IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM.

DEPARTMENT OF MATHEMATICS

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SUBJECT NAME	**: *COMPLEX ANALYSIS***
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UNIT – V

8.1 Residues

1. Define residue.

Definition: Let 'a' be an isolated singularity for f(z) then the residue of $f(z)$ at 'a' is defined to be the co-eff of $\frac{1}{z-a}$ in the Laurent's series expansion of f(z) about 'a' and its denoted by Res $\{f(z); a\}$.

2. Find the residue of $\frac{ze^z}{(z-1)^3}$ at its poles.

Solution: Let f (z) = $\frac{ze^z}{(z-1)^3}$

z = 1 is a pole of order 3 for $f(z)$.

Let $g(z) = ze^z$ so that $g'(z) = e^z(z+1)$, $g''(z) = e^z(z+2)$

Then Res {f(z) ; 1} = $\frac{g''(1)}{2!}$ = 3e / 2

8.2 Cauchy Residue theorem

3. State Cauchy's Residue theorem.

Statement: Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points , $z_1, z_2, \dots, z_n$ inside C then

$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res\{f(z); z_j\}$.

Proof : Let $C_1, C_2, \dots, C_n$ be the circles with centre $z_1, z_2, \dots z_n$ respectively such that all circles are interior to and are disjoint with each other inside C.

By Cauchy's theorem for multiply connected regions we have,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

$$= \sum_{j=1}^{n} 2\pi i \text{ Res } \{f(z), z_j\}$$

4.State Argument Theorem.

Statement: Let f be analytic and on a simple closed curve C except for a finite number of poles inside C also let $f(z)$ have number of zeros on C then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$. Where N is the number of zeros of $f(z)$ inside c and P is the number of poles of f(z) inside C.

Proof: We observe that the singularities of the function $\frac{f'(z)}{f(z)}$ inside C are the poles and zeros of $f(z)$ lying inside C.

Let z_0 be a zero for $f(z)$. Let C_1 be a circle with centre z_0 such that it is the only zero of $f(z)$ inside C.

Then $f(z) = (z - z_0)^n g(z)$

Then $f'(z) = (z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$.

$$\frac{f'(z)}{g'(z)} = \frac{n(z-z_0)^{n-1}\, g(z)}{(z-z_0)^n\, g(z)} + \frac{(z-z_0)^n\, g'(z)}{(z-z_0)^n\, g(z)}$$

Where $g(z)$ is analytic and non zero inside C_1, $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded in Taylor's series about z_0.

Res $\{\frac{f'(z)}{f(z)};\, z_0\} = n$

Similarly if z_1. Is a pole of order p for (z) , then Res $\{\frac{f'(z)}{f(z)};\, z_1\} = -P$

Hence by Cauchy's residue theorem, $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} =$ N$-P$. Where N is the number of zeros and P is the number of poles of $f(z)$ within C

5.State Rouche's theorem.

Statement : If $f(z)$ *and* $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C then $f(z) + g(z)$ and $f(z)$ have the same number of zeros.

Proof: $f(z) + g(z) = f(z)\left[1 + \frac{f(z)}{g(z)}\right] = f(z)\,\varphi(z)$ where $\varphi(z) = \left[1 + \frac{f(z)}{g(z)}\right]$

$[f(z) + g(z)]' = f'(z) + g'(z) = f(z)\,\varphi'(z) + f'(z)\,\varphi(z)$

$$\frac{f'(z)+g'(z)}{f(z)+g(z)} = \frac{f(z)\,\varphi'(z)+f'(z)\,\varphi(z)}{f(z)\,\varphi(z)}$$

$$= \frac{f'(z)}{f(z)} + \frac{\varphi'(z)}{c}$$

$$\frac{1}{2\pi i}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\int_C \frac{\varphi'(z)}{\varphi(z)}\,dz. \rightarrow (1)$$

By hypothesis $|g(z)| < |f(z)|$ and hence $\left|\frac{g(z)}{f(z)}\right| < 1$ on C,

$|\varphi(z)| < 1$ on C.

Hence by maximum modulus theorem, $|\varphi(z) - 1| < 1$ for every point z lies inside C.

Therefore $\varphi(z) \neq 0$ for every point inside C.

