Algebra, Analytical Geometry and Trigonometry

Subject Code: 16SACMM2

Unit I

Binomial, Exponential and Logarithmic Series, Summation & approximation related Problems

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UNIT – I

Binomial, Exponential and Logarithmic series (Formulae only) – summation & approximation related problems only.

Binomial Series:-

$$(x+a)^n = x^n + nc_1\, x^{n-1} a + nc_2\, x^{n-2} a^2 + \dots + nc_r\, x^{n-r} a^r + \dots + n x a^{n-1} + a^n;$$

is called <u>Binomial Series</u>.

<u>General term</u> is $U_{r+1} = nc_r\, x^{n-r} a^r$.

Note:-

(i) The expansion consists of $(n+1)$ terms.

(ii) The numbers $nc_0, nc_1, \dots nc_n$ are called Binomial co-efficients. These are all integers since nc_r is the number of combinations of n things taken r at a time.

(iii) $\because nc_0 = nc_n;\ nc_1 = nc_{n-1} \dots\ nc_r = nc_{n-r}$

the co-efficients of terms equidistant from the begenging and end of the expansion are equal

Problems:

1. Use the binomial series to find 7^{th} power of 11

Soln:

$$11^7 = (10+1)^7$$

$$= 10^7 + 7c_1\, 10^6 + 7c_2\, 10^5\, 1^2 + 7c_3\, 10^4\, 1^3 + 7c_4\, 10^3\, 1^4 + 7c_5\, 10^2\, 1^5 + 7c_6\, 10\, 1^6 + 7c_7\, 1^7$$

$= 1,00,00,000 + 7 \times 1000000 + \frac{7 \times \cancel{6}^{3}}{1 \times \cancel{2}} 100000 + \frac{7 \times 6 \times 5}{1 \times \cancel{2 \times 3}} 1000$

$+ \frac{7 \times \cancel{6} \times 5 \times \cancel{4}}{1 \times \cancel{2 \times 3} \times \cancel{4}} 1000 + \frac{7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times 3}{1 \times 2 \times 3 \times \cancel{4} \times \cancel{5}} 100 + \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 10}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$

$+ 1.$

$= 1,00,00,000 + 70,00,000 + 2100000 + 350000 + 35000$

$+ 2100 + 70 + 1.$

$= 19,4,87,171$

2. write the middle term of $\left(\frac{y\sqrt{x}}{5} - \frac{5}{x\sqrt{y}}\right]^{12}$

Soln:-

$\therefore$ $n = 12$, there are 13 terms.

Middle term $= 7$.

$U_{r+1} = nC_r \; x^{n-r} \; a^{r}.$

put $r = 6$

$u_7 = 12C_6 \left(\frac{y\sqrt{x}}{5}\right)^{12-6} \left(\frac{-5}{x\sqrt{y}}\right)^{6}.$

$= \frac{\cancel{12}^{2} \times 11 \times \cancel{10}^{3} \times \cancel{9}^{4} \times \cancel{8} \times 7}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times \cancel{6}} \left(\frac{y\sqrt{x}}{5}\right)^{6} \left(\frac{-5}{x\sqrt{y}}\right)^{6}$

$= 1848 \cdot \frac{y^6 x^3}{\cancel{5^6}} \frac{\cancel{5^6}}{x^6 y^3}$

$= 1848 \frac{y^3}{x^3}.$

$\frac{24 \times 7}{168 \times 11}$
1848

write down the middle term in the expansion of $\left(2x - \frac{3}{x}\right)^{15}$

Soln:

G.T $n = 15$ there are 16 terms.

