ODE, PDE, Laplace Transforms and Vector Analysis

Subject Code: 16SACMM3

Unit II

Formation of Partial Differential Equations, Claitaut's Form, Lagrange's method of Pp+Qq = R

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Unit - II

Formation of Partial differential equation (PDE)

To form a PDE we use the following notations:

'z' will be taken as a dependent variable which depends on two independent variables x, y so that

$z = f(x, y)$ then,

$$\frac{\partial z}{\partial x} = p \ ; \quad \frac{\partial z}{\partial y} = q \ ; \quad \frac{\partial^2 z}{\partial x^2} = r.$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = t.$$

Definition of PDE: A PDE is one which involves partial derivatives for instance,

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2.$$

(i) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ are all partial differential equations.

Problems based on Formation of Partial differential equations by elimination of arbitrary constants:

Standard types of first order PDE:

Type I $f(p,q) = 0$.

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Procedure: Step(i)

In this case the eqn contains only p, q, then x, y are not occur.

$\therefore$ The complete solution is,

$$z = ax + by + c \quad \text{where } f(a,b) = 0. \text{ then.}$$

$$b = \phi(a) \qquad \left(\because \frac{\partial z}{\partial x} = p = a \text{ and } \frac{\partial z}{\partial y} = q = b \right)$$

$\therefore$ The complete solution is

$$z = ax + \phi(a)\,y + c \quad \text{--- (1)}$$

Step(ii)

Differentiating (1) w.r.to 'a' we get

$$\frac{\partial z}{\partial a} = 0$$

Differentiating (1) w.r.to 'c' we get

$$\frac{\partial z}{\partial c} = 0 \Rightarrow \boxed{0 = 1} \text{ which is absurd.}$$

Next to find the general integral

Put $c = f(a)$.

$$\therefore z = ax + y\,\phi(a) + f(a) \quad \text{--- (2)}$$

Differentiating (2) w.r.to 'a' we get Third eqn eliminating 'a' between (2) and (3) we get the general solution.

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Problems:

(1) Solve $p+q=pq$. ——— (1)

Soln Step(i) The eqn is of the form $f(p,q)=0$ ——— (2)

The solution form is $z=ax+by+c$. ——— (3)

WKT $\frac{\partial z}{\partial x}=p=a$

$\frac{\partial z}{\partial y}=q=b$.

$\therefore$ (1) $\Rightarrow$ $a+b=ab$

$b-ab=-a$

$b(1-a)=-a$

$b=\frac{-a}{1-a}$. $\therefore$ $\boxed{b=\frac{a}{a-1}}$

Subs b in (3)

(3) $\Rightarrow$ $z=ax+\left(\frac{a}{a-1}\right)y+c$. ——— (4)

Step(ii) Differentiating (4) w.r.to a we get $\frac{\partial z}{\partial a}=$

$\frac{\partial z}{\partial a}=(1)a+\left(\frac{(a-1)(1)-a(1)}{(a-1)2}\right)y+0$

$\therefore$ Step 2 Put $c = f(a)$. in (4) page 3

(4) $\Rightarrow$ $z = ax + \left(\frac{a}{a-1}\right)y + f(a)$. ——— (5)

Diff w.r.t 'a'

$$\frac{\partial z}{\partial a} = (1)x + \left(\frac{(a-1)(1) - a(1)}{(a-1)^2}\right)y + f'(a).$$

$$0 = x + \left(\frac{a-1-a}{(a-1)^2}\right)y + f'(a). \quad \left(\because \frac{\partial z}{\partial a} = 0\right)$$ ——— (6)

Eliminating 'a' between (5), (6) gives the General Solution of the given PDE.

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(2) Solve $p^2+q^2 = npq$ (Page 4)

Soln step (i) Given that $p^2+q^2 = npq$ ——— (1)

This eqn is of the form $f(p,q)=0$ ——— (2)

The solution form is $z = ax + by + c$ ——— (3)

wkt $\frac{\partial z}{\partial x} = a = p$; $\frac{\partial z}{\partial y} = q = b$

$\therefore$ (1) $\Rightarrow$ $a^2+b^2 = nab$.

