DIFFERENTIAL GEOMETRY — P16MA42

UNIT-IV

The Second fundamental form:-

The quadratic form $Ldu^2 + 2Mdudv + Ndv^2$ is called the second fundamental form, and the functions of u and v denoted by L, M and N are called the second fundamental coefficients.

where $L = -\bar{N_1} \cdot \bar{r_1}$, $M = -\bar{N_1} \cdot r_2 = -\bar{N_2} \cdot \bar{r_1}$

$N = -\bar{N_2} \cdot \bar{r_2}$.

Meusnier's Theorem:-

If ϕ denotes the angle between the principal normal $\bar{n}$ to a curve on the surface and the surface normal $\bar{N}$, it follows from the relation

$$\bar{r}'' = k_n \bar{N} + \lambda \bar{r_1} + \mu \bar{r_2}$$

$$\therefore \bar{N} \cdot r'' = k_n$$

$$k_n = \bar{N} \cdot k\bar{n} = k \cdot \cos\phi \quad (\because \bar{N} \cdot \bar{n} = 1 \cdot 1 \cos\phi = \cos\phi)$$

a result known as Meusnier's Theorem.

Elliptic, parabolic and Hyperbolic points:

If at a point P on the surface this form is definite, ie) if $LN - M^2 > 0$, then k_n maintains the same sign for all directions at P.

When $LN - M^2 = 0$, then P is called elliptic point.
When $LN - M^2 < 0$, then P is called a Parabolic point.
When $LN - M^2 > 0$, then P is called a Hyperbolic point.
The critical directions are called the asymptotic directions.

Principal curvatures:-

We shall write k for the normal curvature instead of k_n.

The normal curvature at P in a direction specified by direction coefficients (l, m) is given by

$$k = Ll^2 + 2Mlm + Nm^2 \longrightarrow (1)$$

where $El^2 + 2Flm + Gm^2 = 1 \longrightarrow (2)$.

Its extreme values may be found by making use of Langrange's multipliers.

Writing $k = Ll^2 + 2Mlm + Nm^2 - \lambda(El^2 + 2Flm + Gm^2 - 1)$, then when k is stationary.

$$\frac{1}{2}\frac{\partial k}{\partial l} = Ll + Mm - \lambda El - \lambda Fm = 0 \longrightarrow (3)$$

$$\frac{1}{2}\frac{\partial k}{\partial m} = Ml + Nm - \lambda Fl - \lambda Gm = 0 \longrightarrow (4)$$

Multiply (3) by l, (4) by m and add, we get

$$(Ll^2 + 2Mlm + Nm^2) - \lambda(El^2 + 2Flm + Gm^2) = 0.$$

$$k - \lambda = 0, \quad \lambda = k.$$

(3) & (4) become,

$$(L - kE)l + (M - kF)m = 0 \longrightarrow (5)$$

$$\& \quad (M - kF)l + (N - kG)m = 0 \longrightarrow (6)$$

Eliminating l & m between (5) & (6) we obtain

$$k^2(EG - F^2) - k(EN + GL - 2FM) + (LN - M^2) = 0 \longrightarrow (7)$$

The roots k_a, k_b of this equation are called the Principal curvatures.

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Associated with the mean curvature μ defined by

$$\mu = \frac{1}{2}(k_a + k_b) = \frac{EN + GL - 2FM}{2(EG - F^2)} \rightarrow (8).$$

and the Gaussian curvature K defined by

$$K = k_a k_b = \frac{LN - M^2}{EG - F^2} \rightarrow (9)$$

Lines of Curvature:-

A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature.

Theorem: (Rodrigue's Formula)

The necessary and sufficient condition that a curve on a surface be a line of curvature is that $k\,d\bar{r} + d\bar{N} = 0$ at each of its point.

Proof:

Since the principal directions are given by equation (3) & (4) (of previous section), the equation of a line of curvature are

$$\left.\begin{aligned}(L - kE)\,du + (M - kF)\,dv &= 0\\ (M - kF)\,du + (N - kG)\,dv &= 0\end{aligned}\right\} \rightarrow (1)$$

where k is one of the principal curvatures.