$\therefore$ middle term $= 7 \,\&\, 8 \,\&\, 9$.

$$U_{r+1} = nc_r \, x^{n-r} \, a^r$$

put $r = 7$, $U_8 = 15c_7 (2x)^{15-7} \left(-\frac{3}{x}\right)^7$

$$= 15c_7 \, (2x)^8 \left(-\frac{3}{x}\right)^7$$

$$= -15c_7 \, 2^8 \, x^8 \, \frac{3^7}{x^7}$$

$$= -15c_7 \times 2^8 \times 3^7 \times x.$$

put $r = 8$, $U_9 = 15c_8 \, (2x)^{15-8} \left(-\frac{3}{x}\right)^8$

$$= 15c_8 \, 2^7 \, x^7 \, \frac{3^8}{x^8}$$

$$= 15c_8 \times \frac{2^7 \times 3^8}{x}$$

4\. Find the co-efficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Soln:-

G.T $\left(x^4 - \frac{1}{x^3}\right)^{15} \Rightarrow n = 15$

$$U_{r+1} = 15c_r\,(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r$$

$$= 15c_r\, x^{60-4r}\left(\frac{(-1)^r}{x^{3r}}\right).$$

$$= 15c_r\,(-1)^r\, x^{60-7r}.$$

$60-7r = 32$

$7r = 60-32 = 28$

$\boxed{r=4}$

$$\begin{array}{r} 91 \\ 15 \\ \hline \cancel{15} \\ 1365 \end{array}$$

$$\therefore U_5 = 15c_4\,(-1)^4\, x^{32}.$$

$$= \frac{15\times \overset{7}{\cancel{14}}\times 13\times \cancel{12}}{1\times\cancel{2}\times\cancel{3}\times\cancel{4}}\, x^{32}$$

$$= 1365 .\, x^{32}$$

$\therefore$ co.efficient of $x^{32} = 1365.$

5. If a_r be the co.efficient of x^r in the expansion $(1+x)^n$. P.T

$$\frac{a_1}{a_0} + 2\frac{a_2}{a_1} + 3\frac{a_3}{a_2} + \dots + n\frac{a_n}{a_{n-1}} = \frac{n(n+1)}{2}$$

Soln:

General term of $(1+x)^n = U_{r+1} = nc_r\, 1^{n-r}\, x^r$

$$= nc_r\, x^r.$$

co-efficient of $x^r = {}^nc_r$

$$a_r = {}^nc_r$$

$$a_{r-1} = {}^nc_{r-1}$$

$$\frac{a_r}{a_{r-1}} = \frac{{}^nc_r}{{}^nc_{r-1}}$$

$$= \frac{\cancel{n!}}{r!\,(n-r)!}\;\frac{(n-r+1)!\;(r-1)!}{\cancel{n!}}$$

$$= \frac{n-r+1}{r}.$$

$$r\frac{a_r}{a_{r-1}} = n-r+1 \quad \text{——} \; 1$$

put $r = 1, 2, \ldots n$ in 1

$$\frac{a_1}{a_0} = n$$

$$2\frac{a_2}{a_1} = n-1$$

$$3\frac{a_3}{a_2} = n-2.$$

$\vdots$

$$n\frac{a_n}{a_{n-1}} = 1.$$

Add above equations,

$$\frac{a_1}{a_0} + 2\frac{a_2}{a_1} + 3\frac{a_3}{a_2} + \ldots + n\frac{a_n}{a_{n-1}} = n + (n-1) + \ldots + 1$$

$$= \frac{n(n+1)}{2}$$

6\. S.T $2^{2n}-3n-1$ is divisible by 9 for all +ve integeal values of n.

Soln:-

$$2^{2n}-3^{n}-1 = (2^2)^n - 3n - 1.$$
$$= 4^n - 3n - 1$$
$$= (3+1)^n - 3n - 1.$$
$$= 3^n + nC_1\, 3^{n-1} + nC_2\, 3^{n-2} + \dots + nC_{n-2}\, 3^2 + nC_{n-1}\, 3 + 1 - 3n - 1$$
$$= 3^n + nC_1\, 3^{n-1} + nC_2\, 3^{n-2} + \dots + nC_{n-2}\, 3^2 + 3n + 1 - 3n - 1$$
$$= 3^n + nC_1\, 3^{n-1} + nC_2\, 3^{n-2} + \dots + nC_{n-2}\, 3^2$$

∵ Each term is divisible by $3^2 = 9$.

