1+x}{1-x}\right)$  $2y = \log_e\left(\frac{1+\chi}{1-\chi}\right)$  $y = \frac{1}{2} \log \left( \frac{1+\chi}{1-\chi} \right)$  $tanh x = \frac{1}{2} loge(\frac{1+x}{1-x})$ 



Problem :

If  $x + iy = Sin (A + iB) p.T - \frac{x^2}{Sin^2 A} - \frac{x^2}{Cos^2 A} = 1$ 

Solu !

x+iy = Sin(A+iB)

= Sin A CosiB + cosA Sin(iB) x+iy = Sin A cosh BticosA sinh B [cosiB=cosh B siniB=sinh B

X = SinA CoshB, J=CosA SinhB

$$\frac{\chi^2}{SinA} - \frac{\chi^2}{CosA} = \frac{(SinA coshB)^2}{SinA} \frac{(cosASinhB)}{cosA}$$

= SintAcosh2B \_ CostAsinh2B SintA \_ CostA

= cosh2B - sinh2B

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(i) 
$$\frac{x^2}{\cos^2 B} + \frac{y^2}{\sin^2 B} = 1$$
  
Already to find  
 $x = \sin A \cosh B$ .  $y = \cos A \sinh B$   
We have  $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \left[\frac{\sin A \cosh B}{\cosh^2 B}\right]^2 + \left(\frac{\cos A \sinh B}{\sin^2 B}\right]^2$   
 $= \frac{\sin^2 A \cosh^2 B}{\cosh^2 B} + \frac{\cos^2 A \sinh^2 B}{\cosh^2 B}$   
 $= Sin^2 A + \cos^2 A$   
 $= 1$   
 $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$ 

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= tanhB [ Sind + Cosa Cosa + Sind sech B a Colar tanh B = tanhB [ sind + cosd) sind cosd Sech<sup>2</sup>B - (8+A) -= tanhB alistant Sin2d+cosd=) sindcosa card barbs + cotort Sech B tan hB x j Sindlasd 13 sech 2 B SinhB / CoshB Sindcosd cond tarbs + cot of to cosh<sup>2</sup>B - SinhB x 1 x coshB CoshB Singcosd

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- Simp Cosh B  $\therefore x = Seco$ Apply the formula, we get u = cosh (Seco) = loge  $\left[ x + \sqrt{x^2} - 1 \right]$ = loge Seco+ Vsec20-1) = loge Seco + tano] [vsec=0-1 = tano  $2\log\left[\frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}\right]$ = loge (1+sind) coso = loge [ 1 + 2tan0/2 1+tan20/2 1-tan20/2 Sin 20 = 2 tano 1+tan COS20 = 1-tant 1+tan 0/2 (than 0/2+ 2tan 0/2 -1+tan 0/2 = loge 1200 FOR

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, loge 1+tan2 + 2tan2 1-tan2 tang +2tang+1 1-tang 2 = loge (1+tang) (1+tang) (1-tang) tan 0+2+anot) = loge (thang) 1-tang=(1+tan) = loge 1+tano 1-tano Put =1=tan Ty = loge (tan 17/4 + tan 0/2 1-(tan 17/4) (tan 0/2) toen (10/4+0/2)) u = loje Conversely let  $u = \log_e \tan\left(\frac{\pi}{4} + \frac{0}{2}\right)$ Take exponential on both sides e<sup>u</sup> = \$ loge tan (17/4+0/2)

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opal  $e^{u} = tan\left(\frac{\pi}{4} + \frac{0}{2}\right)$  $e^{\frac{u}{2}} \times e^{\frac{u}{2}} = \tan \frac{\pi}{10} + \tan \frac{0}{2}$ 1 - tan I tan Q 02 ... =  $1 + \tan \frac{Q}{2}$ - 4. 1-tan@  $\frac{y}{e^2} - \frac{y}{e^2}$ = X+tan 0/2 - [X-tan 0/2]-02 + 0 Z 1+tan 0/2+[1-tan 0/2 tanh (1/2) = tan 0/2 + tan 0/2 tanh (1/2) = 2/tan 0/2 1al - 13  $\tanh\left(\frac{w}{2}\right) = \tan \frac{\sigma}{2}$  $\cosh \mu = 1 + \tanh^2(\mu/2)$ 1+tan2(0/2)  $\frac{1-\tanh^2(u/2)}{\cosh u} = \frac{1-\tanh^2(0/2)}{1-\tan^2(0/2)}$ 