Since $E = \bar{r}_1^{\,2}$, $F = \bar{r}_1 \cdot \bar{r}_2$, $G = \bar{r}_2^{\,2}$.

$L = -\bar{N}_1 \cdot \bar{r}_1$, $M = -\bar{N}_2 \cdot \bar{r}_1 = -\bar{N}_1 \cdot \bar{r}_2$, $N = -\bar{N}_2 \cdot \bar{r}_2$

Substituting these values in (1) we obtain,

$$(k\bar{r}_1^{\,2} + \bar{N}_1 \cdot \bar{r}_1)\,du + (k\bar{r}_1 \cdot \bar{r}_2 + \bar{N}_2 \cdot \bar{r}_1)\,dv = 0$$

and $(k\bar{r}_1 \cdot \bar{r}_2 + \bar{N}_1 \cdot \bar{r}_2)\,du + (k\bar{r}_2^{\,2} + \bar{N}_2 \cdot \bar{r}_2)\,dv = 0$

ie) $[k(\bar{r}_1 du + \bar{r}_2 dv) + (\bar{N}_1 du + \bar{N}_2 dv)]\bar{r}_1 = 0$

and $[k(\bar{r}_1 du + \bar{r}_2 dv) + \bar{N}_1 du + \bar{N}_2 dv)]\cdot\bar{r}_2 = 0$

(or) $(k d\bar{r} + d\bar{N})\cdot\bar{r}_1 = 0$

and $(k d\bar{r} + d\bar{N})\cdot\bar{r}_2 = 0.$

~~(or)~~ $\cancel{(k d\bar{r} + d\bar{N})\cdot\bar{r}_1 = 0}$

This shows that the vector $k d\bar{r} + d\bar{N}$ is along the surface normal.

Also the vector $(k d\bar{r} + d\bar{N})$ is tangential to the surface.

This is possible only if $k d\bar{r} + d\bar{N} = 0 \rightarrow (2)$.

Conversely if (2) holds along a curve for any function k, then equation (1) follow and curve is thus a line of curvature.

Equation (2) characterises the lines of curvature and is known as Rodrigue's formula.

<u>Euler's Theorem:—</u>

If k is the normal curvature in a direction (l, m) making an angle ψ with the principal direction v = constant then

$k = k_a \cos^2\psi + k_b \sin^2\psi$ where k_a and k_b are principal curvatures at that point.

<u>Proof:</u>

Take the lines of curvatures as the parametric curves. Then we've $F = 0$, $M = 0$ and hence the normal curvature in a direction (l, m) is

$$k = Ll^2 + Nm^2$$

The direction coefficients of the parametric curves v = constant and u = constant are $(1/\sqrt{E}, 0)$ and $(0, 1/\sqrt{G})$

$\therefore$ k_a = normal curvature along v = constant

$k_a = L(1/E) + N(0) = L/E$

and k_b = normal curvature along u = constant.

$= L(0) + N(1/G) = N/G$

ie) $k_b = N/G$ & $k_a = L/E \longrightarrow (4)$

Now ψ is the angle between the direction (l, m) and the principal direction v = constant.

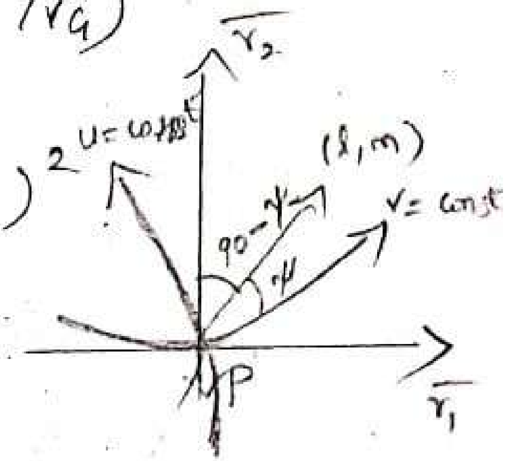
$\therefore \cos\psi = E(1/\sqrt{E}) + 0 + G(m)(0).$