∴ $2^{2n}-3n-1$ is divisible by 9 ∀ +ve integral values of n.

1\. write down the expansion of

(i) $(3x+5y)^5$ (ii) $\left(4x-\frac{3}{y}\right)^7$ (iii) $\left(2x^2-\frac{3}{x}\right)^6$

(iv) $\left(2x^{1/2}-\frac{3}{x^{1/2}}\right)^4$

$n-2)+$

2\. Find

(i) 6th term of $(3-4x)^{33}$

(ii) 10th term of $\left(\frac{a}{2b}-\frac{2b}{a^2}\right)^{14}$

3. Using Binomial thm., find

(i) $(1.01)^5$ to three decimal places

(ii) $(9.999)^4$ to 4 places of decimal.

4. write down the middle term

(i) $\left(3x - \frac{1}{2x^2}\right)^8$ (ii) $\left(x - \frac{2}{x}\right)^{12}$ (iii) $\left(\frac{b}{x} + \frac{x}{b}\right.$

5 write down two middle term

(i) $(x-2y)^{13}$ (ii) $\left(2x^2 + \frac{y}{x}\right)^9$

6. S.T the middle term of $\left(x - \frac{1}{x}\right)^{2n}$ is

$$\frac{(-2)^n \, 1.3.5 \ldots (2n-1)}{n!}$$

soln:-

Here $n = 2n$

$\therefore$ middle term $= (n+1)^{th}$ term

$$u_{r+1} = 2n\,c_r \, x^{2n-r} \left(-\frac{1}{x}\right)^r$$

$$u_{n+1} = 2n\,c_n \, x^{2n-n} \left(-\frac{1}{x}\right)^n$$

$$= \cancel{2n(2n-1)(2n-2)(2n-3)\ldots 4\times3\times2\times}$$

$$= \frac{2n!}{n!\,(2n-n)!} \, x^n \left(-\frac{1}{x}\right)^n$$

$$= (-1)^n \frac{1.2.3 \ldots (2n-1)(2n)}{(1.2.3 \ldots n)\; n!}$$

$$= (-1)^n \frac{2[1.2 \ldots n][1.3.5 \ldots 2n-1]}{1.2 \ldots n \;\; n!}$$

$= (-2)^n \dfrac{(1.3.5 \ldots 2n-1)}{n!}$

7 S.T the middle term in the expansion of $(1+x)^{2n}$

is $\dfrac{1.3.5 \ldots (2n-1)}{n!} 2^n x^n$

8. Find the co-efficient of

(i) x^{11} in the expansion of $\left(x + \frac{2}{x^2}\right)^{17}$

(ii) x^{16} in $(1+x+x^2)(1-x)^{15}$

(iii) x^{10} in $(1+x+x^2+x^3)^n$

(iv) x^5 in $(1-2x+3x^2)^5$

Binomial series

If n is a rational number and $-1<x<1$

then,

$$(1+x)^n = 1 + nc_1 x + nc_2 x^2 + nc_3 x^3 + \ldots + nc_r x^r + \ldots + \infty \quad \text{— (1)}$$

This infinite series is also called Binomial series.

Note:

It can be shown that the series is also convergent to the sum $(1+x)^n$ in the following cases

(i) If $x = 1$ and $n > -1$

(ii) If $x = 1$ and $n > 0$.