$b^2 - nab = -a^2$.

$b^2 - nab + a^2 = 0$.

$\therefore b = \frac{na \pm \sqrt{a^2n^2 - 4a^2}}{2(1)} = \frac{na \pm a\sqrt{n^2-4}}{2}$

$\therefore b = \frac{a}{2}\left[n \pm \sqrt{n^2-4}\right]$

subs b value in (3).

(3) $\Rightarrow$ $z = ax + \frac{a}{2}\left(n \pm \sqrt{n^2-4}\right) + C$ ——— (4)

step (ii) Diff (4) w.r.to 'a' we get $\frac{\partial z}{\partial a} = 0$

$\frac{\partial z}{\partial a} = (1)x + \frac{1}{2}\left(n \pm \sqrt{n^2-4}\right) + 0$.

$\therefore 0 = x + \frac{1}{2}\left(n \pm \sqrt{n^2-4}\right)$.

Diff (4) w.r.to "c" we get $\frac{\partial z}{\partial c} = 0$

(ie) $\boxed{0 = 1}$ which is absurd.

step 2

$\therefore$ Put $c = f(a)$ in (4) Page 5

(4) $\Rightarrow$ $z = ax + \frac{a}{2}\left(n \pm \sqrt{n^2-4}\right) + f(a)$ ——— (5)

Diff (5) w.r.t. 'a'

$$\frac{\partial z}{\partial a} = (1)x + \frac{1}{2}\left(n \pm \sqrt{n^2-4}\right) + f'(a)$$

(i.e.) $0 = x + \frac{1}{2}\left(n \pm \sqrt{n^2-4}\right) + f'(a)$ ——— (6)

eliminating 'a' from (5) and (6) we get the general solution of the given PDE.

Type ii

$$F(x, p, q) = 0.$$

Put $q = a$ and find p.

~~The~~ But $dz = p\,dx + q\,dy$.

(i.e.) $dz = p\,dx + a\,dy$ $(\because q = a)$ ——— (1)

Integrating (1) we get the solution

(i.e.) $z = \int p\,dx + \int a\,dy$.

Type III: $f(y, p, q) = 0$

Put $p = a$ and find q

But $dz = p\,dx + q\,dy$

(ie) $dz = a\,dx + q\,dy \quad (\because p = a)$ — (1)

Integrating (1) we get the solution

(ie) $z = \int a\,dx + \int q\,dy$

Type IV: $f_1(x, p) = f_2(y, q)$.

Let $f_1(x, p) = a$; $f_2(y, q) = a$

To find 'p' and 'q'

$dz = p\,dx + q\,dy$

$\therefore$ The soln is $\int dz = \int p\,dx + \int q\,dy$

(ie) $\boxed{z = \int p\,dx + \int q\,dy}$

Put $c = f(a)$ and find the solution

Problems:

Solve:

1. $P = 2qx$

Soln given $P = 2qx$. —(1)

This is of the form $f(x, P, q) = 0$

Put $q = a$ & find p; $dz = p\,dx + q\,dy$ —(2)

$\therefore$ (1) $\Rightarrow$ $P = 2ax$.

Subs p, q in (2)

(2) $\Rightarrow$ $dz = 2ax\,dx + a\,dy$ $(\because q = a)$

Integrating $\int dz = \int 2ax\,dx + \int a\,dy$

$$z = \frac{2ax^2}{2} + ay + c$$

$$\therefore \boxed{z = ax^2 + ay + C}$$

2. Solve $q = 2px$.

Soln given $q = 2px$ —(1).

This is of the form $f(x, P, q) = 0$

Put $q = a$ & find p; $dz = p\,dx + q\,dy$ —(2)

$\therefore$ (1) $\Rightarrow$ $a = 2px$

$\therefore \frac{a}{2x} = p$

Subs p, q values in (2)

(2) $\Rightarrow$ $dz = \frac{a}{2x} dx + a\,dy$.

