IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

DEPARTMENT OF MATHEMATICS

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SEM	:	II
UNIT	:	V (VECTOR ANALYSIS)
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UNIT-V

PART -A

1. Define gradient.

Let φ(x,y,z) be a scalar point function and is continuously differentiable,then the vector $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$ is called the **gradient** of the scalar function φ. Grad $\varphi = \nabla\varphi$.

2. Define Divergence.

The **divergence** of the vector function $\vec{F}$ is defined as

$$\nabla.\vec{F} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}).(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

3. Define curl.

The **curl or rotation** of $\vec{F}$ is defined by

$$\nabla\times\vec{F} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \times (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

4. Define solenoidal.

A vector $\vec{F}$ is said to be **solenoidal** if its divergence is zero.That is, $\nabla\circ\vec{F} = 0$ or $div\,\vec{F} = 0$.

5. Define irrotational.

A vector $\vec{F}$ is said to be **irrotational** if its curl is zero.That is, $\nabla\times\vec{F} = 0$ or $curl\,\vec{F} = 0$.

PART –B

1. Find the directional derivative of f = xyz at (1,1,1) in the direction of $\vec{i} + \vec{j} + \vec{k}$.

Solution:

Unit normal vector $\hat{n} = \frac{\vec{a}}{|a|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$

grad f = ∇f

$= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$

$= \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$

grad $f_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$

Directional derivative = grad f. $\hat{n}$

$= (\vec{i} + \vec{j} + \vec{k}).\frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$

$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}}$

2. Find the normal derivative of f = xy + yz + zx at (-1,1,1).

Solution:

Normal derivative of f = $|grad\ f|$

Given f = xy + yz + zx

grad f = ∇f

$= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

$= \vec{i}\frac{\partial}{\partial x}(xy + yz + zx) + \vec{j}\frac{\partial}{\partial y}(xy + yz + zx) + \vec{k}\frac{\partial}{\partial z}(xy + yz + zx)$

$= \vec{i}(y+z) + \vec{j}$ (x + z)+ $\vec{k}$(y + x)

grad f $_{(-1,1,1)}$ = $\overrightarrow{2i}$ +0 $\vec{j}$ +0 $\vec{k}$

Normal derivative of f = $|grad\ f|$

$$= \sqrt{2^2 + 0^2 + 0^2} = 2$$

3. Find a unit vector normal to the surface $x^2 + y^2 - z = 10$ at (1,1,1).

Solution:

Unit normal vector = $\hat{n} = \frac{\nabla f}{|\nabla f|}$

f $= x^2 + y^2 - z - 10$

∇f = $\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

$= \vec{i}\frac{\partial}{\partial x}(x^2 + y^2 - z - 10) + \vec{j}\frac{\partial}{\partial y}(x^2 + y^2 - z - 10) + \vec{k}\frac{\partial}{\partial z}(x^2 + y^2 - z - 10)$

= 2x$\vec{i} + 2y\vec{j} - \vec{k}$

(∇f) $_{(1,1,1)}$ = $2\vec{i} + 2\vec{j} - \vec{k}$

$|\nabla f|_{(1,1,1)} = \sqrt{4+4+1} = 3$

Unit normal vector = $\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{\overrightarrow{2i} + \overrightarrow{2j} - \vec{k}}{3}$

4. **If $\nabla\varphi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$, find the scalar potential φ.**

Solution:

Given $\nabla\varphi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$

$\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$

Equating like coefficients on both sides we get ,

$\frac{\partial\varphi}{\partial x}$ =2xyz …(1)

$\frac{\partial\varphi}{\partial y}$ $= x^2 z$ …(2)

$\frac{\partial\varphi}{\partial z}$ $= x^2 y$ …(3)

Integrating (1) we get, $\int \partial\varphi = \int 2xyz\ \partial x \implies \varphi = x^2yz + f(y,z)$ …(4)

Integrating (2) we get, $\int \partial\varphi = \int x^2 z\ \ \partial y \implies \varphi = x^2yz + f(x,z)$ …(5)

Integrating (3) we get, $\int \partial\varphi = \int x^2 y\ \partial z \implies \varphi = x^2yz + f(x,y)$ …(6)

From (4),(5),(6) we get, $\varphi = x^2yz + c$

5. Find $\nabla \circ \overrightarrow{F}$ and $\nabla \times \overrightarrow{F}$ of the vector point function $\overrightarrow{F} = xz^3\,\vec{i} - 2x^2\ yz\,\vec{j} + 2yz^4\,\vec{k}$ at the point (1,-1,1).

Solution:

$\nabla.\ \overrightarrow{F} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}).(xz^3\,\vec{i} - 2x^2\ yz\,\vec{j} + 2yz^4\,\vec{k})$

$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2\ yz) + \frac{\partial}{\partial z}(2yz^4)$

$= z^3 - 2x^2\ z + 8z^3y$

$(\nabla.\ \overrightarrow{F})_{(1,-1,1)} = 1-2-8 = -9$

$$\nabla \times \overrightarrow{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2\ yz & 2yz^4 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2yz^4) + \frac{\partial}{\partial z}(2x^2\ yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-2x^2\ yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= \vec{i}(2z^4 + 2x^2\ y) - \vec{j}(0 - 3xz^2) + \vec{k}(-4xyz - 0)$$

$(\nabla \times \overrightarrow{F})_{(1,-1,1)} = \vec{i}(2-2)) - \vec{j}(0-3) + \vec{k}(4)$

$= 3\,\vec{j} + 4\vec{k}$

6. Show that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal

Solution:

If div $\vec{F}$ = 0, then $\vec{F}$ is solenoidal.

