Vector Calculus and Fourier Series

(16SCCMM7)

Unit - IV.

Fourier-series - definition - Fourier series expansion of periodic functions with period 2π and period $2a$ - use of odd and even functions in Fourier series.

Fourier series:-

Fourier series is an infinite series representation of periodic functions in terms of the trignometric sine and cosine functions.

Periodic functions:-

A function $f(x)$ is said to be periodic function with period $T>0$ if for all x,
$f(x+T) = f(x)$ and T is the least of such values.

Ex: 1) $\sin x$, $\cos x$ are periodic functions with period 2π.

2) $\tan x$, $\cot x$ are periodic functions with period π.

Euler's formulae:-

The fourier series for the function $f(x)$ in the interval $c \le x \le c+2\pi$ is given by defined in $(-l, l)$,

3) ...

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx\,dx.$$

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx\,dx.$$

These values a_0, a_n, b_n are known as Euler's formulae.

Condition for fourier expansion (Dirchlet Condition)

A function $f(x)$ defined in $[0, 2\pi]$ has a valid fourier series expansion of the form.

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, b_n are constants Provided,

1) $f(x)$ is well defined and single valued, expect possibly at a finite number of Point in the interval $[0, 2\pi]$.

2) $f(x)$ has infinite number of finite discontinues in the interval in $[0, 2\pi]$

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3) $f(x)$ has finite number of finite maxima and minima

Definition of Fourier Series:-

* Let $f(x)$ be a function defined in $[0, 2\pi]$

Let $f(x+2\pi) = f(x)$ for all x, then the fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx.$$

Those values of a_0, a_n, b_n are called as fourier co-efficients of $f(x)$ in $[0, 2\pi]$

* Let $f(x)$ be a function defined in $[-\pi, \pi]$

Let $f(x+2\pi) = f(x)$ for all x, then the fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

These values a_0, a_n, b_n are called as fourier co-efficients of $f(x)$ in $[-\pi, \pi]$.

* Let $f(x)$ be a function defined in $[0, 2l]$, let $f(x+2l) = f(x)$ for all x, then the fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx.$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$b_n = 1/l\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

These values a_0, a_n, b_n are called as fourier co efficients of $f(x)$ in $[0, 2l]$.

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* let $f(x)$ be a function defined in $(-l, l)$, let $f(x+2\pi) = f(x)$ for all x, then the fourier Series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$.

$$\text{Where } a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

These values a_0, a_n, b_n are called as fourier Co-efficients of $f(x)$ in $[-l, l]$.

Fourier Series for Even and odd functions:-

We know that $f(x)$ be a function defined in $[-\pi, \pi]$. let $f(x+2\pi) = f(x)$ for all x, then the fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

These values a_0, a_n, b_n are called as fourier Co-efficients of $f(x)$ in $[-\pi, \pi]$.

Case(i) Even function:-

A function is called even if

$f(-x) = f(x)$

then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= 2/\pi \int_{0}^{\pi} f(x)\,dx.$

since $\cos nx$ is an even function, $f(x)$ is an even function.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \, dx.$

In even function $b_n = 0$, because.

If a function $f(x)$ is even in $[-\pi, \pi]$, its fourier series contain only cosine terms.

$$\boxed{\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.}$$

Case (ii) When Odd function:-

A function is called odd if

$f(-x) = -f(x)$

then $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0.$

Since $\cos nx$ is an even function, $f(x)$ is an odd function.

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = 0.$$

Now, $\sin nx$ is an odd function, $f(x)$ is an odd function.

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

$$= 2/\pi \int_{0}^{\pi} f(x) \sin nx\, dx.$$

Thus, if a function $f(x)$ is odd in $[-\pi, \pi]$, its fourier series expansions contains only sine terms.

$$\boxed{\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

Problems on fourier series:-

① find the fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$.

Soln:-

We know that the fourier series of $f(x)$ defined in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Here $f(x) = x^2$.

Now,

Step 1: To find a_0.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx.$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{3\pi}\left[2\pi^3 - 0\right].$$

$$= 8/3\ \pi^2$$

$$\boxed{a_0 = 8/3\ \pi^2}$$

Step 2: To find a_n:-

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx.$$

Apply $\int uv$ formula.

$$\int uv\,dx = u\,dv\,dx + v\,du\,dx.$$

$$\frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left[x^2\int \cos nx\,dx - \left\{\int \cos nx\,dx.\ \frac{d}{dx}(x^2)\,dx\right\}\right.$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \int 2x\left(\frac{\sin nx}{n}\right)dx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left\{\int x \sin nx \, dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \int 1 . \frac{\cos nx}{n} dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\int \cos nx dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \quad \left[\because \begin{array}{l} \cos 2n\pi = 1 \\ \sin 2n\pi = 0 \end{array}\right]$$

$$\Rightarrow \boxed{a_n = 4/n^2}$$

Step 3: To find b_n :-

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx.$$

using $\int uv \, dx$ formula.

$$= \frac{1}{\pi}\left[x^2\int \sin nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \sin nx\,dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \left\{\int 2x\left(-\frac{\cos nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left\{\int x\cos nx\,dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \int 1\cdot\frac{\sin nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\int \sin nx\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}\left(x\sin nx + \frac{1}{n}\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}x\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= -\frac{4\pi}{n} \qquad \left[\because \begin{array}{l}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{array}\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\therefore f(x) = x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx\right]$$

$$\Rightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx\right)$$

This is fourier series for the function $f(x) = x^2$.