Integrating

$$\int dz = \int \frac{a}{2x} dx + \int a\,dy$$

$$\boxed{z = \frac{a}{2} \log x + ay + C}$$ Ans $\left(\because \int \frac{1}{x} dx = \log x\right)$

(3) Solve $q = py + p^2$.

Soln given $q = py + p^2$ —— (1)

This is of the form $f(y, p, q) = 0$

Put $p = a$ and find q.

$dz = p\,dx + q\,dy$ —— (2)

(1) $\Rightarrow$ $q = ay + a^2$.

subs p, q in (2)

$dz = a\,dy + (ay + a^2)\,dy$

Integrating, $\int dz = \int a\,dy + \int (ay + a^2)\,dy$

$$\boxed{z = ay + \frac{ay^2}{2} + a^2 y + C}$$ Ans

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(7) Solve $q = xp + p^2$

Soln given that $q = xp + p^2$ —— (1)

This is of the form $f(x, p, q) = 0$

Put $q = a$ and find p

$dz = p\,dx + q\,dy$ —— (2)

(1) $\Rightarrow$ $a = xp + p^2$

$0 = xp + p^2 - a$

(ie) $p^2 + xp - a = 0$

$p = \dfrac{-x \pm \sqrt{x^2 + 4a}}{2(1)}$ (ie) $p = \dfrac{-x + \sqrt{x^2 + 4a}}{2}$

Subs p and q in (2)

(2) $\Rightarrow$ $dz = \left(\dfrac{-x + \sqrt{x^2 + 4a}}{2}\right) dx + a\,dy$

Integrating, $\int dz = \frac{1}{2}\int \left(-x + \sqrt{x^2 + 4a}\right) dx + \int a\,dy$

$z = \frac{1}{2}\left\{ \int -x\,dx + \int \sqrt{x^2 + 4a}\,dx \right\} + \int a\,dy$

$z = \frac{1}{2}\left\{ -\frac{x^2}{2} + \int \sqrt{x^2 + (2\sqrt{a})^2}\,dx \right\} + ay$

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$$z = \frac{1}{2}\left\{ -\frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+(2\sqrt{a})^2} + \frac{2\sqrt{a}}{2}\sin^{-1}\frac{x}{2\sqrt{a}}\right\}$$

$$+ ay + C$$ // Ans

$$\left(\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a}{2}\sin^{-1}\left(\frac{x}{a}\right) \right.$$

Here $x = x$; $a = 2\sqrt{a}$)

(5) Solve $p+q = x+y$

Soln given that $p+q = x+y \Rightarrow p-x = y-q$ —— (1)

This is of the form $f_1(x,p) = f_2(y,q)$

Here $f_1(x,p) = a$, $f_2(y,q) = a$.

from (1)

$p-x = f_1(x,p)$	$y-q = f_2(x,p)$
$\therefore p-x = a$	$y-q = a$
$\boxed{p = x+a}$	$\boxed{q = y-a}$

$$dz = p\,dx + q\,dy$$

Subs p, q.

$$dz = (x+a)\,dx + (y-a)\,dy$$

$$\int dz = \int (x+a)\,dx + \int (y-a)\,dy.$$

$$z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + C$$

Put $C = f(a)$

$$z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + f(a).$$

Diff w.r.t 'a'

$$\frac{\partial z}{\partial a} = \frac{\cancel{2}(x+a)}{\cancel{2}} + \frac{\cancel{2}(y-a)}{\cancel{2}}(-1) + f'(a)$$

$$0 = (x+a) - (y-a) + f'(a). \quad \left(\because \frac{\partial z}{\partial a} = 0\right)$$

$$\boxed{0 = x - y + 2a + f'(a)}$$ Ans

Clairaut's form $z = px + qy + f(p, q)$

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The form is $z = px + qy + f(p, q)$

The soln is $z = ax + by + f(a, b)$

$(\because p = a,\ q = b)$

1) Solve. $z = px + qy + \sqrt{1+p^2+q^2}$ Page 12

<u>Soln</u>

Given that $z = px + qy + \sqrt{1+p^2+q^2}$ — (1)

WKT the clairauts form is,

$$z = px + qy + f(p,q).$$

The soln form is $z = ax + by + f(a,b)$.

