Bharath college of Science and management,

Thanjavur - 5

Department of mathematics,

Subject : Linear Algebra

subject code : 16SCCMM8

unit : 1 to 5

vector space:

2m (R) U.Q 1. Define vector space.

A non empty set V is said to be a vector space over a field F if.

i. V is an abelian group under an operation called addition which we denote by '+'

ii. For every $\alpha \in F$ and $v \in V$, there is defined an element αv in V subject to the following conditions.

i. $\alpha(u+v) = \alpha u + \alpha v$ for all $u, v \in V$ and $\alpha \in F$.

ii. $(\alpha+\beta)u = \alpha u + \beta u$ for all $u, \in V$ and $\alpha, \beta \in F$

iii. $\alpha(\beta u) = (\alpha\beta)u$ for all $u \in V$ and $\alpha, \beta \in F$.

iv. $1u = u$ for all $u \in V$.

Examples:

1. $R \times R$ is a vector space over R under addition and scalar multiplication defined by

$(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$ and

$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.

→

Proof.

The binary operations '+' is commutative and associative.

(0,0) is the zero element.

The inverse of (x_1, x_2) is $(-x_1, -x_2)$

$\therefore (R \times R)$ is abelian group.

Now, let $u = (x_1, x_2)$

$v = (y_1, y_2)$ and $\alpha, \beta, \in R$

i. $\alpha(u+v) = \alpha((x_1, x_2) + (y_1, y_2))$

$= \alpha((x_1+y_1), (x_2+y_2))$

$= (\alpha(x_1+y_1), \alpha(x_2+y_2))$

$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2)$

$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$

$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$

$\alpha(u+v) = \alpha u + \alpha v$

for $u, v, \in R$.

ii. $(\alpha+\beta)u = (\alpha+\beta)(x_1, x_2)$

$= ((\alpha+\beta)x_1 - (\alpha+\beta)x_2)$

$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2)$

$= ((\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$.

$= \alpha(x_1, x_2) + \beta(x_1, x_2)$

$= \alpha u + \beta u$.

iii. $\alpha(\beta u) = \alpha(\beta(x_1, x_2))$

$= \alpha(\beta x_1, \beta x_2)$

$= \alpha\beta x_1, \alpha\beta x_2$

$= \alpha\beta(x_1, x_2)$

$= \alpha\beta(u).$

iv. $1u = 1\cdot(x_1 - x_2)$

$= (x_1, x_2)$

$= u.$

2. $R^n = \{x_1, x_2 \ldots\ldots x_n / x_i \in R, 1 \le 1 \le n\}$ then, R^n is a vector space over R under addition and scalar multiplication defined by, $(x_1, x_2, \ldots x_n) + (y_1, y_2 \ldots y_n) = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$ and $\alpha(x_1, x_2 \ldots x_n) = (\alpha x_1, \alpha x_2 \ldots\ldots \alpha x_n).$

proof:

Clearly, the Binary operations '+' is commutative and Associative.

$(0, 0, 0 \ldots 0)$ is the zero element.

The inverse of $(x_1, x_2 \ldots x_n)$ is $(-x_1, -x_2 \ldots -x_n)$

Hence, $(R^n, +)$ is an abelian group.

$u = (x_1, x_2 \ldots\ldots x_n)$

$v = (y_1, y_2 \ldots\ldots y_n)$

$\alpha, \beta \in R$

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i. $\alpha(u+v) = \alpha(x_1, x_2, \ldots, x_n) + (y_1, y_2 \ldots\ldots y_n)$

$= \alpha(x_1+y_1, x_2+y_2, \ldots\ldots x_n+y_n)$

$= (\alpha(x_1+y_1), \alpha(x_2+y_2) \ldots \alpha(x_n+y_n))$

$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2 \ldots \alpha x_n + \alpha y_n$

$= \alpha(x_1, x_2 \ldots x_n) + \alpha(y_1, y_2 \ldots y_n)$

$\alpha(u+v) = \alpha u + \alpha v.$

ii. $(\alpha+\beta)u = (\alpha+\beta)(x_1, x_2 \ldots x_n)$

$= (\alpha x_1, \beta x_1, \alpha x_2 + \beta x_2 \ldots\ldots \alpha x_n + \beta x_n)$

$= (\alpha x_1, \alpha x_2 \ldots \alpha x_n) + (\beta x_1, \beta x_2 \ldots \beta x_n)$

$= \alpha(x_1, x_2 \ldots x_n) + \beta(x_1, x_2 \ldots x_n)$

$= \alpha u + \beta u$

$(\alpha+\beta)u = \alpha u + \beta u$

iii. $\alpha(\beta u) = \alpha(\beta(x_1, x_2 \ldots x_n))$

$= \alpha(\beta x_1, \beta x_2 \ldots \beta x_n)$

$= \alpha\beta x_1, \alpha\beta x_2 \ldots \alpha\beta n$

$= \alpha\beta(x_1, x_2 \ldots x_n)$

$= (\alpha\beta)u.$

iv. $1 \cdot u = 1(x_1, x_2 \ldots x_n)$

$= x_1, x_2 \ldots x_n$

$1 \cdot u = u$

$n = u.$

$\mathbb{R}$ is a vector space over $\mathbb{R}$.

3. Let F be a Field, let $F^n = \{x_1, x_2, \ldots x_n \ldots x_n / x_i \in F\}$. In F^n we defined addition and scalar multiplication $(x_1, x_2 \ldots x_n) + (y_1, y_2 \ldots y_n) = (x_1 + y_1, x_2 + y_2 \ldots x_n + y_n)$ and $\alpha(x_1, x_2 \ldots x_n) = (\alpha x_1, \alpha x_2 \ldots \alpha x_n)$. Then F^n is a vector space over a F and we denote the vector space $V_n(F)$.

Soln: Clearly, the Binary operations '+' is commutative and Associative

$(0,0,0)$ is the zero element.

the inverse of $(x_1, x_2 \ldots x_n)$ is $(-x_1, -x_2 \ldots -x_n)$.

$(F, +)$ is a abelian group.

$u = (x_1, x_2 \ldots x_n)$

$V = (y_1, y_2 \ldots y_n)$

$\alpha, \beta \in \mathbb{R}$.

i. $\alpha(U+V) = \alpha(x_1, x_2 \ldots x_n) + (y_1, y_2 \ldots y_n)$

$= \alpha(x_1+y_1, x_2+y_2 \ldots x_n+y_n)$

$= (\alpha(x_1+y_1), \alpha(x_2+y_2) \ldots \alpha(x_n+y_n))$

$= \alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2 \ldots \alpha x_n + \alpha y_n$

$= \alpha(x_1, x_2 \ldots x_n) + \alpha(y_1, y_2 \ldots y_n)$

$= \alpha(U) + \alpha V$

$\alpha(U+V) = \alpha U + \alpha V.$

ii. $(\alpha+\beta)U = (\alpha+\beta)(x_1, x_2 \ldots\ldots x_n)$

$= \alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2 \ldots\ldots \alpha x_n + \beta x_n$

$= \alpha(x_1, x_2 \ldots\ldots x_n) + \beta(x_1, x_2 \ldots\ldots x_n)$

$= \cancel{\alpha} \; \alpha U + \beta U$

$(\alpha+\beta)U = \alpha U + \beta U$

iii. $\alpha(\beta U) = \alpha(\beta(x_1, x_2 \ldots x_n))$

$= \alpha(\beta x_1, \beta x_2 \ldots \beta x_n)$

$= \alpha\beta x_1, \alpha\beta x_2 \ldots \alpha\beta x_n$

$= \alpha\beta(x_1, x_2 \ldots\ldots x_n)$

$= \alpha\beta(u)$

iv. $1.V = 1(x_1, x_2 \ldots\ldots x_n)$

$= x_1, x_2 \ldots x_n$

$= U.$

$n = U.$

F^n is a vector space over a F.

4. C is a vector space over the field R. Here addition is the usual addition is is C and the scalar multiplication is the usual multiplication of real number and a complex number

$(x_1 + ix_2) + (y_1 + iy_2) = (x_1 + y_1) + i(x_2 + y_2)$

$\alpha(x_1 + ix_2) = \alpha x_1 + i\alpha x_2$.

proof :

clearly, the Binary operation '+' is commutative & associative.

$(0, 0)$ is zero element.

and the inverse $(x_1 + ix_2)$ is $(-x_1 - ix_2)$

$\therefore (C, +)$ is an abelian group.

let $U = x_1 + ix_2$

$V = y_1 + iy_2$

$\alpha, \beta \in R$

i. $\alpha(U+V) = \alpha((x_1 + ix_2) + (y_1 + iy_2))$

$= \alpha((x_1 + y_1) + i(x_2 + y_2))$

$= \alpha(x_1 + y_1) + \alpha i(x_2 + y_2)$

$= \alpha x_1 + \alpha y_1 + i(\alpha x_2 + \alpha y_2)$

$= (\alpha x_1 + i\alpha x_2) + (\alpha y_1 + i\alpha y_2)$

$= \alpha(x_1 + ix_2) + \alpha(y_1 + iy_2)$

$= \alpha U + \alpha V$

ii. $(\alpha + \beta)U = (\alpha + \beta)(x_1 + ix_2)$

$= (\alpha + \beta)x_1 + (\alpha + \beta)ix_2$

$= \alpha x_1 + \beta x_1 + i\alpha x_2 + i\beta x_2$

$= \alpha x_1 + \beta x_1 + i(\alpha x_2 + \beta x_2)$.

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$= (\alpha x_1 + i\alpha x_2) + (\beta x_1 + i\beta x_2)$

$= \alpha(x_1 + ix_2) + \beta(x_1 + i\beta x_2)$

$= \alpha U + \beta U$

iii. $\alpha(\beta U) = \alpha(\beta(x + ix_2))$

$= \alpha(\beta x + \beta i x_2)$

$= \alpha\beta x + \alpha\beta i x_2$

$= \alpha\beta(x + ix_2)$

$= \alpha\beta(u).$

iv. $1 \cdot U = 1 \cdot (x_1 + ix_2)$

$= x_1 + ix_2$

$n. = U$

c is a vector group over the field.

Hence, proved.

(X) Let V denote the set of all solutions of the differential equation $2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$

Sn. The V is vector space over R.

→ proof:

Let, $f, g \in V$ and $\alpha \in R$ then;

$\frac{2d^2f}{dx^2} - 7\frac{df}{dx} + 3f = 0$ and $\frac{d^2g}{dx^2} - 7\frac{dg}{dx} + 3a = 0.$

$\therefore 2\left[\frac{d^2f}{dx^2}+\frac{d^2g}{dx^2}\right]-7\left[\frac{df}{dx}+\frac{dg}{dx}\right]+3[f+g]=0$ 9

Hence, $(f+g) \in V$.

Since the operation are usual addition and a scalar multiplication. The axioms of vector space are true,

Hence, V is a vector space over R.

6. R is not a vector space over C.

proof:

clearly, the binary operations '+' is commutative and associative, zero element and and inverse in R.

$\therefore (R,+)$ is an abelian group.

But, the scalar multiplication is not defined,

for eg $\alpha=(a+ib)\in C$ and $u\in R$ then.

$\alpha u = au+ibu \notin R$

$\therefore$ R is not a vector space over C.

7. consider $R\times R$ with usual addition we define scalar multiplication by $\alpha(x,y)=(\alpha x,\alpha^2 y)$

Then, $R\times R$ is not a vector space over R.

proof:

clearly, the binary operation '+' is commutative and associative.

$(0,0)$ is the zero element

The inverse of (x,y) is $(-x,-y)$

Hence, $(R\times R, +)$ is an abelian group.

$$(\alpha+\beta)u = (\alpha+\beta)(x,y)$$
$$= (\alpha+\beta)x, (\alpha+\beta)^2 y)$$
$$= ((\alpha+\beta)x, (\alpha^2+\beta^2+2\alpha\beta)y)$$
$$= (\alpha x+\beta x, \alpha^2 y + \beta^2 y + 2\alpha\beta y)$$

Also,

$$\alpha(x,y)+\beta(x,y) = ((\alpha x, \alpha^2 y)+(\beta x, \beta^2 y))$$
$$= (\alpha x+\beta x, \alpha^2 y+\beta^2 y)$$

$$(\alpha+\beta)u \neq \alpha(x,y)+\beta(x,y).$$

$\therefore$ $R\times R$ is not a vector space over R.

8. Consider $R\times R$ with usual addition to define the scalar multiplication as $\alpha(a,b)=(0,0)$ Then $R\times R$ is not a vector space.

proof:

Clearly, the binary operation '+' is commutative and associative.

$(0,0)$ is the zero element.

The inverse of (a,b) is $(-a,-b)$

Hence, $(R\times R, +)$ is abelian group.

Also,

$\alpha(u+v)=0$ and

$\alpha u+\alpha v=0+0=0$ so that,

$\alpha u+\alpha v=\alpha u+\alpha v$

ii. $(\alpha+\beta)u=0$

$\alpha u+\beta u=0+0-0$

$(\alpha+\beta)u=\alpha u+\beta u$.

iii. $\alpha(\beta u)=0$

$(\alpha\beta)u=0$

$\alpha(\beta u)=(\alpha\beta)u$.

iv. $1u=1(a,b)$

$=(0,0)$

$\therefore R\times R$ is not a vector space.

9. let, V be the set of all ordered pairs of the real numbers addition and multiplication define by $(x,y)+(x_1,y_1)=(x+x_1, y+y_1)$ and $\alpha(x,y)=(x,\alpha y)$ where, x, y, x_1, y_1 and α are real number. Then ~~r~~ V is not a vector Space over R.

proof:

clearly, the binary operation '+' is commulative and Associative.

$(0,0)$ is the zero group.

The Inverse of (x,y) is $(-x,-y)$.

Hence $(R \times R, +)$ is a abelian group

Let, $\alpha, \beta \in R$ and $(x, y) \in V$

Now, $(\alpha+\beta)(x, y) = (x, (\alpha+\beta)y)$

$= (x(\alpha y + \beta y)) \rightarrow ①$

$\alpha(x, y) + \beta(x, y) = (x, \alpha y) + (x, \beta y)$.

$= (x + x, \alpha y + \beta y)$

$= (2x, \alpha y + \beta y) \rightarrow ②$

$(\alpha+\beta)(x, y) \neq \alpha(x, y) + \beta(x, y)$

Hence, V is not a vector space over R.

10 Let, R^+ be the set of all positive real number

Define addition and scalar multiplication as follows:

$u + v = uv$ for all $u, v \in R^+$

$\alpha u = u^\alpha$ for all $u \in R^+$ and $\alpha \in R$.

Then R^+ is a real vector space

Proof:

Clearly, the binary operation '+' is commutative and associative.

$(1, 1)$ is the identity element.

The inverse of u is $-u$.

clearly $(R^+, +)$ is an abelian group with identity 1.

Now,

$$\alpha(u+v) = \alpha(uv)$$
$$= (uv)^\alpha$$
$$= u^\alpha v^\alpha$$
$$= \alpha u + \alpha v.$$

$$(\alpha+\beta)u = u^{\alpha+\beta}$$
$$= u^\alpha u^\beta$$
$$= \alpha u + \beta u.$$

$$\alpha(\beta u) = \alpha u^\beta$$
$$= (u^\beta)^\alpha = u^{\beta\alpha} = u^{\alpha\beta} = (\alpha\beta)u$$

Also, $1u = u^1 = u.$

$\therefore R^+$ is a vector space over R.

U.Q.14. Let V be the vector space over a field F.

Then, i. $\alpha 0 = 0$ for all $\alpha \in F$.

ii. $0v = 0$ for all $v \in V$.

iii. $(-\alpha)v = \alpha(-v) = -(\alpha v)$ for all $\alpha \in F$ and $v \in V$.

iv. $\alpha v = 0 \Rightarrow \alpha = 0$ or $v = 0$.

Proof:

i. $\alpha 0 = \alpha(0+0)$
$= \alpha 0 + \alpha 0$

hence, $\alpha 0 = 0$.

ii. $0v = (0+0)v$

$= 0v + 0v$

hence $0v = 0$.

iii. $0 = [\alpha + (-\alpha)]v = \alpha v + (-\alpha)v$

hence $(-\alpha)v = -\alpha v$

Similarly $\alpha(-v) = \alpha(v)$

hence.

iv. Let, $\alpha v = 0$

if $\alpha = 0$, there is nothing to prove.

$\therefore$ Let $\alpha \neq 0$

Then $\alpha^{-1} \in F$

now,

$$\begin{aligned} v &= 1v \\ &= (\alpha^{-1}\alpha)v \\ &= \alpha^{-1}(\alpha v) \\ &= \alpha^{-1}(0) \\ &= 0. \end{aligned}$$

V.Q.

Subspaces:

Definition:

Let V be a vector space over a field F. A non-empty subset W of V is called a subspace of V if W itself is a vector space over F under the operations of V.

Theorem:

Let V be a vector space over F.

15 A non-empty subset w of V is a subspace of V. if and only iff w is closed with respect to vector addition and scalar multiplication in V.

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proof:

Let w is colse

Let w be a vector subspace of V.

Then, W itself is a vector space and hence W is closed with respect to vector addition and scalar multiplication.

Conversly, let w be a non empty subset of V. such that $u, v \in W \Rightarrow u+v \in W$. and $u \in W$ and $\alpha \in F$.

$\Rightarrow \alpha u \in W$

we prove that, W is a subspace of V.

Since, W is non-empty. there exists an element $u \in W$.

$\therefore 0u = 0 \in W$.

Also, $v \in W \Rightarrow (-1)v = -v \in W$

Thus W Contains 0 and the additive inverse of each of its element.

Hence, W is an additive subgroup of V.

Also, $u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$

Since, the element of W are the elements of V all other axioms of a vector space are true in W.

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Hence w is a subspace of V.

5m (ix) Theorem:

Let V be the vector space over a a field F. A non empty subset w of V is a subspace of V iff $u, v \in w$ and $\alpha, \beta \in F$ $\Rightarrow \alpha u + \beta v \in w$

Proof:

Let w be a subspace of V

Let $u, v \in w$ and $\alpha, \beta \in F$.

Then, by theorem.

αu and $\beta v \in w$.

and hence, $\alpha u + \beta v \in w$.

Conversely,

Let, $u, v \in w$

and $\alpha, \beta \in F$

$\Rightarrow \alpha u + \beta v \in w \rightarrow (1)$

we have to prove w is a subspace of V.

Taking, $\alpha = \beta = 1 \rightarrow$ in (1)

we get, $u, v \in w$,

$\Rightarrow u + v \in w \longrightarrow (2)$

Taking $\beta = 0 \rightarrow$ in (1)

we get

$\alpha \in F$ and $u \in w$

$\Rightarrow \alpha u \in W$.

since, W is non-empty there exists an element $u \in W$.

$\therefore 0u = 0 \in W$.

Also, $v \in W \Rightarrow (-1)v = -v \in W$

Thus W contains 0 and the additive inverse of each of its element.

Hence, W is an additive subgroup of V.

Also, $u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$

since, the element of W are the elements of V the other axioms of a vector space are the true in W.

Hen, W is a subspace of V.

Example:

1. $\{0\}$ and V are subspace of any vector space V. They are called the trivial subspaces of V.

2. $W = \{(a,0,0) / a \in R\}$ is a subspaces of R?

Solution:

Let, $u = (a,0,0)$

and $v = (b,0,0)$

$u, v, \in W$ and $\alpha, \beta \in R$.

Then, $\alpha u + \beta v = \alpha(a,0,0) + \beta(b,0,0)$

$\alpha u+\beta v = (\alpha a, 0, 0) + (\beta b, 0, 0)$

$= (\alpha a+\beta b, 0, 0) \in W.$

$\therefore$ Hence W is subspace of R^3.

3. In R^3, $W = \{(ka, kb, kc / k \in R\}$ is a subspace of R^3.

Solution:

Let, $u = (k_1 a, k_2 b, k_2 c)$

$v = (k_2 a, k_2 b, k_2 c)$.

$u, v \in W$ and $\alpha, \beta \in R$.