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Sinhy - Cosx Sinhy + Los'a cosh problem If cos(x+iy) = coso + ising Prove that  $\cos 2\pi + \cosh 2y = 2$ Soly:  $= \cos(x + iy)$ Loso +isino = cospe) cos(iy) - sin(x) sin(iy) = cosx coshy - sinx (isinhij Cosotisino = coszcoshy - isinz sinhy Take Real of Imaginary parts Coso = Cosx Coshy, Sino =- Sinx sinhy W.K.T  $sin^2 \theta + cos^2 \theta = 1$ (-sinx sinhy)+ (cosxcoshy) = 1 Sin'x Sinh<sup>2</sup>y + cos<sup>2</sup>x cosh<sup>2</sup>y = (1-cos2) sinh2y + cos22 coshy=1

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=> sinhy - cosx sinhy + last coshy = 1 => Sinhy + cosx (coshy - Sinh2y) =1  $\Rightarrow$  sinh<sup>2</sup>y + cos<sup>2</sup>x (1) = 1 = sinh<sup>2</sup>y + cos<sup>2</sup>x = 1 Cosh 2y-1. + Cosh 2y-1. + Cos 2n 2 2 2 2 2 = 1 Coshay - X + X + cosan stag 200 port + = 1 eros  $\cosh 2y + \cos 2x = 1(2)$  $\cosh 2y + \cos 2x = 2$ (-Sinx Sinhy)+ (cosxcosny) = 1 = 15 dead x 200 1. 10 date x 6012 (= Kyson x 500 + Fyris (x)

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(A- B) and 1- (2+B) - 1 Sin (B- B) problem: 3 (J-A)200+(S+A)200 but Acatib. B= a - ib If  $\tan(a+ib)=(x+iy) P.T \frac{x}{y} = \frac{\sin 2a}{\sin h2b}$ Solu ! (di-0)200 (dipp) 20016 xtig = tan (atib) = Sin(a+ib)Cos (atib) Conjugate Nr & Dr by cos(a-ib)  $\frac{\sin(a+ib)}{\cos(a+ib)} \times \frac{\cos(a-ib)}{\cos(a-ib)}$ Sin (a+ib) Cos(a-ib) Cos(a+ib) Cos(a-ib) Pat Sur Multiple of divide by 2  $x + iy = 2i \sin(a + ib)\cos(a - ib)$ 21 cos (atib) cos(a-ib)

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(ad) 2SinA COSA = Sin (A+B) + Sin (A-B) formula 2 COSA COSB = COS(A+B) + COS(A-B) Put A=a+ib, B=a-ib 13 = ton (a+ib)=(x+iy) p. 1 El al x+iy = & sin (a+ib) cos(a-ib) Zicos (artib) cos (a-ib) Sin(A+B) + Sin(A-B)COS (AFB) + COS (A-B) = Sin (a+ib+a-ib)+ Sin (a+ib-a+ib) Cos(a+ib + a-ib) + cos(a+ib) - a+ib = Sin (2a) +1Sin (2ib)  $\cos(2a) + \cos(2ib)$ (di-p)200 (di+p) miz Sin 2 a ti Sin h 2 b Pat Singih= Xtiy = 2 Sught Cos2a + cosh2b Cosdib= Cash2 a Sin (a+1b) Costa-ib) (dip) 200 (dit PILE

x = Sin 29  $y = \frac{\sinh 2b}{\cos 2q + \cosh 2b}$ cos 2a t cosh 2b Sin29 Cossa+cosh2b Sinh 2h Cas2a+cosh2b Costat Cosh2b = Sin2a Cossationshab Sinhab - Sin 2a | X y sinhab (Ri-x) + (Ri+x) [(Ki-x) (Ki+x) ]-1 Problem ! Separate into real and imaginary posts (i) tan (x+iy) (iv) tanh (1+i) (ii) Sun (coso +isino) = AC ast iii) sinh (a+iB)

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Solu 1 and

Case (i) let tan (x+iy) = A+iB

Put tan'(x+iy) = A+iB [(x+iy) = tan(A+iB)] - (D)[(x-iy) = tan(A-iB)] - (D)