$$(\because p = a,\ q = b).$$

(1) ⇒ $z = ax + by + \sqrt{1+a^2+b^2}$ — (2)

$$z = ax + by + (1+a^2+b^2)^{1/2}$$

$$\frac{\partial z}{\partial a} = (1)x + 0 + \frac{1}{2}(1+a^2+b^2)^{-\frac{1}{2}}(0+2a+0)$$

$$\frac{\partial z}{\partial a} = x + \frac{1}{\cancel{2}(1+a^2+b^2)^{1/2}}(\cancel{2}a)$$

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}} \quad \left(\because \frac{\partial z}{\partial a} = 0\right)$$

(ie) $\boxed{x = \frac{-a}{\sqrt{1+a^2+b^2}}}$ — (3)

Diff (2) with respect to b.

$$\frac{\partial z}{\partial b} = 0 + y + \frac{1}{2}(1+a^2+b^2)^{-1/2}(0+0+2b).$$

$$0 = y + \frac{1}{\cancel{2}\sqrt{1+a^2+b^2}}(\cancel{2}b) \quad \left(\because \frac{\partial z}{\partial b} = 0\right)$$

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$$

$\therefore \boxed{y = -b/\sqrt{1+a^2+b^2}}$ — (4)

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$(3)^2 + (4)^2 \Rightarrow$

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$$x^2+y^2 = \frac{a^2}{1+a^2+b^2} + \frac{b^2}{1+a^2+b^2}.$$

$$x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2.} \quad \text{———} \quad (5)$$

Taking reciprocal on Both Sides

$$\frac{1}{x^2+y^2} = \frac{1+a^2+b^2}{a^2+b^2}$$

$$\frac{1}{x^2+y^2} = \frac{1}{a^2+b^2} + \frac{a^2+b^2}{a^2+b^2}$$

$$\frac{1}{x^2+y^2} = \frac{1}{a^2+b^2} + 1$$

$$\frac{1}{x^2+y^2} - 1 = \frac{1}{a^2+b^2.}$$

$$\frac{1-(x^2+y^2)}{x^2+y^2} = \frac{1}{a^2+b^2}$$

$$1-(x^2+y^2) = \frac{x^2+y^2}{a^2+b^2}$$

$$1-x^2-y^2 = \frac{x^2+y^2}{a^2+b^2.}$$

Subs x^2+y^2 from (5)

$$1-x^2-y^2 = \frac{\frac{a^2+b^2}{1+a^2+b^2}}{a^2+b^2}$$

$$1-x^2-y^2 = \frac{\cancel{a^2+b^2}}{(1+a^2+b^2)\cancel{(a^2+b^2)}}$$

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$$\therefore\ 1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

(ie) $\boxed{1+a^2+b^2 = \frac{1}{1-x^2-y^2}}$ —— 6

from 3 and 4

$x = \frac{-a}{\sqrt{1+a^2+b^2}}$ and $y = \frac{-b}{\sqrt{1+a^2+b^2}}$

(ie) $x\sqrt{1+a^2+b^2} = -a$

$\boxed{-x\sqrt{1+a^2+b^2} = a}$

$y\sqrt{1+a^2+b^2} = -b$

$\boxed{-y\sqrt{1+a^2+b^2} = b}$

Subs a, b and eqn 6 in 2

$$2 \Rightarrow z = ax + by + \sqrt{1+a^2+b^2}$$

$$z = \left(-x\sqrt{1+a^2+b^2}\right)x + \left(-y\sqrt{1+a^2+b^2}\right)y + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = -x^2\sqrt{1+a^2+b^2} - y^2\sqrt{1+a^2+b^2} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = -x^2\left(\frac{1}{\sqrt{1-x^2-y^2}}\right) - y^2\left(\frac{1}{\sqrt{1-x^2-y^2}}\right) + \frac{1}{\sqrt{1-x^2-y^2}} \quad \text{(from 6)}$$

$$z = \frac{1}{\sqrt{1-x^2-y^2}}\left\{-x^2-y^2+1\right\} = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$\boxed{z = \sqrt{1-x^2-y^2}}$ Ans.

