Analytical geometry.

unit-5

Condition for the Plane to touch the Quadric cone:.

If condition for the plane $lx+my+nz=0$ to touch the quadric cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

Let (x_1, y_1, z_1) be the point of contact the tangent plane at (x_1, y_1, z_1) is

$$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0.$$

This is identical with the plane $lx+my+nz=0$.

$$\therefore ax_1+hy_1+gz_1-kl=0 \rightarrow (1).$$

$$hx_1+by_1+fz_1-km=0 \rightarrow (2).$$

$$gx_1+fy_1+cz_1-kn=0 \rightarrow (3).$$

Since (x_1, y_1, z_1) lies on $lx+my+nz=0$

$$\therefore lx_1+my_1+nz_1=0 \rightarrow (4).$$

Eliminating x_1, y_1, z_1 from equations (1), (2), (3) and (4)

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Simplifying, we get,

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0 \rightarrow (5)$$

Where A, B, C, F, G, H in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Multiplying (1) by A (2) by H and (3) by G and adding, we get,

$$\Delta x_1 = k(Al+Hm+Gn).$$

Since

$$\Delta = Aa+Hh+Gg$$

$$0 = Ah+Hb+Gf$$

$$0 = Ag+Hf+Gc.$$

III'y,

$$\Delta y_1 = k(Hl+Bm+Fn)$$

$$\Delta z_1 = k(Gl+Fm+Cn).$$

Hence the point of contact,

$$\frac{x_1}{Al+Hm+Gn} = \frac{y_1}{Al+Bm+Fn} = \frac{z_1}{Gl+Fm+Cn}.$$

$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ which is perpendicular to the plane $lx+my+nz=0$ at the origin, is a generator of the cone.

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \rightarrow (6).$$

$$\Delta' = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}, \text{ we get}$$

$$A' = BC-F^2 = a\Delta', \quad F' = GH-AF = f\Delta'$$

$$B' = CA-G^2 = b\Delta', \quad G' = HF-BG = g\Delta'$$

$$C' = AB-H^2 = c\Delta', \quad H' = FG-CH = h\Delta'$$

Hence the perpendicular to the tangent planes to the cone (6) generate cone.

$$A'x^2+B'y^2+C'z^2+2F'yz+2G'zx+2H'xy=0.$$

(ie) $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \rightarrow (7).$

The cones (6) and (7) are said to be reciprocal.

Problems:-

(1) find the equation of the tangent planes to the cone $9x^2+4y^2+16z^2=0$ which contain the line

$$\frac{x}{32} = \frac{y}{72} = \frac{z}{27}$$

Soln:-

The line is the intersection of the planes,

$72x-32y=0$ (ie) $9x-4y=0$

and $27y - 72z = 0$ (i.e) $3y - 8z = 0$.

Hence any plane passing through this line is of the form,

$$9x - 4y + \lambda(3y - 8z) = 0$$

(i.e) $9x + y(3\lambda - 4) - 8\lambda z = 0 \longrightarrow ①$.

This line touches the cone

$$9x^2 - 4y^2 + 16z^2 = 0.$$

Hence the normal to the plane,

$$\frac{x}{9} = \frac{y}{3\lambda - 4} = \frac{z}{-8\lambda} \longrightarrow ③.$$

is a generator of the reciprocal cone of the cone (2).

Equation of the reciprocal cone (2) is

$$\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{16} = 0 \longrightarrow ④.$$

③ is a generator of cone ④.

$$\therefore \frac{9^2}{9} - \frac{(3\lambda - 4)^2}{4} + \frac{(-8\lambda)^2}{16} = 0.$$

$$= 9 - 4(3\lambda - 4)^2 + 64\lambda^2 = 0.$$

$$= 16(9) - 4(9\lambda^2 - 24\lambda + 16) + 64\lambda^2 = 0.$$

$= 144 - 4(9\lambda^2 - 24\lambda + 16) + 64\lambda^2 = 0$

$= 144 - 36\lambda^2 + 96\lambda - 64 + 64\lambda^2 = 0.$

$= 28\lambda^2 + 96\lambda + 80 = 0.$

$\div 4 \quad = 7\lambda^2 + 24\lambda + 20 = 0.$

(i.e) $\lambda = -2$ (or) $\lambda = \frac{-10}{7}.$

Hence the equation of the planes are

$9x - 10y + 16z = 0$ &

$63x - 58y + 80z = 0.$

Ex:2: Find the general equation to a cone which touches the co-ordinate plane.

