VECTOR CALCULUS AND FOURIER SERISES
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UNIT IV
TOPIC:FOURIER SERIES

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Fourier Series

Full range — Half range

ii) Half range

Interval	Cosine Series	Sine series
$(0, \pi)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$	$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$
	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$
	$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$	$a_0 = a_n = 0$
	$b_n = 0$	
$(0, l)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$	$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$
	$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$	$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$
	$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	$a_0 = a_n = 0$
	$b_n = 0$	

i) Full range

$(0, 2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Determine the fourier expansion $f(x) = x$ where $-\pi < x < \pi$

Let $f(x) = x$ $\quad$ x odd fn

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx \Rightarrow \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = 0$$

→ if limit is $2\int_0^{\pi}$ The Ans will be π

if $\int_{-\pi}^{\pi}$ Ans is 0

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$\int u\,dv = uv - \int v\,du$

$u = x \quad du = dx$

$dv = \cos nx\,dx$

$v = \frac{\sin nx}{n}$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{u}{x}\,\overset{v}{\cos nx}\,dx \Rightarrow \frac{1}{\pi}\left[\left[\frac{x\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{n^2\pi}\left[\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{n^2\pi}\left[\cos n\pi - \cos n(-\pi)\right] \Rightarrow \frac{1}{n^2\pi}\left[\cos n\pi - \cos n\pi\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$\cos x = \sin x$

$\sin x = -\cos x$

$$= \frac{1}{\pi}\left[\left[\frac{x - (\cos nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-\cos nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{n\pi}\left[-\pi\cos n\pi - (-\pi\cos n(-\pi))\right]$$

$$= -\frac{1}{n\pi}\left[\pi\cos n\pi + \pi\cos n\pi\right]$$

$$= \frac{-2\cos n\pi}{n} \Rightarrow \frac{-2(-1)^n}{n} \Rightarrow \frac{2(-1)^{n+1}}{n}$$

$$\therefore\ x = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin nx$$

$$\therefore\ x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right].$$

3π 5π

① Find the fourier series for the function $f(x) = x^2$ where $-\pi \le x \le \pi$ and deduce that.

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol

Let $f(x) = x^2$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx \quad (\text{since } x^2 \text{ is an even } fn)$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^3}{3\pi} - 0 = \frac{2\pi^2}{3}$$

$u = x^2$

$du = 2x\,dx$

$u_2 = 2$

$dv = \cos nx\,dx$

$v = \frac{\sin nx}{n}$

$v_1 = \frac{-\cos nx}{n^2}$

$v_2 = \frac{-\sin nx}{n^3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left[x^2 \cdot \frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin nx}{n}\cdot 2x\,dx\right]$$

$$= \left[\frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}\right.$$

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$$= \frac{2}{\pi}\left[0 + \frac{2\pi \cos n\pi}{n} - 0\right] \Rightarrow \frac{4\cos n\pi}{n^2} \Rightarrow \frac{4(-1)^n}{n^2}$$

III ly

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= 0$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

when $x=0$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$$

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \cdots \frac{1}{n^2}\right]$$

$$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right] \quad \text{--- (1)}$$

when $x=\pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{2}{2^2} + \cdots \frac{1}{n^2}\right]$$

$$\therefore 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots \frac{1}{n^2}\right] = \pi^2 - \frac{\pi^2}{3}$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \cdots \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{--- (2)}$$

Adding (1) & (2)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left[\frac{1}{1^2} + \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \Rightarrow \frac{\pi^2}{8}$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots //$$

5m

Properties of odd and even functions

i) $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd

ii) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, if $f(x)$ is even

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{--- (1)}$$

In the first Integral put $x=-y$ $\quad dx = -dy$

$x=-a \quad y=a$

$x=0 \quad y=0$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y)\,(-dy) = -\int_{a}^{0} f(-y)\,dy$$

$$= \int_{0}^{a} f(-y)\,dy \Rightarrow \int_{0}^{a} f(-x)\,dx$$

$$= \int_{0}^{a} f(x)\,dx \quad \text{--- (2)}$$

Sub (2) in (1)

