DIFFERENTIAL EQUATIONS
AND
LAPLACE TRANSFORMS

SUBJECT CODE: 16SCCMM3

Unit- v

TOPIC: LAPLACE TRANSFORMS

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UNIT – V
LAPLACE TRANSFORMS
IMPORTANT QUESTION & ANSWERS

Result : First shifting Property

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

Proof :

By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s) \quad \text{— (1)}$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-st} \left[e^{-at} f(t)\right] dt$$

$$= \int_0^\infty e^{-st} e^{-at} \left[f(t)\right] dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\, dt \quad \text{— (2)}$$

$= F(s+a)$, ∵ By comparing (1) & (2)

$$L\{e^{-at} f(t)\} = F(s+a)$$

(2) Find $L(\sin^2 2t)$

Solu:

Since $\sin^2 2t = \left(\frac{1-\cos 4t}{2}\right)$

We have

$$L(\sin^2 2t) = L\left(\frac{1-\cos 4t}{2}\right)$$

$$= \frac{1}{2}\left[L\,(1-\cos 4t)\right]$$

$$= \frac{1}{2}\,L(1) - \frac{1}{2}\,L(\cos 4t)$$

$$= \frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{s}{s^2+4^2}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+16-s^2}{s(s^2+16)}\right]$$

$$= \frac{1}{2}\left[\frac{16}{s(s^2+16)}\right]$$

$$L(\sin^2 2t) = \frac{8}{s(s^2+16)}$$

① example :

Find $L^{-1}\left[\frac{s}{s^2-16}\right]$

Solu :

$$L^{-1}\left[\frac{s}{s^2-16}\right] = L^{-1}\left[\frac{s}{s^2-4^2}\right]$$

$$= \cosh 4t \qquad \left(L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh$$

$$\therefore L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t$$

② example :

Find $L^{-1}\left[\frac{s}{s^2+9}\right]$

Solu :

$$L^{-1}\left[\frac{s}{s^2+9}\right] = L^{-1}\left[\frac{s}{s^2+3^2}\right]$$

$$= \cos 3t \qquad \left(L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at\right)$$

$$L^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t$$

(3) example :

$$L^{-1}\left[\frac{1}{(s+2)^2+16}\right] = e^{-at} L^{-1}[F(s)]$$

$$= e^{-2t} L^{-1}\left[\frac{1}{s^2+4^2}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{1}{s^2+4^2}\right]$$

$$= e^{-2t}\left[\frac{\sin 4t}{4}\right]$$

$$L^{-1}\left[\frac{1}{(s+2)^2+16}\right] = e^{-2t}\left[\frac{\sin 4t}{4}\right]$$

Problem Identification

If $L^{-1}\left\{\frac{s + \text{any term}}{(\text{Quadratic equation})^2}\right\}$

We have $L^{-1}\{F'(s)\} = -t\, L^{-1}\{F(s)\}$

④ Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Solu:

Assume that $F'(s) = \frac{s}{(s^2+a^2)^2}$

$\int$ing on bothsides w.r.to s

$$\int F'(s)\,ds = \int \frac{s}{(s^2+a^2)^2}\,ds$$

$$F(s) = \int \frac{s}{(s^2+a^2)^2}\,ds$$

Put $s^2+a^2 = u$

$$2s = \frac{du}{ds}$$

$$2s\,ds = du$$

$$s\,ds = \frac{du}{2}$$

$$\therefore F(s) = \int \frac{1}{u^2}\,\frac{du}{2}$$

$$= \frac{1}{2}\int \frac{du}{u^2}$$

$$= \frac{1}{2}\left[-\frac{1}{4}\right]$$

$$F(S) = -\frac{1}{2}\left[\frac{1}{s^2+a^2}\right]$$

we have $L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -t\,L^{-1}\left[\frac{-1}{2(s^2+a^2)}\right]$$

$$= -t \times -\frac{1}{2}\,L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{t}{2}\,L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{t}{2}\left[\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]\right]$$ (Multible & divide by 'a')

$$= \frac{t}{2a}\,L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{t}{2a}\left[\sin at\right]$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

