IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

DEPARTMENT OF MATHEMATICS

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UNIT V

HYPERBOLIC FUNCTION RELATION BETWEEN HYPERBOLIC FUNCTION-INVERSE HYPERBOLIC FUNCTION- SEPARATE INTO REAL AND IMAGINARY

PART A

1.Write the formula for Euler's exponential values sin θ and cos θ

sin θ $=\frac{e^{i\theta}-e^{-i\theta}}{2}$ and cos $\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$, tan $\theta=\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$

2.Hyperbolic function

The hyperbolic function of a real number x are defined as

sin$h\theta$ $=\frac{e^{\theta}-e^{-\theta}}{2}$ and cosh$\theta=\frac{e^{\theta}+e^{-\theta}}{2}$, tan $h\theta=\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$

3.Prove that $cosh^2x - sinh^2x = 1$

Proof:

We know that $cos^2x + sin^2x = 1$ → (1)

Put x=ix in this equation ,we get

$cos^2(ix) + sin^2(ix) = 1$

Since Sin(ix) = iSin(hx) and Cos(ix)=Cos (hx)

$cosh^2(x) + i^2sinh^2(x) = 1$

$cosh^2(x) - sinh^2(x) = 1$

Hence proved.

4.Prove that Sinh2x=2Sinhx coshx

Proof:

We know that Sin 2 $\theta = 2sin\theta\cos\theta$

Put $\theta = ix$, Sin 2(ix) $= 2\sin(ix)\cos(ix)$

iSin 2(hx) $= i2\sin(hx)\cos(hx)$

Sin 2(hx) $= 2\sin(hx)\cos(hx)$

Hence proved.

5.Prove that tanh(x+y)=$\frac{tanhx+tanhy}{1+tanhx+tanhy}$

Proof:

We know that: tan (A+B)=$\frac{tanA+tanB}{1+tanA+tanB}$

Put A=ix , B=iy

itanh(x+y)=$\frac{\tan(ix)+\tan(iy)}{1-\tan(ix)+\tan(iy)}$

itanh(x+y) =$\frac{\mathrm{i}(\tan(hx)+\tan hy)}{1-i^2\tan(hx)+\tan(hy)}$

tanh(x+y) =$\frac{(\tan(hx)+\tan hy)}{1+\tan(hx)+\tan(hy)}$

Hence proved.

6.Separate into real and imaginary parts of sin(x+iy)

Solution:

sin(x+iy)=sinx cosiy+ cosx siniy

sin(x+iy) =sinxcoshy+ i cosxsinhx

Real part: sinxcoshy , Imaginary = cosxsinhx

PART B

1.Separate into real and imaginary parts of tanh(x+iy)

Solution:

Since tan(ix)= itanhx , tanhx=-ix

tan(x+iy)= -itan(ix-y)

=- $\frac{\sin(ix-y)}{\cos(ix-y)}$

=-i$\frac{2\sin(ix-y)\cos(ix-y)}{2\cos(ix-y)\cos(ix-y)}$

itanh (x+iy) =-i$\frac{-isin2y+isinh2x}{cos2y+cosh2x}$

tanh (x+iy) =$\frac{\sinh 2x}{cos2y+cosh2x}+i\frac{isin2y}{cos2y+cosh2x}$

Real part=$\boldsymbol{\frac{\sinh 2x}{cos2y+cosh2x}}$

Imaginary Part=$\boldsymbol{\frac{isin2y}{cos2y+cosh2x}}$

2.If $\boldsymbol{\sin(\theta+i\varphi)=\tan(x+iy)}$ show that $\boldsymbol{\frac{\tan\theta}{tanh\varphi}=\frac{\sin 2x}{sinh2y}}$

Solution:

$\sin(\theta+i\varphi)$=sin$\theta\ cosi\varphi+cos\theta sini\varphi$

$\sin(\theta+i\varphi$ =sin $\theta cosh\varphi+icos\theta sinh\varphi$ → (1)

tan(x+iy) =$\frac{\sin(x+iy)}{\cos(x+iy)}$

$=\frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$

$=\frac{sin2x+sini2y}{cos2x+cos2iy}$

tan(x+iy) = $\frac{sin2x+isin2hy}{cos2x+cosh2y}$ → (2)

From (1) and (2) equating real and imaginary parts

sin $\theta cosh\varphi$=$\frac{sin2x}{cos2x+cosh2y}$ → (3)