Hence $\int_C \frac{\varphi'(z)}{\varphi(z)}\,dz$ = Number of zeros of $\varphi(z)$ within C.

= 0

Hence from (1) , we have $\frac{1}{2\pi i}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz.$

Therefore $N_1 = N_2$ where N_1 and N_2 denote respectively the number of zeros of $f(z) + g(z)$ and $f(z)$ inside C.

6. State Fundamental theorem of algebra.

Statement : A polynomial of degree n with complex coefficients has n zeros in **C.**

8.3 Evaluation of Definite Integrals

Type : 1

$\int_0^{2\pi} f(cos\theta, sin\theta)d\theta$ *where* $f(cos\theta, sin\theta)$*rational function of* $cos\theta$ *and* $sin\theta$.

To evaluate this type of integral we substitute $z = e^{i\theta}$. ***As*** θ ***varies from*** 0 ***to*** 2π, z ***describes the unit circle*** $|z| = 1$.

Also , $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$ ***and*** $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$

Substituting these values in the given integrand is transformed into $\int_C \theta(z)dz$ ***where*** $\theta(z) = f\left[\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right]$ ***and C is the positively oriented unit circle*** $|z| = 1$. ***The integral*** $\int_C \theta(z)dz$ ***can be evaluated using the residue theorem.***

Problem 1. Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\, sin\theta}$.

Solution : Let $I = \int_0^{2\pi} \frac{d\theta}{5+4\, sin\theta}$

Put $z = e^{i\theta}$. Then $dz = id\theta$ and $\sin\theta = \frac{z-z^{-1}}{2i}$.

The given integral is transformed to I = $\int_C \frac{dz}{iz\left[5+4\left(\frac{z-z^{-1}}{2i}\right)\right]}$ where C is the unit circle $|z| = 1$.

$$= \int_C \frac{dz}{2z^2+5iz+2}$$

Let $f(z) = \frac{dz}{2z^2+5iz+2} = \frac{dz}{2(z+2i)+(z+i/2)}$

Therefore $-2i$ *and* $-i/2$ are simple poles of $f(z)$ and the pole $-i/2$ lies inside C.

Also $Res\{f(z); -i/2\}$ = $\lim_{z\to -i/2} \frac{1}{2(z+2i)} = \frac{1}{3i}$

Hence by Cauchy's residue theorem $I = 2\pi i \left(\frac{1}{3i}\right) = \frac{2\pi}{3}$.

Problem 2. Prove that $\int_0^{2\pi} \frac{d\theta}{1+a\, sin\theta} = \frac{2\pi}{\sqrt{1+a^2}}$, $(-1 < a < 1)$

Solution : Let $I = \int_0^{2\pi} \frac{d\theta}{1+a\, sin\theta}$

Put $z = e^{i\theta}$. Then $dz = id\theta$ and $\sin\theta = \frac{z-z^{-1}}{2i}$.

The given integral is transformed to $I = \int_C \frac{dz}{iz\left[1+a\left(\frac{z-z^{-1}}{2i}\right)\right]}$ where C is the unit circle $|z| = 1$.

$$= \int_C \frac{2dz}{az^2+2iz-a}$$

Let $f(z) = \frac{dz}{az^2+2iz-a}$

The poles of $f(z)$ are given by $z = \frac{-2i \pm\sqrt{-4+4a^2}}{2a}$

$$= \frac{-i \pm\sqrt{1-a^2}}{a} \text{ (since } (-1 < a < 1)$$

Let $z_1 = \frac{-i+\sqrt{1-a^2}}{a}$ and $z_2 = \frac{-i-\sqrt{1-a^2}}{a}$

We note that $|z_2| = \frac{-i-\sqrt{1-a^2}}{a} > 1$.

$z_1 = \frac{-i+\sqrt{1-a^2}}{a}$ is the only simple pole lies inside C.

