ABSTRACT ALGEBRA

Subject Code: 16SCCMM12

Unit V

Important questions

Prepared by,

Mrs. H. SABITHA BEGUM, M.Sc.,B.Ed.,M.Phil.,SET.,

Assistant Professor,

Department of Mathematics,

AIMAN College of Arts and Science for Women,

Trichy 21.

Unit - 5

IMPORTANT QUESTIONS.

① Define.

(i) Maximal Ideal.

IS a maximal ideal in z.

Let U be an ideal properly contain.

U contains an odd integer say, $2n+1$

$1 = (2n+1) - 2n \in U$

$U = Z$

Thus there is no proper ideal of z propaly containing is a Maximal ideal of z.

(ii) Prime Ideal.

Let p be any prime. The (P) is maximal ideal in z.

Let U be any ideal of z s.T $(P) \subseteq U$

every ideal of z is a principal ideal

$U = (n) \quad , n \in z$

$P \in (P) \subseteq U \Rightarrow P \in U = (n)$.

$p = nm$ for some integer m

P is prime either $n = 1$ or $n = p$

$n = 1$ Then $U = Z$

$n = P$ " $U = (P)$.

(iii) Unique factorization domain.

Let R be commutative ring.

Let $a, b \in R$ and $a \neq 0$. We say that a divides b and write a/b if there exist an element $c \in R$ such that $b = ac$. If a/b we say that a is a divisor or a factor of b.

Example :-

In Z, $2/6$ since $6 = 2 \times 3$. However in 2Z, 2 does not divide 6 since there is no element $c \in 2Z$ such that $6 = 2c$.

(iv) Eulidean domain.

Let R be a commutative ring with zero-divisors. R is called an Euclidean domain & Euclidean ring if for every non-zero element is defined a non-negative integer $d(a)$.

Example :-

For any two non-zero elements $a, b \in R$, there exist $q, r \in R$ such that $a = qb + r$ either $r = 0$ or $d(r) < d(b)$.

(V) Quotient rings.

Let R be a ring. Let $(I, +)$ be subgroup of $(R, +)$. Since addition is commutative in R, I is a normal subgroup of $(R, +)$.

$R/I = (I + a / a \in R)$ is a group under the operation $(I+a) + (I+b) = I + (a+b)$.

$$(I+a)(I+b) = \cancel{I+(a+b)}\ I + ab.$$

(2) Let R be a commutative ring with identity. An ideal M of R is maximal iff R/M is a field.

Proof:

Let M be a maximal ideal in R.

Since R is a commutative ring with identity $M \neq R$, R/M is also a commutative ring with identity.

Let $M + a$ be an non-zero element in R/M so that $a \notin M$.

$M+a$ has a multiplicative inverse in R/M.

Let $U = \{ra + m / r \in R \text{ and } m \in M\}$

Scanned with CamScanner

$(r_1a+m_1)-(r_2a+m_2)=(r_1-r_2)a+(m_1-m_2)\in U$

$r(r_1a+m_1)=(rr_1)a+rm_1\in U.$

U is an ideal of R

Let $m\in M$. Then $m=0a+m\in U$

$\therefore M\subseteq U$

$a=1a+0\in U$ and $a\notin M$

$M\neq U$

U is an ideal of R properly containing M

M is a maximal ideal of R.

$U=R$. Hence $1\in U$

$1=ba+m$ for some $b\in R$.

$$M+1=M+ba+m=M+ba$$
$$=(M+b)(M+a)$$

$M+b$ is the inverse of $M+a$.

R/M is a field.

Let U be any ideal of R properly containing M.

There exist an element $a\in U$ such that $a\notin M$

$\therefore M+a$ is a non-zero element of R/M

Since R/M is a field $M+a$ has an inverse

$M+b$.

$\therefore (M+a)(M+b)=M+1$

$M+ab=M+1$

$1-ab\in M$

$1 = (1-ab) + ab \in U$

$1 \in U$

$U = R$. This There is no proper ideal of R properly containing M. M is a maximal ideal in R.