Soln: If the co-ordinate planes touch a cone, the perpendicular to co-ordinate plane touch the reciprocal cone.

Hence the cone touching the co-ordinate planes is reciprocal to the cone passing through the co-ordinate axes.

The direction cosines of the co-ordinate axes are $(1,0,0;\ 0,1,0;\ 0,0,1)$

The equation of the cone passing through the axis is of the form

$$2fyz + 2gzx + 2hxy = 0$$

The required cone is the reciprocal cone of this cone and its equation is

$$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0.$$

This equation can be put in the form

$$\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$$

The angle between the lines in which the plane $ux + vy + wz = 0$ cuts the cone:-

$$f(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

The plane meets the cone in two lines which pass through the origin and the cone. So the equation of the lines are of the form,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

The line lies in the plane and in the cone,

$$\therefore ul + vm + wn = 0 \longrightarrow (1)$$

$$\text{and } al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \longrightarrow (2).$$

Eliminating between (1) and (2), we get

$$l^2(cu^2+aw^2-2gwn)+2lm(hw^2+cuv-fwv-gvw)$$
$$+m^2(cv^2+bw^2-2fvw)=0 \rightarrow (3).$$

The direction cosines of the two line satisfy the equation (3) and if they are l_1, m_1, n_1 and l_2, m_2, n_2 we have

$$\frac{l_1}{m_1}+\frac{l_2}{m_2} = -\frac{2(hw^2+cuv-fuw-gvw)}{cu^2+aw^2-2gwu}$$

$$\frac{l_1 l_2}{m_1 m_2} = \frac{cv^2+bw^2-2fvw}{cu^2+aw^2-2gwv.}$$

$$\therefore \frac{l_1 l_2}{bw^2+cv^2-2fvw} = \frac{m_1 m_2}{cu^2+aw^2-2gwn.}$$

$$= \frac{l_1 m_2 - l_2 m_1}{\pm\{(hw^2+cuv-fuw-gvw)^2-(bw^2+cv^2-2fvw)(cu^2+uw^2-2gwn)\}^{1/2}}$$

$$= \frac{l_1 m_2 - l_2 m_1}{\pm 2w(-Au^2-Bv^2-Cw^2-2fvw-2Gwu-2Huv)\}^{1/2}}$$

$$= \frac{l_1 m_2 - l_2 m_1}{\pm 2wp} \rightarrow (4).$$

Where $p^2 = -(Au^2 + Bv^2 + Cw^2 + 2Fuw + 2Gwu + 2Huv)$.

and A, B, C, F, G, H are the co-factors of a, b, c, f, g, h in the determinant.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

from the symmetry, we get the expression in (4) is equal to

$$\frac{n_1 n_2}{av^2 + bu^2 - 2huv} = \frac{m_1 n_2 - m_2 n_1}{\pm 2up} = \frac{n_1 l_2 - n_2 l_1}{\pm 2vp} \rightarrow (5)$$

Each expression in (4) and (5) is

$$= \frac{\{\Sigma(m_1 n_2 - m_2 n_1)^2\}^{1/2}}{\pm 2(u^2+v^2+w^2)^{1/2} p.}$$

If θ is the angle between the lines,

$$\frac{\cos\theta}{l_1 l_2 + m_1 m_2 + n_1 n_2} = \frac{\sin\theta}{(\Sigma(m_1 n_2 - m_2 n_1)^2\})^{1/2}}$$

$$\therefore \frac{\cos\theta}{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)} = \frac{\sin\theta}{\pm 2(u^2+v^2+w^2)^{1/2}p} \rightarrow (6).$$

Condition that the cone has three mutually Perpendicular generators:-

The condition that the plane should cut the cone in perpendicular generators is that $\theta = 90°$.

In that case by (6) of the previous section

$$(a+b+c)(u^2+v^2+w^2) = f(u,v,w).$$

The third generator is perpendicular to these two generators. Hence it is normal to the plane containing these perpendicular generators.

If the normal to the plane containing $ux+vy+wz=0$ lies on the cone, we have

$$f(u,v,w)=0$$

$$\therefore a+b+c=0.$$

Example:-

Find the equation to the Cone through the Co-ordinate axes and the lines in which the plane $lx+my+nz=0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Soln:-

Let the equation of the Cone passing through the Co-ordinates axes by $Fyz+Gzx+Hxy=0$.