(5) Find $L^{-1}\left[\frac{S+2}{(S^2+4S+5)^2}\right]$

Solu:-

Assume that $F'(S) = \frac{S+2}{(S^2+4S+5)^2}$

$\int$ing on both sides w.r.to S

$$\int F'(S)\,dS = \int \frac{S+2}{(S^2+4S+5)^2}\,dS$$

$$F(S) = \int \frac{S+2}{(S^2+4S+5)^2}\,dS$$

Put $u = S^2+4S+5$

$$\frac{du}{dS} = (2S+4)$$

$$\frac{du}{dS} = 2(S+2)$$

$$\frac{du}{2} = (S+2)\,dS$$

$$\therefore F(S) = \int \frac{1}{(S^2+4S+5)^2}\,(S+2)\,dS$$

$$\therefore F(S) = \int \frac{1}{u^2} \frac{du}{2}$$

$$= \frac{1}{2} \int \frac{du}{u^2}$$

$$= \frac{1}{2} \left(-\frac{1}{u} \right)$$

$$= -\frac{1}{2} \left[\frac{1}{u} \right]$$

$$= -\frac{1}{2} \left[\frac{1}{S^2+4S+5} \right]$$

$$F(S) = \frac{-1}{2(S^2+4S+5)}$$

we have $L^{-1}[F'(S)] = -t\, L^{-1}[F(S)]$

$$L^{-1}\left[\frac{S+2}{(S^2+4S+5)^2} \right] = -t\, L^{-1}\left[\frac{-1}{2(S^2+4S+5)} \right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{S^2+4S+5} \right]$$

$$= \frac{t}{2} L^{-1}\left\{ \frac{1}{(S+2)^2 - 2^2 + 5} \right\}$$

$$= \frac{t}{2} L^{-1}\left\{ \frac{1}{(S+2)^2 + 1} \right\}$$

$$= \frac{te^{-2t}}{2} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{te^{-2t}}{2} \sin t$$

$$\therefore L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \frac{te^{-2t}\sin t}{2}$$

) Find $L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$

Solu:

Here $F'(s) = \dfrac{s}{(s^2-1)^2}$

Put $u = s^2-1$

$$\frac{du}{ds} = 2s$$

$$\frac{du}{2} = s\,ds$$

$$\int F'(s)\,ds = \int \frac{s}{(s^2-1)^2}\,ds$$

$$F(s) = \int \frac{1}{(s^2-1)^2} s\,ds$$

$$= \int \frac{1}{u^2} \times \frac{du}{2}$$

$$= \frac{1}{2}\int \frac{du}{u^2}$$

$$= \frac{1}{2}\left(-\frac{1}{u}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{u}\right)$$

$$= -\frac{1}{2}\left[\frac{1}{s^2-1}\right]$$

$$F(s) = \frac{-1}{2(s^2-1)}$$

We have $L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$

$$L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = -t\,L^{-1}\left[-\frac{1}{2(s^2-1)}\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{s^2-1}\right]$$

$$= \frac{t}{2}\left[\sinh t\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = \frac{t\sinh t}{2}$$

Result :

If $L^{-1}\left[\frac{s}{(\text{Quadratic equation})}\right]$, then $L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$

(7) Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

solu:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt}\left[e^{-2t} L^{-1}\left(\frac{1}{s^2}\right)\right]$$

$$= e^{-2t} \frac{d}{dt}\left[L^{-1}\left(\frac{1}{s^2}\right)\right]$$

$$= e^{-2t} \frac{d}{dt}[t]$$

$$= \frac{d}{dt}\left[t e^{-2t}\right]$$

$$= \left[t e^{-2t}(-2) + e^{-2t}(1)\right]$$

$$= \left[-2te^{-2t} + e^{-2t}\right]$$

$$= \left[e^{-2t} - 2te^{-2t}\right]$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = e^{-2t}\left[1-2t\right]$$