Then, $\alpha u+\beta v = \alpha(k_1 a, k_2 b, k_1 c) + \beta(k_2 a, k_2 b, k_2 c)$

$\alpha u+\beta v = (\alpha k_1 a, \alpha k_1 b, \alpha k_1 c) + (\beta k_2 a, \beta k_2 b, \beta k_2 c)$

$= (\alpha k_1 a+\beta k_2 a), (\alpha k_1 b+\beta k_2 b), (\alpha k_1 c+\beta k_2 c) \in W$

$= \alpha u+\beta u \in W.$

Hence W is subspace of R^3.

4. $W = \{(a, b, 0) / a, b \in R\}$ is a subspace of R^3.

Solution:

Let, $u = (a_1, b_1, 0)$

$v = (a_2, b_2, 0)$

$u, v \in W$ and $\alpha, \beta \in R$

Then, $\alpha u+\beta v = \alpha(a_1, b_1, 0) + \beta(a_2, b_2, 0)$

$= (\alpha a_1, \alpha b_1, 0) + (\beta a_2, \beta b_2, 0) + (0, 0)$

$= (\alpha a_1 + \beta a_2 + 0), (\alpha b_1 + \beta b_2 + 0 \ldots) \in W$

$= \alpha u + \beta \beta 0 \in W$

Hence, W is subspace of R^3

5. Let W be the set of all points in R^3 satisfying the equation $lx + my + nz = 0$. W is a subspace of R^3.

Solution:

Let, $u = (a_1, b_1, c_1)$

$v = (a_2, b_2, c_2)$

Hence, $u, v \in W$ and $\alpha, \beta \in R$

Then we have,

$la_1 + mb_1 + nc_1 = 0 = la_2 + mb_2 + nc_2$

$\alpha(la_1 + mb_1 + nc_1) + \beta(la_2 + mb_2 + nc_2) = 0$

$(\alpha la_1 + \alpha mb_1 + \alpha nc_1) + (\beta la_2 + \beta mb_2 + \beta nc_2) = 0$

$(\alpha la_1 + \beta la_2) + (\alpha mb_1 + \beta mb_2) + (\alpha nc_1 + \beta nc_2) = 0.$

$l(\alpha a_1 + \beta a_2) + m(\alpha b_1 + \beta b_2) + n(\alpha c_1 + \beta c_2) = 0$

(i.e) $\alpha u + \beta v \in W$

$\therefore$ W is subspace of R^3.

6. Let $W = \{ f \mid f \in F[x]$ and $f(a) = 0 \}$ or W is the set of all polynomials in $F[x]$ having 'a' a root where $a \in F$. Then W is a vector space over F.

Soln:

we know that,

$x - a \in W$ and

hence W is non empty

let $f, g \in F[x]$ and $\alpha, \beta \in F$.

To prove that $\alpha f + \beta g \in W$ or we have to show that a is a root of $\alpha f + \beta g$.

now, $(\alpha f + \beta g)(a) = \alpha f(a) + \beta g(1)$

$= \alpha 0 + \beta 0$

$= 0$.

Hence, 'a' is a root of $\alpha f + \beta g$.

$\therefore \alpha f + \beta g \in W$. and hence W is a subspace of F[x].

7. Let $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / a, b \in \mathbb{R} \right\}$ is a subspace of $M_2(\mathbb{R})$.

problems:

U.Q. prove that the intersection two subspaces of a vector space is a subspace.

$\rightarrow$

Solution:

Let A and B be two subspace of a vector space V over a field F.

we prove that $A \cap B$ is a subspace of V.

clearly,

$0 \in A \cap B$ and hence $A \cap B$ is non-empty.

Now, let, $u, v \in A \cap B$ and $\alpha, \beta \in F$.

Then, $u, v \in A$ and $u, v \in B$.

$\alpha u + \beta v \in A$ and $\alpha u + \beta v \in B$.

$\therefore$ Since, A and B are subspaces.

$\therefore \alpha u + \beta v \in A \cap B$.

Hence, $A \cap B$ is a subspace of V.

2. prove that the union of two subspaces of a vector space need not be a subspace.

Solution:

Let, A and B be two subspace of a vector space V over a field F.

We prove that, $A \cup B$ is not a subspace of V.

Let, $A = \{(a, 0, 0) / a \in R\}$

and $B = \{(0, b, 0) / b \in R\}$

Clearly, A and B are subspace of R^3.

However, $A \cup B$ is not a subspace of R^3.

For $(1, 0, 0)$ and $(0, 1, 0) \in A \cup B$.

But, $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin A \cup B$.

Hence, proved.

5. if A and B are subspaces of V. prove that $A+B = \{v / v = a+b, a \in A, b \in B\}$ is a subspace of V. Futher, show that $A+B$ is the smallest subspace contairleding A and B. (i.e) if W is any subspace of V containing A and B then, W contains $A+B$.

$\rightarrow$

Solution:

let, $v_1, v_2 \in A+B$ and $\alpha \in F$.

then, $v_1 = a_1 + b_1$

and $v_2 = a_2 + b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Now, $v_1 + v_2 = (a_1+b_1) + (a_2+b_2)$

$= (a_1+a_2) + (b_1+b_2) \in A+B$

$v_1 + v_2 \in A+B \rightarrow (1)$

$\alpha(a_1+b_1) = \alpha a_1 + \alpha b_1 \in A+B$

$\therefore \alpha v_1 \in A+B$

$\therefore A+B$ is a subspace of V.

clearly, $A \subseteq A+B$, and $B \subseteq A+B$

now, let, W be any subspace of V

Containing A and B.

we have to prove that $A+B \subseteq W$.

Let, $v \in A+B$. Then $v = a+b$, where, $a \in A$ and $b \in B$.

Since $A \subseteq W$, $a \in W$, Similarly $b \in W$.

$A+B \subseteq W$, So that $A+B$ is the smallest subspace of V. Containing A and B.

Hence, proved.

\# ~~Theore~~ problem:

1. Let A and B be subspace of a vector space V. Then $A \cap B = \{0\}$ iff every vector $v \in A+B$. Can be uniquely Expressed in the form $v = a+b$ where $a \in A$ and $b \in B$.

Solution:

Given, that,

$$A \cap B = \{0\},$$

and $v \in A+B$.

Let us Consider $v = a_1 + b_1 = a_2 + b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Then, $a_1 - a_2 = b_2 - b_1$

but, $a_1 - a_2 \in A$ and $b_2 - b_1 \in B$

Hence $a_1 - a_2 \in A \cap B$, $b_2 - b_1 \in A \cap B$

Since, $A \cap B = \{0\}$, $a_1 - a_2 = 0$ and $b_2 - b_1 = 0$

So that, $a_1 = a_2$ and $b_1 = b_2$. Hence the expression of v in the form $a+b$, where $a \in A$ and $b \in B$ is unique.

Conversely, Suppose that any element in $A+B$ can be uniquely expressed in the form $a+b$, where $a \in A$ and $b \in B$.

we have to prove that $A \cap B = \{0\}$.

If $A \cap B \neq \{0\}$, Let $v \in A \cap B$ and $v \neq 0$.

Then $0 = v - v = 0 + 0$

thus 0 has been expressed in the form $a+b$ in two different way which is Contradiction.

Hence, $A \cap B = \{0\}$.

$\therefore$ Hence, proved.

Define Direct Sum:

Let A and B be subspaces of a vector space V. Then V is called the direct sum of A and B. if (i) $A+B = V$.

(ii). $A \cap B = \{0\}$

if V is the direct sum of A and B we write $A \oplus B$.

Example:

1. if $V_3(R)$, let $A = \{(a, b, 0) / a, b \in R\}$ and $B = \{(0, 0, c) / c \in R\}$. find if, then $V_3(R)$ is Direct sum.

proof:

clearly, A and B are subspaces of V. and

$A \cap B = \{0\}$.

Let, $v = (a, b, c) \in V_3(R)$

Then, $v = (a, b, 0) + (0, 0, c)$

So, that, $A + B = V_3(R)$

Hence, $A \oplus B = V_3(R)$.

2. In $M_2(R)$, let A be the set of all matrixs of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and B be the set of all matrices of the form $\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$

Solution:

clearly, A and B are subspace of $M_2(R)$

and $A \cap B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

~~Let, $v = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in M_2(R)$~~

Let, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$

Then, $M = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$

$M = A+B$

$\therefore A+B \in M_2(R)$

$\therefore A \oplus B = M_2(R)$.

10 mark UQ 3.

Theorem:

Let V be a vector space over F and W be a subspace of V. Let $V/W = \{W+V / v \in V\}$. Then V/W is a vector space over F. under the following ~~of~~ operations.

i. $(W+V_1)+(W+V_2) = W+V_1+V_2$.

ii. $\alpha(W+V_1) = W+\alpha V_1$

proof:

Since, W is a subspace of V it is a subgroup of $(V,+)$.

Since $(V,+)$ is abelian. W is the normal subgroup of $(V,+)$.

So, that (i) is well defined operations.

Now, we have to prove that $\alpha(W+V_1) = W+\alpha V_1$ is well defined operation.

$W+V_1 \neq W+V_2$

$V_1 - V_2 \in W$ → since W is a subspace.

$\alpha(V_1 - V_2) \in W$

$\alpha(V_1 - V_2) \in W$

$\alpha V_1 - \alpha V_2 \in W$

$\alpha v_1 \in W + \alpha v_2$

$W + \alpha v_1 = W + \alpha v_2$

Hence, $\alpha(W + v_1) = W + \alpha v_1$ is well defined operation.

Now, let $W+v_1, W+v_2$ & $W+v_3 \in V/W$

Then,

$(W+v_1)+(W+v_2)+(W+v_3) = (W+v_1)+(W+v_2+v_3)$

$= W + v_1 + v_2 + v_3$

$= (W+v_1+v_2)+(W+v_3)$

$= [(W+v_1)+(W+v_2)] + (W+v_3)$

Hence, + is associative.

$W+0 = W \in V/W$ is the additive identity element for $(W+v_1)+(W+0) = W+v_1 = (W+0)+(W+v_1)$ for all $v_1 \in V$.

Also, $W-v_1$ is the additive inverse of $W+v_1$

Hence, V/W is a group under +

Further, $(W+v_1)+(W+v_2) = W+v_1+v_2$

$= W+v_2+v_1$

$= (W+v_2)+(W+v_1)$

Hence, V/W is an abelian group.

Now, let $\alpha, \beta \in F$

and $U = W+v_1$

$V = W+v_2$

i. $\alpha(U+V) = \alpha((W+V_1)+(W+V_2))$

$= \alpha[W+V_1+V_2]$

$= W+\alpha(V_1+V_2)$

$= W+\alpha V_1+\alpha V_2$

$= (W+\alpha V_1)+(W+\alpha V_2)$

$= \alpha(W+V_1)+\alpha(W+V_2)$

$= \alpha U+\alpha V.$

ii. $(\alpha+\beta)U = (\alpha+\beta)(W+V_1) = W+(\alpha+\beta)V_1$

$= W+\alpha V_1+\beta V_1$

$= (W+\alpha V_1)+(W+\beta V_1)$

$= \alpha(W+V_1)+\beta(W+V_1)$

$= \alpha V+\beta V$

iii. $\alpha[\beta U] = \alpha[\beta(W+V_1)]$

$= \alpha[W+\beta V_1]$

$= W+\alpha\beta V_1$

$= \alpha\beta[W+V_1]$

$= (\alpha\beta)U.$

iv. $1\cdot U = 1\cdot(W+V_1)$

$= (W+V_1)$

$= U.$

Hence, V/W is a vector space.

$\therefore$ The vector space V/W is called quotient space of V by W.

Linear tranfarmation: 29

1. Define. homomarphism?

Let, V and W be vector space over a field F. A morphing $T: V \to W$ is called a homomarphism. if i. $T(U+V) = T(U) + T(V)$ and

ii. $T(\alpha U) = \alpha \cdot T(U)$ where $\alpha \in F$ and $U, V, \in V$.

2. Define linear tranfarmation?

Let V and W be a vector space over a field F. A maping $T: V \to W$ is called a homomarphism if

i. $T(U+V) = T(U) + T(V)$ and

ii. $T(\alpha U) = \alpha \cdot T(U)$ where $\alpha \in F$ and $U, V \in V$.

A homomarphism T of vector space is also called linear transfarmation.

Result:

i. if T is 1-1 is called monomarphism.

ii. if T is onto then T is called epimarphism

iii. if T is 1-1 and onto then T is called an isomarphism.

iv. Two vectors space V and W are sat to be isomarphic if there exist

an isomorphism T from V to W and
we write $V \cong W$
iv. A linear transformation $T: V \to F$ is
called a linear functional.

Examples:

1. $T: V \to W$ defined by $T(v) = 0$ for all $v \in V$
is a trivial linear transformation.

2. $T: V \to V$ defined by $T(v) = v$ for all $v \in V$
is the identity linear transformation.

Q.1 3. Let V be a vector space over a field F
and W a subspace of V. Then $T: V \to V/W$.
defined by $T(v) = W + v$ is a linear transformation

Soln:

Given that,
V be vector space over a field F.
and W be subspace of V.
We have to prove T is linear transformation

Since, $T(v) = W + v$

Let, i. $T(v_1 + v_2) = W + v_1 + v_2$
$= (W + v_1) + (W + v_2)$
$= T(v_1) + T(v_2) \to (1)$

ii. $T(\alpha v_1) = W + \alpha v_1$
$= \alpha(W + v_1)$.
$= \alpha\, T(v_1) \to (2)$

31 This is called the natural homomorphism from V to V/W. clearly T is onto and and hence T is an epimorphism. Therefore T is linear transformation

Example:

1. ~~V_3~~ In (R) the vectors $(1,2,1)$, $(2,1,0)$ and $(1,-1,2)$ are linearly independent.

Example:

2. Let V be a set of all polynomials of degree $\leq n$ in $R[x]$. then $T: V \to V_{n+1}(R)$ defined by $T(a_0 + a_1x + \cdots + a_0x^n) = (a_0, a_1, \ldots a_n)$ is a linear transformation.

Solution:

Let,

$$f = a_0 + a_1x + \cdots + a_nx^n$$

and $g = b_0 + b_1x + \cdots + b_nx^n$

Then, $f+g = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$

$$\therefore T(f+g) = [(a_0+b_0), (a_1+b_1), \ldots (a_n+b_n)]$$

$$= (a_0, a_1, \ldots a_n) + (b_0, b_1, \ldots b_n)$$

$$= T(f) + T(g)$$

Also, $T(\alpha f) = (\alpha a_0, \alpha a_1, \ldots, \alpha a_n)$

$$= \alpha(\alpha a_0, \alpha a_1, \ldots \alpha a_n)$$

$= \alpha(a_0, a_1, \ldots, a_n)$

$= \alpha T(f)$

clearly T is 1-1 and onto and hence T [is] an isomorphism.

3. $T: R^2 \to R^2$ defined by $T(a,b) = (2a-3b, a+4b)$ is a linear transformation.

Solution:

Let, $u = (a,b)$ and $v = (c,d)$ and $\alpha \in R$.

$T(u+v) = T[(a,b)+(c,d)]$

$= T[(a+c), (b+d)]$

$= T[2(a+c) - 3(b+d), (a+c)+4(b+d)$

$= [2a+2c-3b-3d, a+c+4b+4d]$

$= [2a-3b, a+4b] + [2c-3d, c+4d]$

$= T(a,b) + T(c,d)$

$= T(u) + T(v)$.

Also, $T(\alpha u) = (\alpha(a,b))$

$= T(\alpha a, \alpha b)$

$= (2\alpha a - 3\alpha b, \alpha a + 4\alpha b)$

$= \alpha(2a - 3b, a+4b)$

$= \alpha(T(a,b))$

$= \alpha T(u)$.

$\therefore$ T is linear transformation.

4. Let $T: V \to W$ be a linear transformation. Then, $T(V) = \{T(v) / v \in V\}$ is a Subspace of W.

Proof:

Let, $w_1 \in T(V)$ and $w_2 \in T(V)$ and $\alpha \in F$.

Then there exists $v_1 - v_2 \in V$ Such. That

$T(v_1) = w_1$ and $T(v_2) = w_2$.

Hence,

$$w_1 + w_2 = T(v_1) + T(v_2)$$
$$= T(v_1 + v_2) \in T(V)$$

Similarly,

$$\alpha w_1 = \alpha T(v_1) = T_1(\alpha v_1) \in T(V)$$

Hence, $T(V)$ is Subspace of W.

V.Q ⊗ 2mark Definition:

Let, V and W be vector space over a field F and $T: V \to W$ be a linear transformation, Then the kernal of T is defined to be $\{v / v \in V$ and $T(v) = 0\}$ and is denoted by ker T.

Thus, $\ker T = \{v / v \in V$ and $T(v) = 0\}$

Example:

1. $\ker T = V$
2. $\ker T = \{0\}$
3. ker T is the set of all constant polynomial.

Result:

~~let T such that~~

let $T: V \to W$ be a linear transform

the T is monomorphism iff kernal of T=

10m 1. State and prove fundamental theorem of homomorphism.

Let V and W be vector space over field F and $T: V \to W$ be an epimorphism then i. $\ker T = V_1$ is a subspace of V and

ii. $\frac{V}{V_1} \cong W$.

To proof:

Given,

i. $V_1 = \ker T = \{v / v \in V \text{ and } T(v) = 0\}$

clearly, $T(0) = 0$

Hence, $0 \in \ker T = V_1$

$\therefore V_1$ is non-empty subset of V.

Let $u, v \in \ker T$ and $\alpha, \beta \in F$.

$T(u) = 0$, and $T(v) = 0$

Now, $T(\alpha u + \beta v) = T(\alpha u) + T(\beta v)$

$= \alpha T(u) + \beta T(v)$

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0$

$\therefore \alpha u + \beta v \in \ker T$.

$\therefore \ker T$ is Subspace of V.

ii. We define a map $\phi : \frac{V}{V_1} \to W$

by $\phi(V_1 + v) = T(v)$

ϕ is well defined. Let $V_1 + v = V_1 + w$

$\therefore v \in V_1 + w$

$v = v_1 + w$ where $v_1 \in V_1$

$\therefore T(v) = T(v_1 + w) = T(v_1) + T(w)$

$= 0 + T(w)$

$= T(w)$

$\therefore \phi(V_1 + v) = \phi(V_1 + w)$

$\therefore \phi$ is 1-1

$\phi(V_1 + v) = \phi(V_1 + w)$

$\Rightarrow T(v) = T(w)$

$T(v) + T(-w) = 0$

$T(v - w) = 0$

$v - w = 0$

$v \in V_1 + w$

$V_1 + v = V_1 + w$

ϕ is onto.

Let $w \in W$. Since T is onto there Exists $v \in V$ such that $T(v) = w$.

$\therefore \phi(V_1 + v) = w$

$\phi(V_1 + v) = w$

$\therefore \phi$ is a homomorphism.

$\phi[(V_1 + v) + (V_1 + w)] = \phi[V_1 + (v + w)]$

$= T(v_1 + w)$

$= T(v) + T(w)$

$= \phi(V_1 + v) + \phi(V_1 + w)$

Also $\phi[\alpha(V_1 + v)] = \phi(V_1 + \alpha v)$

$= T(\alpha v)$

$= \alpha T(v)$

$= \alpha\phi(V_1 + v)$

ϕ is an isomorphism from $\frac{V}{V_1}$ onto w.

$\frac{V}{V_1} \cong W$

Hence proved.

Theorem:

Let V be a vector space over a field F. Let A and B be subspace of V. Then $\frac{A+B}{A} \cong \frac{B}{A \cap B}$.

proof:

W.K.T,

57 $A+B$ is a subspace of V containing A.
Hence, $\frac{A+B}{A}$ is also a vector space over F.

An element of $\frac{A+B}{A}$ is of the form $A+(a+b)$, where $a \in A$ and $b \in B$.

But, $A+a = A$.

Hence, An element of $\frac{A+B}{A}$ is of the form $A+b$.

Consider $f : B \to \frac{A+B}{A}$ defined by

$$f(b) = A+b.$$

clearly, f is onto.