$cos\theta sinh\varphi$ =$\frac{sin2hy}{cos2x+cosh2y}$ → (4)

(3)/(4) gives

tanθcothφ= $\frac{sin2x}{Sinh2y}$

$$\frac{\boldsymbol{tan\theta}}{\boldsymbol{tanh\varphi}} = \frac{\boldsymbol{sin2x}}{\boldsymbol{Sinh2y}}$$

3.If sin(A+iB)=x+iy prove that

i) $\frac{x^2}{sin^2A} - \frac{y^2}{cos^2A} = 1$

ii) $\frac{x^2}{cosh^2A} + \frac{y^2}{sinh^2A} = 1$

Solution:

x+iy = sin(A+iB)

=sinAcosiB+cosAsiniB

x+iy =sinAcoshB+icosAsinhB

Equating real and imaginary parts

x= sinAcoshB , y=cosAsinhB

$$\frac{x}{sinA} = coshB \ , \ \frac{y}{cosA} = sinhB$$

Squaring and subtracting $\frac{x}{sin^2A} + \frac{y}{cos^2A} = cosh^2B + sinh^2B$

Again $\frac{x}{coshB} = sinA \ , \ \frac{y}{sinhB} = cosA$

Squaring and adding $\frac{x}{cosh^2B} += \frac{y}{sinh^2B} sin^2A + cos^2A$

4.If $\sin(\theta + i\varphi) = \tan\alpha + i\sec\alpha$ prove that $\cos 2\theta \cosh 2\varphi = 3$

Solution:

Given $\sin(\theta + i\varphi) = \tan\alpha + i\sec\alpha$

$\sin\theta \cos i\varphi + \cos\theta \sin i\varphi = \tan\alpha + i\sec\alpha$

$\sin\theta \cosh\varphi + i\cos\theta \sinh\varphi = \tan\alpha + i\sec\alpha$

Equating real and imaginary parts

$\sin\theta \cosh\varphi = \tan\alpha$ → (1)

$\cos\theta \sinh\varphi = \sec\alpha$ → (2)

Squaring and Subtracting

$\sin^2\theta \cosh^2\varphi - \cos^2\theta \sinh^2\varphi = \tan^2\alpha - \sec^2\alpha$

$(\frac{1+cos2\theta}{2}) \sinh^2\varphi - \left(\frac{1-cos2\theta}{2}\right) cosh^2\varphi = 1$

$$(sinh^2\varphi - cosh^2\varphi) + cos2\theta(cosh^2\varphi + sinh^2\varphi) = 2$$

$$\mathbf{cos2\theta cosh2\varphi = 3}$$

5.Separate into real and imaginary parts of sinh(x+iy)

Solution:

We know sin ix=isinhx

Sinhx=-isinix

Sinh(x+iy)=-isin(x+iy)

=-isin(ix-y)

=-i[sin(ix) cosy-cos(ix) siny]

=-i[isin(hx) cosy-cos(x) siny]

Sinh(x+iy) = sinhx cosy+i coshx siny

Real Part= sinhx cosy

Imaginary Part = coshx siny

PART C

If sinhx =y , then x=$sinh^{-1}y$. Then sympol $sinh^{-1}$ is called the inverse hyperbolic function.

Similarly $cosh^{-1}$, $tanh^{-1}$ denote the inverse hyperbolic cosine and inverse hyperbolic tangent function.

1.Show that $\boldsymbol{sinh^{-1}x = log_e(x + \sqrt{x^2 + 1})}$

Solution:

Let y= $sinh^{-1}x$, x=sinhy =$\frac{e^y - e^{-y}}{2}$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2x = e^y - \frac{1}{e^y}$$

$$e^{2y} - 2xe^y - 1 = 0$$

Solving for e^y $\rightarrow$ $e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$\rightarrow$ $e^y = x \pm \sqrt{x^2+1}$

Since $e^y > 0$ always, the value $x - \sqrt{x^2+1}$ is not possible.

$e^y = x + \sqrt{x^2+1}$, y= $log_e\left(x + \sqrt{x^2+1}\right)$,

y=$\boldsymbol{sinh^{-1}x = log_e\left(x + \sqrt{x^2+1}\right)}$

2.Show that $\boldsymbol{cosh^{-1}x = log_e(x + \sqrt{x^2-1})}$

Solution:

Let y= $cosh^{-1}x$, x=coshy $= \frac{e^y + e^{-y}}{2}$

$$x = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + e^{-y}$$

$$2x = e^y + \frac{1}{e^y}$$

$$e^{2y} - 2xe^y + 1 = 0$$

Solving for e^y $\rightarrow$ $e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$

$\rightarrow$ $e^y = x \pm \sqrt{x^2-1}$

Since $e^y > 0$ always, the value $x - \sqrt{x^2-1}$ is not possible.