$\because \cos\theta = E l l' + F(lm' + l'm) + Gmm'$

$= l\sqrt{E}.$

and $\cos(90 - \psi) = E(l)(0) + G(m)(1/\sqrt{G})$

$\therefore \sin\psi = m\sqrt{G}$



Thus $k = Ll^2 + Nm^2$

$= L(1/\sqrt{E}\cos\psi)^2 + N(1/\sqrt{G}\sin\psi)^2$

$= L/E \cos^2\psi + N/G \sin^2\psi$

$k = k_a \cos^2\psi + k_b \sin^2\psi$ //

Developables:-

A developable is a surface enveloped by a one-parameter family of planes. Such a family is given by the equation $\bar{r}.\bar{a} = p$, where $\bar{a}$ and p are functions of a real parameter u.

Developable associated with space curves:-

At each point of a curve we have the three planes, viz osculating plane, normal plane and the rectifying plane.

Each of these planes contains only one parameter viz, the arc lengths.

The envelope of these planes are respectively called the osculating developable, polar developable and rectifying developable.

*) To prove that the edge of regression of the rectifying developable has equation

$$\bar{R} = \bar{r} + k\,\frac{(\tau\bar{t} + k\bar{b})}{k'\tau - k\tau'}$$

<u>Solution:</u>

The position vector $\bar{R}$ of any point on the rectifying developable is satisfies the equation given by $(\bar{R}-\bar{r}).\bar{n} = 0 \longrightarrow (1)$

Shows that the principal normal to the curve at $P(\bar{r})$ coincides with the normal to the developable surface.

Thus by the normal property we see that any curve is a geodesic on its rectifying developable.

Now we shall obtain the equation to the edge of regression of rectifying developable.

Differentiating (1) w.r.t. s,

$$-\bar{t}.\bar{n} + (\bar{R}-\bar{r}).(-k\bar{t}+\tau\bar{b}) = 0$$

$$(\bar{R}-\bar{r}).(-k\bar{t}+\tau\bar{b}) = 0 \longrightarrow (2)$$

Differentiating (2) w.r.to s.

$$-\bar{t}(-k\bar{t}+\tau\bar{b}) + (\bar{R}-\bar{r}).[-k^2\bar{n} - k'\bar{t} - \tau^2\bar{n} + \tau'\bar{b}] = 0$$

Using (1) this reduces to

$$k + (\bar{R}-\bar{r}).(k'\bar{t} + \tau'\bar{b}) = 0 \longrightarrow (3)$$

The point of intersection of the planes (1), (2) and (3) is the characteristic point whose locus is the edge of regression.

From ① and ② we find that $(\bar{R}-\bar{r})$ is $\perp^r$ to both $\bar{n}$ and $(-k\bar{t}+\tau\bar{b})$ and thus parallel to $\bar{n}\times(-k\bar{t}+\tau\bar{b})$. ie. $(\tau\bar{t}+k\bar{b})$

Thus we can write $(\bar{R}-\bar{r}) = \lambda(\tau\bar{t}+k\bar{b}) \rightarrow$ ④

To find λ: Using ④ in ③ we obtain

$$k+\lambda(\tau\bar{t}+k\bar{b})\cdot(-k't+\tau'b)=0.$$

ie) $k+\lambda(-k'\tau+k\tau')=0$

(or) $\lambda = \dfrac{k}{k'\tau-k\tau'}$

Hence the equation to the edge of regression of the rectifying developable is given by $\bar{R} = \bar{r}+k\dfrac{(\tau\bar{t}+k\bar{b})}{k'\tau-k\tau'}$ //

Developables associated with Curves on surfaces:-

Monge's Theorem:-

A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

Proof:

Consider the curve $\bar{r}=\bar{r}(s)$.