Find $L^{-1}\left(\frac{s-3}{s^2+4s+13}\right)$

solu

$$L^{-1}\left(\frac{s-3}{s^2+4s+13}\right) = L^{-1}\left(\frac{s}{s^2+4s+13}\right) - 3L^{-1}\left(\frac{1}{s^2+4s+13}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{s^2+4s+13}\right) - 3L^{-1}\left(\frac{1}{s^2+4s+3}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{s^2+4s+13}\right) - 3L^{-1}\left(\frac{1}{s^2+4s+3}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2-2^2+13}\right) - 3L^{-1}\left(\frac{1}{(s+2)^2-2^2+13}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2+9}\right) - 3L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= \frac{d}{dt} \times \frac{1}{3} L^{-1}\left(\frac{3}{(s+2)^2+9}\right) - 3 \times \frac{1}{3} L^{-1}\left[\frac{3}{(s+2)^2+9}\right]$$

$$= \frac{d}{dt}\left[-\frac{e^{2t}}{3} L^{-1}\left(\frac{3}{s^2+3^2}\right)\right] - e^{-2t} L^{-1}\left(\frac{3}{s^2+3^2}\right)$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{3} \sin 3t\right] - e^{-2t} \sin 3t$$

$$= \frac{1}{3} \frac{d}{dt}\left[e^{-2t} \sin 3t\right] - e^{-2t} \sin 3t$$

$$= \frac{1}{3}\left[e^{-2t} (3\cos 3t) + \sin 3t\,(e^{-2t}(-2))\right] - e^{-2t} \sin 3t$$

$$= \frac{1}{3}\left[3e^{-2t} \cos 3t - 2e^{-2t} \sin 3t\right] - e^{-2t} \sin 3t$$

$$= e^{-2t}\left[\cos 3t - \frac{2}{3} \sin 3t - \sin 3t\right]$$

$$= e^{-2t}\left[\cos 3t - \frac{2\sin 3t - 3\sin 3t}{3}\right]$$

$$= e^{-2t}\left[\cos 3t - \frac{5\sin 3t}{3}\right]$$

$$L^{-1}\left(\frac{s-3}{s^2+4s+13}\right) = e^{-2t}\left(\cos 3t - \frac{5\sin 3t}{3}\right)$$

Result :

If $L\{f(t)\} = F(s)$, then $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(x)dx$

$$\boxed{\therefore L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}\{f(s)\}\,dt}$$

examples :

(9) Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

solu :

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left\{\frac{1}{s+a}\right\}dt$$

$$L^{-1}\left\{\frac{1}{s}\left\{\frac{1}{s+a}\right\}\right\} = \int_0^t e^{-at}\,dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{1}{a}\left[-e^{-at}\right]_0^t$$

$$= \frac{1}{a}\left[-e^{-at}\right]_0^t$$

$$= \frac{1}{a}\left[-e^{-at} - (-e^{0})\right]$$

$$= \frac{1}{a}\left[-e^{-at} + 1\right]$$

$$= \frac{1}{a}\left[1 - e^{-at}\right]$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1}{a}(1-e^{-at})$$

Find the inverse transform using Partial fractions.

10 Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Solu:

Let $F(s) = \frac{1}{s(s+1)(s+2)}$

Let us first resolve F(s) into Partial fractions.

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\frac{1}{\cancel{s(s+1)(s+2)}} = \frac{A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)}{\cancel{s(s+1)(s+2)}}$$

$$1 = A(s+1)(s+2) + B\,s(s+2) + C\,s(s+1)$$

put $s=0$, we get

$$1 = A(2) \Rightarrow 1 = 2A, \boxed{A = 1/2}$$

put $s=-1$

$$1 = B(-1)(1)$$

$$\boxed{B = -1}$$

put $s=-2$

$$1 = C(-2)(-1)$$

$$1 = 2C$$

$$\boxed{1/2 = C}$$

$$\therefore \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2}(1) - e^{-t} + \frac{1}{2}e^{-2t}$$

$$= \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$$

$$= \frac{1}{2}\left[1 - 2e^{-t} + e^{-2t}\right]$$

$$= \frac{1}{2}\left[e^{-2t} - 2e^{-t} + 1\right]$$

$$= \frac{1}{2}\left[(e^{-t} - 1)^2\right]$$

$$\therefore L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}\left[(e^{-t} - 1)^2\right]$$

(ii) Find $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$

Solu:

Let $F(s) = \frac{1+2s}{(s+2)^2(s-1)^2}$

Let us first resolve F(s) into partial fractions

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{(s+2)^2} + \frac{B}{(s-1)^2}$$

$$\frac{1+2s}{\cancel{(s+2)^2(s-1)^2}} = \frac{A(s-1)^2 + B(s+2)^2}{\cancel{(s+2)^2(s-1)^2}}$$

$$1+2s = A(s-1)^2 + B(s+2)^2$$

Put $s=1$, we get $1+2 = 3^2(B)$

$3 = 9B$

$3/9 = B \Rightarrow \boxed{B = \frac{1}{3}}$

Put $s=-2$, we get $1+2(-2) = 9A$

$1-4 = 9A$

$-3 = 9A \Rightarrow -3/9 = A$

$\boxed{A = -1/3}$

$$\therefore \frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{(s+2)^2} + \frac{B}{(s-1)^2}$$

$$\therefore \frac{1+2s}{(s+2)^2(s-1)^2} = -\frac{1}{3(s+2)^2} + \frac{1}{3(s-1)^2}$$

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = -\frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= +\frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{3}e^{t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3}e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= \frac{1}{3}e^{t}t - \frac{1}{3}e^{-2t}t$$

$$= \frac{t}{3}\left[e^{t} - e^{-2t}\right]$$

$$\therefore L^{-1}\left[\frac{1+2s}{(s+2)^2(s+1)^2}\right] = \frac{t}{3}\left[e^{t} - e^{-2t}\right]$$

Solve ordinary differential equations with constant co-efficients.

If $L[y] = \bar{y}$ then (i) $L(y') = sL(y) - y(0)$

(ii) $L(y'') = s^2 L(y) - sy(0) - y'(0)$

(12) Solve the equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$

given that $y = \frac{dy}{dt} = 0$ when $t = 0$

Solu:

The equation can be written in the form

$$y'' + 2y' - 3y = \sin t$$

Applying Laplace transforms on both-sides, we have

$$L[y'' + 2y' - 3y] = L(\sin t)$$

$$L(y'') + 2L(y') - 3L(y) = L(\sin t)$$

$$s^2 L(y) - sy(0) - y'(0) + 2(sL(y) - y(0)) - 3L(y) = L(\sin t)$$

$$s^2 L(y) - s(y(0)) - y'(0) + 2\{sL(y) - y(0)\} - 3L(y) = \frac{1}{s^2+1}$$

substituting the values of $y(0)$ & $y'(0)$ in the equations

$$s^2 L(y) + 2\{sL(y)\} - 3L(y) = \frac{1}{s^2+1}$$

where $\bar{y} = L(y)$

$$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$$

$$(s^2+2s-3)\bar{y} = \frac{1}{s^2+1}$$

$$\bar{y} = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$\bar{y} = \frac{1}{(s^2+1)(s-1)(s+3)}$$

Let $F(s) = \frac{1}{(s^2+1)(s-1)(s+3)}$

Let us first resolve F(s) into partial fractions

$$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{Cs+D}{s^2+1} + \frac{A}{s-1} + \frac{B}{s+3}$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)$$

Put $s=1$, we get $1 = A(4)(2)$

$$1 = 8A \Rightarrow \boxed{\tfrac{1}{8} = A}$$

Put $s=-3$, we get $1 = B(-3-1)(9+1)$

$$1 = B(-4)(10)$$

$$1 = -40B$$

$$\boxed{-\tfrac{1}{40} = B}$$

Equating the co-efft of s^3

$$A+B+C = 0$$

$$\frac{1}{8} - \frac{1}{40} + C = 0 \Rightarrow C = -\frac{1}{8} + \frac{1}{40}$$

$$C = \frac{-5+1}{40}$$

$$C = -\frac{4}{40}$$

$$\boxed{C = -\frac{1}{10}}$$

Equating the co-efft of s^2

$$A + B + 2D - 3C = 0$$

$$A + B - 3C = -2D$$

$$\frac{1}{8} - \frac{1}{40} - 3\left(-\frac{1}{10}\right) = -2D$$

$$\frac{1}{8} - \frac{1}{40} + \frac{3}{10} = -2D$$

$$\frac{5-1+12}{40} = -2D$$

$$\frac{16}{40} = -2D \Rightarrow \frac{16}{-2 \times 40} = D$$

$$\therefore D = -\frac{1}{5}$$

$$\frac{1}{(s-1)(s-3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$\frac{1}{(s-1)(s-3)(s^2+1)} = \left[\frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{\frac{1}{10}s - \frac{1}{5}}{s^2+1}\right]$$

$$y = L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right] = \frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$- \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

$$\therefore y = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

UNIT - V Completed