$e^y = x + \sqrt{x^2-1}$, y= $log_e\left(x + \sqrt{x^2-1}\right)$, y=$\boldsymbol{cosh^{-1}x = log_e\left(x + \sqrt{x^2-1}\right)}$

3.Show that $\boldsymbol{tanh^{-1}x = \frac{1}{2} log_e(\frac{1+x}{1-x})}$

Solution:

Let $y = tanh^{-1}x$

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$= \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$x(e^{2y} + 1) = e^{2y} - 1$$

$$x = \frac{e^{2y} - 1}{(e^{2y} + 1)}$$

$$2y = log_e(\frac{1+x}{1-x})$$

$$y = \frac{1}{2} log_e(\frac{1+x}{1-x})$$

$$\boldsymbol{y = tanh^{-1}x = \frac{1}{2} log_e(\frac{1+x}{1-x})}$$

4.If $\boldsymbol{\tanh(\frac{x}{2}) = \tan(\frac{\theta}{2})}$. Show that $\boldsymbol{x = \log \tan(\frac{\pi}{4} + \frac{\theta}{2})}$

Solution:

Given $\tanh(\frac{x}{2}) = \tan(\frac{\theta}{2})$

$$(\frac{x}{2}) = \tanh^{-1}(\tan(\frac{\theta}{2}))$$

$$\left(\frac{x}{2}\right) = \frac{1}{2}\log(\frac{1+\tan(\frac{\theta}{2})}{1-\tan(\frac{\theta}{2})})$$

$$x = \log \frac{tan\frac{\pi}{4} + \tan(\frac{\theta}{2})}{1 - tan\frac{\pi}{4} + \tan(\frac{\theta}{2})}$$

$$x = \log \tan(\frac{\pi}{4} + \frac{\theta}{2})$$

$$\boldsymbol{x = \log \tan(\frac{\pi}{4} + \frac{\theta}{2})}$$

5.Separate into real and imaginary parts $\tan^{-1}(x+iy)$

Solution:

Let $\tan^{-1}(x+iy)=A+iB$

$(x+iy)=\tan(A+iB)$

$(x-iy)=\tan(A-iB)$

$2A=A+iB+A-iB$

$\tan 2A=\tan(A+iB+A-iB)$

$=\frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB).\tan(A-iB)}$

$=\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$

$\tan 2A=\frac{2x}{1-x^2-y^2}$

$2A=\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$ or $A=\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$

Further $2iB=(A+iB)-(A-iB)$

$\tan 2iB=\tan[(A+iB)-(A-iB)]$

$=\frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB).\tan(A-iB)}$

$=\frac{(x+iy)-(x-iy}{1+(x+iy)(x-iy)})$

$i\tanh 2B=\frac{2iy}{1+x^2+y^2}$

$\tanh 2B=\frac{2y}{1+x^2+y^2}$

$2B=\tan^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$ or $B=\frac{1}{2}\tan^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$

Real Part= $A=\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$

Imaginary Part= B=$\frac{1}{2}$ tan^{-1} ($\frac{2y}{1+x^2+y^2}$)

6.Show that cosh$^5\theta$=$\frac{1}{16}[\cos h5\,\theta + 5cosh3\theta + 10cosh\theta]$

Solution:

$\cosh^5\theta = (\frac{e^{\theta}+e^{-\theta}}{2})^5$

$= \frac{1}{2^5}[e^{5\theta} + 5c_1e^{4\theta}e^{-\theta} + 5c_2e^{3\theta}e^{-2\theta} + +5c_3e^{2\theta}e^{-3\theta} + 5c_4e^{\theta}e^{-4\theta} + e^{-5\theta}$

$=\frac{1}{2^5}[\ (e^{5\theta} + e^{-5\theta}) + 5(e^{3\theta}+e^{-3\theta})+10(e^{\theta}+e^{-\theta})$

$=\frac{1}{16}[\cos h5\,\theta + 5cosh3\theta + 10cosh\theta]$

cosh$^5\theta$=$\frac{1}{16}[\cos h5\,\theta + 5cosh3\theta + 10cosh\theta]$