Q) Determine the fourier series expansion of the function $f(x) = x$, where $-\pi \leq x \leq \pi$.

Soln:-

Check whether it is odd or even.

$$f(-x) = -x = -f(x)$$

It is an odd function.

Hence $a_0 = 0$, $a_n = 0$. for all $x, n \geq 0$.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Here $f(x) = x$.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x^2\int \sin nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \sin nx\,dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \left\{\int 2x\left(-\frac{\cos nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left\{\int x\cos nx\,dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \int 1\cdot\frac{\sin nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\int \sin nx\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}\left(x\sin nx + \frac{1}{n}\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}x\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= -\frac{4\pi}{n} \quad \left[\because \begin{array}{l}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{array}\right]$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cdot \sin nx \, dx$$

$$\int u\,dv = uv - \int v\,du$$

$u = x$ $\qquad dv = \sin nx \, dx$

$du = dx$ $\qquad v = -\frac{\cos nx}{n}$

$$\frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi}\left[x \frac{-\cos nx}{n} - \int \frac{-\cos nx}{n} . dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x \frac{-\cos nx}{n} - \frac{1}{n}\int - \cos nx \, dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi \frac{\cos n\pi}{n} - \frac{\sin n\pi}{n^2} - (0+0)\right]$$

$\therefore \cos n\pi = (-1)^n$

$\sin n\pi = 0 .$

$\cos 0 = 1$

$\sin 0 = 1 .$

$$= \frac{2}{\pi}\left[-\pi \frac{(-1)^n}{n}\right]$$

$$= 2\left[\frac{(-1)(-1)^n}{n}\right]$$

$$b_n = 2\left[\frac{(-1)^{n+1}}{n}\right]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= \sum_{n=1}^{\infty} 2\left[\frac{(-1)^{n+1}}{n}\right] \sin nx.$$

(3) find the fourier series for $f(x) = x^2$ in $-\pi \leq x \leq \pi$ and deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \pi^2/6.$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$

Soln:-

Given $f(x) = x^2$,

$$f(-x) = (-x)^2 = f(x).$$

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Therefore $f(x)$ is an even function in $(-\pi, \pi)$.

Hence $b_n = 0$.

$\therefore$ The fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = 2/\pi \int_0^{\pi} x^2\,dx.$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} dx$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$= 2/\pi \left(\pi^{\cancel{3}\,2}/3\right)$$

$$= \frac{2\pi^2}{3}.$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= 2/\pi \int_0^{\pi} x^2 \cos nx\,dx$$

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$\int u\,dv = uv - \int v\,du$

$u = x^2 \qquad dv = \cos nx\,dx.$

$du = 2x\,dx \qquad v = \dfrac{\sin nx}{n}.$

$$\frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - \int \frac{\sin nx}{n} \cdot 2x\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - \int \frac{\sin nx}{n} 2x\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\int x \sin nx\,dx\right]_0^{\pi}.$$

$u = x \qquad dv = \sin nx\,dx$

$du = dx \qquad v = -\dfrac{\cos nx}{n}.$

$$= 2/\pi\left[x^2 \frac{\sin nx}{n} - \frac{2}{n}\left[x - \frac{\cos nx}{n} - \int -\frac{\cos nx}{n}\,dx\right]_0^{\pi}\right.$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - \frac{2}{n}\left[x \frac{-\cos nx}{n} - \frac{1}{n}\int -\cos nx\,dx\right]_0^{\pi}\right.$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - \frac{2}{n}\left[x \frac{-\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right]_0^{\pi}\right.$$

$\sin n\pi = 0$

$\sin 0 = 0.$

$$= \frac{2}{\pi}\left[\left(\pi^2 \frac{\sin n\pi}{n} - \frac{2}{\pi}\left[\frac{-\pi \cos n\pi}{n} + \frac{1}{n}\frac{\sin n\pi}{n}\right]\right) - \right.$$

$$\left. 0\,\frac{\sin n0}{n} - \frac{2}{n}\left[\frac{0\cos n0}{n} + \frac{1}{n}\frac{\sin n0}{n}\right]\right).$$

$$= 2/\pi \left[\frac{2}{n}\left(\frac{\pi \cos n\pi}{n}\right)\right] \qquad \cos n\pi = (-1)^n.$$

$$= \frac{4\pi}{\pi n^2}(-1)^n$$

$$a_n = 4/n^2\,(-1)^n.$$

$$a_0 = \frac{2\pi^2}{3}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$f(x) = \frac{2\pi^2}{2.3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos nx$$