Also, $f(b_1+b_2) = A+(b_1+b_2)$
$= (A+b_1)+(A+b_2)$
$= f(b_1)+f(b_2)$

and $f(\alpha b_1) = A+\alpha b_1$
$= \alpha(A+b_1)$
$= \alpha f(b_1)$

Hence, f is a linear transformation.

Let, K be the kernal of f.

then, $K = \{ b / b \in B.\ A+b = A \}$

Now, $A+b = A$ iff $b \in A$.

Hence, $K = A \cap B$

∴ W.K.T.

$$\frac{B}{A \cap B} \cong \frac{A+B}{A}$$

Hence, proved.

Theorem:

Let V and W be vector space over a field F. Let L(V, W) represent the set of all linear transformations from V to W. Then L(V, W) itself is a vector space over F and under addition and scalar multiplication defined by $(f+g)v = f(v)+g(v)$ and $(\alpha f)v = \alpha f(v)$

Proof:

Let, $f, g \in L(V, W)$ and

$v_1, v_2 \in V$

Then $(f+g)(v_1+v_2) = f(v_1+v_2)+g(v_1+v_2)$

$= f(v_1)+f(v_2)+g(v_1)+g(v_2)$

$= f(v_1)+g(v_1)+f(v_2)+g(v_2)$

$= (f+g)v_1 + (f+g)v_2$

Also, $(f+g)(\alpha v) = f(\alpha v)+g(\alpha v)$

$= \alpha f(v) + \alpha \cdot g(v)$

$= \alpha[f(v)+g(v)]$

$= \alpha(f+g)(v)$

Hence, $(f+g) \in L(V, W)$.

Now, $(\alpha f)(v_1 + v_2) = (\alpha f)(v_1) + (\alpha f)(v_2)$

$$= \alpha f(v_1) + \alpha f(v_2)$$

$$= \alpha(f(v_1) + f(v_2))$$

$$= \alpha \cdot [f(v_1 + v_2)]$$

Also, $(\alpha f)(\beta v) = \alpha[f(\beta v)]$

$$= \alpha[\beta f(v)]$$

$$= \beta[\alpha f(v)]$$

$$= \beta[(\alpha f)(v)]$$

Hence, $\alpha f \in L(V, W)$

Addition defined on $L(V, W)$ is obviously commutative and associative.

The function $f: V \to W$ defined by $f(v) = 0$ for all $v \in V$ is clearly a linear transformation and is the additive identity of $L(V, W)$

Further $(-f): V \to W$ defined by $(-f)(v) = -f(v)$ is the additive inverse of f.

Thus, $L(V, W)$ is an abelian group under addition. The remaining axiom for a vector space can be easily verified.

Hence, $L(V, W)$ is a vector space over F.

Span of a set:

Definition:

Let V be a vector space over a field F. Let, $v_1, v_2 \dots v_n \in V$. Then an element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$. where $\alpha_i \in F$ is called linear combination of the vectors $v_1, v_2 \dots v_n$.

Definition : Linear span:

Let S be a non-empty subset of a vector space V. Then the set of all linear combination of finite sets of elements of S is called the linear span of S denoted by $L(S)$.

Note:

Any element of $L(S)$ is of the form,

$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ where $\alpha_1, \alpha_2 \dots \alpha_n \in F$.

(7) **Theorem:**

V.Q 10marks

Let V be a vector space over a field F and S be a non-empty subset of V. Then,

i. $L(S)$ is a subspace of V.

ii. $S \subseteq L(S)$.

iii. If W is any subspace of V such that $S \subseteq W$, then $L(S) \subseteq W$. (i.e) $L(S)$ is the smallest subspace of V containing S.

Proof:

Let, $v, w \in L(S)$ and $\alpha, \beta \in F$.

Then, $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \rightarrow (1)$

where, $v_i \in S$ and $\alpha_i \in F$.

Also, $w = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n \rightarrow (2)$

where $w_j \in S$ and $\beta_j \in F$

Now,

$$\alpha v + \beta w = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) + \beta(\beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n)$$

$$= (\alpha\alpha_1) v_1 + (\alpha\alpha_2) v_2 + \cdots + (\alpha\alpha_n) v_n + (\beta\beta_1) w_1 + (\beta\beta_2) w_2 + \cdots + (\beta\beta_n) w_n$$

$\therefore \alpha v + \beta w$ is also a linear combination of a finite number of element of S.

Hence, $\alpha v + \beta w \in L(S)$

$\therefore L(S)$ is a Subspace of

ii. Let $v \in S$

Then $1 \cdot v = v \in L(S)$.

Hence, $S \subseteq L(S)$

iii. Let W be any Subspace of V Such

$S \subseteq W$.

we prove that,

$L(S) \subseteq W$.

Let $u \in L(S)$

Then, $u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

where, $v_i \in S$ and $\alpha_i \in F$

Since, $S \subseteq W$, we have $v_1, v_2 \ldots v_n \in W$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in W$

(since W is a Subspace of V).

$\therefore u \in W$.

Hence, $L(S) \subseteq W$

Definition:

Subspace Spanned.

Let, $L(S)$ is called Subspace

Scanned with CamScanner

spanned (generated) by the set S.

Examples:

1. In $V_3(R)$, let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$

$\Rightarrow$ a. let $S = \{e_1, e_2\}$, Then, $L(S) = \{\alpha e_1 + \beta e_2$.

$$L(S) = \{\alpha e_1 + \beta e_2 / \alpha, \beta \in R\}$$
$$= \{(\alpha, \beta, 0) / \alpha, \beta \in R\}$$

b. let $S = \{e_1, e_2, e_3\}$ then,

$$L(S) = \{\alpha e_1 + \beta e_2 + \gamma e_3 / \alpha, \beta, \gamma \in R\}$$
$$= \{(\alpha, \beta, \gamma) / \alpha, \beta, \gamma \in R\}$$
$$= V_3(R)$$

Thus, $V_3(R)$ is spanned by $\{e_1, e_2, e_3\}$

2. In $V_n(R)$, let, $e_1 = (1, 0, \ldots\ldots 0)$, $e_2 = (0,1,0 \ldots\ldots 0) \ldots\ldots$ $e_n = (0, 0, \ldots\ldots 1)$.

Soln:

let $S = \{e_1, e_2 \ldots\ldots e_n\}$,

Then, $L(S) = \{\alpha_1 e_1 + \alpha_2 e_2 + \ldots\ldots + \alpha_n e_n / \alpha_i \in R\}$

$$= \{(\alpha_1, \alpha_2 \ldots\ldots \alpha_n) / \alpha_i \in R\}$$
$$= V_n(R).$$

thus, $V_n(R)$ is spanned by $\{e_1, e_2 \ldots e_n\}$.

Theorem:

Remark. Let V be a vector space over a field F, let $S, T \subseteq V$ then,

a. $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

b. $L(S \cup T) = L(S) + L(T)$

c. $L(S) = S$ iff S is a subspace of

Proof:

a). Let, $S \subseteq T$, let $s \in L(S)$

Then, $s = \alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n \rightarrow$

where, $s_i \in S$ and $\alpha_i \in F$.

Now, since, $S \subseteq T$, $s_i \in T$

Hence, $\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n \in L(T)$

Thus $L(S) \subseteq L(T)$

b). Let $s \in L(S \cup T)$

Then, $s = \alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n$, where $s_i \in S \cup T$ and $\alpha_i \in F$.

We can assume that,

$s_1, s_2 \ldots\ldots s_m \in S$

and, $s_{m+1} \ldots\ldots s_n \in T$

Hence,

$$\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_m s_m \in L(S)$$

and

$$\alpha_{m+1} s_{m+1} + \alpha_{m+2} s_{m+2} + \cdots + \alpha_n s_n \in L(T)$$

$$\therefore s = (\alpha_1 s_1 + \cdots + \alpha_m s_m) + (\alpha_{m+1} s_{m+1} + \cdots + \alpha_n s_n)$$

$$\in L(S) + L(T)$$

Hence, $L(S \cup T) \subseteq L(S) + L(T)$

Also, by (a) $L(S) \subseteq L(S \cup T)$

and $L(T) \subseteq L(S \cup T)$

Hence, and $L(S) + L(T) \subseteq L(S \cup T) \rightarrow (3)$

Hence, from (2) & (3), $L(S \cup T) = L(S) + L(T)$

c. Let $L(S) = S$,

W.K.T, $L(S) = S$ is a Subspace of V.

Conversely,

let S be a Subspace of V. Then the Smallest Subspace containing S is S itself.

Hence, $L(S) = S$.

UNIT - II

Linear Independence :-

Definition :-

Let V be a vector space over a field F. V is said to be <u>finite dimensional</u> if there exists a finite subset S of V such that $L(S) = V$.

Examples :-

1. $V_3(R)$ is a finite dimensional vector space
2. $V_n(R)$ is a finite dimensional vector space, since $S = \{e_1, e_2 \ldots e_n\}$ is a finite subset of $V_n(R)$ such that $L(S) = V_n(R)$. In general if F is any field $V_n(F)$ is a finite dimensional vector space over F.
3. Let V be the set of all polynomials in $F[x]$ of degree $\leq n$. Let $S = \{1, x, x^2 \ldots x^n\}$. Then $L(S) = V$ and hence V is finite dimensional.
4. C is a finite dimensional vector space over R. Since $L(\{1, i\}) = C$.
5. In $M_2(R)$ consider the set S consisting of the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = aA + bB + cC + dD$.

Hence $L(S) = M_2(R)$ so that $M_2(R)$ is finite dimensional.

Definition : Let V be a vector space over a field F. A finite set of vectors $v_1, v_2 \ldots v_n$ in V is said to be linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Def: linearly dependent:-

If $v_1, v_2 \ldots v_n$ are not linearly independent, then they are said to be linearly dependent.

Examples:-

1. In $V_n(F)$, $\{e_1, e_2 \ldots e_n\}$ is a linearly independent set of vectors, for

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$$

$$\Rightarrow \alpha_1(1,0, \ldots 0) + \alpha_2(0,1, \ldots 0) + \cdots \alpha_n(0,0 \ldots 1) = (0, \ldots 0)$$

$$(\alpha_1, 0, \ldots 0) + (0, \alpha_2, \ldots 0) + (0, 0, \ldots \alpha_n) = (0, 0, \ldots 0)$$

$$(\alpha_1, \alpha_2, \ldots \alpha_n) = (0, 0 \ldots 0)$$

$$\Rightarrow \alpha_1 = \alpha_2 \cdots = \alpha_n = 0.$$

2. In $V_3(R)$ the vectors $(1,2,1)$, $(2,1,0)$ and $(1,-1,2)$ are linearly independent.

Solution :-

let $e_1 = (1,2,1)$

$e_2 = (2,1,0)$

and $e_3 = (1,-1,2)$

then, $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$

$$\alpha_1(1,2,1)+\alpha_2(2,1,0)+\alpha_3(1,-1,2)=0$$

$$(\alpha_1+2\alpha_2+\alpha_3,\ 2\alpha_1+\alpha_2-\alpha_3,\ \alpha_1+2\alpha_3)=0$$

$\therefore$ $\alpha_1+2\alpha_2+\alpha_3=0 \rightarrow (1)$

$2\alpha_1+\alpha_2-\alpha_3=0 \rightarrow (2)$

$\alpha_1+2\alpha_3=0 \rightarrow (3)$

$(1) \Rightarrow \alpha_1+2\alpha_2+\alpha_3=0$

$2\times(2) \Rightarrow \underline{4\alpha_1+2\alpha_2-2\alpha_3=0}$

$-3\alpha_1+3\alpha_3=0$

$-\alpha_1+\alpha_3=0 \rightarrow (4)$

$(3) \Rightarrow \underline{\alpha_1+2\alpha_3=0}$

$3\alpha_3=0$

$\boxed{\alpha_3=0}$

Put $\alpha_3=0$ in (4), we get,

$-\alpha_1+0=0$

$\boxed{\alpha_1=0}$

Put $\alpha_1=0$ & $\alpha_3=0$ in (1) we get

$\boxed{\alpha_2=0}$

$\therefore$ $\alpha_1=\alpha_2=\alpha_3=0$

$\therefore$ The given vectors are linearly independent

3. In $V_3(R)$ the vectors $(1,4,-2)$, $(-2,1,3)$ and $(-4,11,5)$ are linearly dependent..

<u>Solution</u> :-

Consider $e_1=(1,4,-2)$

$e_2=(-2,1,3)$

and $e_3 = (-4, 11, 5)$

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$$

$$\alpha_1 (1, 4, -2) + \alpha_2 (-2, 1, 3) + \alpha_3 (-4, 11, 5) = 0$$

$$(\alpha_1 - 2\alpha_2 - 4\alpha_3 , 4\alpha_1 + \alpha_2 + 11\alpha_3 , -2\alpha_1 + 3\alpha_2 + 5\alpha_3) = 0$$

$\therefore$
$$\alpha_1 - 2\alpha_2 - 4\alpha_3 = 0 \rightarrow (1)$$
$$4\alpha_1 + \alpha_2 + 11\alpha_3 = 0 \rightarrow (2)$$
$$-2\alpha_1 + 3\alpha_2 + 5\alpha_3 = 0 \rightarrow (3)$$

From (1) & (2),

$$\frac{\alpha_1}{\begin{vmatrix} -2 & -4 \\ 1 & 11 \end{vmatrix}} = \frac{-\alpha_2}{\begin{vmatrix} 1 & -4 \\ 4 & 11 \end{vmatrix}} = \frac{\alpha_3}{\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix}}$$

$$\frac{\alpha_1}{-22+4} = \frac{-\alpha_2}{11+16} = \frac{\alpha_3}{1+8}$$

$$\frac{\alpha_1}{-18} = \frac{\alpha_2}{-27} = \frac{\alpha_3}{9}$$

$\therefore \alpha_1 = -18, \alpha_2 = -27, \alpha_3 = 9$ as a non trivial solution.

$\therefore$ Hence the three vectors are linearly dependent.

Thm:- Any Subset of a linearly independent set is linearly independent.

Proof:-

Let V be a vectorspace over a field F.

Let $S = \{v_1, v_2 \dots v_n\}$ be a linearly independent set.

Let S' be a subset of S. without ~~loss~~ loss of generality We take $S' = \{v_1, v_2 \dots v_k\}$ where $k \leq n$.

Suppose S' is a linearly dependent set. Then there exists $\alpha_1, \alpha_2 \dots \alpha_n$ in F not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0$$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k + \dots + 0 v_n = 0$ is a non trivial linear combination giving the zero vector.

Hence S is linearly dependent set which is a contradiction

Hence S' is linearly independent.

Thm:- Any set ~~of~~ containing a linearly dependent set is also linearly dependent.

Proof:-

Let V be a vector space.

Let S be a linearly dependent set.

Let $S' \supset S$

If S' is linearly independent S is also linearly independent. which is contradiction

Hence S' is linearly dependent.

Thm:- Let $S = \{v_1, v_2 \ldots v_n\}$ be a linearly independent set of vectors in a vector space V over a field F. Then every element of $L(S)$ can be uniquely written in the form.

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \text{ where } \alpha_i \in F.$$

Proof:-

By definition every elements of $L(S)$ is of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Now, Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots \beta_n v_n$.

Hence $(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \cdots + (\alpha_n - \beta_n)v_n = 0$

Since S is a linearly independent set, $\alpha_i - \beta_i = 0$

$\therefore \alpha_i = \beta_i$ for all i.

Hence the theorem.

Thm:- $S = \{v_1, v_2 \ldots v_n\}$ is a linearly dependent set of vectors in V iff there exists a vector $v_k \in S$ such that v_k is a linear combination of the preceding vectors $v_1, v_2 \ldots v_{k-1}$.

Proof

Suppose $v_1, v_2 \ldots v_n$ are linearly dependent.

Then there exists $\alpha_1, \alpha_2 \ldots \alpha_n \in F$, not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Let k be the largest integer for which $\alpha_k \neq 0$.

Then $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$

$$\alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_{k-1} v_{k-1}$$

$$\therefore v_k = (-\alpha_k^{-1}\alpha_1) v_1 + \cdots + (-\alpha_k^{-1}\alpha_{k-1}) v_{k-1}$$

$\therefore v_k$ is a linear combination of the preceding vectors.

Conversely, suppose there exists a vector v_k such that $v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1}$.

Hence $-\alpha_1 v_1 - \cdots - \alpha_{k-1} v_{k-1} + v_k + 0 v_{k+1} + \cdots + 0 v_n = 0$.

Since the co-efficient of $v_k = 1$, we have $S = \{v_1, \ldots, v_n\}$ is linearly dependent.

Example:

In $V_3(R)$, let $S = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$.

Here $(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$.

Thus $(1,1,1)$ is a linear combination of the preceding vectors. Hence S is a linearly dependent set.

Theorem 5.15

Lemma

Let V be a vector space over F. Let $S=\{v_1, v_2, \ldots, v_n\}$ and $L(S)=W$. Then there exists a linearly independent subset S' of S such that $L(S')=W$.

Proof:

Let $S=\{v_1, v_2, \ldots, v_n\}$.

If S is linearly independent there is nothing to prove.

If not, let v_k be the first vector in S which is a linear combination of the preceding vectors.

Let $S_1=\{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$.

(i.e) S_1 is obtained by deleting the vector v_k from S.

We claim that $L(S_1)=L(S)=W$.

Since $S_1 \subseteq S$, $L(S_1) \subseteq L(S)$. [ref thm 5.10].

Not, let $v \in L(S)$.

Then $v = \alpha_1 v_1 + \cdots + \alpha_k v_k + \cdots + \alpha_n v_n$.

Now, v_k is a linear combination of the preceding vectors.

Let $v_k = \beta_1 v_1 + \dots + \beta_{k-1} v_{k-1}$.

Hence $v = \alpha_1 v_1 + \dots + \alpha_{k-1} v_{k-1} + \alpha_k (\beta_1 v_1 + \dots + \beta_{k-1} v_{k-1})$
$+ \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n$.

$\therefore$ v can be expressed as a linear combination of the vectors of S_1, so that $v \in L(S_1)$. Hence $L(S) \subseteq L(S_1)$.

Thus $L(S) = L(S_1) = W$.

Now, if S_1 is linearly independent, the proof is complete.

If not, we continue the above process of removing a vector from S_1, which is linear combination of the proceeding vectors until we arrive at a line early independent subset S' of S such that $L(S') = W$.

Basis and Dimension

Defination :- Basis : A linearly independent subset S of a vector space V which spans the whole space V is called a basis of the vector space.

Theorem:-

Any finite-dimensional vector space V contains a finite number of linearly independent vectors which span V (ie) A ~~lin~~ finite dimensional vector space has a basis consisting of a finite number of vectors.

Proof:-

Since V is finite dimensional there exists a finite subset S of V such that $L(S) = V$. This ~~set~~ 'S' contains a linearly independent subset $S' = \{v_1, v_2, \dots v_n\}$ such that

$$L(S') = L(S) = V$$

Hence S' is a basis for V.

Theorem:-

Let V be a vector space over a field F. Then $S = \{v_1, v_2, \dots v_n\}$ is a basis for V iff every element of V can be uniquely expressed as a linear combination of element of S.

Proof:-

Let S be a basis for V.

Then by defination S is linearly independent and $L(S) = V$. Hence every element of V can be uniquely expressed as a linear combination of elements of S.

Conversely, suppose every element of V can be uniquely expressed as a linear combination of elements of S.

Clearly $L(S) = V$

Now, let $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$

Also $0v_1 + 0v_2 + \dots + 0v_n = 0$

Thus we have expressed 0 as a linear combination of vectors of S into two ways.

$\therefore$ By hypothesis $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Hence S is linearly independent. Hence S is a basis.

Examples :- 2marks

1) $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $V_3(R)$ for, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$\therefore$ Any vector (a,b,c) of $V_3(R)$ has been expressed uniquely as a linear combination of the elements of S and hence S is a basis for $V_3(R)$.

2] $S = \{e_1, e_2, \ldots, e_n\}$ is a basis for $V_n(F)$. This is known as the standard basis for $V_n(F)$.