The general term of the series is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

Formulae:-

1. $(1-x)^{-1} = 1+x+x^2+\dots+x^n+\dots\infty$

2. $(1+x)^{-1} = 1-x+x^2-\dots+(-1)^n x^n+\dots\infty$

3. $(1-x)^{-2} = 1+2x+3x^2+\dots+(n+1)x^n+\dots\infty$

4. $(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots+(-1)^n(n+1)x^n+\dots\infty$

5. $(1-x)^{-3} = \frac{1}{2}[1.2+2.3x+3.4x^2+\dots+(n+1)(n+2)x^n+\dots+\infty$

6. $(1+x)^{-3} = \frac{1}{2}[1.2-2.3x+3.4x^2+\dots+(-1)^n(n+1)(n+2)x^n$

7. $(1-x)^{-4} = \frac{1}{6}[1.2.3+2.3.4x+3.4.5x^2+\dots+(n+1)(n+2)(n+3)x^n+\dots\infty]$

8. $(1+x)^{-4} = \frac{1}{6}[1.2.3+2.3.4x+3.4.5x^2+\dots+(-1)^n(n+1)(n+2)(n+3)x^n\dots$

In general,

$$(1-x)^{-p/q} = 1+\frac{(+p/q)}{1}\left(\frac{+x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3\dots$$

$$(1+x)^{-p/q} = 1-\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2-\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\dots\infty$$

$$(1-x)^{-n} = 1+\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\dots\infty$$

$$(1+x)^{-n} = 1-\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\dots\infty$$

$$(1+x)^n = 1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

$$(1-x)^n = 1 - \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Thm:-

If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \infty$ is absolutely convergent for $|x| < 1$ then $a_0 + a_1 + a_2 + \cdots + a_r$ is equal to the co.efficient of x^r in the expansion of $\frac{f(x)}{1-x}$.

Summation of Binomial series:

Given a binomial series we can find its sum by expressing the given series in the form

$$1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \cdots$$

$$= (1-x)^{-p/q}$$

Standard form:-

i) It has to begin with 1 as its first term.

ii) In each term starting from the second, the factors in the numerator form an A.P and the factors in the Dr also form an A.P

iii) The no. of factors in the Nr should be the same as the no. of factors in Dr

(iv) ~~Every~~ Express the denominators in factorial term.

v) Then identify the given series with the series expansion for $(1-x)^{-p/q}$ to get the required sum

~~In this~~ chapter ~

Problem:

1. Find the sum to infinity of the series

$$1+\frac{2}{6}+\frac{2\cdot 5}{6\cdot 12}+\frac{2\cdot 5\cdot 8}{6\cdot 12\cdot 18}+\dots = 4^{1/3}$$

2. Sum to infinity the series $\frac{2\cdot 4}{3\cdot 6}+\frac{2\cdot 4\cdot 6}{3\cdot 6\cdot 9}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 6\cdot 9\cdot 12}$

$p+q=4$, $q=1$, $\frac{x}{q}=\frac{1}{2}$ $= 4/3$

3. Sum to ∞ : $\frac{1}{3}\left[\frac{3\cdot 4}{2\cdot 4}+\frac{4\cdot 5}{2\cdot 4\cdot 6}+\frac{4\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 8}+\dots\right] = \frac{11}{6}$

4. Sum to ∞ : $\frac{11\cdot 14}{10\cdot 15\cdot 20}+\frac{11\cdot 14\cdot 17}{10\cdot 15\cdot 20\cdot 25}+\dots\infty =$

$$= \frac{1}{8}\left[\frac{5}{2}\left(\frac{25}{4}\right)^{1/3}-\frac{254}{75}\right.$$

$$\frac{3\cdot 4}{2^2\cdot 2!}+\frac{3\cdot 4\cdot 5}{2^3\cdot 3!}+\frac{3\cdot 4\cdot 5\cdot 6}{2^4\cdot 4!}$$

5. Sum to ∞ series $2+\sum_{n=1}\frac{1}{3}\cdot\frac{1}{6^{n-1}}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{\lfloor n}$