$x + y + \ldots$

2) Find singular integral $z = px + qy - p^2 - q^2$

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soln:

colauraits form $z = px + qy + f(p,q)$

complete integral $z = ax + by + f(a,b)$

let $z = ax + by + a^2 - b^2 \rightarrow (1)$

Now

$$\frac{\partial z}{\partial a} = x + 2a$$

$$\frac{\partial z}{\partial b} = y - 2b$$

Now,

condition to find singula integral

i) $\frac{\partial z}{\partial a} = 0$

$x + 2a = 0$

$2a = -x$

$a = -\frac{x}{2} \rightarrow (2)$

(ii) $\frac{\partial z}{\partial b} = 0$

$y - 2b = 0$

$-2b = -y$

$b = y/2 \rightarrow (3)$

Now,

Sub (2), (3) in (1)

$(1) \Rightarrow z = ax + by + a^2 - b^2$

$$z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(-\frac{x^2}{2}\right) - \left(\frac{y^2}{2}\right)$$

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$z = \frac{-2x^2+2y^2+x^2-y^2}{4}$$

$$z = \frac{-x^2+y^2}{4}$$

$$4z = -x^2+y^2$$

$$x^2-y^2+4z=0$$

3) find singular integral $z = px+qy+pq$

Colauraits form $z = px+qy+f(p,q)$

complete integral $z = ax+by+f(a,b)$

Let $z = ax+by+ab$.

Now,

$$\frac{\partial z}{\partial a} = x+0+b = x+b$$

$$\frac{\partial z}{\partial b} = 0+y+a = y+a$$

Now the conditions to find singular integral

i) $\frac{\partial z}{\partial a} = 0$ ii) $\frac{\partial z}{\partial b} = 0$

$x+b=0$ $y+a=0$

$b=-x$; $a=-y$

Now, subs ②, ③ in ①

① $\Rightarrow ax+by+ab$

$z \Rightarrow -yx + xy \Rightarrow -yx - xy + xy$

$z = -xy$

$z = -xy$

4) $z = px + qy + p^2$. find its solution.

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soln:

colazuraits form $z = px + qy + p^2$

simple form of integral $z = ax + bqy + a^2$

Now, let $z = ax + by + a^2 \rightarrow ①$

The defferonciation,

$$\frac{\partial z}{\partial a} = x + 2a$$

$$\frac{\partial z}{\partial b} = y$$

conditions to find singular integrals,

i) $\frac{\partial z}{\partial a} = 0$ (ii) $\frac{\partial z}{\partial b} = 0$

$x + 2a = 0$ $y = 0 \rightarrow ③$

$a = -\frac{x}{2} \rightarrow ②$

Now, sub ②, ③ in ①.

$$z = ax + 0 + a^2$$

$$z = -\frac{x^2}{2} + a^2$$

$$z = -\frac{x^2}{2} + \frac{x^2}{4}$$

$$z = \frac{-4x^2 + 2x^2}{8} \Rightarrow \frac{-\cancel{2}^{1}x^2}{\cancel{8}_{4}}$$

$$z = -\frac{x^2}{4}$$

$$4z = x^2$$

$$x^2 + 4z = 0$$

$$z = -\frac{x^2}{4}$$

5) Find singular Integral of $z = px + qy + 2\sqrt{pq}$

colaurraits form $z = px + qy + 2\sqrt{pq}$ (18)

simple integral $z = ax + by + 2\sqrt{ab}$

Let $z = ax + by + 2\sqrt{ab} \rightarrow (1)$

$z = ax + by + 2a^{1/2} b^{1/2}$

Now,

$\frac{\partial z}{\partial a} = x + 2 \frac{1}{2} a^{-1/2} b^{1/2}$

$\frac{\partial z}{\partial a} = x + \sqrt{b/a}$

$\frac{\partial z}{\partial b} = y + 2 \frac{1}{2} a^{1/2} b^{-1/2}$.

