# Analytical Geometry 3D

## Subject Code: 16SCCMM4

## Unit IV

### Important Exercise Problems

# Prepared by,

**Mrs. H. SABITHA BEGUM,** M.SC.,B.Ed.,M.Phil.,SET.,

**Assistant Professor,**

**Department of Mathematics,**

**AIMAN College of Arts and Science for Women,**

**Trichy 21.**

Ex-15.

1 Find the eqn. of the cone of the second degree which passes through the axes.

(or)

S.T the general eqn. to a cone which passes through the 3 axes is

$$fyz + gzx + hxy = 0,$$

f, g, h being parameters.

Soln.

The general eqn. of a cone with its vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{--- (1)}$$

Now x-axis is a generator

$\Rightarrow$ its d.c's (1,0,0) satisfy (1)

$\Rightarrow$ $a = 0$

IIIly $b = 0$ & $c = 0$.

Subs in eqn (1)

$$2fyz + 2gzx + 2hxy = 0$$

$$fyz + gzx + hxy = 0.$$

(2) Find the eqn. of the quadric cone which passes through the coordinate axes and 3 lines

$$\frac{x}{3} = \frac{y}{5} = \frac{z}{1} \text{ --- (1)}, \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{2} \text{ --- (2)}, \quad \frac{x}{-11} = \frac{y}{5} = \frac{z}{8} \text{ --- (3)}$$

Soln.

The general eqn. of a cone through the 3 axes is

$fyz + gzx + hxy = 0$, f, g, h are parameters.

$l_1 = 3 \quad m_1 = 5 \quad n_1 = 1$

$l_2 = 1 \quad m_2 = -1 \quad n_2 = 2$

$l_3 = -11 \quad m_3 = 5 \quad n_3 = 8$

It will contain the lines ① & ③ as a generator, if

$$fyz + gzx + hxy = 0$$

$$f(5)(1) + g(1)(3) + h(3)(5) = 0$$

$$5f + 3g + 15h = 0 \quad \text{— ④}$$

$$f(-1)(2) + g(2)(1) + h(1)(-1) = 0$$

$$-2f + 2g - h = 0 \quad \text{— ⑤}$$

The $f(5)(8) + g(8)(-11) + h(-11)(5) = 0$

$$40f - 88g - 55h = 0 \quad \text{— ⑥}$$

④ & ⑤

$$\frac{f}{\begin{vmatrix} 3 & 15 \\ 2 & -1 \end{vmatrix}} = \frac{-g}{\begin{vmatrix} 5 & 15 \\ -2 & -1 \end{vmatrix}} = \frac{h}{\begin{vmatrix} 5 & 3 \\ -2 & 2 \end{vmatrix}}.$$

$$\frac{f}{-3-30} = \frac{g}{-(-5+30)} = \frac{h}{(10+6)}$$

$$\frac{f}{-33} = \frac{g}{-25} = \frac{h}{16}$$

$$-33yz - 25zx + 16xy = 0$$

$$33yz + 25zx - 16xy = 0$$

—— X ——

3. S.T A cone of the second degree can be found to pass through any five concurrent lines.

<u>Soln.</u>

Let origin be the pt. of concurrence of 5 lines which are


$$\frac{x}{l_r} = \frac{y}{m_r} = \frac{z}{n_r} \quad ; \quad r = 1,2,3,4,5$$

General Second degree eqn. of the cone with Vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

$$x^2 + \frac{b}{a}y^2 + \frac{c}{a}z^2 + \frac{2f}{a}yz + \frac{2g}{a}zx + \frac{2h}{a}xy = 0.$$

$$x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0.$$

This contains 5 arbitrary constants & therefore, can be determined by five independent conditions.

Since dc's of generators satisfy the eqn. of a cone, so any 5 lines (passing thro' origin) are sufficient to determine the 5 arbitrary constants.

$\therefore$ A cone of second degree can be found to pass thro' 5 concurrent lines.

— x —

Excercise - 15

(11) P.T $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$$

<u>Soln</u>.

Let

$$f(x,y,z,t)=ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0$$

$\frac{\partial f}{\partial x}=0$ for $t=1$ gives

$2ax+2u=0$ or $x=-u/a$ — (1)

$\frac{\partial f}{\partial y}=0$ for $t=1$ gives

$2by+2v=0$ or $y=-v/b$ — (2)

$\frac{\partial f}{\partial z}=0$ for $t=1$ gives

$2cz+2w=0$ or $z=-\frac{w}{c}$ — (3)

$\frac{\partial f}{\partial t}=0$ for $t=1$ gives

$ux+vy+wz+d=0$ — (4)