3] $S = \{(1,0,0), (0,1,0), (1,1,1)\}$ is a basis for $V_3(R)$.

Proof:-

We shall show that any element (a,b,c) of $V_3(R)$ can be uniquely expressed as a linear combination of vectors of S.

Let $(a,b,c) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(1,1,1)$

Then $\alpha + \gamma = a, \beta + \gamma = b, \gamma = c$.

Hence $\alpha = a - c$ and $\beta = b - c$.

Thus $(a,b,c) = (a-c)(1,0,0) + (b-c)(0,1,0) + c(1,1,1)$

$\therefore$ S is a basis for $V_3(R)$. 684,039

4] $S = \{1\}$ is a basis for the vector space R over R. 5250

5] $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is a basis for $M_2(R)$, since any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be uniquely written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$+ c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

6] $\{1, i\}$ is a basis for the vector space C ~~for~~ over R

7] Let V be the set of all polynomials of degree $\leq n$ in $R[x]$. Then $\{1, x, x^2, \ldots, x^n\}$ is a basis for V.

8] $\{(1,0), (i,0), (0,1)$ ~~an~~ $(0,i)\}$ is a basis, for vector space $C \times C$ over R,

for $(a+ib, c+id) = a(1,0) + b(i,0)$
$+ c(0,1) + d(0,i)$

9] $S = \{(1,0,0), (0,1,0), (1,1,1), (1,1,0)\}$ Spans the vector space $V_3(R)$ but is not a basis.

Proof:-

Let $S' = \{(1,0,0), (0,1,0), (1,1,1)\}$

Then $L(S') = V_3(R)$

Now, Since $S \subseteq S'$,

We get, $L(S) = V_3(R)$

Thus S Spans $V_3(R)$.

But S is linearly dependent since -

$$(1,1,0) = (1,0,0) + (1,1,0)$$

Hence S is not a basis.

(10). $S = \{(1,0,0)\ (1,1,0)\}$ is linearly independent but not a basis of $V_3(R)$..

Proof.

Let $\alpha(1,0,0) + \beta(1,1,0) = (0,0,0)$.

Then $\alpha + \beta = 0$, and $\beta = 0$

$\therefore \alpha = \beta = 0$.. Hence S is linearly independent

Also $L(S) = \{(a,b,0) / a, b \in R\} \neq V_3(R)$

$\therefore S$ is not a basis.

Thm:-

Let V be a vector space over a field F. Let $S = \{v_1, v_2 \cdots v_n\}$ span V. Let $S = \{w_1, w_2 \cdots w_m\}$ be a linearly independent set of vectors in V. Then $m \leq n$.

Proof:-

Since $L(S) = V$, every vector in V and in particular w_1, is a linear combination of $v_1, v_2 \ldots v_n$..

Hence $S_1 = \{w_1, v_1, \ldots v_n\}$ is a linearly dependent set of vectors.

Hence there exists a vector $v_k \neq w_1$ in S_1 which is a linear combination of the preceding vectors.

Let $S_2 = \{w_1, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$

Clearly, $L(S_2) = V$

Hence w_2 is a linear combination of the vectors in S_2

Hence $S_3 = \{w_2, w_1, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$ is linearly dependent. Hence there exists a vector in S_3 which is a linear combination of the preceding vectors.

Since the w_i's are linearly independent, this vector cannot be w_2 or w_1 and hence must be some v_j where $j \neq k$. Deletion of v_j from the set S_3 gives the set

$$S_4 = \{w_2, w_1, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_{j-1}, \ldots, v_{j+1}, \ldots, v_n\}$$ of n vectors spanning V.

In this process, at each step we insert one vector from $\{w_1, w_2, \ldots, w_m\}$ and delete one vector from $(v_1, v_2, \ldots, v_n)$.

If $m > n$ after repeating this process n times, we arrive at the set $\{w_n, w_{n-1}, \ldots, w_1\}$ which span V.

Hence W_{n+1} is a linear combination

of $w_1, w_2 \dots w_n$. Hence $\{w_1, w_2 \dots w_n, w_{n+1}, \dots w_m\}$ is linearly dependent which is a contradiction

Hence $m \leq n$.

Thm :-

Any two bases of a finite dimensional vector space V have the same number of elements.

Proof :-

Since V is finite dimensional, it has a basis,

$S = \{v_1, v_2 \dots v_n\}$.

Let $S' = \{w_1, w_2, \dots w_m\}$ be any other basis for V.

Now, $L(S) = V$ and S' is a set of m linearly dependent vector's.

Hence $m \leq n \longrightarrow (1)$

Also, Since $L(S') = V$ and S is a set of n linearly independent vectors,

Hence $n \leq m \longrightarrow (2)$

From (1) & (2), we get

$n = m$.

$\therefore$ Any two bases of a finite dimensional vector space V have the same number of elements.

Hence proved.

Definition: Dimension:-

Let V be a finite dimensional vector space over a field F. The number of elements in any basis of V is called the dimension of V and is denoted by $\dim V$.

Ex:-

1. $\dim V_n(R) = n$. Since $\{e_1, e_2 \ldots e_n\}$ is a basis of $V_n(R)$.

2. $M_2(R)$ is a vector space of dimension 4 over R Since $\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}$ is a basis for $M_2(R)$.

3. C is a Vector space of dimension 2 over R Since $\{1, i\}$ is a basis for C.

4. Let V be the set of all polynomials of degree $\leq n$ in $R[x]$. V is a vector space over R having dimension $n+1$, Since $\{1, x, x^2, \ldots x^n\}$ is a basis for V.

Thm:-

Let V be a vector space of dimension n. Then (i) any set of m vectors where $m > n$ is linearly dependent

(ii) any set of m vectors where $m < n$ Cannot span V.

Proof:-

(i) Let $S = \{v_1, v_2 \ldots v_n\}$ be a basis for V. Hence $L(S) = V$.

Let S' be any set consisting of m vectors where $m > n$. Suppose that S' is linearly independent. Since S spans V. By known thm,

$$m \leq n,$$

which is contradiction.

Hence S' is linearly dependent.

(ii) Let S' be a set consisting of m vectors where $m < n$. Suppose $L(S') = V$.

Now, $S = \{v_1, v_2 \ldots v_n\}$ is a basis for V and hence linearly independent. Hence by known theorem $n \leq m$. which is contradiction

Hence S' cannot span V.

<u>Theorem :-</u>

Let V be a finite dimensional vector space over a field F. Any linearly independent set of vectors in V is part of a basis.

<u>Proof :-</u>

Let $S = \{v_1, v_2 \ldots v_r\}$ be a linearly independent set of vectors.

If $L(S) = V$ then S itself is a basis.

If $L(S) \neq V$, choose an element

$$v_{r+1} \in V - L(S).$$

Now, consider $S_1 = \{v_1, v_2 \ldots v_r, v_{r+1}\}$

We shall prove that S_1 is linearly independent by showing that no vector in S_1

is a linear combination of the preceeding vectors.

Since $\{v_1, v_2 \dots v_r\}$ is linearly independent v_i where $1 \le i \le r$ is not a linear combination of the preceeding vectors.

Also, $v_{r+1} \notin L(S)$ and hence v_{r+1} is not a linear combination of $v_1, v_2 \dots v_r$.

Hence S_1 is linearly independent.

If $L(S_1) = V$, then S_1 is a basis of V. If not we take an element $v_{r+2} \in V - L(S_1)$ and proceed as before. Since the dimension of V is finite. This process must stop at a certain stage giving the required basis containing S.

Thm:-

Let V be a finite dimensional vector space over a field F. Let A be a Subspace of V. Then there exists a Subspace B of V such that $V = A \oplus B$.

Proof

Let $S = \{v_1, v_2 \dots v_r\}$ be a basis of A.

We find $w_1, w_2 \dots w_s \in V$. Such that $S' = \{v_1, v_2, \dots\dots v_r, w_1, \dots w_s\}$ is a basis of V.

Now, let $B = L\{(w_1, w_2, \dots w_s)\}$

We claim that

$A \cap B = \{0\}$ and $V = A + B$.

Now, Let $v \in A \cap B$. Then $v \in A$ and $v \in B$.

Hence $v = \alpha_1 v_1 + \dots + \alpha_r v_r$

$= \beta_1 w_1 + \dots + \beta_s w_s$

$\therefore \alpha_1 v_1 + \dots + \alpha_r v_r - \beta_1 w_1 - \dots - \beta_s w_s = 0$

Now, Since S' is linearly independent $\alpha_i = 0 = \beta_j$ for all i and j.

Hence $v = 0$. Thus $A \cap B = \{0\}$.

Now, let $v \in V$.

$v = (\alpha_1 v_1 + \dots + \alpha_r v_r) + (\beta_1 w_1 + \dots + \beta_s w_s)$

$\in A + B$.

Hence $A + B = V$. So that

$V = A \oplus B$

Hence proved.

__Definition:__ Maximal linearly independent set:-

Let V be a vector space and $S = \{v_1, v_2 \dots v_n\}$ be a set of independent vectors in V. Then S is called a maximal linearly independent set if for every $v \in V - S$, the set $\{v, v_1, v_2 \dots v_n\}$ is linearly dependent.

__Definition:__ Minimal Generating Set:-

Let $S = \{v_1, v_2 \dots v_n\}$ be a set of vectors in V and Let $L(S) = V$. Then S is called a minimal generating set if for any $v_i \in S$, $L[(S) - \{v_i\}] \neq V$.

Thm.

Let V be a vector space over a field F. Let $S = \{v_1, v_2, \ldots v_n\} \subseteq V$. Then the following are equivalent

(i) S is a basis for V.

(ii) S is a maximal linearly independent set.

(iii) S is a minimal generating set.

Proof :-

(i) $\Rightarrow$ (ii)

Let $S = \{v_1, v_2, \ldots v_n\}$ be a basis for V. Then any n+1 vectors in V are linearly dependent and hence S is a maximal linearly independent set.

(ii) $\Rightarrow$ (i).

Let $S = \{v_1, v_2, \ldots v_n\}$ be a maximal linearly independent set. Now To prove that S is a basis for V.

We show that $L(S) = V$.

Obviously, $L(S) \subseteq V$.

Now, Let $v \in V$.

If $v \in S$, then $v \in L(S)$.

If $v \notin S$, $S' = \{v_1, v_2, \ldots v_n, v\}$ is a linearly dependent set.

∴ There exists a vector in S' which is a linear combination of the preceeding vectors.

Since $v_1, v_2 \ldots v_n$ are linearly independent. This vectors must be v. Thus v is a linear combination of $v_1, v_2 \ldots v_n$.

$\therefore v \in L(S)$.

Hence $V \subseteq L(S)$

Thus $V = L(S)$

Set.

(i) $\Rightarrow$ (iii)

Let $S = \{v_1, v_2 \ldots v_n\}$ be a basis. Then $L(S) = V$.

n

If S is not minimal, there exists $v_i \in S$ such that $L(S - \{v_i\}) = V$.

Since S is linearly independent, $S - \{v_i\}$ is also linearly independent. Thus $S - \{v_i\}$ is a basis consisting on $n-1$ elements which is a contradiction.

Hence S is a minimal generating set.

(iii) $\Rightarrow$ (i).

Let $S = \{v_1, v_2, \ldots v_n\}$ be a minimal generating set.

To prove that S is a basis, we have to show that S is linearly independent

If S is linearly dependent, there exists a vector v_k which is a linear combination of the preceeding vectors.

clearly $L(S - \{v_k\}) = V$ contradicting the minimality of S.

Thus S is linearly independent and since $L(S) = V$. S is a basis for V.

V.Q Thm:- Any vector space of dimension n over a field F is isomorphic to $V_n(F)$.

Proof:-

Let V be a vector space of dimension n. Let $\{v_1, v_2, \dots v_n\}$ be a basis for V.

Then we known that if $v \in V$, v can be written uniquely as $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, where $\alpha_i \in F$.

Now, consider the map $f : V \to V_n(F)$ given by

$$f(\alpha_1 v_1 + \dots + \alpha_n v_n) = (\alpha_1, \alpha_2 \dots \alpha_n)$$

clearly f is 1-1 and onto.

Let $v, w \in V$.

Then $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ and

$$w = \beta_1 v_1 + \dots + \beta_n v_n.$$

$$f(v+w) = f[(\alpha_1 + \beta_1) v_1 + \dots + (\alpha_n + \beta_n) v_n]$$

$$= ((\alpha_1 + \beta_1), (\alpha_2 + \beta_2), \dots (\alpha_n + \beta_n))$$

$$= (\alpha_1, \alpha_2 \dots \alpha_n) + (\beta_1, \beta_2 \dots \beta_n)$$

$$= f(v) + f(w).$$

Also,

$$f(\alpha v) = f(\alpha \alpha_1 v_1 + \dots + \alpha \alpha_n v_n)$$
$$= (\alpha\alpha_1, \alpha\alpha_2, \dots \alpha\alpha_n)$$
$$= \alpha(\alpha_1, \alpha_2 \dots \alpha_n)$$
$$= \alpha f(v)$$

Hence f is an isomorphism of V to $V_n(F)$

Hence proved.

__Thm:-__ Let V and W be vector spaces over a field F. Let $T: V \to W$ be an isomorphism. Then T maps a basis of V onto a basis of W.

__Proof:-__

Let $\{v_1, v_2 \dots v_n\}$ be a basis for V.

We prove that $T(v_1), T(v_2), \dots T(v_n)$ are linearly independent and that they span W.

Now, $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_n (T(v_n)) = 0$

$$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \dots + T(\alpha_n v_n) = 0$$
$$T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = 0$$
$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \quad (\text{Since } T \text{ is } 1\text{-}1)$$
$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

(Since $v_1, v_2, \dots v_n$ are linearly independent).

$\therefore T(v_1), T(v_2), \dots T(v_n)$ are linearly independent.

Now, let $w \in W$. Then since T is onto, there exists a vector $v \in V$ such that $T(v) = w$

Let $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$

Then $w = T(v)$

$= T(\alpha_1 v_1 + \cdots + \alpha_n v_n)$

$= \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$

Thus w is a linear combination of the vectors $T(v_1), \ldots, T(v_n)$.

$T(v_1), \ldots, T(v_n)$ span W and hence is a basis for W.

Corollary.

Two finite dimensional vector spaces V and W over a field F are isomorphic iff they have the same dimension.

Theorem 5.26 :

Let V and W be finite dimensional vector spaces over a field F. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and let $w_1, w_2, \ldots, w_n$ be any n vectors in W (not necessarily distinct). Then there exists a unique linear transformation $T: V \to W$ such that $T(v_i) = w_i, i = 1, 2, \ldots, n$.

proof :

Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in V$.

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We define $T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$.

Now, let $x, y \in V$.

Let $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$ and

$y = \beta_1 v_1 + \cdots + \beta_n v_n$.

$\therefore x + y = (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_n + \beta_n) v_n$.

$\therefore T(x+y) = (\alpha_1 + \beta_1) w_1 + \cdots + (\alpha_n + \beta_n) w_n$.

$= (\alpha_1 w_1 + \cdots + \alpha_n w_n) + (\beta_1 w_1 + \cdots + \beta_n w_n)$.

$= T(x) + T(y)$.

|||ly $T(\alpha x) = \alpha T(x)$.

Hence T is linear transformation.

Also, $v_1 = 1 v_1 + 0 v_2 + \cdots + 0 v_n$.

Hence $T(v_1) = 1 w_1 + 0 w_2 + 0 w_n = w_1$.

|||ly $T(v_i) = w_i$ for all $i = 1, 2, \ldots, n$.

Now, to prove the uniqueness, let $T' : V \to W$ be any other linear transformation such that $T'(v_i) = w_i$.

Let $v = \alpha_1 v_1 + \cdots + \alpha_n v_n \in V$.

$T'(v) = \alpha_1 T'(v_1) + \cdots + \alpha_n T'(v_n)$

$= \alpha_1 w_1 + \cdots + \alpha_n w_n = T(v)$.

Hence $T = T'$.

Remark:

the above theorem shows that a linear transformation is completely determined by its value on the elements of a basis.

Theorem 5.27

Let V be a finite dimensional vector space over a field F. Let W be a subspace of V.

Then, (i) $\dim W \le \dim V$.

(ii) $\dim V/W = \dim V - \dim W$.

proof :

(i) Let $S = \{w_1, w_2, \ldots, w_m\}$ be a basis for W. Since W is a subspace of V, S is a part of a basis for V.

Hence $\dim W \le \dim V$.

(ii) Let $\dim V = n$ and $\dim W = m$.

Let $S = \{w_1, w_2, \ldots, w_m\}$ be a basis for W. clearly S is a linearly independent set of vectors in V.

Hence S is a part of a basis in V. Let $\{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_r\}$ be a basis for V.

Then $m + r = n$.

Now, we claim $S' = \{W+v_1, W+v_2, \ldots, W+v_r\}$ is a basis for V/W.

$\alpha_1(W+v_1) + \alpha_2(W+v_2) + \ldots + \alpha_r(W+v_r) = W+0$.

$\Rightarrow (W+\alpha_1 v_1) + (W+\alpha_2 v_2) + \dots + (W+\alpha_r v_r) = W.$

$\Rightarrow W + \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r = W.$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r \in W.$

Now, since $\{w_1, w_2, \dots, w_n\}$ is a basis for W.

$\therefore \alpha_1 v_1 + \dots + \alpha_r v_r = \beta_1 w_1 + \dots + \beta_m w_m.$

$\therefore \alpha_1 v_1 + \dots + \alpha_r v_r - \beta_1 w_1 - \dots - \beta_m w_m = 0.$

$\therefore \alpha_1 = \alpha_2 = \alpha_r = \beta_1 = \beta_2 = \dots = \beta_m = 0.$

$\therefore S'$ is a linearly independent set.

Now, let $W+v \in V/W.$

Let $v = \alpha_1 v_1 + \dots + \alpha_r v_r + \beta_1 w_1 + \dots + \beta_m w_m$

Then $W+v = W + (\alpha_1 v_1 + \dots + \alpha_r v_r + \beta_1 w_1 + \dots + \beta_m w_m)$

$= W + (\alpha_1 v_1 + \dots + \alpha_r v_r);$

[since $\beta_1 w_1 + \dots + \beta_m w_m \in W$].

$= (W+\alpha_1 v_1) + \dots + (W + \alpha_r v_r).$

$= \alpha_1 (W+v_1) + \dots + \alpha_r (W+v_r).$

Hence S' spans V/W so that S' is a basis for V/W.

$\therefore \dim V/W = r = n-m.$

$= \dim V - \dim W.$

Theorem 5.28.

Let V be a finite dimensional vector space over a field F. Let A and B be subspaces of V. Then $\dim(A+B) = \dim A + \dim B - \dim(A \cap B).$

proof:

A and B are subspaces of V. Hence $A \cap B$ is subspaces of V.

Let $\dim(A \cap B) = r$.

Let $S = \{v_1, v_2, \ldots, v_r\}$ be a basis for $A \cap B$.

Since $A \cap B$ is a subspace of A and B, S is a part of a basis for A and B.

Let $\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots u_s\}$ be a basis for A and $\{v_1, v_2, \ldots v_r, w_1, w_2, \ldots, w_t\}$ be a basis for B.

We shall prove that $S' = \{v_1, \ldots v_r, u_1, \ldots u_s, w_1, \ldots w_t\}$ is a basis for $A + B$.

Let $\alpha_1 v_1 + \ldots + \alpha_r v_r + \beta_1 u_1 + \ldots + \beta_s u_s + \gamma_1 w_1 + \ldots + \gamma_t w_t = 0$.

Then $\beta_1 u_1 + \cdots + \beta_s u_s = -(\gamma_1 w_1 + \cdots + \gamma_t w_t) - (\alpha_1 v_1 + \cdots + \alpha_r v_r) \in B$.

Hence $\beta_1 u_1 + \cdots + \beta_s u_s \in B$.

Also $\beta_1 u_1 + \ldots + \beta_s u_s \in A$.

Hence $\beta_1 u_1 + \ldots + \beta_s u_s \in A \cap B$.