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \left[\begin{array}{c} f(x) \text{ is an even fn} \\ f(-x) = f(x) \end{array}\right]$$

$$= 2\int_{0}^{a} f(x)\,dx$$

case (i) $f(x)$ is an odd function then $f(x)\cos nx$ is also an odd function

$$\therefore \int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Hence $a_n = 0$

$f(x)$ is an odd function then $f(x)\sin nx$ is an even fn.

$$\therefore \int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$\therefore \text{Hence } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

case (ii)

If $f(x)$ is an even function, then $f(x)\cos nx$ is an even function. hence

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$\therefore a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

If $f(x)$ is an even function, then $f(x)\sin nx$ is an odd function, hence

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

$$\therefore b_n = 0$$

If $f(x) = x + x^2 \ (-\pi < x < \pi)$ P.T $f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \cdots\right) + 2\left(\frac{\sin nx}{1} - \frac{\sin 2x}{2} + \cdots\right)$

Reduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

fourier series = a_0, a_n, b_n $a_0 = x^2$

P.T → all step

The fourier Series is

∴ cosine series followed

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \, dx \Rightarrow \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$

$$= \frac{2}{\cancel{\pi}}\left(\frac{\pi^{\cancel{3}}}{3}\right) \Rightarrow a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$\int x \cos nx \, dx = 0$

$u = x^2$ $dv = \cos nx \, dx$

$u' = 2x$ $v = \frac{\sin nx}{n}$

$u'' = 2$ $v_1 = \frac{-\cos nx}{n^2}$

$u''' = 0$ $v_2 = -\frac{\sin nx}{n^3}$

$$a_n = \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} + 2x\,\frac{\cos nx}{n^2} - 2\,\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[0 + 2\pi\,\frac{\cos n\pi}{n^2} - 0\right] - [0]\right\}$$

$$= \frac{2}{\pi}\left[2\pi\,\frac{(-1)^n}{n^2}\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$u = x$ $dv = \sin nx\, dx$

$u' = 1$ $v = -\frac{\cos nx}{n}$

$u'' = 0$ $v_1 = -\frac{\sin nx}{n^2}$

$$b_n = \frac{2}{\pi}\left[\left(-x \cdot \frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\pi \cdot \frac{\cos n\pi}{n} + 0\right) - (0)\right] \Rightarrow \frac{2}{\pi}\left(\frac{-\pi(-1)^n}{n}\right)$$

$$\boxed{b_n = \frac{-2(-1)^n}{n}}$$

$$f(x) = \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n}\sin nx$$

$$= \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cdot\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-2\left[\frac{-1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right]$$

$$f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

$$F(x) = x + x^2$$

$$f(\pi) = \pi + \pi^2$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin nx$$

Put $x = \pi$, a point of discontinuity

$$\frac{f(\pi) + f(-\pi)}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos n\pi + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin n\pi$$

$$\frac{\pi + \pi^2 - \pi + \pi^2}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}(-1)^n + 0$$

$$\frac{2\pi^2}{2} - \frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} 1/n^2$$

$$\frac{2\pi^2}{12} = \sum_{n=1}^{\infty} 1/n^2$$

$$\pi^2/6 = \sum 1/n^2$$

Develope f(x) as a fourier series in the Interval $(-\pi, \pi)$

Q.a if $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ \pi & \text{for } 0 < x < \pi \end{cases}$