Eliminating between $lx+my+nz=0$ and

$$ax^2+by^2+c\frac{(lx+my)^2}{n^2}-\frac{2fy(lx+my)}{n}-\frac{2gx(lx+my)}{n}+2hxy=0.$$

(i.e) $x^2(an^2+cl^2-2gln)+\ldots+y^2(cm^2+bn^2-2fmn)=0.$

|||ly eliminate between $lx+my+nz=0$.

and $Fyz+Gzx+Hxy=0$

$$\therefore -\frac{Fy(lx+my)}{n}-\frac{Gx(lx+my)}{n}+Hxy=0$$

(i.e) $Glx^2+\ldots+Fmy^2=0$

Since the two cones have common generator we get,

$$\frac{an^2+cl^2-2gln}{Gl} = \frac{cm^2+bn^2-2fmn}{Fm}$$

Similarly eliminating gx we get the condition

$$\frac{bl^2+am^2-2hml}{Hm} = \frac{an^2+cl^2-2gnl}{Gn}$$

$$\therefore \frac{an^2+cl^2-2gln}{Gnl} = \frac{bl^2+am^2-2hlm}{Hlm}$$

$$= \frac{cm^2+bn^2-2fmn}{Fmn}$$

Hence

$$\frac{F}{l(cm^2+bn^2-2fmn)} = \frac{G}{m(cl^2+an^2-2gln)} =$$

$$= \frac{H}{n(am^2+bl^2-2hlm)}.$$

Hence the equation of the required cone is

$$l(cm^2+bn^2-2fmn)yz+m(cl^2+an^2-2gln)zx$$
$$+n(am^2+bl^2-2hlm)xy=0.$$

Central quadrics:-

Definition:- If $P(x_1, y_1, z_1)$ lies on the surface

$$Ax^2 + By^2 + Cz^2 = 1 \longrightarrow (1).$$

$Q(-x_1, -y_1, -z_1)$ also lies on the surface and O is the origin and mid point of PQ.

Hence all chords of (1) passes through O and bisected at O. for this reason (1) is called a central quadrics, O is called its centre and a chord through O is called a diameter.

Case (i) Let A, B, C be all positive

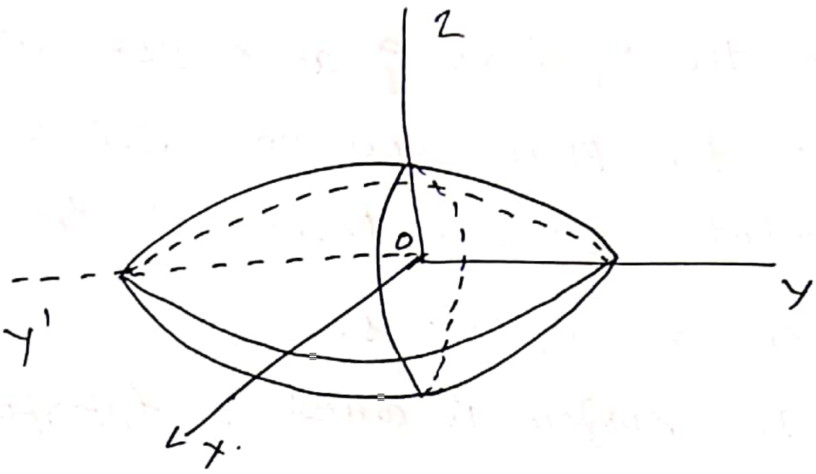
Put $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$ and $C = \frac{1}{c^2}$ and

the equation (1) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \longrightarrow (ii)$$

Put $z = k$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{1-k^2}{c^2}, z = k$ and these are the equation of an ellipse when $k^2 \leq c^2$.

When $k^2 > c^2$, the plane does not cut the surface in real points.



Case (ii)

Let A and B be positive and C be negative and then the $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = -\frac{1}{c^2}$

equation ① becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1+k^2}{c^2}, \quad z = k$$

and for all values of k, this is an ellipse. This surface is called a hyperboloid of one sheet.

Case (iii) Let C be positive and A and B are negative.

Put $C = \frac{1}{c^2}$, $A = -\frac{1}{a^2}$, $B = -1/b^2$ and the equation ① becomes,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The equation of the section of these surface by the plane $z = k$

These are the equations of an ellipse when $k^2 > c^2$.
When $k^2 < c^2$ the plane does not cut the surface in real points. Sections parallel to the yz and zx planes are hyperbolas.