$\therefore \beta_1 u_1 + \cdots \beta_s u_s = \delta_1 v_1 + \ldots + \delta_r v_r$.

$\therefore \beta_1 u_1 + \cdots \beta_s u_s - \delta_1 v_1 - \ldots - \delta_r v_r = 0$.

$\therefore \beta_1 = \ldots = \beta_s = \delta_1 = \ldots = \delta_r = 0$.

[since $\{u_1, \ldots u_s, v_1, \ldots v_r\}$ is linearly Independent].

III'ly we can prove $\gamma_1 = \gamma_2 = \ldots = \gamma_t = 0$.

$\therefore \alpha_i = \beta_j = \gamma_k = 0$ for $1 \leq i \leq r$.

$1 \leq j \leq s$; $1 \leq k \leq t$.

Thus S' is a linearly independent set.

Clearly S' spans $A+B$.

$\therefore S'$ is a basis for $A+B$.

$\therefore$ Hence $\dim(A+B) = r+s+t$.

Also $\dim A = r+s$; $\dim B = r+t$ and $\dim(A \cap B) = r$.

$\dim A + \dim B - \dim A \cap B = (r+s)+(r+t) - r$

$= r+s+t$

$= \dim(A+B)$

Alter: By thm 5.7 $\frac{A+B}{A} = \frac{B}{A \cap B}$

Hence $\dim\left[\frac{A+B}{A}\right] = \dim\left[\frac{B}{A \cap B}\right]$

$\therefore \dim(A+B) - \dim A = \dim B - \dim(A \cap B)$

$\therefore \dim(A+B) = \dim A + \dim B - \dim(A \cap B)$.

Corollary:

If $V = A \oplus B$, $\dim V = \dim A + \dim B$.

proof:

$V = A \oplus B \Rightarrow A+B = V$ and

$A \cap B = \{0\}$.

$\therefore \dim(A \cap B) = 0$.

Hence $\dim V = \dim A + \dim B$.

Rank and Nullity:

Definition:

Let $T: V \rightarrow W$ be a linear transformation. Then the dimension of $T(V)$ is called the rank of T. The dimension of $\ker T$ is called the nullity of T.

Theorem 5.29.

Let $T: V \rightarrow W$ be a linear transformation. Then $\dim V =$ rank T + nullity T.

Proof:

W.K.T $V/\ker T \cong T(V)$.

$\therefore \dim V - \dim(\ker T) = \dim(T(V))$.

$\therefore \dim V -$ nullity $T =$ rank T.

$\therefore \dim V =$ nullity $T +$ rank T.

Result: $\ker T$ is also called null space of T.

Example:

Let V denote the set of all polynomials of degree $\leq n$ in $R[x]$. Let $T: V \rightarrow V$ be defined by $T(f) = df/dx$. W.K.T T is a linear transformation.

Since $\frac{df}{dx} = 0 \Leftrightarrow f$ is constant, ker T consists of all constant polynomials. The dimension of this subspace of V is 1. Hence nullity T is 1. Since $\dim V = n+1$, rank $T = n$.

Definition :-

A linear transformation $T : V \to W$ is called non-singular if T is 1-1 ; otherwise T is called singular.

Matrix of a Linear Transformation :-

Let V and W be finite dimensional vector spaces over a field F. Let $\dim V = m$ and $\dim W = n$. Fix an ordered basis $\{v_1, v_2, \ldots, v_m\}$ for V and an ordered basis $\{w_1, w_2, \ldots, w_n\}$ for W.

Let $T : V \to W$ be a linear transformation. We have seen that T is completely specified by the elements $T(v_1), T(v_2), \ldots, T(v_m)$. Now, let

$$\begin{aligned} T(v_1) &= a_{11} w_1 + a_{12} w_2 + \cdots + a_{1n} w_n \\ T(v_2) &= a_{21} w_1 + a_{22} w_2 + \cdots + a_{2n} w_n \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \\ T(v_m) &= a_{m1} w_1 + a_{m2} w_2 + \cdots + a_{mn} w_n. \end{aligned} \qquad (1)$$

Hence $T(v_1), T(v_2), \ldots, T(v_m)$ are completely specified by the mn elements a_{ij} of the field F. These a_{ij} can be conveniently arranged in the form of m rows and n columns as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Such an array of mn elements of F arranged in m rows and n columns is known as $m \times n$ matrix over the field F and is denoted by (a_{ij}). Thus to every linear transformation T there is associated with it an $m \times n$ matrix over F. conversely any $m \times n$ matrix over F defines a linear transformation $T: V \to W$ given by the formula (1).

Note: The $m \times n$ matrix which we have associated with a linear transformation $T: V \to W$ depends on the choice of the basis for V and W.

For example, consider the linear transformation $T: V_2(R) \longrightarrow V_2(R)$ given by $T(a,b) = (a, a+b)$. choose $\{e_1, e_2\}$ as a basis both for the domain and the range.

Then $T(e_1) = (1,1) = e_1 + e_2$

$T(e_2) = (0,1) = e_2$.

Hence the matrix representing T is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Now, we choose $\{e_1, e_2\}$ as a basis for the domain and $\{(1,1), (1,-1)\}$ as a basis for the range.

Let $w_1 = (1,1)$ and $w_2 = (1,-1)$.

Then $T(e_1) = (1,1) = w_1$,

and $T(e_2) = (0,1) = (1/2)w_1 - (1/2)w_2$.

Hence the matrix representing T is $\begin{bmatrix} 1 & 0 \\ 1/2 & -1/2 \end{bmatrix}$

Solved problems:

problem 1. obtain the matrix representing the linear transformation $T: V_3(R) \to V_3(R)$ given by $T(a,b,c) = (3a, a-b, 2a+b+c)$ w.r.t the standard basis $\{e_1, e_2, e_3\}$

③

Solution:

$T(e_1) = T(1,0,0) = (3,1,2) = 3e_1 + e_2 + 2e_3.$

$T(e_2) = T(0,1,0) = (0,-1,1) = -e_2 + e_3.$

$T(e_3) = T(0,0,1) = (0,0,1) = e_3.$

Thus the matrix representing T is $\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 2. Find the linear transformation, $T: V_3(R) \to V_3(R)$ determined by the matrix

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ w.r.t. the standard basis $\{e_1, e_2, e_3\}$.

Solution:

$T(e_1) = e_1 + 2e_2 + e_3 = (1,2,1)$

$T(e_2) = 0e_1 + e_2 + e_3 = (0,1,1)$

$T(e_3) = -e_1 + 3e_2 + 4e_3 = (-1,3,4)$

Now, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$= ae_1 + be_2 + ce_3.$

$\therefore T(a,b,c) = T(ae_1 + be_2 + ce_3).$

$= aT(e_1) + bT(e_2) + cT(e_3).$

$= a(1,2,1) + b(0,1,1) + c(-1,3,4)$

$\therefore T(a,b,c) = (a-c, 2a+b+3c, a+b+4c).$

This is the required linear transformation.

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Definition:

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. We define the sum of these two matrices by $A + b = (a_{ij} + b_{ij})$.

Note that we have defined addition only for two matrices having the same number of rows and the same number of columns.

Definition :-

Let $A = (a_{ij})$ be an arbitrary matrix over a field F. Let $\alpha \in F$. We define $\alpha A = (\alpha a_{ij})$.

Theorem 5.30.

These set $M_{m \times n}(F)$ of all $m \times n$ matrices over the field F is a Vector space of dimension mn over F under matrix addition and scalar multiplication defined above.

proof:

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices over the field F. The addition of $m \times n$ matrices is a binary operation which is both commutative and associative. The $m \times n$ matrix whose entries are 0 is the identity matrix and $(-a_{ij})$ is the inverse matrix of (a_{ij}). Thus the set of all $m \times n$ matrices over the field F is an abelian group with respect to addition. The verification of the following axioms are straight forward.

(a) $\alpha(A+B) = \alpha A + \alpha B$.

(b) $(\alpha+\beta)A = \alpha A + \beta A$.

(c) $(\alpha\beta)A = \alpha(\beta A)$

(d) $IA = A$.

Hence the set of all $m \times n$ over F is a vector space over F.

Now, we shall prove that the dimension of this vector space is mn. Let E_{ij} be the matrix with entry 1 in the $(i,j)^{th}$ place and 0 in the other places. We have mn matrices of this form. Also any matrix $A = (a_{ij})$ can be written as $A = \sum a_{ij} \cdot E_{ij}$. Hence A is a linear combination of the matrices E_{ij}. Further these mn matrices E_{ij} are linearly independent. Hence these mn matrices form a basis for the space of all $m \times n$ matrices. Therefore the dimension of the vector space is mn.

__Theorem 5.31__

Let V and W be two finite dimensional vector spaces over a field F. Let $\dim V = m$ and $\dim W = n$. Then $L(V,W)$ is a vector space of dimension mn over F.

__Proof:__

By thm 5.8, $L(V,W)$ is a vector space over F. Now, we shall prove that the vector space $L(V,W)$ is isomorphic to the vector space $M_{m\times n}(F)$. Since $M_{m\times n}(F)$ is of dimension mn, it follows that $L(V,W)$ is also of dimension mn.

Fix a basis $\{v_1, v_2, \ldots\ldots, v_m\}$ for V and a basis $\{w_1, w_2, \ldots\ldots, w_n\}$ for W.

We know that any linear transformation $T \in L(V,W)$ can be represented by an $m\times n$ matrix over F.

Let T be represented by $M(T)$. This function $M : L(V,W) \longrightarrow M_{m\times n}(F)$ is clearly 1-1 and onto.

Let $T_1, T_2 \in L(V,W)$ and $M(T_1) = (a_{ij})$ and $M(T_2) = (b_{ij})$.

$$M(T_1) = (a_{ij}) \Rightarrow T_1(v_i) = \sum_{j=1}^{n} a_{ij} w_j.$$

$$M(T_2) = (b_{ij}) \Rightarrow T_2(v_i) = \sum_{j=1}^{n} b_{ij} w_j.$$

$$\therefore (T_1 + T_2)(v_i) = \sum_{j=1}^{n} (a_{ij} + b_{ij}) w_j.$$

$$\therefore M(T_1 + T_2) = (a_{ij} + b_{ij}) = (a_{ij}) + (b_{ij}) = M(T_1) + M(T_2)$$

Similarly $M(\alpha T_i) = \alpha M(T_i)$

Hence M is the required isomorphism from $L(V,W)$ to $M_{m\times n}(F)$.

Inner Product Spaces:-

Definition of Inner product:-

Let V be a vector space over F. An inner product on V is a function which assigns to each ordered pair of vectors u,v in V a scalar in F denoted by $\langle u,v\rangle$ satisfying the following conditions.

[i] $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$

[ii] $\langle \alpha u, v\rangle = \alpha\langle u,v\rangle$

[iii] $\langle u,v\rangle = \overline{\langle v,u\rangle}$, where $\overline{\langle v,u\rangle}$ is the complex conjugate of $\langle u,v\rangle$.

[iv] $\langle u,u\rangle \geq \langle u,u\rangle = 0$ iff $u=0$.

A vector space with an inner product defined on it is called an inner product space. An inner product space is called an Euclidean space or unitary space according as F is the field of real numbers or complex numbers.

Result 1:

If F is the field of real numbers then condition (iii) takes the form $\langle u,v\rangle = \langle v,u\rangle$. Further (iii) asserts that $\langle u,u\rangle$ is always real and hence (iv) is meaningful whether F is the field of real or complex numbers.

Result 2:

$$\langle u, \alpha v\rangle = \bar{\alpha}\langle u,v\rangle.$$

for, $\langle u, \alpha v\rangle = \overline{\langle \alpha v, u\rangle}$

$= \overline{\alpha\langle v,u\rangle}$

$= \bar{\alpha}\langle u,v\rangle.$

Result 3:

$$\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle.$$

for, $\langle u, v+w\rangle = \overline{\langle v+w, u\rangle}$

$= \overline{\langle v,u\rangle + \langle w,u\rangle}$

$= \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle}$

$= \langle u,v\rangle + \langle u,w\rangle.$

Result 4:

$$\langle u,0\rangle = \langle 0,v\rangle = 0.$$

for, $\langle u,0\rangle = \langle u, 00\rangle = 0\langle u,0\rangle$

$= 0$

Similarly $\langle 0,v\rangle = 0.$

Examples:-

1. $V_n(R)$ is a real inner product space with inner product defined by.

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n \text{ where.}$$

$$x = (x_1, x_2, \dots, x_n) \text{ and}$$

$$y = (y_1, y_2, \dots, y_n).$$

This is called the standard inner product on $V_n(R)$.

proof:

Let $x, y, z \in V_n(R)$ and $\alpha \in R$.

(i) $\langle x+y, z\rangle = (x_1+y_1)z_1 + (x_2+y_2)z_2 + \dots + (x_n+y_n)z_n$

$= (x_1z_1 + x_2z_2 + \dots + x_nz_n) + (y_1z_1 + y_2z_2 + \dots + y_nz_n)$

$= \langle x, z\rangle + \langle y, z\rangle.$

(ii) $\langle \alpha x, y\rangle = \alpha x_1y_1 + \alpha x_2y_2 + \dots + \alpha x_ny_n.$

$= \alpha(x_1y_1 + x_2y_2 + \dots + x_ny_n).$

$= \alpha\langle x, y\rangle.$

(iii) $\langle x, y\rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n.$

$= y_1x_1 + y_2x_2 + \dots + y_nx_n.$

$= \langle y, x\rangle.$

(iv) $\langle x, x\rangle = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$ and

$\langle x, x\rangle = 0$ iff $x_1 = x_2 = \dots = x_n = 0.$

$\therefore \langle x, x\rangle = 0$ iff $x = 0.$

2. $V_n(C)$ is a complex inner product space with inner product defined by

$$\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n} \text{ where}$$

$$x = (x_1, x_2, \ldots, x_n) \text{ and}$$

$$y = (y_1, y_2, \ldots, y_n)$$

<u>proof</u>:-

Let $x, y, z \in V_n(C)$ and $\alpha \in C$.

(i) $\langle x+y, z\rangle = (x_1+y_1)\overline{z_1} + (x_2+y_2)\overline{z_2} + \cdots + (x_n+y_n)\overline{z_n}$

$$= (x_1\overline{z_1} + x_2\overline{z_2} + \cdots + x_n\overline{z_n}) + (y_1\overline{z_1} + y_2\overline{z_2} + \cdots + y_n\overline{z_n})$$

$$= \langle x, z\rangle + \langle y, z\rangle.$$

(ii) $\langle \alpha x, y\rangle = \alpha x_1\overline{y_1} + \alpha x_2\overline{y_2} + \cdots + \alpha x_n\overline{y_n}$.

$$= \alpha(x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n})$$

$$= \alpha\langle x, y\rangle.$$

(iii) $\overline{\langle y, x\rangle} = \overline{y_1\overline{x_1} + y_2\overline{x_2} + \cdots + y_n\overline{x_n}}$.

$$= \overline{y_1}x_1 + \overline{y_2}x_2 + \cdots + \overline{y_n}x_n.$$

$$= x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$$

$$= \langle x, y\rangle.$$

(iv) $\langle x, x\rangle = x_1\overline{x_1} + \cdots + x_n\overline{x_n}$

$$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0 \text{ and,}$$

$\langle x, x\rangle = 0$ iff $x = 0$.

3. Let V be the set of all continuous real valued functions defined on the closed interval $[0,1]$. V is real inner product space with inner product defined by,

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Proof:-

Let $f, g, h \in V$ and $\alpha \in \mathbb{R}$.

(i) $\langle f+g, h\rangle = \int_0^1 [f(t)+g(t)]h(t)\,dt.$

$= \int_0^1 f(t)h(t)\,dt + \int_0^1 g(t)h(t)\,dt.$

$= \langle f, h\rangle + \langle g, h\rangle.$

(ii) $\langle \alpha f, g\rangle = \int_0^1 \alpha f(t)g(t)\,dt.$

$= \alpha \int_0^1 f(t)g(t)\,dt.$

$= \alpha\langle f, g\rangle.$

(iii) $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$

$= \int_0^1 g(t)f(t)dt$

$= \langle g, f\rangle.$

(iv) $\langle f, f\rangle = \int_0^1 [f(t)]^2\,dt \geqslant 0$ and

$\langle f, f\rangle = 0$ iff $f = 0.$

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Definition:-

Let V be an inner product space and let $x \in V$. The norm or length of x, denoted by $\|x\|$, is defined by $\|x\| = \sqrt{\langle x, x\rangle}$.

x is called a unit vector if $\|x\| = 1$.

Solved problems:

problem 1:

Let V be the vector space of polynomials with inner product given by $\langle f, g\rangle = \int_0^1 f(t) g(t)\, dt$.

Let $f(t) = t+2$ and $g(t) = t^2 - 2t - 3$.

Find, (i) $\langle f, g\rangle$ (ii) $\|f\|$.

Solution:

(i) $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$.

$$= \int_0^1 (t+2)\,(t^2 - 2t - 3)\,dt.$$

$$= \int_0^1 (t^3 - 7t - 6)\, dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$= -\frac{37}{4}.$$

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(ii) $\|f\|^2 = \langle f, f \rangle$.

$$= \int_0^1 [f(t)]^2 dt$$

$$= \int_0^1 (t+2)^2 dt.$$

$$= \int_0^1 (t^2+4t+4) dt.$$

$$= \left[\frac{t^3}{3} + 2t^2 + 4t\right]_0^1$$

$$= \frac{1}{3} + 2 + 4.$$

$$= 19/3$$

$$\therefore \|f\| = \sqrt{19}/\sqrt{3}$$

Theorem 6.1

The norm defined in an inner product space V has the following properties.

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff. $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$.

(iii) $|\langle x, y \rangle| \leq \|x\| \, \|y\|$ [Schwartz's Inequality]

(iv) $\|x+y\| \leq \|x\| + \|y\|$ [Triangle inequality]

proof :-

(i) $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$. and $\|x\| = 0$ iff $x = 0$.

(ii) $\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle$.

$$= \alpha \langle x, \alpha x \rangle$$

$$\alpha\bar{\alpha} = \langle x, x\rangle.$$

$$= |\alpha|^2 \|x\|^2.$$

Hence $\|\alpha x\| = |\alpha|\,\|x\|$.

(iii) The inequality is trivially true when $x=0$ or $y=0$.

Hence let $x \neq 0$ and $y \neq 0$.

consider $z = y - \dfrac{\langle y, x\rangle}{\|x\|^2} x$.

Then $0 \leq \langle z, z\rangle$.

$$= \left\langle y - \frac{\langle y, x\rangle}{\|x\|^2} x,\ y - \frac{\langle y, x\rangle}{\|x\|^2} x \right\rangle.$$

$$= \langle y, y\rangle - \frac{\overline{\langle y, x\rangle}}{\|x\|^2}\langle y, x\rangle - \frac{\langle y, x\rangle}{\|x\|^2}\langle x, y\rangle$$

$$+ \frac{\langle y, x\rangle\overline{\langle y, x\rangle}}{\|x\|^2 \cdot \|x\|^2}\langle x, x\rangle.$$

$$= \|y^2\| - \frac{\overline{\langle y, x\rangle}\langle y, x\rangle}{\|x\|^2} - \frac{\langle y, x\rangle\langle x, y\rangle}{\|x\|^2}$$

$$+ \frac{\langle y, x\rangle\overline{\langle y, x\rangle}}{\|x\|^2}.$$

$$= \|y\|^2 - \frac{\overline{\langle x, y\rangle}\langle x, y\rangle}{\|x\|^2}.$$

$$\therefore\ 0 \leq \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2.$$

$$\therefore\ |\langle x, y\rangle \leq \|x\|\,\|y\|.$$

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(iv) $\|x+y\|^2 = \langle x+y, x+y\rangle$.

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle.$$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2.$$

$$= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2.$$

$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2.$$

$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \quad (\text{by (iii)}).$$

$$\leq (\|x\| + \|y\|)^2.$$

$$\therefore \|x+y\| \leq \|x\| + \|y\|.$$

Orthogonality:-

Define orthogonal:-

Let V be an inner product space and let $x, y \in V$. x is said to be orthogonal to y if $\langle x, y\rangle = 0$.