Position vec $\bar{R} = \bar{r}(s)+v\bar{N}(s) \rightarrow$ ①

Differentiation w.r.to s and v by suffix 1 & 2 respectively, we've,

$$\bar{R}_1 = \bar{t}+v\bar{N}'$$

$$\bar{R}_2 = \bar{N}$$

$$R_{11} = \bar{t}'+v\bar{N}''$$

$$R_{12} = R_{21} = \bar{N}'$$

$$R_{22} = 0$$

Surface Normal $\bar{N} = \dfrac{\bar{R}_1\times\bar{R}_2}{|\bar{R}_1\times\bar{R}_2|}$

$$\bar{N} = \frac{(\bar{t} \times \bar{N}) + v(\bar{N}' \times \bar{N})}{H}$$ where $H = |\bar{R}_1 \times \bar{R}_2|$

Thus $M = \bar{N} \cdot \bar{R}_1 = \frac{1}{H}[(\bar{t} \times \bar{N}) + v(\bar{N}' \times \bar{N})] \cdot \bar{N}'$

$= \frac{1}{H}(\bar{t}, \bar{N}, \bar{N}')$, $\because (\bar{N}' \times \bar{N}) \cdot \bar{N}' = 0$

$\& \; N = \bar{N} \cdot \bar{R}_{22} = 0$.

Hence the Gaussian Curvature $K = \frac{LN - M^2}{EG - F^2}$ of the surface will be zero if and only if $LN - M^2 = 0$.

ie) $M = 0$, if and only if $[\bar{t}, \bar{N}, \bar{N}'] = 0$

The surface normals along the curve form a developable if and only if $[\bar{t}, \bar{N}, \bar{N}'] = 0$

Now prove that this condition is satisfied if and only if the curve is a line of curvature.

Since $\bar{t} \times \bar{N}'$ is normal to the given surface, the equation $[\bar{t}, \bar{N}, \bar{N}'] = 0$ implies that $\bar{t} \times \bar{N}' = 0$.

ie) $\bar{N}' = -k\bar{t}$ for some function k

$$\frac{d\bar{N}}{ds} = -k \frac{d\bar{r}}{ds}$$

(or) $d\bar{N} + k\,d\bar{r} = 0$ Hence by Rodrigre's formula the curve is a line of curvature.

Conversely if $d\bar{N} + k\,d\bar{r} = 0$

(or) $\frac{d\bar{N}}{ds} = -k \frac{d\bar{r}}{ds}$

ie) $\bar{N}' = -k\bar{t}$

$\therefore [\bar{t}, \bar{N}, \bar{N}'] = 0$

This completes the proof the theorem.

Minimal surfaces:-

Surfaces whose mean curvature is zero at all points are called minimal surface.

We've, the mean curvature μ is given by

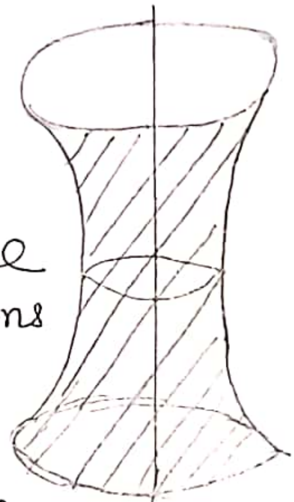
$$\mu = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

$$= \frac{EN + GL - 2FM}{2H^2}$$

Thus for a minimal surface $\mu = 0$.

$$\Rightarrow EN + GL - 2FM = 0.$$

Ruled surfaces:—



A ruled surface is generated by the motion of a straight line with one degree of freedom, the various positions of the line being called generators.

The developable surfaces discussed in to the family of ruled surfaces, are very special and have properties not characteristic of ruled surfaces in general.

An example of a ruled surface which is not a developable is hyperboloid of revolution.

Let C be any curve on a ruled surface having the property that it meets each generator precisely once.

Such a curve will be called a base curve. It is clear that such a curve is by no means uniquely determined.

Then the surface is determined by any base curve C and the direction of the generators at each point of C.

Example:-1.

Show that the surface $xy = (z-c)^2$ is developable.