Here $f(x)$ is neither even nor odd

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$f_1(-x) = f_2(x)$ even

$= -f_2(x)$ odd

where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \pi\,dx \right\} \Rightarrow \frac{1}{\pi} \cdot \pi (x)_0^{\pi}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \left\{ \int_{\pi}^{0} 0 \cos nx\,dx + \int_{0}^{\pi} \pi \cos nx\,dx \right\}$$

$$= \frac{1}{\pi} \cdot \pi \left(\frac{\sin nx}{n} \right)_0^{\pi}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \Rightarrow \frac{1}{\pi} \left\{ \int_{-\pi}^{0} 0 \sin nx\,dx + \int_{0}^{\pi} \pi \sin nx\,dx \right\}$$

$$= \frac{1}{\pi} \cdot \pi \left[\frac{-\cos nx}{n} \right]_0^{\pi} \Rightarrow \frac{1}{n} \left\{ -(-1)^n + 1 \right\}$$

$$= \begin{cases} \frac{1}{n}(-1+1), & n \text{ is even} \\ \frac{1}{n}(1+1), & n \text{ is odd} \end{cases}$$

$$b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n}, & n \text{ is odd} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} 0 \cos nx + \sum_{n=1,2,3} \frac{2}{n} \sin nx$$

$$f(x) = \frac{\pi}{2} + 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad //$$

S.T in the range $-\pi$ to π. e^x as a fourier series $e^x = \frac{\sinh n\pi}{\pi}\left\{1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right.$ deduce that from that

$$\frac{\pi}{\sinh n\pi} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

Q e^x is neither even nor odd. The fourier's series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx \quad \left[\because \frac{e^x - e^{-x}}{2} = \sinh x\right]$$

$$= \frac{1}{\pi}(e^x)_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}\, e^{\pi} - e^{-\pi}$$

$$a_0 = \frac{1}{\pi}\, 2\sinh nx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi} \quad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + \sin bx)\right]$$

$$= \frac{1}{\pi}\left\{\frac{e^{\pi}}{n^2+1}(\cos n\pi) - \frac{e^{-\pi}}{n^2+1}(\cos n\pi)\right\}$$

$$= \frac{1}{\pi}\,\frac{\cos n\pi}{n^2+1}(e^{\pi} - e^{-\pi}) \Rightarrow \frac{1}{\pi}\,\frac{(-1)^n}{n^2+1}\, 2\sinh n\pi$$

$$\boxed{a_n = \frac{2(-1)^n}{\pi(n^2+1)}\sinh n\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{n^2+1}(\sin nx - \cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{n^2+1}(-n\cos n\pi) - \frac{e^{-x}}{n^2+1}(-n\cos n\pi)\right]$$

$$= \frac{1}{\pi}\left(\frac{-n\cos n\pi}{n^2+1}\right)(e^{\pi} - e^{-\pi})$$

$$\boxed{b_n = \frac{-n(-1)^n}{\pi(n^2+1)}\, 2\sinh n\pi}$$

$$f(x) = \frac{\frac{2\sinh n\pi}{\pi}}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi(n^2+1)}\sinh n\pi\cos nx + \sum_{n=1}^{\infty}\frac{-n(-1)^n}{\pi(n^2+1)}\, 2\sinh n\pi \sin nx$$

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$$= \frac{\sinh \pi}{\pi}\left\{1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2+1}\cos nx + \sum_{n=1}^{\infty} \frac{(-n)(-1)^n}{n^2+1} 2 \sin nx\right\}$$

$$= \frac{\sinh \pi}{\pi}\left\{1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2+1}(\cos nx - n \sin nx)\right\}$$ combination of series

Put $x=0$ point of continuity

$$e^0 = \frac{\sinh \pi}{\pi}\left\{1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}(\cos 0 - n \sin 0)\right\}$$

$$1 = \frac{\sinh \pi}{\pi}\left\{1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}\right\}$$

$$\frac{\pi}{\sinh \pi} = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$= 1 + 2\left[\frac{(-1)^1}{1^2+1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}\right]$$

$$= 1 + 2\left\{-\frac{1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}\right\}$$

$$= 1 - 1 + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{\sinh \pi} = 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} \;//$$

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