Now

Parite

$$= \frac{\tan (A + iB) + \tan (A - iB)}{1 - \tan (A + iB) \tan (A - iB)}$$

$$(x+iy) + (x-iy)$$

22

- [n2+y2] (Mitro) land

2 (gi+ip) dais

tan 2A =

HINDOW

(it) dast (r)

: mald

Pasiate

27

 $2A = \tan^{-1} \frac{2x}{1-x^2-y^2}$ Real Part A is  $A = \frac{1}{2} \tan \left[ \frac{2\chi}{1-\chi^2-y^2} \right]$ Now  $\tan 2(iB) = \tan ((A+iB) - (A-iB))$ (pits)=istan (A+iB) - tan (A-iB) A-iB) tan (A+iB) tan (A+iB) tan (A-iB)  $= (\chi + i y) - (\chi - i y)$ 1 + (x+iy) (x-iy) sog sed predes = iy+iy (1102 x 200) + (1200 1 + (2+42) 2 + 920) 1 + (2+42)  $\tan 2(iB) = \frac{2iy}{1+x^2+y^2}$ 

 $tanh \ 2B = 2g$ 1+2+42  $2B = \tanh \left[ \frac{2y}{1+x^2+y^2} \right]$ The Imaginary Part Bis B = 1 tanh [1+2+1/2] (ai. A) - (Bi+A) and - (A-iB) - (A-iB) Case (ii) Let Sin ( Coso + i sino) = xitiy (COSO + isino) = sin (x+iy) Loso tisino = sinx coshy + i cosx sin hy : Caso = Sinx cashy , Sino = Casa sinhy (Hi-x) (Pi+x) + 1 (1) + (2) squaring & adding we get losto + sinto = (Sinx coshy) + (cosx sinhy) = surt coshy+ cosx sunty = sin2x (Itsinhy)+(I-sizz) sinhy

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1= Sintx + Sinte sinhy + Sinhy - Singsinh 1 = Sinter + Sinhey 1-Sin2x = Sinhy Cosn = Sinhy V cos 2 = sin hy Cosx = Sinhy ] From (2) of (3) We get Take Z-Sin Q = Cosx Sin hy [-: sin hy = cosn = Cosn Cosn  $\sin \theta = \cos^2 \pi$ : cotre = Sino Cosn = Vsino Hence the real part is n= cost vision Swholass + 1 coshq

Take eqn (3) we get  
Sin hy = cosx.  
Sin hy = 
$$\sqrt{8in0}$$
  
Hence the Imaginary part is  
 $y = sinh^{-1}(\sqrt{sin0})$   
Cobe(111) Let  $x+iy = sinh(x+ip)$   
 $= -i sin(ix+ip)$   
 $= -i sin(ix+ip)$   
 $= -i sin(ix+ip)$   
 $= -i (sin(ix)cosp - cos(ix)sin)$   
 $= -i (sinhd cosp - coshar sinp)$   
 $= -i (sinhd cosp + i coshar sinp)$   
 $= -i -1) sinhd cosp + i coshar sinp$ 

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x + iy = sichd cosp + i coshd sing Real Part is x = Sinha Cosp (in) and (I+I) d miz Imaginary part is y = cosha sing 200

Conse:  
(iv) 
$$\tanh(1+i) = x+iy$$
  
 $x+iy = \tanh(1+i)$   
 $= sin h(1+i)$   
 $cosh(1+i)$   
 $= -i \frac{sin i(1+i)}{cos i(1+i)}$   
 $= -i \frac{sin (i-i) \cos(i+i)}{2cos(i-i)cos(i+i)}$   
 $= -i \frac{sin 2i - sin 2}{cos2i + cos2}$   
 $= -i \frac{(i sin h 2 - sin 2)}{cosh 2 + cos 2}$   
 $= -i \frac{2}{cosh 2 + cos 2}$   
 $= -(-1) sin h 2 + isin 2$   
 $cosh 2 + cos 2$   
 $= -(-1) sin h 2 + isin 2$ 

Sinh2 tisin2 ntiy CoSh2 + COS2 Sinh 2 Real Part Cosh2+ cos2 Sin 2 Imaginary Part coshet cose Uni Completed E + 200 +