$Res\{f(z);\, z_1\} = \lim_{z\to z_1}(z - z_1)\left[\frac{2/a}{(z-z_1)(z-z_2)}\right]$

$$= \frac{2/a}{(z_1-z_2)}$$

$$= \frac{1}{i\sqrt{1-a^2}}$$

By residue theorem $\int_0^{2\pi} \frac{d\theta}{1+a\ sin\theta}$ $=2\pi i\ [\frac{1}{\sqrt{1+a^2}}]$

$$=\frac{2\pi}{\sqrt{1+a^2}}.$$

Type : 2

$\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{g(x)}{h(x)}$ and $g(x)$, $h(x)$ are polynomials in x and the degree of $h(x)$

exceeds that of $g(x)$ by atleast two.

Problem 1. Evaluate $\int_0^{\infty} \frac{dx}{1+x^4}$.

Solution: *Let* $f(z) = \frac{dz}{1+z^4}$

The poles of $f(z)$ are given by the roots of the equation $1 + z^4 = 0$, which are the four fourth roots of -1.

By Demoivre's theorem they are given by $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ and all are simple poles.

We choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi-circle $|z| = r$ which we denoted by C_1.

Therefore $\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz.$ $\rightarrow (1)$

The poles of $f(z)$ lying inside the contour C are obviously $e^{i\pi/4}$ *and* $e^{i3\pi/4}$ only.

We find the residues of $f(z)$ at these points.

$Res\left\{f(z); e^{\frac{i\pi}{4}}\right\} = \frac{h\left(e^{\frac{i\pi}{4}}\right)}{k'\left(e^{\frac{i\pi}{4}}\right)}$ where $h(z) = 1$ *and* $k(z) = 1 + z^4$ so that $k'(z) = 4z^3$.

$$\therefore\ Res\left\{f(z); e^{\frac{i\pi}{4}}\right\} = \frac{e^{-i3\pi/4}}{4}$$

Similarly $Res\left\{f(z); e^{\frac{i3\pi}{4}}\right\} = \frac{e^{-i9\pi/4}}{4}$

By residue theorem,

$\int_C f(z)dz = 2\pi i \left[\frac{e^{-\frac{i3\pi}{4}}}{4} + \frac{e^{-\frac{i9\pi}{4}}}{4}\right]$

$= \frac{\pi}{\sqrt{2}}.$

From (1), $\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{C_1} f(z)dz = \frac{\pi}{2}.$

$As\ r \to \infty$, $\int_{C_1} f(z)dz \to 0$

$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$

$2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$

$\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$

Problem 2 . Prove that $\int_0^\infty \frac{x^4 dx}{x^6-1} = \frac{\pi\sqrt{3}}{6}.$

Solution: *Let* $f(z) = \frac{z^4}{z^6-1}.$

The poles of $f(z)$ are given by the sixth roots of unity namely $e^{2n\pi i/6}, n = 0,1,..,5$

Therefore $f(z)$ has 2 simple poles on the real axis, viz., $1\ and - 1$ and the two poles $e^{\pi i/3}$ and $e^{2\pi i/3}$ lie on the upper half of the plane.

Now choose the contour C , $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{-r}^{-1-\epsilon_1} f(x)dx + \int_{C_2} f(z)dz + \int_{-1+\epsilon_1}^{-1-\epsilon_2} f(x)dx + \int_{C_3} f(z)dz + \int_{1+\epsilon_2}^{-r} f(x)dx.$ $\rightarrow (1)$

Now, $\int_{C_2} f(z)dz = -\pi i\ Res\ \{f(z);\ -1\}$

$= -\pi i \left(\frac{h(-1)}{k'(-1)}\right)$ where $h(z) = z^4$ and $k(z) = z^6 - 1$

$= \pi i/6.$ $\rightarrow (2)$

Similarly, $\int_{C_3} f(z)dz = -\pi i\ Res\ \{f(z);\ 1\}$

$= -\pi i \left(\frac{h(1)}{k'(1)}\right)$ where $h(z) = z^4$ and $k(z) = z^6 - 1$