Now, div $\vec{F}$ = $\nabla . \vec{F}$

$$= (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}).(z\vec{i} + x\vec{j} + y\vec{k})$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0+0+0 = 0$$

Hence $\vec{F}$ is solenoidal.

7. Show that the vector $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \vec{i}(x - x)) - \vec{j}(y - y) + \vec{k}(z-z)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

$$\nabla \times \vec{F} = 0.$$

Therefore,the vector $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

PART-C

1. **Find the value of the constant a,b,c so that the vector $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.**

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right] + \vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= \vec{i}(c+1)) - \vec{j}(4-a) + \vec{k}(b-2)$$

Given $\vec{F}$ is irrotational.

That is, $\nabla \times \vec{F} = 0$.

That is, $\vec{i}(c+1)) - \vec{j}(4-a) + \vec{k}(b-2) = 0$.

That is,each component should be zero.

That is, $(c+1) = 0$; (a - 4) = 0; (b-2) = 0.

That is, c = -1, a = 4, b = 2.

2. **(i)Prove that $\nabla.(\vec{F} \pm \vec{G}) = \nabla.\vec{F} \pm \nabla.\vec{G}$**
   (ii)Prove that $\nabla \times (\vec{F} \pm \vec{G}) = \nabla \times \vec{F} \pm \nabla \times \vec{G}$

Proof:

(i) $\nabla.(\vec{F} \pm \vec{G}) = \nabla.\vec{F} \pm \nabla.\vec{G}$

$$\nabla.(\vec{F} \pm \vec{G}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(\vec{F} \pm \vec{G})$$

$$= \vec{i}\frac{\partial}{\partial x}(\vec{F} \pm \vec{G}) + \vec{j}\frac{\partial}{\partial y}(\vec{F} \pm \vec{G}) + \vec{k}\frac{\partial}{\partial z}(\vec{F} \pm \vec{G})$$

$$= \vec{i}.\left(\frac{\partial \vec{F}}{\partial x} \pm \frac{\partial \vec{G}}{\partial x}\right) + \vec{j}.\left(\frac{\partial \vec{F}}{\partial y} \pm \frac{\partial \vec{G}}{\partial y}\right) + \vec{k}.\left(\frac{\partial \vec{F}}{\partial z} \pm \frac{\partial \vec{G}}{\partial z}\right)$$

$$= \left(\vec{i}.\frac{\partial \vec{F}}{\partial x} + \vec{j}.\frac{\partial \vec{F}}{\partial y} + \vec{k}.\frac{\partial \vec{F}}{\partial z}\right) \pm \left(\vec{i}.\frac{\partial \vec{G}}{\partial x} + \vec{j}.\frac{\partial \vec{G}}{\partial y} + \vec{k}.\frac{\partial \vec{G}}{\partial z}\right)$$

$$\nabla.(\vec{F} \pm \vec{G}) = \nabla.\vec{F} \pm \nabla.\vec{G}$$

(ii) $\boldsymbol{\nabla \times (\vec{F} \pm \vec{G}) = \nabla \times \vec{F} \pm \nabla \times \vec{G}}$

$$\nabla \times (\vec{F} \pm \vec{G}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (\vec{F} \pm \vec{G})$$

$$= \vec{i} \times \frac{\partial}{\partial x}(\vec{F} \pm \vec{G}) + \vec{j} \times \frac{\partial}{\partial y}(\vec{F} \pm \vec{G}) + \vec{k} \times \frac{\partial}{\partial z}(\vec{F} \pm \vec{G})$$

$$= \vec{i} \times \left(\frac{\partial \vec{F}}{\partial x} \pm \frac{\partial \vec{G}}{\partial x}\right) + \vec{j} \times \left(\frac{\partial \vec{F}}{\partial y} \pm \frac{\partial \vec{G}}{\partial y}\right) + \vec{k} \times \left(\frac{\partial \vec{F}}{\partial z} \pm \frac{\partial \vec{G}}{\partial z}\right)$$

$$= \vec{i} \times \frac{\partial \vec{F}}{\partial x} \pm \vec{i} \times \frac{\partial \vec{G}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} \pm \vec{j} \times \frac{\partial \vec{G}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z} \pm \vec{k} \times \frac{\partial \vec{G}}{\partial z}$$

$$= \left(\vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}\right) \pm \left(\vec{i} \times \frac{\partial \vec{G}}{\partial x} + \vec{j} \times \frac{\partial \vec{G}}{\partial y} + \vec{k} \times \frac{\partial \vec{G}}{\partial z}\right)$$

$$\nabla \times (\vec{F} \pm \vec{G}) = \nabla \times \vec{F} \pm \nabla \times \vec{G}$$