This surface is called a hyperboloid of two sheets.

The intersection of a line and a quadric:-

Let the equation of the straight line be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}.$$

and the quadric be $ax^2+by^2+cz^2=1$.

The co-ordinates of any point on the line are of the form,

$$(x_1+lr,\ y_1+mr,\ z_1+nr).$$

If this point lies on the quadric

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1$$

(ie) $$r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)$$
$$+ax_1^2+by_1^2+cz_1^2-1=0 \quad \rightarrow (1).$$

Tangent and Tangent Planes:-

Any line through $P(x_1, y_1, z_1)$ is of the form

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \rightarrow (1)$$

and if the line meets the Conicoid $ax^2+by^2+cz^2=1$ at the point $\{x_1+lr, y_1+mr, z_1+nr\}$, the parameters r is given by the equation

$$(al^2+bm^2+cn^2)r^2 + 2r(alx_1+bmy_1+cnz_1) + ax_1^2+by_1^2+cz_1^2-1 = 0 \rightarrow (2)$$

Then the equation (2) becomes

$$(al^2+bm^2+cn^2)r^2 + 2r(alx_1+bmy_1+cnz_1) = 0 \rightarrow (3)$$

If line (1) is a tangent to the Conicoid, line (1) will meet the conicoid in two coincident points.

Hence equation (3) has two zero roots

$$\therefore alx_1 + bmy_1 + cnz_1 = 0. \rightarrow (4)$$

(4) is the condition that the line (1) is perpendicular to the line whose direction cosines are proportional to $ax_1 : by_1 : cz_1$.

Hence all tangent lines at $P(x_1, y_1, z_1)$ to the Conicoid is perpendicular to the line whose direction ratios are (ax_1, by_1, cz_1).

Hence all tangent lines at P lie in a Plane through P perpendicular to this direction.

This plane is known as the tangent plane at P and its equation is

$$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)=0.$$

(i.e) $axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2$

(i.e) $axx_1+byy_1+czz_1=1.$

Condition for the plane to touch the conicoid:-

Let the plane touch the conicoid at (x_1, y_1, z_1).

The equation of the tangent plane at (x_1, y_1, z_1) is

$$axx_1+byy_1+czz_1=1 \longrightarrow (1).$$

This plane is also represented by the equation.

$$lx+my+nz=p \longrightarrow (2)$$

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{cz_1}{n}=\frac{1}{p}.$$

(i.e) $x_1=\frac{l}{ap},\ y_1=\frac{m}{bp}=z_1=\frac{n}{cp}$

Since (x_1, y_1, z_1) lies on the conicoid

$$ax_1^2+by_1^2+cz_1^2=1.$$

$$\therefore a\left(\frac{l}{ap}\right)^2+b\left(\frac{m}{bp}\right)^2+c\left(\frac{n}{cp}\right)^2=1.$$

(i.e) $p^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}.$

Example 1:-

If OD is the diameter parallel to a secant APQ through A meeting the conicoid at P and Q show that $\frac{AP \cdot AQ}{OD^2}$ is constant.

Soln:- Let the conicoid be $ax^2+by^2+cz^2=1$ A be (α, β, γ) and the direction cosines of the line APQ be l, m, n.

The equation APQ is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

The co-ordinates of a point at a distance r from A are $(\alpha+lr, \beta+mr, \gamma+nr)$. This point lies on the conicoid.

$$\therefore a(\alpha+lr)^2 + b(\beta+mr)^2 + c(\gamma+nr)^2 = 1.$$

(ie) $r^2(al^2+bm^2+cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0.$

$$\therefore AP \cdot AQ = \frac{a\alpha^2 + b\beta^2 + c\gamma^2 - 1}{al^2+bm^2+cn^2}$$

The direction cosines of the line OD are also l, m, n. D is the point (lk, mk, nk), where $k = OD$

Since D lies on the conicoid $al^2k^2 + bm^2k^2 + cn^2k^2 = 1$

$$\therefore k^2 = \frac{1}{al^2+bm^2+cn^2}$$

Hence $\frac{AP \cdot AQ}{OP^2} = \frac{AP \cdot AQ}{\alpha^2} = a\alpha^2 + b\beta^2 + c\gamma^2 = 1$