$\frac{\partial z}{\partial b} = y + \sqrt{a/b}$

Now,

conditions to find singular Integral.

i) $\frac{\partial z}{\partial a} = 0$ (ii) $\frac{\partial z}{\partial b} = 0$.

$x + \sqrt{b/a} = 0$ $y + \sqrt{a/b} = 0$

$x = -\sqrt{b/a} \rightarrow (2)$ $y = -\sqrt{a/b} \rightarrow (3)$

Eliminate a & b,

$xy = \left(-\sqrt{\cancel{b/a}}\right)\left(-\sqrt{\cancel{a/b}}\right)$

$xy = 1$

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b) Find singular Integral of $z = px+qy+p^2+pq+q^2$

Soln:-

Colaurait's form $z = px+qy+p^2+pq+q^2$

Simple integral $z = ax+by+a^2+ab+b^2$

Now,

Let $z = ax+by+a^2+ab+b^2 \rightarrow (1)$

Now,

$$\frac{\partial z}{\partial a} = x+2a+b$$

$$\frac{\partial z}{\partial b} = y+2b+a$$

conditions to find singular Integral,

i) $\frac{\partial z}{\partial a} = 0$

$x+2a+b=0$

$2a = x-b$

$a = \frac{-(x+b)}{2} \rightarrow (2)$

ii) $\frac{\partial z}{\partial b} = 0$

$y+a+2b=0 \rightarrow (*)$

$2b = -y-a$

$b = -\frac{(y+a)}{2} \rightarrow (3)$

$(*) \Rightarrow y+a+2b=0$

$y-\left(\frac{x+b}{2}\right)+2b=0$

$y-\frac{x}{2}-\frac{b}{2}+2b=0$

$\frac{3b}{2} = \frac{x}{2}-y$

$\frac{3}{\cancel{2}}b = \frac{x-2y}{\cancel{2}}$

$b = \frac{x-2y}{3}$

$(2) \Rightarrow a = -\left(\frac{x+b}{2}\right)$

$= \frac{-1}{2}(x+b)$

$= \frac{-1}{2}\left[x+\frac{x-2y}{3}\right]$

$= \frac{-1}{2}\left[\frac{3x+x-2y}{3}\right]$

$= -\frac{1}{2}\left[\frac{4x-2y}{3}\right]$

$= \frac{-1}{2}\left[\frac{4x-2y}{3}\right]$

$= \frac{-\cancel{2}}{\cancel{2}}\left[\frac{2x-y}{3}\right)$

$a = \frac{y-2x}{3}$

Subs a & b in (1)

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$(1) \Rightarrow z = ax + by + a^2 + ab + b^2$

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$z = \frac{xy-2x^2}{3} + \frac{xy-2y^2}{3} + \frac{y^2+4x^2-4xy}{9} + \frac{xy-2y^2-2x^2+4xy}{9} + \frac{x^2+4y^2-4xy}{9}$$

$$z = \frac{3xy - 6x^2 + 3xy - 6y^2 + y^2 + 4x^2 - \cancel{4xy} + \cancel{xy} - 2y^2 - 2x^2 + 4\cancel{xy} + x^2 + 4y^2 - \cancel{4xy}}{9}$$

$$z = \frac{3xy - 3x^2 - 3y^2}{9}$$

$$z = \frac{\cancel{3}(xy - x^2 - y^2)}{\cancel{9}\,3}$$

$$\boxed{3z = xy - x^2 - y^2}$$

Ans

Lagrangian multiplier 10m

Langrange's type :

$Pp + Qq = R$, The soln is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

(2)

1) solve $x(y-z)p + y(z-x)q = z(x-y)$

Soln :-

Lagrange's type : $Pp + Qq = R$

$P = x(y-z)$

$Q = y(z-x)$

$R = z(x-y)$

The S.E is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

i) choose 1, 1, 1 are Lagrangian multiply

$$\frac{dx+dy+dz}{xy-zx+yz-xy+zx-yz} = \frac{dx+dy+dz}{0}$$

$\therefore dx+dy+dz=0$

Integrating

$\int dx + \int dy + \int dz = 0$

$x+y+z=0$

let $u = x+y+z \rightarrow (1)$

$\boxed{\rightarrow}$ $\frac{\frac{1}{x}dx = \frac{1}{y}dy = \frac{1}{z}dz}{\frac{1}{x}x(y-z) \quad \frac{1}{y}y(z-x) \quad \frac{1}{z}z(x-y)}$