Subs (1), (2), (3) in (4), we get the required equation

$$u\left(-u/a\right)+v\left(-v/b\right)+w\left(-\frac{w}{c}\right)+d=0$$

$$\Rightarrow -\frac{u^2}{a}-\frac{v^2}{b}-\frac{w^2}{c}+d=0$$

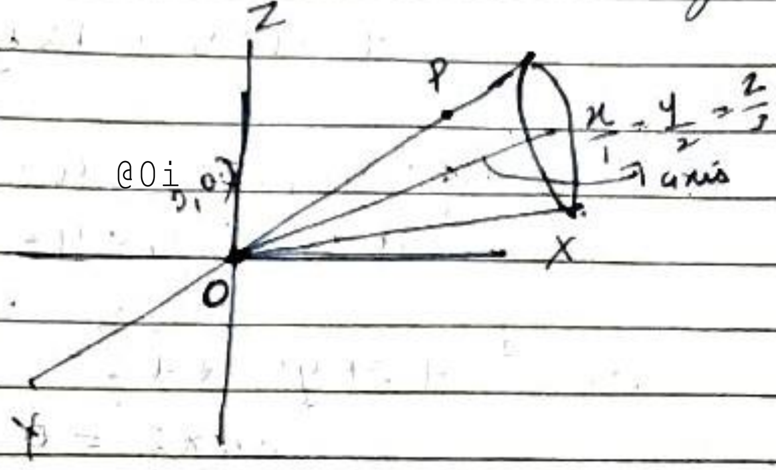
$$\Rightarrow \frac{u^2}{a^2}+\frac{v^2}{b^2}+\frac{w^2}{c}=d$$

//

4)

Find the eqn to the right circular cone whose vertex is at the origin, whose axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and which has a vertical angle of $60°$.

<u>Soln.</u>




Let $P(x, y, z)$ be any point on the surface of the cone, so that the d.r's of the line OP are $x, y, z$ (since $x-0, y-0, z-0$) and O being the origin).

The d.c's of the line (axis) $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are $(\overset{l}{1}, \overset{m}{2}, \overset{n}{3})$.

$$\therefore \cos\alpha = \frac{lp + mq + nr}{\sqrt{l^2+m^2+n^2}\sqrt{p^2+q^2+r^2}}$$

$$\cos 30^0 = \frac{x(1) + y(2) + z(3)}{\sqrt{x^2+y^2+z^2}\sqrt{1^2+2^2+3^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{x + 2y + 3z}{\sqrt{x^2+y^2+z^2}\sqrt{14}}$$

$$\sqrt{14} \cdot \sqrt{3} \sqrt{x^2+y^2+z^2} = 2(x+2y+3z)$$

sq. on sides

$$\therefore 14 \cdot 3 \cdot (x^2+y^2+z^2) = 4(x+2y+3z)^2$$

$$\Rightarrow 42x^2 + 42y^2 + 42z^2 = 4(x^2 + 4y^2 + 9z^2 + 4xy + 12yz + \quad) = 0$$

$$\Rightarrow 42x^2 + 42y^2 + 42z^2 - 4x^2 - 4y^2 - 3$$
$$- 16xy - 48yz - 24xz = 0.$$

$$\Rightarrow 38x^2 + 26y^2 + b\,a'' - 16xy - 48yz$$
$$- 24xz = 0$$

$\div$ by 2

$$\Rightarrow 19x^2 + 13y^2 + 3z^2$$
$$- 8xy - 24yz - 12xz = 0$$

—X—

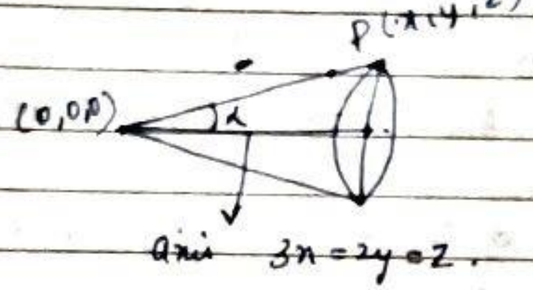
14 a) S.T $33x^2+13y^2-95z^2-144yz-96zx-48xy=0$ represents a right circular cone whose axis is the line $3x=2y=z$. Find its vertical angle.

Soln:

General eqn. of the right circular cone is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

ie) Vertex is origin.




Let $\alpha$ be the semi vertical angle.

The d.c's of the axis of the line $3x=2y=z$ is $(1/3, 1/2, 1)$.

The d.r's of any line is $(x-0, y-0, z-0)$ ie $x, y, z$.

$$\cos\alpha = \frac{x(1/3)+y(1/2)+z(1)}{\sqrt{x^2+y^2+z^2}\sqrt{(1/3)^2+(1/2)^2+1^2}}$$

$$\cos\alpha = \frac{x/3+y/2+z}{\sqrt{x^2+y^2+z^2}\sqrt{1/9+1/4+1}}$$

$$\cos\alpha = \frac{1/6(2x+3y+6z)}{\sqrt{x^2+y^2+z^2}\sqrt{49/36}}$$

$$\cos\alpha = \frac{2x+3y+6z}{\cancel{6}\sqrt{x^2+y^2+z^2}\;\frac{7}{\cancel{6}}}$$

$$\Rightarrow 7\cos\alpha = \frac{2x+3y+6z}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow 7\cos\alpha\sqrt{x^2+y^2+z^2} = 2x+3y+6z$$