Soln:

$$S = 2+\sum_{n=1}^{\infty}\frac{1}{3}\cdot\frac{1}{6^{n-1}}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$$

$$= 2+\frac{1}{3}\left[\frac{1}{1!}+\frac{1\cdot 3}{2!}\,\frac{1}{6}+\frac{1\cdot 3\cdot 5}{3!}\,\frac{1}{6^2}+\dots\right]$$

$$= 2 + \frac{6^2}{3}\left[\frac{1}{1!}\,\frac{1}{6} + \frac{1\cdot 3}{2!}\,\frac{1}{6^2} + \frac{1\cdot 3\cdot 5}{3!}\,\frac{1}{6^3} + \cdots\right]$$

$$= 2\left[1 + \frac{1}{1!}\left(\tfrac{1}{6}\right) + \frac{1\cdot 3}{2!}\left(\tfrac{1}{6}\right)^2 + \cdots\right].$$

$p = 1$ $\quad p+q=3$ $\quad q = 2 \qquad x/q = 1/6 \qquad x = q/6 = 2/6 = 1/3$

$$S = 2\,(1-x)^{-p/q}$$

$$= 2\left(1 - \tfrac{1}{3}\right)^{-1/2}$$

$$= 2\left(\tfrac{2}{3}\right)^{-1/2} = 2\sqrt{3}/\sqrt{2} = \sqrt{3/2}$$

6. S.T

$$1 - \frac{n+x}{1!\,(1+x)} + \frac{(n+2x)(n-1)}{2!\,(1+x)^2} \mp \frac{(n+3x)(n-1)(n-2)}{3!\,(1+x)^3} + \cdots$$

Soln:

$$S = 1 - \frac{n+x}{1!\,(1+x)} + \frac{(n+2x)(n-1)}{2!\,(1+x)^2} \mp \frac{(n+3x)(n-1)(n-2)}{3!\,(1+x)^3} + \cdots$$

$$= \left[1 - \frac{n}{1!\,(1+x)} + \frac{n(n-1)}{2!\,(1+x)^2} \mp \frac{n(n-1)(n-2)}{3!\,(1+x)^3} + \cdots\right]$$

$$- \frac{x}{1!\,(1+x)} + \frac{2x(n-1)}{2!\,(1+x)^2} \mp \frac{3x(n-1)(n-2)}{3!\,(1+x)^3} + \cdots$$

$$= \left(1 - \tfrac{1}{1+x}\right)^n - \frac{x}{1+x}\left[1 - \frac{n-1}{1+x} + \frac{(n-1)(n-2)}{2!}\left(\frac{1}{1+x}\right)^2 - \cdots\right]$$

$$= \left(1 - \frac{1}{1+x}\right)^n - \frac{x}{1+x}\left(1 - \frac{1}{1+x}\right)^{n+1}$$

$$= \left(\frac{x}{1+x}\right)^n - \frac{x}{1+x}\left(\frac{x}{1+x}\right)^{n-1} = 0.$$

7. Assuming the square and highest powers of x ma...
be neglected S.T

$$\frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = 4 - \frac{10x}{3}$$

Soln:-

$$\frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = \frac{(1+x)^{1/2}\, 4^{3/2}\left(1-\frac{3x}{4}\right)^{3/2}}{8^{1/3}\left(1+\frac{5}{8}x\right)^{1/3}}$$

$$= \frac{\overset{4}{\cancel{8}}\,(1+x)^{1/2}\,(1-3x/4)^{3/2}}{\cancel{2}\,(1+5/8\,x)^{1/3}}$$

$$= 4\left[1+\frac{1}{2}x\right]\left[1-\frac{9}{8}x\right]\left[1-\frac{5}{24}x\right]$$

$$= 4\left[1+\frac{x}{2}-\frac{9x}{8}\right]\left[1-5/24\,x\right]$$

$$= 4\left[1\cancel{+}-\frac{5x}{8}\right]\left[1-5/24\,x\right]$$

$$= 4\left[1-\frac{5x}{8}-\frac{5x}{24}\right]$$

$$= 4\left[1-\frac{\overset{10}{\cancel{20}}x}{\underset{12}{\cancel{24}}}\right]$$

$$= 4 - \frac{10x}{3}$$

8. If c be small in comparision with l show that

$$\left(\frac{l}{l+c}\right)^{1/2} + \left(\frac{l}{l-c}\right)^{1/2} = 2 + \frac{3c^2}{4l^2} \text{ approximately.}$$