$= \text{Constant}$

(2) Find the equation of the tangent planes to $x^2+y^2+4z^2=1$ which intersect in the line whose equations are

$$12x - 3y - 5 = 0, \; z = 1.$$

Soln:-

Any plane which passes through the line is given by

$$12x - 3y - 5 + \lambda(z-1) = 0.$$

(i) $12x - 3y - 5 + z\lambda - \lambda = 0.$

$$12x - 3y + \lambda z - (\lambda + 5) = 0 \rightarrow (1).$$

Let this plane touch the conicoid at (x_1, y_1, z_1)

The equation of the tangent plane at (x_1, y_1, z_1) is

$$x x_1 + y y_1 + 4 z z_1 = 1 \rightarrow (2).$$

Eqn (1) & (2) represents the same plane.

$$\therefore \frac{x_1}{12} = \frac{y_1}{-3} = \frac{4z_1}{\lambda} = \frac{1}{\lambda+5}$$

$$\therefore x_1 = \frac{12}{\lambda+5} \; ; \; y_1 = \frac{-3}{\lambda+5} \; ; \; z_1 = \frac{\lambda}{4(\lambda+5)}$$

Since (x_1, y_1, z_1) lies on the conicoid $x_1^2 + y_1^2 + 4z_1^2 = 1.$

$$\therefore \left(\frac{12}{\lambda+5}\right)^2 + \left(\frac{-3}{\lambda+5}\right)^2 + 4 \cdot \left\{\frac{\lambda}{4(\lambda+5)}\right\}^2 = 1.$$

$$\frac{12^2}{(\lambda+5)^2} + \frac{(-3)^2}{(\lambda+5)^2} + \frac{4}{4^2}\left[\frac{\lambda^2}{(\lambda+5)^2}\right] = 1.$$

$$144 + 9 + \frac{4\lambda^2}{4^2} = (\lambda+5)^2$$

$$576 + 36 + \lambda^2 = 4(\lambda+5)^2$$

$$= 4(\lambda^2 + 25 + 10\lambda)$$

$$= 4\lambda^2 + 100 + 40\lambda.$$

$$576 + 36 + \lambda^2 - 4\lambda^2 - 100 - 40\lambda = 0.$$

$$-3\lambda^2 - 40\lambda + 512 = 0.$$

(ie) $3\lambda^2 + 40\lambda - 512 = 0.$

by solving $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Here $a = 3$, $b = 40$, $c = -512$

$$\lambda = \frac{-40 \pm \sqrt{40^2 - 4(3)(-512)}}{2(3)}$$

$$= \frac{-40 \pm \sqrt{1600 + 6144}}{6}$$

$$\lambda = \frac{-40 \pm \sqrt{7744}}{6}.$$

$$\lambda = \frac{-40 \pm 88}{6}$$

$$\lambda = 48/6 \quad ; \quad \lambda = \frac{-128}{6}$$

$$\lambda = 8 \quad , \lambda = -64/3$$

Hence the equations of the tangent planes are sub λ values in eqn ①.

$$12x - 3y - 5 + \lambda(z-1) = 0.$$

sub $\lambda = 8$

$$12x - 3y - 5 + 8(z-1) = 0$$

$$12x - 3y - 5 + 8z - 8 = 0$$

$$12x - 3y + 8z - 13 = 0.$$

and.

Sub $\lambda = -64/3$

$$12x - 3y - 5 + 64/3\,(z-1) = 0.$$

$$36x - 9y - 15 - 64z + 64 = 0$$

$$36x - 9y - 64z + 49 = 0.$$

③ write down the equation of tangent plane at (x_1, y_1, z_1) to the cone.

Soln:-

If $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is a tangent to the cone at

$$l(ax_1 + hy_1 + gz_1) + m(hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1) = 0.$$

(4) write down the two tangent Planes to a conicoid (27)

Parallel to plane $lx + my + nz = 0$.

Soln:-

The condition for the Plane $lx + my + nz = 0$ to touch the conicoid $ax^2 + by^2 + cz^2 = 1$

Let the plane touch the conicoid at (x, y, z). The equation of tangent plane at (x, y, z)

$$axx_1 + byy_1 + czz_1 = 1 \longrightarrow (1)$$

This plane is also represent by the equation

$$lx + my + nz = p \longrightarrow (2)$$

$$x_1 = \frac{l}{ap} ; \quad y_1 = \frac{m}{bp} ; \quad z = \frac{n}{cp}$$