(ii) choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are legrangian multiply (x^{-1}, y^{-1}, z^{-1})

$\boxed{\rightarrow}$
$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

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Integrating

$\left(\because \log m + \log n = \log(mn) \right.$
$\left. \log m - \log n = \log\left(\frac{m}{n}\right) \right)$

$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$

$\log x + \log y + \log z = \log c_2$

$\log(x \cdot y \cdot z) = \log c_2$

$xyz = c_2$

let

$V = xyz \rightarrow (2)$

soln of PDE $\phi[u, v] = 0$

$\phi[x+y+z, xyz] = 0$

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2) solve $x(y^2-z^2)p = y(z^2-x^2)q = z(x^2-y^2)$

soln:-

Lagrange's type : $Pp + Qq = R$

$P = x(y^2-z^2)$

$Q = y(z^2-x^2)$

$R = z(x^2-y^2)$

The SE is

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\boxed{\rightarrow}$ $\frac{x\,dx}{x^2(y^2-z^2)} = \frac{y\,dy}{y^2(z^2-x^2)} = \frac{z\,dz}{z^2(x^2-y^2)}$

$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$

i) choose x, y, z are Lagrangian multiplier.

$\boxed{\rightarrow}$

$\frac{x\,dx + y\,dy + z\,dz}{x^2y^2 - z^2x^2 + y^2z^2 - x^2y^2 + z^2x^2 - y^2z^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$

$\therefore x\,dx + y\,dy + z\,dz = 0$

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Integrating

$\int x\,dx + \int y\,dy + \int z\,dz = 0.$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = 0 \Rightarrow \frac{x^2+y^2+z^2}{2} = 0$

(2B)

Let $u = x^2 + y^2 + z^2 \rightarrow (1).$

(ii) choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as lagrangian multiplier.

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$

Integrating,

$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0.$

$\log x + \log y + \log z = \log c_2$

$\log(xyz) = \log c_2$

$xyz = c_2$

Let

$V = xyz \rightarrow (2)$

Soln of PDE

$\phi[u, v] = 0$

$\phi[x^2 + y^2 + z^2, xyz] = 0$

3) solve $x^2(y-z)p = y^2(z-x)q = z^2(x-y)$

Soln:-

Lagrange's type: $Pp + Qp = R$.

$P = x^2(y-z)$

$Q = y^2(z-x)$

$R = z^2(x-y)$

The S.E is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

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$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

i) choose $1/x^2, 1/y^2, 1/z^2$ are lagrangian multiplier.

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y - z + z + x - y} = \frac{\frac{1}{x^2}dx + 1/y^2\, dy + \frac{1}{z^2}dz}{0}$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating,

$$\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = 0 \Rightarrow \int x^{-2}dx + \int y^{-2}dy + \int z^{-2}dz = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_1 \qquad -x^{-1} = -\frac{1}{x}$$

$$-\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \cancel{-}c_1$$

let $u = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \rightarrow ①$

ii) choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are lagrangian multiplier.

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy - zx + yz - xy + zx - yz} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\cancel{\log}(xyz) = \cancel{\log}\, c_2$$

$c = xyz$

let $v = xyz \rightarrow (2)$

Soln of PDE

$$\phi[u, v] = 0$$

$$\phi\left[\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right] = 0$$

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4) Solve $(x-2z)p = (2z-y)q = y-x$

soln:-

lagrange type $Pp + Qq = R$

$P = x - 2z$

$Q = 2z - y$

$R = y - x$

The SE is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x}$$

$x_c \; y_c$

$x^2 - 2xz + 2yz - y^2 + yz - xz$

$yx - 2zy + 2xz - xy$

$2zy - 2zx$

i) choose 1, 1, 1 are lagrangian multiplier.

$$\frac{dx + dy + dz}{x - 2z + 2z - y + y - x} = \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

Integrating,

$$\int dx + \int dy + \int dz = 0.$$

$$x + y + z = c_1$$

let $u = x + y + z \longrightarrow (1)$

(ii) choose $y, x, 2z$ Lagrangian multiplier.