Solution:-

Given $xy = (z-c)^2$

$$\sqrt{xy} = (z-c)$$

$$z = c + \sqrt{xy}$$

$$\frac{\partial z}{\partial x} = p = \frac{1}{2}\sqrt{y/x} \; ; \; \frac{\partial z}{\partial y} = q = \frac{1}{2}\sqrt{x/y}.$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x} = -\frac{1}{4} y^{1/2} x^{-3/2},$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y} = -\frac{1}{4} x^{1/2} y^{-3/2}.$$

$$s = \frac{\partial p}{\partial y} = -\frac{1}{4} x^{-1/2} y^{-1/2}$$

$$rt - s^2 = \left(-\frac{1}{4} y^{1/2} x^{-3/2}\right)\left(-\frac{1}{4} x^{1/2} y^{-3/2}\right) - \left(-\frac{1}{4} x^{-1/2} y^{-1/2}\right)^2$$

$$= \frac{1}{16} y^{-1} x^{-1} - \frac{1}{16} x^{-1} y^{-1}$$

$$= \frac{1}{16} \frac{1}{xy} - \frac{1}{16} \frac{1}{xy} = 0$$

Hence the given surface is developable.

Example: 2

Determine whether the surface $xyz = a^3$ is developable.

Solution:-

Given $xyz = a^3$, $z = a^3/xy$.

$$p = \frac{\partial z}{\partial x} = -a^3/x^2y \; ; \; q = \frac{\partial z}{\partial y} = -a^3/xy^2.$$

$$r = \frac{\partial p}{\partial x} = \frac{2a^3}{x^3y} \; ; \; t = \frac{\partial q}{\partial y} = \frac{2a^3}{xy^3}.$$

$$s = \frac{\partial q}{\partial y} = a^3/x^2y^2$$

$$rt - s^2 = \frac{3a^6}{x^4y^4} \neq 0.$$

Hence the surface is not developable.

Example : 3

Show that the surface $e^z \cos x = \cos y$ is minimal.

Solution :-

Given $e^z \cos x = \cos y$

$\therefore z + \log(\cos x) = \log \cos y$

$z = \log(\cos y) - \log(\cos x)$

$\therefore p = \frac{\partial z}{\partial x} = \tan x, \quad q = \frac{\partial z}{\partial y} = -\tan y.$

$r = \frac{\partial p}{\partial x} = \sec^2 x, \quad s = \frac{\partial p}{\partial y} = 0, \quad t = \frac{\partial q}{\partial y} = -\sec^2 y.$

$E = 1+p^2 = 1+\tan^2 x = \sec^2 x.$

$G = 1+q^2 = 1+\tan^2 y = \sec^2 y.$

$F = pq = -\tan x \tan y.$

$L = \frac{r}{H} = \frac{\sec^2 x}{H}, \quad M = \frac{s}{H} = 0, \quad N = \frac{t}{H} = \frac{-\sec^2 y}{H}$

The condition for the surface to be minimal is

$EN - 2FM + GL = 0$

$EN - 2FM + GL = \sec^2 x \cdot \frac{-\sec^2 y}{H} - 2(-\tan x \tan y)(0) + \sec^2 y \cdot \frac{\sec^2 x}{H}.$

$= -\frac{\sec^2 x \sec^2 y}{H} + \frac{\sec^2 x \sec^2 y}{H}$

$= 0$

Hence the given surface is minimal.

Example : 4

Show that the ruled surface generated by the binormals of a space curve has the curve itself as the line of striction.

Solution :-

Taking the given space curve C as the base curve the equation to the ruled surface can be written as

$\bar{R}(s,v) = \bar{r}(s) + v\bar{g}(s) \longrightarrow (1)$

Where $\bar{r}(s)$ is the position vector of the point P on C

and $\bar{g}(s)$ is the unit along the generator at P.
Since the ruled surface is generated by the binormals to C we have $\bar{g} = \bar{b}$

Let v be the distance from P of the central point of the generator at P.

Then from equation $(\bar{g}' \cdot \bar{r}') + v\bar{g}'^2 = 0$.
where we've used dashes instead of dots since the parameter of the curve is taken as the arc length s.

Thus $(\bar{b}' \cdot \bar{t}) + v\bar{b}'^2 = 0 \quad (\because \bar{b} = \bar{g}, \bar{r}' = \bar{t})$

$$-\tau\bar{n}\cdot t + v(-\tau\bar{n})^2 = 0$$

$$0 + \tau^2 v n^2 = 0$$

$$\tau^2 v(1) = 0$$

$$\tau^2 v = 0$$

$$v = 0$$

This shows that the central point of the generator at P is P itself.

Thus the given curve itself is the line of striction of the ruled surface.

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