Result 1:

x is orthogonal to $y \Rightarrow \langle x, y\rangle = 0$.

$$\Rightarrow \overline{\langle x, y\rangle} = \bar{0}$$

$$\Rightarrow \langle y, x\rangle = 0$$

$\Rightarrow y$ is orthogonal to x.

Thus x and y are orthogonal iff $\langle x, y\rangle = 0$.

Result 2:

x is orthogonal to $y \Rightarrow \alpha x$ is orthogonal to y.

Result 3:

x_1 and x_2 are orthogonal to $y \Rightarrow x_1 + x_2$ is orthogonal to y.

Result 4:

0 is orthogonal to every vector in V and is the only vector with this property.

Definition orthogonal set :-

Let V be an inner product space. A set S of vectors in V is said to be an orthogonal set if any two distinct vectors in S are orthogonal.

Define orthonormal set :-

S is said to be an orthonormal set if S is orthogonal and $\|x\| = 1$ for all $x \in S$.

Example:

The standard basis $\{e_1, e_2, \ldots, e_n\}$ in R^n or C^n is an orthogonal set with respect to the standard inner product.

Theorem 6.2

Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of non-zero vectors in an inner product space V. Then S is linearly independent.

proof:

Let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$.

Then $\langle \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n, v_1 \rangle = \langle 0, v_1 \rangle = 0$.

$\therefore \alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \langle v_2, v_1 \rangle + \ldots + \alpha_n \langle v_n, v_1 \rangle = 0$.

$\therefore \alpha_1 \langle v_1, v_1 \rangle = 0$ [since S is orthogonal).

$\therefore \alpha_1 = 0$ [since $v_1 \neq 0$].

Similarly $\alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$.

Hence S is linearly independent.

Theorem 6.3.

Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal of set of non-zero vectors in V. Let $v \in V$ and $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$. Then $\alpha_k = \dfrac{\langle v, v_k \rangle}{\|v_k\|^2}$.

proof:

$\langle v, v_k \rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n, v_k \rangle$.

$= \alpha_1 \langle v_1, v_k \rangle + \alpha_2 \langle v_2, v_k \rangle + \ldots + \alpha_k \langle v_k, v_k \rangle + \ldots + \alpha_n \langle v_n, v_k \rangle$

$= \alpha_k \langle v_k, v_k \rangle$ [since S is orthogonal].

$$= \alpha_k \|v_k\|^2.$$

$$\therefore \alpha_k = \frac{\langle v, v_k \rangle}{\|v_k\|^2}$$

Theorem 6.4.

Every finite dimensional inner product space has an orthonormal basis.

proof:.

Let V be a finite dimensional inner product space. Let $\{v_1, v_2, \ldots\ldots v_n\}$ be a basis for V. From this basis we shall construct an orthonormal basis $\{w_1, w_2, \ldots\ldots, w_n\}$ by means of a construction known as Gram-schmidt orthogonalisation process.

First we take $w_1 = v_1$.

$$\text{Let } w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1.$$

We claim that $w_2 \neq 0$. For, if $w_2 = 0$ then v_2 is a scalar multiple of w_1 and hence of v_1 which is a contradiction since v_1, v_2 are linearly independent.

$$\text{Also, } \langle w_2, w_1 \rangle = \left\langle v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1, w_1 \right\rangle.$$

$$= \left\langle v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1, v_1 \right\rangle \quad [\because w_1 = v_1].$$

$$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_1 \rangle.$$

$= \langle v_2, v_1 \rangle - \langle v_2, v_1 \rangle$

$= 0.$

Now, suppose that we have constructed non-zero orthogonal vectors $w_1, w_2, \ldots, w_k$. Then put

$$w_{k+1} = v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} w_j .$$

We claim that $w_{k+1} \neq 0$. For, if $w_{k+1} = 0$, then v_{k+1} is a linear combination of $w_1, w_2, \ldots, w_k$ and hence is a linear combination of $v_1, v_2, \ldots, v_k$ which is a contradiction since $v_1, v_2, \ldots, v_{k+1}$ are linearly independent.

Also,

$$\langle w_{k+1}, w_i \rangle = \langle v_{k+1}, w_i \rangle - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} \langle w_j, w_i \rangle .$$

$$= \langle v_{k+1}, w_i \rangle - \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} \langle w_i, w_i \rangle$$

$$= \langle v_{k+1}, w_i \rangle - \langle v_{k+1}, w_i \rangle .$$

$$= 0$$

Thus, continuing in this way we ultimately obtain a non-zero orthogonal set $\{w_1, w_2, \ldots, w_n\}$.

By theorem 6.2 this set is linearly independent and hence a basis.

To obtain an orthonormal basis we replace each w_i by $\frac{w_i}{\|w_i\|}$.

Solved problems:-

problem 1.

Apply Gram - schmidt process to construct an orthonormal basis for $V_3(R)$ with the standard inner product for the basis $\{v_1, v_2, v_3\}$ where $v_1 = (1,0,1)$; $v_2 = (1,3,1)$ and $v_3 = (3,2,1)$.

Solution:-

Take $w_1 = v_1 = (1,0,1)$

Then $\|w_1\|^2 = \langle w_1, w_1 \rangle = 1^2 + 0^2 + 1^2 = 2$.

and $\langle w_1, v_2 \rangle = 1 + 0 + 1 = 2$.

put, $w_2 = v_2 - \dfrac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$.

$= (1,3,1) - (1,0,1)$

$= (0,3,0)$.

$\therefore \|w_2\|^2 = 9$.

Also, $\langle w_2, v_3 \rangle = 0 + 6 + 0 \Rightarrow 6$ and $\langle w_1, v_3 \rangle = 3 + 0 + 1 \Rightarrow 4$

Now,

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2.$$

$= (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9}(0,3,0)$

$= (3,2,1) - 2(1,0,1) - \frac{2}{3}(0,3,0)$

$= (1,0,-1)$

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$\therefore \|w_3\|^2 = 2$

$\therefore$ The orthogonal basis is

$\{(1,0,1), (0,3,0), (1,0,-1)\}$.

Hence the orthognormal basis is,

$$\left\{\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], (0,1,0), \left[\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right]\right\}.$$

Problem 2.

Let V be the set of all polynomials of degree ≤ 2 together with the zero polynomial. V is a real inner product space with inner product defined by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$. Starting with the basis $\{1, x, x^2\}$, obtain an orthonormal basis for V.

Solution :-

Let $V_1 = 1$; $V_2 = x$ and $V_3 = x^2$.

Let $w_1 = V_1$.

Then $\|w_1\|^2 = \langle w_1, w_1\rangle = \int_{-1}^{1} 1\,dx$

$= 2$.

Hence $\|w_1\| = \sqrt{2}$

$$w_2 = V_2 - \frac{\langle V_2, w_1\rangle}{\|w_1\|^2} w_1.$$

$$= x - \frac{1}{2}\int_{-1}^{1} x\,dx$$

$$= x.$$

$$\therefore \|w_2\|^2 = \langle w_2, w_2\rangle$$

$$= \int_{-1}^{1} x^2\,dx \Rightarrow 2/3.$$

Now ; $w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2.$

$$= x^2 - \tfrac{1}{2}\int_{-1}^{1} x^2\,dx - \left[\frac{3x}{2}\right]\int_{-1}^{1} x^3\,dx.$$

$$= x^3 - \tfrac{1}{3}.$$

$$\therefore \|w_3\|^2 = \langle w_3, w_3\rangle = \int_{-1}^{1}\left(x^2 - \tfrac{1}{3}\right)^2 dx \Rightarrow 8/45$$

Hence the orthogonal basis is $\{1, x, x^2 - 1/3\}$

$\therefore$ The required orthonormal basis is

$$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}x, \frac{\sqrt{10}}{4}(3x^2 - 1)\right\}.$$

Problem 3:

Find a vector of unit length which is orthogonal to $(1, 3, 4)$ in $V_3(R)$ with standard inner product.

Solution:

Let $x = (x_1, x_2, x_3)$ be any vector orthogonal to $(1, 3, 4)$. Then $x_1 + 3x_2 + 4x_3 = 0$. Any solution of this equation gives a vector orthogonal to $(1,3,4)$. For example $x = (1,1,-1)$ is orthogonal to $(1,3,4)$.

Also $\|x\| = \sqrt{3}$. Hence a unit vector orthogonal to $(1,2,3)$ is given by $\left(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)$.

Result :-

The set of all vectors orthogonal to $(1,3,4)$ are the points lying on the plane $x+3y+4z=0$, which is a two dimensional subspace of $V_3(R)$.

Problem 4:

Find an orthogonal basis containing the vector $(1,3,4)$ for $V_3(R)$ with the standard inner product.

Solution:

$(1,1,-1)$ is a vector orthogonal to $(1,3,4)$

[refer problem 3 above].

Now, let $y=(y_1,y_2,y_3)$ be a vector orthogonal to both $(1,3,4)$ and $(1,1,-1)$.

Then $y_1+3y_2+4y_3=0$.

$y_1+y_2-y_3=0$.

Any solution of this system of equations gives a vector orthogonal to $(1,3,4)$ and $(1,1,-1)$.

For example $(7, -5, 2)$ is one such vector. [By cross multiplication method].

Hence $\{(1,3,4), (1,1,-1), (7,-5,2)\}$ is an orthogonal basis containing $(1,3,4)$.

Orthogonal complement :-

Definition:-

Let V be an inner product space. Let S be a subset of V. The orthogonal complement of S, denoted by $S^\perp$, is the set of all vectors in V which are orthogonal to every vector of S.

(i.e) $S^\perp = \{x / x \in V \text{ and } \langle x, u \rangle = 0 \text{ for all } u \in S\}$.

Examples :-

(1) $V^\perp = \{0\}$ and $\{0\}^\perp = V$ since 0 is the only vector which is orthogonal to every vector.

(2) Let $S = \{(x, 0, 0) / x \in R\} \subseteq V_3(R)$ with standard inner product. Then

$$S^\perp = \{(0, y, z) / y, z \in R\}.$$

(i.e) The orthogonal complement of the x-axis is the yz plane.

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Theorem 6.5:

If S is any subset of V then $S^{\perp}$ is a subspace of V.

proof:

clearly $0 \in S^{\perp}$ and hence $S^{\perp} \neq \Phi$.

Now, let $x, y \in S^{\perp}$ and $\alpha, \beta \in F$.

Then $\langle x, u \rangle = \langle y, u \rangle = 0$ for all $u \in S$.

$\therefore \langle \alpha x + \beta y, u \rangle = \alpha \langle x, u \rangle + \beta \langle y, u \rangle = 0$ for all $u \in S$.

$\therefore \alpha x + \beta y \in S^{\perp}$. Hence $S^{\perp}$ is a subspace of V.

Theorem 6.6

Let V be a finite dimensional inner product space. Let W be a subspace of V. Then V is the direct sum of W and $W^{\perp}$ (i.e) $V = W \oplus W^{\perp}$.

proof:

We shall prove that,

(i) $W \cap W^{\perp} = \{0\}$, and

(ii) $W + W^{\perp} = V$.

(i) Let $v \in W \cap W^{\perp}$. then $v \in W$ and $v \in W^{\perp}$.

Now, $v \in W^{\perp} \Rightarrow v$ is orthogonal to every vector in W.

In particular, v is orthogonal to itself.

$\therefore \langle v, v \rangle = 0$ and hence $v = 0$.

Hence $W \cap W^{-1} = \{0\}$.

(ii) Let $\{v_1, v_2, \ldots\ldots, v_r\}$ be an orthonormal basis for W.

Let $v \in V$.

consider,

$$v_0 = v - \langle v, v_1\rangle v_1 - \langle v, v_2\rangle v_2, \ldots\ldots, -\langle v, v_r\rangle v_r.$$

$$\therefore \langle v_0, v_i\rangle = \langle v, v_i\rangle - \langle v, v_i\rangle \langle v_i, v_i\rangle.$$

[since $\langle v_i, v_j\rangle = 0$ if $i \neq j$].

$$= \langle v, v_i\rangle - \langle v, v_i\rangle$$

[since $\langle v_i, v_i\rangle = 1$].

$$= 0.$$

$\therefore$ v_0 is orthogonal to each of $v_1, v_2, \ldots\ldots; v_r$ and hence is orthogonal to every vector in W.

Hence $v_0 \in W^{\perp}$ and.

$$v = [\langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + \cdots\cdots + \langle v, v_r\rangle v_r] + v_0 \in W + W^{\perp}.$$

$$\therefore V = W \oplus W^{\perp}.$$

Hence the theorem.

<u>corollarly</u>. $\dim V = \dim W + \dim W^{\perp}$.

<u>proof</u>. $\dim V = \dim(W \oplus W^{\perp}) = \dim W + \dim W^{\perp}$.

Theorem 6.7.

Let V be a finite dimensional inner product space. Let W be a subspace of V. Then $(W^{\perp})^{\perp} = W$.

Proof:-

Let $w \in W$. Then for any $u \in W^{\perp}$, $\langle w, u \rangle = 0$.

Hence $w \in ((W^{\perp})^{\perp})$. Thus $W \subseteq (W^{\perp})^{\perp}$ ..... (1).

Now by theorem 6.6, $V = W \oplus W^{\perp}$.

Also $V = W^{\perp} \oplus (W^{\perp})^{\perp}$.

Hence $\dim W = \dim (W^{\perp})^{\perp}$ ..... (2)

From (1) and (2) we get $W = (W^{\perp})^{\perp}$.

Solved problems:

Problem 1:

Let V be an inner product space and let S_1 and S_2 be subsets of V. Then $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$.

Solution:-

Let $u \in S_2^{\perp}$.

Then $\langle u, v \rangle = 0$ for all $v \in S_2$.

But $S_1 \subseteq S_2$. Hence $\langle u, v \rangle = 0$ for all $v \in S_1$.

Hence $u \in S_1^{\perp}$. Thus $S_2^{\perp} \subseteq S_1^{\perp}$.

Problem 2:

Let W_1 and W_2 be subspaces of a finite dimensional inner product space. Then,

(i) $(W_1+W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

(ii) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$.

Solution:

(i) We know that $W_1 \subseteq W_1 + W_2$.

$\therefore (W_1+W_2)^{\perp} \subseteq W_1^{\perp}$ (by the problem 1).

Similarly, $(W_1+W_2)^{\perp} \subseteq W_2^{\perp}$.

Hence $(W_1+W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$ ..... (1)

Now, let $w \in W_1^{\perp} \cap W_2^{\perp}$.

$\therefore \langle w, u\rangle = 0$ for all $u \in W_1$ and W_2.

Now, let $v \in W_1 + W_2$.

Then $v = v_1 + v_2$ where $v_1 \in W_1$ and $v_2 \in W_2$

$\therefore \langle w, v\rangle = \langle w, v_1+v_2\rangle$

$= \langle w, v_1\rangle + \langle w, v_2\rangle$

$= 0 + 0$ [since $v_1 \in W_1$ and $v_2 \in W_2$]

$= 0$.

Hence $w \in (W_1+W_2)^{\perp}$

$\therefore \; W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1+W_2)^{\perp} \ldots\ldots (2)$

From (1) and (2) we get,

$(W_1+W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

(ii) proof is similar to that of (i).

ALL THE BEST

Unit - V

Characteristic Equation and Cayley Hamilton Theorem -

Defn:- Matrix polynomial:-

An expression of the form $A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$ where $A_0, A_1, A_2 \cdots A_n$ are square matrices of the same order and $A_n \neq 0$ is called a matrix polynomial of degree n

For example.

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} x^2$$ is a matrix polynomial of degree 2 and it is simply the matrix.

$$\begin{pmatrix} 1+x+2x^2 & 2+x \\ 2x+3x^2 & 3+x+x^2 \end{pmatrix}$$

Def:- Characteristic matrix: A 1

Let A be any square matrix of order n and let I be the identity matrix of order n. Then the matrix polynomial given by $A - xI$ is called the characteristic matrix of A.

Defn: characteristic polynomial

The determinant $|A - Ix|$ which is an ordinary polynomial in x of degree n is called the characteristic polynomial of A.

Defn: Characteristic equation

The equation $|A - Ix| = 0$ is called the characteristic equation of A.

Example 1: Find the characteristic equation of matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution:-

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Then the characteristic matrix of A is $A - Ix$ given by

$$A - xI = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-x & 2 \\ 3 & 4-x \end{pmatrix}$$

$\therefore$ The characteristic polynomial of A is

$$|A - xI| = \begin{vmatrix} 1-x & 2 \\ 3 & 4-x \end{vmatrix}$$

$$= (1-x)(4-x) - 6$$

$$= 4 - x - 4x + x^2 - 6$$

$$= x^2 - 5x - 2$$

$\therefore$ The characteristic equation of A is $|A - Ix| = 0$

$\therefore$ $x^2 - 5x - 2 = 0$.

$\therefore$ $x^2 - 5x - 2 = 0$ is the characteristic equation of A.

Ex. 2 Find the characteristic equation of $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.

Solution:-

Given that $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

The characteristic matrix of A is $A - xI$ given by

$$A - xI = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} - x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{pmatrix}$$

The characteristic polynomial of A is

$$|A - xI| = \begin{vmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{vmatrix}$$

$$= (1-x)[(1-x)(-x) - 4] + 2(-(1-x))$$

$$= (1-x)[-x + x^2 - 4] + 2(-1+x)$$

$$= (1-x)[x^2 - x - 4] - 2 + 2x$$

$$= x^2 - x - 4 - x^3 + x^2 + 4x - 2 + 2x$$

$$= -x^3 + 2x^2 + 5x - 6$$

The characteristic equation of A is $|A - xI| = 0$

$$|A - xI| = x^3 - 2x^2 - 5x + 6 = 0.$$

$\therefore$ $x^3 - 2x^2 - 5x + 6 = 0$ is the characteristic equation of A.

Theorem:- State and prove Cayley Hamilton theorem.

Statement: Any square matrix A satisfies its characteristic equation

(ie) if $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is the characteristic polynomial of degree n of A then

$$a_0I + a_1A + a_2A^2 + \dots + a_nA^n = 0$$

Proof:-

Let A be a square matrix of order n.

Let $|A - xI| = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \longrightarrow (1)$

be the characteristic polynomial of A.

Now, adj$(A - xI)$ is a matrix polynomial of degree $n-1$. Since each entry of the matrix adj$(A - xI)$ is a cofactor of $A - xI$ and hence is a polynomial of degree $\leq n-1$.

Let adj$(A - xI) = B_0 + B_1x + B_2x^2 + \dots + B_{n-1}x^{n-1}$

$\longrightarrow (2)$

Now, $(A - xI)\,\text{adj}(A - xI) = |A - xI|\, I$.

By using equ (1) & (2), we get. ($\because$ (adj A)$A = A$(adj A) $= |A|I$)

$(A - xI)(B_0 + B_1x + B_2x^2 + \dots + B_{n-1}x^{n-1}) = (a_0 + a_1x + \dots + a_nx^n)I$

$AB_0 + AB_1x + AB_2x^2 + \dots + AB_{n-1}x^{n-1}$

$- B_0xI - B_1x^2I - B_2x^3I - \dots - B_{n-1}x^nI = (a_0 + a_1x + \dots + a_nx^n)I$

Equating the coefficients of the corresponding powers of x we get.

$AB_0 + (AB_1 - B_0I)x + (AB_2 - B_1I)x^2 + \dots$

$+ (AB_{n-1} - B_nI)x^{n-1} - B_{n-1}Ix^n = a_0I + a_1Ix + \dots + a_nx^nI.$

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Equaling the co-effiuents of the corresponding powers of x we have.

$$AB_0 = a_0 I$$
$$AB_1 - B_0 = a_1 I$$
$$AB_2 - B_1 = a_2 I$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$
$$AB_{n-1} - B_n = a_{n-1} I$$
$$-B_{n-1} = a_n I.$$

Pre multiplying the above equation by $I, A, A^2, \ldots A$ ~~respec~~ respectivaly and adding we get,

$$A\cancel{B_0} = a_0 I$$
$$\cancel{A^2 B_1} - \cancel{AB_0} = a_1 A$$
$$A^3 \cancel{B_2} - \cancel{A^2 B_1} = a_2 A^2.$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$
$$A^n \cancel{B_{n-1}} - A^{n-1} B_n = a_{n-1} A^{n-1}$$
$$-A^n \cancel{B_{n-1}} = a_n A^n$$
$$0 = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

$$\therefore \quad a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0.$$

Note:-

The inverse of a non singular matrix can be calculated by using the Cayley Hamilton theorem as follows.