③ Let R be any commutative ring with identity. Let P be an ideal of R. Then P is a prime ideal $\Leftrightarrow R/P$ is an integral domain.

Pf:

Let P be a prime ideal

R is commutative ring with identity R/P is commutative ring with identity

$(P+a)(P+b) = P+0$

$\Rightarrow P+ab = P$

$\Rightarrow ab \in P$

$\Rightarrow a \in P$ or $b \in P$

$\Rightarrow P+a = P$ or $P+b = P$

R/P has no zero divisors

$\therefore R/P$ is integral domain

P is a prime ideal of R

. Let $ab \in P$. Then $P+ab = P$

$\therefore (P+a)(P+b) = P$

$\therefore P+a = P$ or $P+b = P$

Scanned with CamScanner

$\therefore a \in P$ or $b \in P$

$\therefore P$ is a prime ideal of R

(4) Fundamental thm of Homomorphism.

Let $f: G \to G'$ be an epimorphism

Let K be the kernel of f. $G/K \cong G'$.

Prf:

Define $\phi: G/K \to G'$ by $\phi(Ka) = f(a)$.

Step (i). ϕ is well defined.

Let $Kb = Ka$. Then $b \in Ka$.

Hence $b = ka$ where $k \in K$.

$f(b) = f(ka) = f(k) f(a) = e' f(a) = f(a)$

$\therefore \phi(Kb) = f(b) = f(a) = \phi(Ka)$

Hence $\phi(Ka) = \phi(Kb)$.

Step (ii) ϕ is 1-1

$$\begin{aligned}\phi(Ka) = \phi(Kb) &\Rightarrow f(a) = f(b)\\ &\Rightarrow f(a)[f(b)]^{-1} = e'\\ &\Rightarrow f(ab^{-1}) = e'\\ &\Rightarrow ab^{-1} \in K\\ &\Rightarrow a \in Kb\\ &\Rightarrow Ka = Kb\end{aligned}$$

Step (iii) ϕ is onto

Let $a' \in G'$

f is onto, there exists $a \in G$ s.t

$f(a) = a'$

Hence $\phi(Ka) = f(a) = a'$

Step (iv) ϕ is homomorphism

$\phi(KaKb) = \phi(Kab) = f(ab) = f(a) f(b)$

$= \phi(Ka) \phi(Kb)$

ϕ is an isomorphism from G/K

onto G'

$\therefore G/K \cong G'$.

(5) Any Euclidean domain R is a U.F.D

Pf:

First we shall P.T any element a in R is either a unit or can be expressed as the Product of a finite number of prime element of R.

induction of $d(a)$

If $d(a) = d(1)$ then a is a unit in R.

assertion is true.

Assume that, the resultant is true for all $x \in R$

S.T $d(x) < d(a)$, and prove that the result extends

for a.

If a is prime there is nothing to prove

If not, $a = bc$ where neither b or c is a unit in a.

$\therefore d(b) < d(a)$ & $d(c) < d(a)$

The product of a finite of prime elements

Let $a = p_1 p_2 \dots p_r = q_1 q_2 \dots q_s$ where p_i's &

q_i's are prime elements of R.

$\therefore p_1 \backslash q_1 q_2 \dots q_s$.

p_1 / q_i for some i. assume the p_1 / q_1

since p_1 & q_1 are both prime elements of R,

p_1 & q_1.

$\therefore q_1 = u_1 p_1$ where u_1 is a unit in R.

$\therefore p_1 p_2 \dots p_r = u_1 q_2 q_3 \dots q_s$

If $r < s$. r times the left side becomes 1

and the right side contains a impossible.

$r \geq s$.

$s \geq r$ and hence $r = s$.

p_i is an associate of some q_i. Hence them

—— . X . ——

Scanned with CamScanner