Soln:

$$\left(\frac{l}{l+c}\right)^{1/2} + \left(\frac{l}{l-c}\right)^{1/2}$$

$$= \left(\frac{l+c}{l}\right)^{-1/2} + \left(\frac{l-c}{l}\right)^{-1/2}.$$

$$= \left(1+\frac{c}{l}\right)^{-1/2} + \left(1-\frac{c}{l}\right)^{-1/2}$$

$$= 1 - \frac{c}{2l} + \frac{(-1/2)(-3/2)}{2!}\left(\frac{c}{l}\right)^2 + \cdots$$

$$+ 1 + \frac{c}{2l} + \frac{1/2\,(3/2)}{2!}\left(c/l\right)^2 + \cdots$$

$$= 2 + \cancel{2} \times 3/4 \times \frac{1}{\cancel{2}}\left(c/l\right)^2$$

$$= 2 + \frac{3c^2}{l^2}$$

9. If $p-q$ is small compared to p or q s.t

$$\sqrt[n]{p/q} = \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} \text{ approximately.}$$

Find the seventh root of $\frac{131}{132}$

Soln.

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \frac{np+p+nq-q}{np-p+nq+q}$$

$$= \frac{n(p+q) + p-q}{n(p+q) - (p-q)}$$

$$= \frac{n(p+q)\left[\cancel{p+q}\,1 + \frac{p-q}{n(p+q)}\right]}{n(p+q)\left[1 - \frac{p-q}{(p+q)n}\right]}$$

$$= \frac{1 + \frac{p-q}{n(p+q)}}{1 - \frac{(p-q)}{n(p+q)}}$$

$$= \frac{\left[1 + \left(\frac{p-q}{p+q}\right)\right]^{1/n}}{\left(1 - \frac{p-q}{p+q}\right)^{1/n}}$$

$$= \frac{\left(p+q+p-q / p+q\right)^{1/n}}{\left(p+q-p+q / p+q\right)^{1/n}}$$

$$= \left(\frac{\cancel{2}p}{\cancel{2}q}\right)^{1/n}$$

$$\sqrt[7]{\frac{131}{183}} = \frac{8\times 131 + 6\times 133}{6\times 131 + 8\times 133} = \frac{923}{925}$$

Exponential Series.

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \infty$ is called the exponential series.

Note :-

i) $2 < e < 3$

Formullae :

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \dots$$

$$\frac{e - e^{-1}}{2} = 1 - \frac{1}{3!} + \frac{1}{5!} - \dots$$

5. Sum to ∞ the series $\frac{1^2}{1!} + \frac{1^2+2^2}{2!} + \frac{1^2+2^2+3^2}{3!} + \dots$

Soln:-

$$T_n = \frac{1+2^2+3^2+\dots+n^2}{n!}$$

$$= \frac{n(n+1)(2n+1)}{6\,\lfloor n}$$

$$= \frac{(n+1)(2n+1)}{6(n-1)!}$$

A

$$\boxed{(n+1)(2n+1) = A + B(n-1) + C(n-1)(n-2)}$$

put $n=1$ $\boxed{A=6}$

$\cancel{6 = A}$

$n=2$, $15 = A+B$

$B = 15-6 = 9$

put $n=0$, $1 = A - B + 2C$

$2C = 1-6+9$

$2C = 4 \Rightarrow C = 2.$

$$\therefore \frac{(n+1)(2n+1)}{6(n-1)!} = \frac{6}{6(n-1)!} + \frac{9(n-1)}{6(n-1)!} + \frac{2(n-1)(n-2)}{6(n-1)!}$$