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Deduction(i):

Put $x = \pi$, Here $x = \pi$ is a point of continuity

Sub $x = \pi$, $f(x) = \pi^2$

$\therefore f(x)$ at $x = \pi = \pi^2$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n . \cos n\pi .$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n . (-1)^n .$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\frac{2\pi^2}{3 . 4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \rightarrow (1).$$

Deduction(ii):

Put $x = 0$,

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \cos n(0).$$

$$-\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \qquad \because \cos 0 = 1$$

$$\frac{-\pi^2}{3.4} = \sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n.$$

$$\frac{-\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots \rightarrow (2).$$

Deduction 3:-

Add (1) & (2) we get

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] = \frac{\pi^2}{6} + \frac{\pi^2}{12}.$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] = \frac{2\pi^2 + \pi^2}{12}$$

$$= \frac{3\pi^2}{12}$$

$$= \pi^2/4.$$

Hence the result.

(4) Express $f(x) = \frac{1}{2}(\pi - x)$ a fourier series with period 2π to be valid in an interval 0 to 2π.

Soln:-

$$f(2\pi) = \frac{1}{2}(\pi - 2\pi)$$

$$= \frac{1}{2}(-\pi)$$

$$= -\pi/2.$$

It is odd function.

$\therefore a_0 = 0, \quad a_n = 0.$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,dx.$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} \pi \sin nx\,dx - \int_0^{2\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\pi\int_0^{2\pi} \sin nx\,dx - \int_0^{2\pi} x\sin nx\,dx\right].$$

$u = x \qquad dv = \sin nx\,dx$

$du = dx \qquad v = -\frac{\cos nx}{n}.$

$$= \frac{1}{2\pi}\left[\pi\left(-\frac{\cos nx}{n}\right) - \left[x\left(-\frac{\cos nx}{n}\right) - \int -\frac{\cos nx}{n}\,dx\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n}(-\cos nx) - \left[\frac{x}{n}(-\cos nx) + \frac{1}{n}\left[\frac{\sin nx}{n}\right]\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-\pi}{n} + \frac{2\pi}{n} + \frac{\pi}{n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

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$$b_n = \frac{1}{n}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

$$= \sin x + \frac{1}{2} \sin 2x + \cdots$$

(5) Find the fourier series of Periodicity 2π for

$$f(x) = \begin{cases} x, & (0, 2\pi) \\ 2\pi - x & (\pi, 2\pi). \end{cases}$$

Soln:-

Let the fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi} \int_0^{\pi} x\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} (2\pi - x)\, dx.$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx \right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi \cdot 2\pi - \frac{(2\pi)^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right].$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{4\pi^2}{2} - \frac{3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right]$$

$x.$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\, dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\, dx\right]$$

$u = x. \qquad dv = \cos nx\, dx$

$du = dx \qquad v = \frac{\sin nx}{n}.$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n} - \int\frac{\sin nx}{n}dx\right]\right.$$

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$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right)+\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[(2\pi-x)\left(\frac{\sin nx}{n}\right)-\left(\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\pi\left(\frac{\sin n\pi}{n}\right)+\left(\frac{\cos n\pi}{n^2}\right)+0\right]+\right.$$

$$\frac{1}{\pi}\left[(2\pi-2\pi)\left(\frac{\sin 2\pi}{n}\right)-\left(\frac{\cos n\pi}{n^2}\right)+\right.$$

$$\left.(2\pi-\pi)\left(\frac{\sin n\pi}{n}\right)-\left(\frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(0+\frac{(-1)^n}{n^2}\right)-\left(0+\frac{1}{n^2}\right)\right]+\frac{1}{\pi}\left[\left(0-\frac{1}{n^2}\right)-\left(0-\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]+\frac{1}{\pi}\left[\frac{-1}{n^2}+\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{(-1)^n}{n^2}\right)-\frac{1}{n^2}-\frac{1}{n^2}+\left(\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}-\frac{1}{n^2}+\left(\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(x-2\pi)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x\sin nx\,dx - \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n}\right)\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n}\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}$$

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$$= \frac{1}{\pi}\left[\left(\frac{-\pi(-1)^n}{n} + 0\right) - (0+0)\right] +$$

$$\frac{1}{\pi}\left[(0-0) - \left[\frac{-\pi(-1)^n}{n} - 0\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$= 0.$$

$$b_n = 0.$$

Hence $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \cos nx$$