Squaring

$$49\cos^2\alpha\,(x^2+y^2+z^2) = (2x+3y+6z)^2$$

$$\Rightarrow 49\cos^2\alpha\,(x^2+y^2+z^2) - 4x^2 - 9y^2 - 36z^2 - 12xy - 36yz - 24xz = 0.$$

$$\Rightarrow (49\cos^2\alpha - 4)x^2 + (49\cos^2\alpha - 9)y^2 \; (49\cos^2\alpha - 36)z^2 - 12xy - 36yz - 24xz = 0 \quad \text{--- (5)}$$

If we multiply (2) by 4, we get $xy$, $yz$ and $xz$ terms ~~are~~ in (2) are equal to (1).

$\therefore$ By comparing $x^2$ or $y^2$ or $z^2$ term, we will get $\alpha$.

$$4\,(49\cos^2\alpha - 4) = 33$$

$$\Rightarrow 186\cos^2\alpha - 16 = 33$$

$$\Rightarrow 186\cos^2\alpha = 33 + 16 = 49$$

$$\Rightarrow \cos^2\alpha = \frac{49}{186}.$$

$$\Rightarrow \cos\alpha = \pm\frac{7}{14} = \pm\frac{1}{2}$$

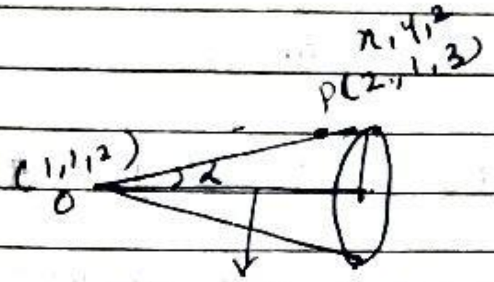
Semi vertical angle, $\alpha = 60^\circ$.

Vertical angle $2\alpha = 120^\circ$ //

14(b) S.T the eqn. of a right cone which passes through $(2,1,3)$ and has its Vertex at the point $(1,1,2)$ and axis the line $\frac{x-1}{2}=\frac{y-1}{-4}=\frac{z-2}{3}$ is

$$17x^2-7y^2+7z^2+48yz-24zx+32xy-18x-114y-52z+118=0$$

Soln:




$$\frac{x-1}{2}=\frac{y-1}{-4}=\frac{z-2}{3}$$

The d.r's of OP is

$(2-1,\ 1-1,\ 3-2) \Rightarrow (1,0,1)$

The d.c's of the line is $(2,-4,3)$

$$\therefore \cos\alpha = \frac{1(2)+0(-4)+1(3)}{\sqrt{1^2+0^2+1^2}\ \sqrt{2^2+4^2+3^2}}$$

$$\cos\alpha = \frac{2+0+3}{\sqrt{2}\ \sqrt{29}} = \frac{5}{\sqrt{2}\sqrt{29}}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{5}{\sqrt{2}\sqrt{29}}\right)$$

If $(x,y,z)$ be any pt on the cone, then its eqn. will be.

$$\cos\alpha = \frac{(x-1)(2)+(y-1)(-4)+(z-2)(3)}{\sqrt{(x-1)^2+(y-1)^2+(z-2)^2}\ \sqrt{2^2+4^2+3^2}}$$

$$\cos\alpha = \frac{2x-2-4y+4+3z-6}{\sqrt{(x-1)^2+(y-1)^2+(z-2)^2}\sqrt{29}}$$

$$\frac{5}{\sqrt{2}\cancel{\sqrt{29}}} = \frac{2x-4y+3z-4}{\sqrt{(x-1)^2+(y-1)^2+(z-2)^2}\cancel{\sqrt{29}}}$$

$$5\sqrt{(x-1)^2+(y-1)^2+(z-2)^2} = \sqrt{2}(2x-4y+3z-4)$$

Squaring on both sides,

$$25\{(x-1)^2+(y-1)^2+(z-2)^2\} = 2(2x-4y+3z-4)^2$$

$$\Rightarrow 25\{x^2-2x+1+y^2-2y+1+z^2-4z+4\}$$
$$= 2\{(2x-4y+3z)^2-2(4)(2x-4y+3z)+4^2\}$$

$$\Rightarrow 25\{x^2+y^2+z^2-2x-2y-4z+6\}$$
$$= 2\{4x^2+16y^2+9z^2-16xy-24yz+12xz-16x+32y-24z+16\}$$

$$\Rightarrow 25x^2 + 25y^2 + 25z^2 - 50x - 50y - 100z + 150$$
$$- 8x^2 - 32y^2 - 18z^2 + 32xy + 48yz$$
$$- 24xz + 32x - 64y + 48z - 32 = 0$$

$$\Rightarrow 17x^2 - 17y^2 + 7z^2 - 18x - 114y - 52z$$
$$+ 32xy + 48yz - 24xz + 118 = 0$$

—— X ——