$= \pi i\,/6.$ $\rightarrow (3)$

Also $\int_C f(z)dz = 2\pi i\left[Res\left\{f(z);\ e^{\frac{\pi i}{3}}\right\} + Res\left\{f(z);\ e^{\frac{2\pi i}{3}}\right\}\right]$

$$=2\pi i\left[\frac{h\left(e^{\frac{\pi i}{3}}\right)}{6e^{\frac{5\pi i}{3}}} + \frac{e^{\frac{8\pi i}{3}}}{6e^{\frac{10\pi i}{3}}}\right]$$

$$=2\pi i\left[\frac{e^{\frac{4\pi i}{3}}}{6e^{\frac{5\pi i}{3}}} + \frac{e^{\frac{8\pi i}{3}}}{6e^{\frac{10\pi i}{3}}}\right]$$

$$= \frac{\pi i}{3}\left(e^{\frac{-\pi i}{3}} + e^{\frac{-2\pi i}{3}}\right)$$

$$=\frac{\pi\sqrt{3}}{3}. \qquad \rightarrow (4)$$

Substituting (2), (3), (4) in (1) and taking limits as $\in_1, \in_2 \rightarrow 0$ and $r \rightarrow \infty$, we get

$$\int_{-\infty}^{\infty}\frac{x^4dx}{x^6-1} + \frac{\pi i}{6} - \frac{\pi i}{6} = \frac{\pi\sqrt{3}}{3}.$$

$$2\int_0^{\infty}\frac{x^4dx}{x^6-1} = \frac{\pi\sqrt{3}}{3}.$$

$$\int_0^{\infty}\frac{x^4dx}{x^6-1} = \frac{\pi\sqrt{3}}{6}.$$

Type : 3

$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)}$ cosax dx or $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)}$ sinax dx, where $g(x)$ and $h(x)$ are real polynomials such that the degree of $h(x)$ exceeds that of $g(x)$ by atleast one and $a > 0$.

Problem 1 . Prove that $\int_0^{\infty} \frac{cosx\ dx}{1+x^2} = \frac{\pi}{2e}$.

Solution: Let $f(z) = \frac{e^{iz}}{1+z^2}$.

The poles of $f(z)$ are given by $z^2 + 1 = 0$.

$z = -i$ and i .

The poles of $f(z)$ that lies within C is i.

Hence by residue theorem,

$$\int_C f(z)dz = 2\pi i\ Res\ \{f(z);\ i\}$$

$$= 2\pi i\ \frac{h(i)}{k'(i)} \text{ where } h(z) = e^{iz} \text{ and } k(z) = 1 + z^2$$

$$= \frac{2\pi i\, e^{-1}}{2i}$$

$$= \frac{\pi}{e}.$$

$$\int_{-r}^{r} \frac{e^{iax}}{x^2+1} dx + \int_{C_1} \frac{e^{iaz}}{z^2+1} dz = \frac{\pi}{e}$$

when $r \to \infty$, the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \frac{\pi}{e}.$$

Equating real parts we get, $\int_{-\infty}^{\infty} \frac{cosx\ dx}{1+x^2} = \frac{\pi}{e}$.

$$\int_0^{\infty} \frac{cosx\ dx}{1+x^2} = \frac{\pi}{2e}.$$

Problem 2 . Prove that $\int_0^\infty \frac{sinx\ dx}{x} = \frac{\pi}{2}$.

Solution: *Let* $f(z) = \frac{e^{iz}}{z}$.

The only singular point of $f(z)$ is 0 which is a simple pole and it lies on the real axis. Now choose the contour C

Then , $\int_C f(z)dz = \int_{-r}^{-\in} f(x)dx + \int_{C_2} f(z)dz + \int_{\in}^{r} f(x)dx + \int_{C_1} f(z)dz$. $\rightarrow (1)$

Since $f(z)$ is analytic within ,$\int_C f(z)dz = 0$ $\rightarrow (2)$

Also , $\int_{C_2} f(z)dz = -\pi i\, Res\,\{f(z);\ 0\}$

$= -\pi i\, e^0 = -\pi i$ $\rightarrow (3)$

when $r \rightarrow \infty$, the integral over C_1 tends to zero.

Substituting $(2), (3)$ in (1) and taking limit $r \rightarrow \infty$, we get

$0 = \int_{-\infty}^{0} f(x)dx - \pi i + \int_0^\infty f(x)dx$

$\int_{-\infty}^{\infty} f(x)dx = \pi i$

Equating the imaginary parts we get, $\int_{-\infty}^{\infty} \frac{sinx\ dx}{x} = \pi$

, $\int_0^\infty \frac{sinx\ dx}{x} = \frac{\pi}{2}$

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