Let $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be the characteristic polynomial of A.

Then we have

$$a_0I + a_1A + a_2A^2 + \cdots + a_nA^n = 0 \rightarrow (1)$$

Since $|A - xI| = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ we get $a_0 = |A|$ (by putting $x=0$)

$a_0 \neq 0$ ($\because$ A is a non singular matrix).

$$\therefore I = \frac{-1}{a_0}[a_1A + a_2A^2 + \cdots + a_nA^n]$$

$$A^{-1} = \frac{-1}{a_0}[a_1I + a_2A + \cdots + a_nA^{n-1}].$$

Problems:-

1. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:-

The characteristic equation of A is given by $|A - \lambda I| = 0$.

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$(8-\lambda)\,[(3-\lambda)(7-\lambda)-16]+6[-6(3-\lambda)+8]+2[24-2(7-\lambda)]=0$

$(8-\lambda)\,[21-3\lambda-7\lambda+\lambda^2-16]+6[-18+6\lambda+8]+2[24+2\lambda-14]=0$

$8-\lambda\,[\lambda^2-10\lambda+5]+6[6\lambda-10]+2[2\lambda+10]=0$

$8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0$

$-\lambda^3+\lambda^2(8+10)+\lambda(-80-5+36+4)=0$

$-\lambda^3+18\lambda^2-45\lambda=0$

$\lambda^3-18\lambda^2+45\lambda=0$

$\therefore \lambda^3-18\lambda^2+45\lambda=0$, which represents the characteristic equation of A.

2. Show that the non-singular matrix $A=\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}$ satisfies the equation $A^2-2A-5I=0$. Hence evaluate A^{-1}.

Solution

Given that

$$A=\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}$$

The characteristic equation is

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 2\\ 3 & 1-\lambda\end{vmatrix}=0$$

$$(1-\lambda)(1-\lambda)-6=0$$

$$1-\lambda-\lambda+\lambda^2-6=0$$

$$\lambda^2-2\lambda-5=0$$

Put $\lambda = A$ and using Cayley-Hamilton theorem

$$A^2-2A-5I=0$$

$\therefore$ ~~अतः~~ or

$$A^2-2A=5I$$

$$I=\frac{1}{5}(A^2-2A)$$

$$I=\frac{1}{5}A(A-2I)$$

$$A^{-1}=\frac{1}{5}(A-2I)$$

$$=\frac{1}{5}\left[\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}-2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]$$

$$=\frac{1}{5}\left[\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}-\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\right]$$

$$A^{-1}=\frac{1}{5}\begin{bmatrix}-1 & 2\\ 3 & -1\end{bmatrix}$$ //

3. Show that the matrix $A=\begin{bmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{bmatrix}$ satisfies the equation $A(A-I)(A+2I)=0$.

Solution

Given that $A=\begin{bmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{bmatrix}$

The characteristic equation is

$$|(A-\lambda I)|=0$$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(1-\lambda)(-4-\lambda) - 6\right] + 3\left[3(-4-\lambda)+15\right] + 1\left[6+5(1-\lambda)\right] = 0$$

$$(2-\lambda)\left[-4-\lambda+4\lambda+\lambda^2-6\right] + 3\left[-12-3\lambda+15\right] + (6+5-5\lambda) = 0$$

$$(2-\lambda)\left[\lambda^2+3\lambda-10\right] + 3\left[-3\lambda+3\right] + 11-5\lambda = 0$$

$$2\lambda^2+6\lambda-20-\lambda^3-3\lambda^2+10\lambda-9\lambda+9+11-5\lambda=0$$

$$-\lambda^3+\lambda^2(2-3)+\lambda(6+10-9-5)=0$$

$$-\lambda^3-\lambda^2+2\lambda=0.$$

$$\lambda^3+\lambda^2-2\lambda=0$$

By cayley Hamilton's theorem,

replace $\lambda = A$

$$A^3+A^2-2A=0$$

$$A(A^2+A-2I)=0 \qquad +2 \;\; -1$$

$$A(A+2I)(A-I)=0.$$

Hence proved.

4. Using Cayley-Hamilton theorem find the inverse of matrix $\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$

Solution :-

Given that $A = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & 1 \end{pmatrix}$

The Characteristic equation is

$$|A-\lambda I|=0.$$

$1+A+$ [illegible]

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)[(-1-\lambda)(-1-\lambda) - 4)] - 2[-6(-1-\lambda) - 12]$$
$$-2[-12 - 6(-1-\lambda)] = 0.$$

$$(7-\lambda)[1+\lambda+\lambda+\lambda^2-4] - 2[6+6\lambda-12] - 2[-12+6+6\lambda]$$

$$(7-\lambda)[\lambda^2+2\lambda-3] - 2[6\lambda-6] - 2[-6+6\lambda] = 0$$

$$7\lambda^2+14\lambda-21-\lambda^3-2\lambda^2+3\lambda-12\lambda+12+12-12\lambda = 0$$

$$-\lambda^3+\lambda^2(7-2)+\lambda(14+3-12-12)+3 = 0$$

$$-\lambda^3+5\lambda^2-7\lambda+3 = 0$$

$$\lambda^3-5\lambda^2+7\lambda-3 = 0$$

By Cayley Hamilton's theorem

$$A^3-5A^2+7A-3I = 0$$

$$3I = A^3-5A^2+7A$$

$$I = \frac{1}{3}(A^3-5A^2+7A)$$

$$A^{-1} = \frac{1}{3}(A^2-5A+7I). \longrightarrow (1)$$

Now, $$A^2 = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 49-12-12 & 14-2-4 & -14+4+2 \\ -42+6+12 & -12+1+4 & 12-2-2 \\ 42-12-6 & 12-2-2 & -12+4+1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{pmatrix}$$

$$① \Rightarrow A^{-1} = \frac{1}{3}\left[\begin{pmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{pmatrix} - 5\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{3}\left\{\begin{pmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{pmatrix} - \begin{pmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}\right\}$$

$$A^{-1} = \frac{1}{3}\left\{\begin{pmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{pmatrix}\right\} //.$$

Find the inverse of the matrix $\begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ using Cayley - Hamilton theorem.

Solution

Given that $A = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0.$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 3 & 4 \\ 2 & -3-\lambda & 4 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$= (3-\lambda)[(-3-\lambda)(1-\lambda)+4] - 3[2(1-\lambda)] + 4[-2]$

$= (3-\lambda)[-3+3\lambda-\lambda+\lambda^2+4] - 6 + 6\lambda - 8$

$= (3-\lambda)[\lambda^2+2\lambda+1] + 6\lambda - 14$

$= 3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda + 6\lambda - 14$

$= -\lambda^3 + \lambda^2(3-2) + \lambda(6-1+6) - 11$

$= -\lambda^3 + \lambda^2 + 11\lambda - 11$

$|A-\lambda I| = \lambda^3 - \lambda^2 - 11\lambda + 11$

By using cayley Hamilton's theorem,

$A^3 - A^2 - 11A + 11I = 0$

$11I = -(A^3 - A^2 - 11A)$

$11I = -A(A^2 - A - 11I)$

$A^{-1} = \frac{-1}{11}[A^2 - A - 11I] \longrightarrow (1)$

$\therefore A^2 = \begin{pmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 9+6 & 9-9-4 & 12+12+4 \\ 6-6 & 6+9-4 & 8-12+4 \\ -2 & 3-1 & -4+1 \end{pmatrix}$

$= \begin{pmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{pmatrix}$

$$A^{-1} = \frac{-1}{11}\left[\begin{pmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{-11}\begin{bmatrix} 1 & -7 & 24 \\ -2 & 3 & -4 \\ -2 & 3 & -15 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1/11 & 7/11 & -24/11 \\ 2/11 & -3/11 & 4/11 \\ 2/11 & -3/11 & 15/11 \end{bmatrix} //.$$

8 (A)

5. Verify Cayley Hamilton's theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Solution

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

By Cayley Hamilton's theorem A satisfies its characteristic equation we have,

$$A^2 - 4A - 5 = 0$$

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$$A^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$$

$$4A = 4\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix}$$

$$A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 9-4-5 & 8-8-0 \\ 16-16-0 & 17-12-5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore A^2 - 4A - 5I = 0$$

(8) (9) (7). Using Cayley's Hamilton's theorems for the matrix $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ find (i) A^{-1} (ii) A^4.

Solution

Given that

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

The characteristic equation of A is

$|A-\lambda I|=0.$

$$\therefore |A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(2-\lambda)-0]-2[0]=0$$

$$(1-\lambda)(4-2\lambda-2\lambda+\lambda^2)=0$$

$$(1-\lambda)(\lambda^2-4\lambda+4)=0.$$

$$\lambda^2-4\lambda+4-\lambda^3+4\lambda^2-4\lambda=0.$$

$$-\lambda^3+5\lambda^2-8\lambda+4=0$$

$$\lambda^3-5\lambda^2+8\lambda-4=0$$

by cayley Hamilton's theorem

$$A^3-5A^2+8A-4I=0 \longrightarrow (1)$$

$$4I=A^3-5A^2+8A \longrightarrow (2)$$

) To find A^{-1}.

Multiply A^{-1} in (2). We get $\qquad A^{-1}=\frac{1}{A}$

$$AA^{-1}=A^2-5A+8I$$

$$A^{-1}=\frac{1}{4}(A^2-5A+8I) \longrightarrow (3)$$

$$A^2=\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 0 & -2-4 \\ [illegible] & [illegible] & -4+8+8 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{4}[A^2 - 5A + 8I]$$

$$= \frac{1}{4}\left[\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{4}\left\{\begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 & -10 \\ +10 & 10 & 20 \\ 0 & 0 & 10 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right\}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 4 \\ 4 & 2 & -8 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & +\frac{1}{2} \end{pmatrix}$$

(ii) To find A^4.

From eqn (29),

$$A^3 = 5A^2 - 8A + 4I$$

Multiplying A on both sides, we get.

$$A^4 = 5A^3 - 8A^2 + 4A$$

$$A^4 = 5[5A^2 - 8A + 4I] - 8A^2 + 4A$$

$$= 25A^2 - 40A + 20I - 8A^2 + 4A$$

$$A^4 = 17A^2 - 36A + 20I$$

$$A^4 = 17\begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix} - 36\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} + 20\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 17 & 0 & -102 \\ 102 & 68 & 204 \\ 0 & 0 & 68 \end{bmatrix} - \begin{bmatrix} 36 & 0 & -72 \\ 72 & 72 & 144 \\ 0 & 0 & 72 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 17-36+20 & 0 & -102+72+0 \\ 102-72+0 & 68-72+20 & 204-144+0 \\ 0 & 0 & 68-72+20 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & -30 \\ 30 & 16 & 60 \\ 0 & 0 & 16 \end{bmatrix}_{//}$$

(4)

Eigen values and Eigen vectors:-

Defn:- Eigen value and Eigen vector:-

Let A be an $n \times n$ matrix. A number λ is called an eigen value of A if there exists a non zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $AX = \lambda X$ and X is called an eigenvector corresponding to the eigen value λ.

Remark:-

1. If X is an eigen vector corresponding to the eigen value λ of A, then αX where α is any non zero number, is also an eigen vector corresponding to λ.

2. Let λ be an eigen vector corresponding the eigen value λ of A. Then $AX = \lambda X$ so that $(A - \lambda I)X = 0$. Thus X is a non-trivial solution of the system of homogeneous linear equations $(A - \lambda I)X = 0$.

Hence $|A - \lambda I| = 0$, which is the characteristic polynomial of A.

Let $|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n$

The roots of this polynomial give the eigen value of A. Hence the eigen values are also called characteristic roots.

(b) Properties of Eigen Values:-

(1) Let X be an eigen vector corresponding to the eigen values λ_1 and λ_2. Then $\lambda_1 = \lambda_2$.

Proof:-

Let X be an eigen value vector corresponding to the eigen values λ_1 and λ_2. Then

$$AX = \lambda_1 X,$$

and $AX = \lambda_2 X, \quad X \neq 0.$

$$\therefore AX = \lambda_1 X$$

$$\lambda_2 X = \lambda_1 X$$

$$0 = \lambda_1 X - \lambda_2 X$$

$$\therefore (\lambda_1 - \lambda_2)X = 0$$

Since $X \neq 0$, $\lambda_1 = \lambda_2$.

Hence proved.

1 (a)

[Le]t A be a square matrix. Then (i) the sum of the [ei]gen value of A is equal to the sum of the diagonal [el]ements (trace of A). (ii) Product of eigen value of A is [...]

[Pro]of.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

The eigen value of A are the roots of the [ch]aracteristic equation

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \longrightarrow (1)$$

Let $|A-\lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n \longrightarrow (2)$

[Fr]om (1) and (2) we get,

$a_0 = (-1)^n;$

$a_1 = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}); \longrightarrow (2)$

$\vdots$

Also by putting $\lambda = 0$ in (2) we get $a_n = |A|$

[N]ow let $\lambda_1, \lambda_2 \cdots \lambda_n$ be the eigen value of A.

$\lambda_1, \lambda_2 \cdots \lambda_n$ are the roots of (2).

$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \dfrac{-a_1}{a_0}$

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$= a_{11} + a_{22} + \cdots + a_{nn}$ (using ③).

Sum of the eigen values = trace of A.

(ii) Product of the eigen values = product of the root

$= \lambda_1 . \lambda_2 \ldots \lambda_n$

$= (-1)^n \frac{a_n}{a_0}$

$= \frac{(-1)^n a_n}{(-1)^n}$

$= a_n$

$= |A|$.

3. The eigen value of A and its transpose A^T are the same.

Proof

It is enough if we prove that A and A^T have the same characteristic polynomial. Since for any square matrix M, $|M| = |M^T|$, we have.

$$|A - \lambda I| = |(A - \lambda I)^T|$$

$$= |A^T - (\lambda I)^T|$$

$$= |A^T - \lambda I|$$

$\therefore$ The eigen values of A and its transpose A^T are the same.

Hence proved.

proved.

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If λ is an eigen value of a non singular matrix then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof:

Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$.

Since A is non singular A^{-1} exists.

$$\therefore A^{-1}(AX) = A^{-1}(\lambda X)$$

$$IX = \lambda A^{-1} X$$

$$\left(\frac{1}{\lambda}\right) X = A^{-1} X$$

$$\therefore A^{-1} X = \left(\frac{1}{\lambda}\right) X$$

$\therefore$ $1/\lambda$ is an eigen value of A^{-1}.

Corollary:-

If $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigen values of a non singular matrix A then $1/\lambda_1, \frac{1}{\lambda_2}, 1/\lambda_3 \cdots 1/\lambda_n$ are the eigen value of A^{-1}.

If λ is an eigen value of A then $k\lambda$ is an eigen value of KA where K is a scalar

Proof.

Let X be an eigen ~~value~~ vector corresponding to λ.

then $AX = \lambda X \longrightarrow (1)$ Corresponding to λ

Now, $(KA)X = K(AX)$

$= K(\lambda X)$ (by 1)

$= (k\lambda) X$

$\therefore$ $k\lambda$ is an eigen value of kA.

If λ is eigen value of A then λ^k is an eigen value of A^k where k is any positive integer.

Proof: Let X be an eigen ~~value~~ vector corresponding to λ. Then

$$AX = \lambda X. \longrightarrow (1)$$

Now,
$$\begin{aligned} A^2X &= (AA)X \\ &= A(AX) \\ &= A(\lambda X) \\ &= \lambda(AX) \\ &= \lambda(\lambda X) \\ &= \lambda^2(X) \\ &= \lambda^2 X \end{aligned}$$

$\therefore$ λ^2 is an eigen value of A^2.

Similarly.

λ^k is an eigen value of A^k for any positive integer.

Corollary:-

If $\lambda_1, \lambda_2 \cdots \lambda_n$ are eigen value of A then $\lambda_1^k, \lambda_2^k \cdots \lambda_n^k$ are eigen value of A^k for any positive integer k.

gen vectors corresponding to distinct eigen values a matrix are linearly independent

Proof:-

Let $\lambda_1, \lambda_2 \cdots \lambda_K$ be distinct eigen values of a matrix and Let X_i be the eigen vector corresponding λ_i.

Hence $AX_i = \lambda_i X_i \quad (i = 1, 2 \cdots) \longrightarrow (1)$

Now, Suppose $X_1, X_2, \cdots X_K$ are linearly dependent. Then there exist real numbers $\alpha_1, \alpha_2 \cdots \alpha_K$ not all zero, such that $\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_K X_K = 0$. Among all such relations, we choose one of shortest length, say j.

By rearranging the vectors $X_1, X_2 \cdots X_K$ we may assume that

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_j X_j = 0 \longrightarrow (2)$$

$$\therefore A(\alpha_1 X_1) + A(\alpha_2 X_2) + \cdots + A(\alpha_j X_j) = 0$$

$$\alpha_1 (AX_1) + \alpha_2 (AX_2) + \cdots + \alpha_j (AX_j) = 0$$

$$\alpha_1 \lambda_1 X_1 + \alpha_2 \lambda_2 X_2 + \cdots + \alpha_j \lambda_j X_j = 0 \longrightarrow (3)$$

Multiplying (2) by λ_1 and subtracting from (3), we get.

$$\alpha_1 \lambda_1 X_1 + \alpha_2 \lambda_1 X_2 + \cdots + \alpha_j \lambda_1 X_j = 0 \longrightarrow (4)$$

$$\alpha_1 (\lambda_1 - \lambda_1) X_1 + \alpha_2 (\lambda_1 - \lambda_2) X_2 + \cdots + \alpha_j \cdot (\lambda_1 - \lambda_j) X_j = 0 \longrightarrow (5)$$

and since $\lambda_1, \lambda_2 \cdots \lambda_j$ are distinct and

$\alpha_2, \alpha_3 \cdots \alpha_j$ are non zero element we have

$$\alpha_i(\lambda_1 - \lambda_i) \neq 0, \quad i = 2, 3, \ldots, j$$

Thus (5) gives the relation whose length is $j-1$, giving a contradiction.

Hence $x_1, x_2 \ldots x_k$ are linearly independent.

8. The characteristic roots of a Hermitian matrix are all real.

Proof:-

Let A be a Hermitian matrix

Hence

$$A = \bar{A}^T \longrightarrow (1)$$

Let λ be a characteristic root of A and let X be a characteristic vector corresponding to λ.

$$\therefore AX = \lambda X \longrightarrow (2)$$

Now,

$$AX = \lambda X \Rightarrow \bar{X}^T A X = \lambda \bar{X}^T X$$

$$\Rightarrow (\bar{X}^T A X)^T = \lambda \bar{X}^T X$$

(since $X^T A X$ is a 1×1 matrix)

$$\Rightarrow X^T A^T (\bar{X}^T)^T = \lambda \bar{X}^T X$$

$$\Rightarrow X^T A^T \bar{X} = \lambda \bar{X}^T X$$

$$\Rightarrow \overline{X^T A^T . \bar{X}} = \overline{\lambda X^T X}$$

$$\Rightarrow \bar{X}^T \bar{A}^T X = \bar{\lambda} X^T \bar{X}$$

$\Rightarrow \bar{X}^T A X = \bar{\lambda} X^T \bar{X}$ (using 1)

$\Rightarrow \bar{X}^T \lambda X = \bar{\lambda} X^T \bar{X}$ (using 2)

$\Rightarrow \lambda(\bar{X}^T X) = \bar{\lambda}(X^T \bar{X}) \longrightarrow (3)$

$$\bar{X}^T X = X^T \bar{X}$$
$$= \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots + \bar{x}_n x_n$$
$$= |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$$
$$\neq 0$$

From (3) we get $\lambda = \bar{\lambda}$

Hence λ is real.