$$= \frac{1}{(n-1)!} + \frac{3}{2(n-2)!} + \frac{\cancel{(n-2)}}{3(n-3)!}$$

$T_1 = 1$

$T_2 = \frac{1}{1!} + 3/2$

$$T_3 = \frac{1}{2!} + \frac{3}{2 \cdot 1!} + \frac{1}{3}$$

$$T_4 = \frac{1}{3!} + \frac{3}{2} \cdot \frac{1}{2!} + \frac{1}{3 \cdot 1!}$$

$$\therefore S = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + \left(3/2 + \frac{3}{2} \cdot \frac{1}{1!} + 3/2 \cdot \frac{1}{2!} + \cdots\right)$$

$$+ \left(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{1!} + \cdots\right)$$

$$= e^1 + \frac{3}{2}\left(1 + 1/1! + 1/2! + \cdots\right) + 1/3\left(1 + \frac{1}{1!} + \cdots\right)$$

$$= e + \frac{3}{2} e + \frac{1}{3} e$$

$$= \frac{6e + 9e + 2e}{6} = \frac{17e}{6}.$$

6. Sum to ∞ the series $5 + \frac{2 \cdot 6}{1!} + \frac{3 \cdot 7}{2!} + \frac{4 \cdot 8}{3!} + \cdots$

soln:-

$$S = 5 + \frac{2 \cdot 6}{1!} + \frac{3 \cdot 7}{2!} + \frac{4 \cdot 8}{3!} + \cdots$$

$$= \frac{1 \cdot 5}{0!} + \frac{2 \cdot 6}{1!} + \frac{3 \cdot 7}{2!} + \cdots$$

$$T_n = \frac{n(n+4)}{(n-1)!}$$

$$n(n+4) = A + B(n-1) + C(n-1)(n-2)$$

put $n=1$, $\boxed{5 = A}$

$n=2$, $12 = A+B \Rightarrow \boxed{B=7}$

$n=0$, $0 = A-B+2C$

$2C = -5+7 \Rightarrow C=1$

$$\frac{n(n+4)}{(n-1)!} = \frac{5}{(n-1)!} + \frac{7(n-1)}{(n-1)!} + \frac{(n-1)(n-2)}{(n-1)!}$$

$$T_n = 5\frac{1}{(n-1)!} + \frac{7}{(n-2)!} + \frac{1}{(n-3)!}$$

$$S_n = 5\sum_{n=1}^{\infty}\frac{1}{(n-1)!} + 7\sum_{n=2}^{\infty}\frac{1}{(n-2)!} + \sum_{n=3}^{\infty}\frac{1}{(n-3)!}$$

$$= 5e + 7e + e = 13e.$$

8) Sum to infinity the series $\sum_{n=0}^{\infty} \frac{5n+1}{\lfloor 2n+1}$

soln -

$$T_n = \frac{5n+1}{\lfloor 2n+1}$$

Let $5n+1 = A + B(2n+1)$

put $n = -\frac{1}{2}$, $-\frac{3}{2} = A$.

$n = 0$, $1 = A + B$ $\quad B = 1 + 3/2 = 5/2$

$$5n+1 = -\frac{3}{2} + \frac{5}{2}(2n+1)$$

$$\frac{5n+1}{\lfloor 2n+1} = -\frac{3}{2(2n+1)!} + \frac{5}{2}\frac{2n+1}{\lfloor 2n+1}$$

$$= -\frac{3}{2}\frac{1}{(2n+1)!} + \frac{5}{2}\cdot\frac{1}{2n!}$$

$$S_n = -\frac{3}{2}\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right] +$$

$$\frac{5}{2}\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{4!} + \cdots\right]$$

$$= -\frac{3}{2}\left(\frac{e-e^{-1}}{2}\right) + \frac{5}{2}\left(\frac{e+e^{-1}}{2}\right)$$

$$= -\frac{3}{4}e + \frac{3}{4}e^{-1} + \frac{5}{4}e + \frac{5}{4}e^{-1}$$

$$= \frac{2}{4}e + \frac{8}{4}e^{-1} = \frac{e}{2} + 2e^{-1}$$