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$$\frac{y\,dx + x\,dy + 2z\,dz}{xy - 2yz + 2zx - xy + 2yz - 2zx} = \frac{y\,dx + x\,dy + 2z\,dz}{0}$$

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$$y\,dx + x\,dy + 2z\,dz = 0$$

Integrating,

$$\int y\,dx + \int x\,dy + \int 2z\,dz = 0$$

$$yx + xy + \frac{2z^2}{2} = c_2$$

$$2xy + z^2 = c_2$$

let $v = 2xy + z^2$

Solution of PDE

$$\phi[u, v] = 0$$

$$\phi\left[x+y+z,\ 2xy+z^2\right] = 0$$

5) solve $(y - xz)p = (yz - x)q = (x+y)(x-y)$

Soln:-

lagrange type $Pp + Qq = R$

$P = y - xz$

$Q = yz - x$

$R = (x+y)(x-y) = x^2 - xy + yx - y^2 \Rightarrow x^2 - y^2 = R$

The SE is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$x\cancel{yz} - y^2 zx + \cancel{xyz} - x^2 + x^2 - y^2$

$$\frac{dx}{y\,xz} = \frac{dy}{yz - x} = \frac{dz}{x^2 - y^2}$$

choose $y, x, 1$ are lagrangian multiplier.

$$\frac{y\,dx + x\,dy + 1\,dz}{y^2 - xyz + xyz - x^2 + x^2 - y^2} = \frac{y\,dx + x\,dy + 1\,dz}{0}$$

$$ydx + xdy + 1 \cdot dz = 0$$

On Integrating,

$$\int y dx + \int x dy + \int 1 \cdot dz = 0.$$

$$xy + xy + z = c_1$$

$$2xy + z = c_1$$

let $u = 2x + z \rightarrow ①$

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ii) Choose x, y, z lagrangian multiplier.

$$\frac{xdx + ydy + zdz}{xy - x^2z + y^2z - xy + x^2z - y^2z} = \frac{xdx + ydy + zdx}{0}$$

$$xdx + ydx + zdx = 0$$

On Integration,

$$\int x dx + \int y dy + \int z dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2^2}{2}$$

let $c_2 = x^2 + y^2 + z^2 \Rightarrow v \rightarrow ②$

Soln of PDE

$$\phi[u, v] = 0$$

$$\phi[2xy + z, \; x^2 + y^2 + z^2] = 0.$$

b) solve $(y^2 + z^2)p - xyq + xz = 0$

let $(y^2 + z^2)p - xyq = -xz$

Lagrange type $Pp = Qq = R$

$$P = y^2 + z^2$$

$$Q = -xy$$

$$R = -xz$$

The SE is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

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$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

i) choose x, y, z are Lagrangian multiplier.

$$\frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^2y - xz^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

On Integration,

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c^2}{2}$$

$$x^2 + y^2 + z^2 = c^2$$

Let $u = x^2 + y^2 + z^2 \rightarrow (1)$

ii) choose method of grouping

$$\frac{dy}{-xy} = \frac{dz}{-xz}$$

$$\frac{dy}{\cancel{+}y} = \frac{dz}{\cancel{+}z}$$

On integration

$$\int \frac{1}{y}\,dy = \int \frac{1}{z}\,dz + 0$$

$$\log y = \log z \cancel{+ \log c_2}$$

$$\log y - \log z = \cancel{\log c_2} \quad (2)$$

$$\log\left(\frac{y}{z}\right) = \log c_2$$

$$\frac{y}{z} = c_2$$

Let $v = \frac{y}{z} \rightarrow (2)$

Soln of PDE: $\phi[u, v] = 0$ (ie) $\boxed{\phi\left[x^2+y^2+z^2, \frac{y}{z}\right] = 0}$ // Ans