Corollary:— The characteristic root of real symmetric matrix are real.

The characteristic roots of a skew Hermitian matrix are either purely imaginary or zero.

Proof

Let A be a skew Hermitian matrix and λ be a characteristic root of A.

$$\therefore |A - \lambda I| = 0$$
$$\therefore |iA - i\lambda I| = 0$$

$\therefore$ $i\lambda$ is characteristic root of iA.

Since A is skew Hermitian iA is Hermitian.

(10) (1X) let λ be a characteristic root of an unitary matrix A. Then $|\lambda|=1$. (ie) the characteristic roots of a unitary matrix are all the unit modulus.

Proof.

let λ be a characteristic root of an unitary matrix A and X be a characteristic vector corresponding to λ.

$$\therefore AX=\lambda X \longrightarrow (1)$$

Taking conjugate and transpose in (1) we get,

$$(\overline{AX})^T=(\overline{\lambda X})^T$$

$$\overline{X}^T\cdot\overline{A}^T=\overline{\lambda}\,\overline{X}^T \longrightarrow (2)$$

Multiplying (1) and (2) we get

$$(\overline{X}^T\overline{A}^T)(AX)=(\overline{\lambda}\,\overline{X}^T)\lambda X$$

$$\overline{X}^T(\overline{A}^TA)X=\overline{\lambda}\lambda(\overline{X}^TX)$$

Now, Since A is an unitary matrix $\overline{A}^TA=I$

Hence $\overline{X}^TX=(\overline{\lambda}\lambda)\overline{X}^TX$

Since X is non zero vector $\overline{X}^T$ is also nonzero vector and

$$\overline{X}^TX=|x_1|^2+|x_2|^2+\cdots+|x_n|^2\neq 0.$$

We get

$$\lambda\overline{\lambda}=1$$

Hence $|\lambda|^2=1$

$$\therefore |\lambda|=1.$$

Corollary:– Let λ be a characteristic root of an Orthogonal matrix A. Then $|\lambda| = 1$.

Zero is an eigen value of A if and only if A is a singular matrix.

Proof:–

The eigen value of A are the roots of the characteristic equation $|A - \lambda I| = 0$.

Now, 0 is an eigen value of $A \Leftrightarrow |A - 0I| = 0$

$\Leftrightarrow |A| = 0$

$\Leftrightarrow A$ is singular matrix.

If A and B are two Square matrices of the same order then AB and BA have the same eigen values.

Solution:–

Let λ be an eigen value of AB and X be an eigen vector corresponding to λ

$\therefore (AB)X = \lambda X$

$\therefore B(AB)X = B(\lambda X)$

$= \lambda(BX)$

$\therefore (BA)(BX) = \lambda(BX)$

$(BA)Y = \lambda Y$

where $Y = BX$

Hence λ is an eigen value of BA.

Also BX is the corresponding eigen vector

If P and A are $n \times n$ matrices and P is a nonsingular matrix then A and $P^{-1}AP$ have the same eigen values.

Proof:-

let $B = P^{-1}AP$.

To prove A and B have the same eigen values. It is enough to prove that the characteristic polynomials of A and B are the same.

Now,

$$|B-\lambda I| = |P^{-1}AP - \lambda I|$$
$$= |P^{-1}AP - P^{-1}(\lambda I)P|$$
$$= |P^{-1}(A-\lambda I)P|$$
$$= |P^{-1}|\,|A-\lambda I|\,|P|$$
$$= |P^{-1}|\,|P|\,|A-\lambda I|$$
$$= |P^{-1}P|\,|A-\lambda I|$$
$$= |I|\,|A-\lambda I|$$
$$|B-\lambda I| = |A-\lambda I|$$

$\therefore$ The characteristic equation of A and $P^{-1}AP$ are the same.

If λ is a characteristic root of A then $f(\lambda)$ is a characteristic root of the matrix $f(A)$ where $f(x)$ is any polynomial.

Proof:-

Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$.

Where $a_0 \neq 0$, and $a_1, a_2, \ldots a_n$ are all real numbers.

$$\therefore f(A) = a_0 A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I$$

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Since λ is characteristic root of A. λ^n is a characteristic root of A^n for any positive integer n.

$$\left.\begin{aligned} \therefore\ A^n X &= \lambda^n X \\ A^{n-1} X &= \lambda^{n-1} X \\ &\cdots\cdots \\ &\cdots\cdots \\ AX &= \lambda X \end{aligned}\right\} \longrightarrow (1)$$

$$\left.\begin{aligned} \therefore\ a_0 A^n X &= a_0 \lambda^n X \\ a_1 A^{n-1} X &= a_1 \lambda^{n-1} X \\ &\cdots\cdots \\ &\cdots\cdots \\ a_{n-1} A X &= a_{n-1} \lambda X \end{aligned}\right\} \longrightarrow (2)$$

Adding the above quation we have

$\cancel{(a_0 A^n + a_1 A^{n-1} + \cdots + a_{n-1} A) X = (a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}) X}$

$$(a_0 A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I) X = (a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n I) X$$

$$\therefore\ f(A) X = f(\lambda) X.$$

Hence $f(\lambda)$ is a characteristic root of $f(A)$.

Solved problems:–

If X_1, X_2 are eigen vectors corresponding to an eigen value λ then $aX_1 + bX_2$ (a, b non zero scalars) is also an eigen vector corresponding to λ.

Solution :- Since X_1 and X_2 are given vectors corresponding to λ, we have

$$AX_1 = \lambda X_1$$

and $$AX_2 = \lambda X_2$$

Hence $A(aX_1) = \lambda(aX_1)$

and $A(bX_2) = \lambda(bX_2)$

$\therefore A(aX_1 + bX_2) = \lambda(aX_1 + bX_2)$

$\therefore aX_1 + bX_2$ is an eigen vector corresponding to λ.

2. If the eigen value of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ are 2, 2, 3.

Find the eigen values of A^{-1} and A^2.

Solution :-

Since 0 is not an eigen value of A, A is a non singular matrix and hence A^{-1} exists.

Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$

and Eigen value of A^2 are $2^2, 2^2, 3^2$.

3. Find the eigen values of A^5 when $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

Solution :-

The characteristic equation of A is obviously $(3-\lambda)(4-\lambda)(1-\lambda) = 0$

Hence the eigen value of A are 3, 4, 1.

$\therefore$ The eigen value of A^5 are , $3^5, 4^5, 1^5$.

Find the sum and product of the eigen values of the matrix $\begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ without actually finding the eigen values.

Solution :-

$$\text{Let } A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

Sum of the eigen values = trace of A

$= 3 + (-2) + 3$

$= 4.$

Product of eigen values $= |A|$

Now $|A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(-6+4) + 4(3-4) - 4(-1+2)$

$= -6 - 4 + 4$

$= \cancel{-14} \quad = -6$

$\therefore$ product of the eigen values $= \cancel{-14}. -6$

(5) Find the characteristic roots of the matrix
$\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

5m

Solution

let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

The characteristic equation of A is given by

$$|A-\lambda I| = 0$$

$$\therefore \begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ -\sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$$

$$(\cos\theta-\lambda)^2 - \sin^2\theta = 0$$

$$(\cos\theta-\lambda-\sin\theta)(\cos\theta-\lambda+\sin\theta) = 0$$

$$(\lambda-(\cos\theta-\sin\theta))(\lambda-(\cos\theta+\sin\theta)) = 0$$

$\therefore$ The two characteristic root of the matrix are $(\cos\theta-\sin\theta)$ and $(\cos\theta+\sin\theta)$.

(6) Find the characteristic roots of the matrix
$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$.

5m

Solution :-

The characteristic equation of A is given by $|A-\lambda I| = 0$.

i.e. $\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ -\sin\theta & -\cos\theta-\lambda \end{vmatrix} = 0$

$$(\cos\theta-\lambda)(-\cos\theta-\lambda) - \sin^2\theta = 0$$

$$-\cos^2\theta - \cancel{\lambda\cos\theta} + \cancel{\lambda\cos\theta} + \lambda^2 - \sin^2\theta = 0$$

$$-\cos^2\theta + \lambda^2 - \sin^2\theta = 0$$

$$\lambda^2 - (\sin^2\theta + \cos^2\theta) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$\therefore \lambda = 1, \lambda = -1.$

$\therefore$ The characteristic roots are 1 and -1.

Q. Find the sum and product of the eigen value of matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ without finding the roots of the characteristic equation.

<u>Solution</u>

Sum of the eigen value = trace of A

$$= a_{11} + a_{22}.$$

Product of the eigen value of A = $|A|$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}.$$

B. Verify the statement that the sum of the elements in the diagonal of a matrix is the sum of the eigen value of the matrix.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution:-

The characteristic equation of A is

$|A-\lambda I|=0$

i.e. $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$(-2-\lambda)[(1-\lambda)(-\lambda)-12]-2[-2\lambda-6]+(-3)[-4+1-\lambda] = 0$

$(-2-\lambda)[-\lambda+\lambda^2-12]+4\lambda+12+9+3\lambda=0$

$+2\lambda-2\lambda^2+24+\lambda^2-\lambda^3+12\lambda+4\lambda+12+9+3\lambda=0$

$-\lambda^3+\lambda^2(-2+1)+\lambda(2+12+4+3)+45=0$

$-\lambda^3-\lambda^2+21\lambda+45=0$

$\lambda^3+\lambda^2-21\lambda-45=0.$

This is a cubic equation in λ and hence it has 3 roots and the roots are the three eigen values of the matrix.

The Sum of the eigen values $= -\dfrac{\text{co-eff. of } \lambda^2}{\text{co-eff. of } \lambda^3}$

$= -\dfrac{1}{1}$

$= -1.$

The sum of the elements ~~of~~ on the diagonal of the matrix

$$A = -2+1+0 = -1.$$

$\therefore$ the sum of the elements in the diagonal of a matrix is the sum of the eigen values of the matrix.

9. The product of two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigen value. What is sum of eigen values of A?

Solution:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given that the product of two eigen value is 16.

i.e, $\lambda_1 \lambda_2 = 16.$

We known that the product of eigen value is $|A|$

$$\text{ie, } \lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$16 \lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$16 \lambda_3 = 48 - 8 - 8$$

$$\lambda_3 = \frac{32}{16}.$$

$$\boxed{\lambda_3 = 2}$$

$\therefore$ The third eigen value is 2.

$\therefore$ The Sum of the eigen value of A

$= \text{trace of } A$

$= 6+3+3$

$= 12.$

10. The product of two eigen values of the

(A) matrix $A = \begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$ is -12. Find the eigen

(11) values of A.

Solution :-

let λ_1, λ_2 and λ_3 be the eigen values of A.

Given that the product of two eigen values is -12.

ie, $\lambda_1 \lambda_2 = -12. \longrightarrow (1)$

W.K.T

The product of the eigen value $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$(-12)\lambda_3 = -12$$

$$\lambda_3 = \frac{-12}{-12}$$

$$\boxed{\lambda_3 = 1} \longrightarrow (2).$$

W.K.T.

The sum of the eigen values = Trace of A.

$$\lambda_1+\lambda_2+\lambda_3 = 2+1-3$$

$$= 0$$

$$\lambda_1+\lambda_2+1 = 0$$

$$\lambda_1+\lambda_2 = -1 \longrightarrow (5)$$

using (3) in (1) we get $\boxed{\lambda_2 = -1-\lambda_1}$

$$\lambda_1\lambda_2 = -12$$

$$\lambda_1(-1-\lambda_1) = -12$$

$$-\lambda_1 - \lambda_1^2 = -12$$

$$\lambda_1^2+\lambda_1-12 = 0$$

$$(\lambda_1+4)(\lambda_1-3) = 0$$

$$\lambda_1 = -4, 3$$

put $\lambda_1 = -4$, in eqn (1) we get

$$(-4)\lambda_2 = -12$$

$$\boxed{\lambda_2 = 3}$$

(or) put $\lambda_1 = 3$ in eqn (1) we get

$$3\lambda_2 = -12$$

$$\boxed{\lambda_2 = -4}$$

$$\therefore \boxed{\lambda_1 = -4, \lambda_2 = 3, \lambda_3 = 1}$$

11. Find the sum of the squares of the eigen values of $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

W.K.T $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigen values of A^2.

$$\therefore A^2 = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 3+2 & 12+6+20 \\ 0 & 4+0 & 12+30 \\ 0 & 0 & 25 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{pmatrix}$$

$\therefore$ Sum of the squares of the eigen value of A^2 = Trace of A^2

$= 9 + 4 + 25$

$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 38$ //.

Sum of the squares of the eigen values of A = 38 //.

(A) (12)

Find the eigen values and eigen vectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution

Given that,

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right] - 1\left[1-\lambda-3\right] + 3\left[1-3(5-\lambda)\right] = 0$$

$$(1-\lambda)\left[5-5\lambda-\lambda+\lambda^2-1\right] - 1+\lambda+3+3-45+9\lambda = 0$$

$$5-5\lambda-\lambda+\lambda^2-1-5\lambda+5\lambda^2+\lambda^2-\lambda^3+\lambda-1+\lambda+3+3-45+9\lambda = 0$$

$$-\lambda^3+\lambda^2(1+5+1)+\lambda(-5-1-5+1+1+9)+$$

$$(1-\lambda)\left[5-5\lambda-\lambda+\lambda^2-1\right] - 1+\lambda+3+3\left[1-15+3\lambda\right] = 0$$

$$(1-\lambda)\left[\lambda^2-6\lambda+4\right] - 1+\lambda+3+3\left[3\lambda-14\right] = 0$$

$$\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda-1+\lambda+3+9\lambda-42 = 0$$

$$-\lambda^3+\lambda^2(1+6)+\lambda(-6-4+1+9)-36 = 0$$

$$\lambda^3-7\lambda^2+36 = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$\begin{array}{c|cccc} & 1 & -7 & 0 & 36 \\ 6 & 0 & 6 & -6 & -36 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$$(\lambda - 6)(\lambda^2 - \lambda - 6) = 0$$

$$\begin{array}{c} -6 \\ -3 \quad 2 \end{array}$$

$$(\lambda - 6)(\lambda - 3)(\lambda + 2) = 0$$

$\lambda = -2, 3, 6$ are the three eigen values.

Case (i). Eigen vector corresponding to $\lambda = -2$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = -2$.

Hence,

$$AX = \lambda X$$

$$AX = -2X$$

i.e.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{bmatrix}$$

$$x_1 + x_2 + 3x_3 = -2x_1$$

$$x_1 + 5x_2 + x_3 = -2x_2$$

$$3x_1 + x_2 + x_3 = -2x_3$$

$3x_1 + x_2 + 3x_3 = 0 \longrightarrow (1)$

$x_1 + 7x_2 + x_3 = 0 \longrightarrow (2)$

$3x_1 + x_2 + 3x_3 = 0 \rightarrow (3)$

We take eqn (1) & (2) only.

$$3x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 7x_2 + x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}} = k.$$

$$\frac{x_1}{(1-21)} = \frac{-x_2}{(3-3)} = \frac{x_3}{(21-1)} = k.$$

$$\frac{x_1}{-20} = \frac{-x_2}{0} = \frac{x_3}{20} = k.$$

$\div 10$

$$\frac{x_1}{-2} = \frac{-x_2}{0} = \frac{x_3}{2} = k.$$

$\therefore x_1 = -2k, \; x_2 = 0, \; x_3 = 2k \qquad \because k = 1$

$\therefore (-2, 0, 2)$ is an eigen ~~value~~ vector corresponding to $\lambda = -2$.

Case ii. Eigen vector corresponding to $\lambda = 3$.

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Then $AX = \lambda X$

$AX = 3X$.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$$

$x_1 + x_2 + 3x_3 - 3x_1 = 0$

$x_1 + 5x_2 + x_3 - 3x_2 = 0$

$3x_1 + x_2 + x_3 - 3x_3 = 0$

$-2x_1 + x_2 + 3x_3 = 0 \longrightarrow (1)$

$x_1 + 2x_2 + x_3 = 0 \longrightarrow (2)$

$3x_1 + x_2 - 2x_3 = 0 \longrightarrow (3)$

Taking the first two equations we get

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{(1-6)} = \frac{-x_2}{(-2-3)} = \frac{x_3}{(-4-1)}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$\div 5$ $$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

t

$\therefore$ $(-1, 1, -1)$ is an eigen vector corresponding to $\lambda = 3$

Case iii

Eigen vector corresponding to $\lambda = 6$

we have

$$AX = \lambda X$$

$$AX = 6X$$

$$x_1 + x_2 + 3x_3 = 6x_1$$

$$x_1 + 5x_2 + x_3 = 6x_2$$

$$3x_1 + x_2 + x_3 = 6x_3$$

$$-5x_1 + x_2 + 3x_3 = 0 \longrightarrow (1)$$

$$x_1 - x_2 + x_3 = 0 \longrightarrow (2)$$

$$3x_1 + x_2 - 5x_3 = 0 \longrightarrow (3)$$

We take first two, then

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{(1+3)} = \frac{-x_2}{(-5-3)} = \frac{x_3}{(5-1)}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$(1, 2, 1)$ is an eigen vector corresponding to $\lambda = 6$.

13. Find the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Solution

The characteristic equation of A is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(6-\lambda)[(3-\lambda)(3-\lambda)-1] + 2[(-2)(3-\lambda)+2] + 2[2-2(3-\lambda)] = 0$

$(6-\lambda)[9-3\lambda-3\lambda+\lambda^2-1] + 2[-6+2\lambda+2] + 2[2-6+2\lambda] = 0$

$(6-\lambda)[\lambda^2-6\lambda+8] + 2[2\lambda-4] + 2[2\lambda-4] = 0$

$6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$

$-\lambda^3 + \lambda^2(6+6) + \lambda(-36-8+4+4) + 32 = 0$

$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\begin{array}{r|rrrr} & 1 & -12 & 36 & -32 \\ 2 & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$(\lambda-2)(\lambda^2 - 10\lambda + 16) = 0$

$$(\lambda-2)(\lambda-2)(\lambda-8)=0$$

16

-8 -2

$$\lambda = 2, 2, 8$$

$\therefore$ The eigen values are 2, 2, 8.

To find eigen vector.

Case(i)

let $\lambda = 2$.

$$AX = \lambda X$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$6x_1 - 2x_2 + 2x_3 = 2x_1$$

$$-2x_1 + 3x_2 - x_3 = 2x_2$$

$$2x_1 - x_2 + 3x_3 = 2x_3$$

$$4x_1 - 2x_2 + 2x_3 = 0 \longrightarrow (1)$$

$$-2x_1 + x_2 - x_3 = 0 \longrightarrow (2)$$

$$2x_1 - x_2 + x_3 = 0 \longrightarrow (3)$$

The above 3 equations are equivalent to the single equation $2x_1 - x_2 + x_3 = 0$, the independent Eigen vectors can be obtained by giving orbitary values to any two of the unknows x_1, x_2, x_3. put $x_1 = 1, x_2 = 2$ then, $2(1) - 2 + x_3 = 0$ $\Rightarrow x_3 = 0$. put, $x_1 = 3$, then $x_3 \Rightarrow 2(3) - 4 + x_3 = 0 \Rightarrow 6 - 4 + x_3$ $x_3 = -2$. The independent vectors corresponding to $\lambda = 2$ or $(1, 2, 0)$ and $(3, 4, -2)$.

Case ii:- $\lambda = 8$.

$$AX = \lambda X$$

$$AX = 8X$$

$$6x_1 - 2x_2 + 2x_3 = 8x_1$$

$$-2x_1 + 3x_2 - x_3 = 8x_2$$

$$2x_1 - x_2 + 3x_3 = 8x_3$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \longrightarrow (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \longrightarrow (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \longrightarrow (3)$$

From (1) & (2) we get,

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{2+10} = \frac{-x_2}{2+4} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

We get an eigen vector corresponding to 8 as $(2, -1, 1)$.

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Find the eigen values and eigen vector of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$ (H.W).