IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

DEPARTMENT OF MATHEMATICS

CLASS : III B.SC., MATHS

SUBJECT NAME : DYNAMICS

SUBJECT CODE : 16SCCMM14

SEM : VI

UNIT : V (MOTION UNDER THE ACTION OF CENTRAL FORCES)

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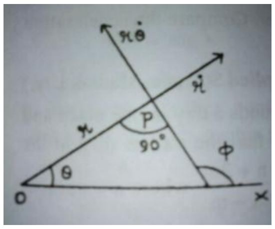
UNIT - V

PART- A

1. Define radial and transverse components of v.

Let P be the position of a moving particle at time t. Taking O as the pole and OX as the initial line, let the polar coordinates of P be (r,θ). Hence the velocity vector v at P has components $\dot{r}$ along OP in the direction in which r increases and r $\dot{\theta}$ perpendicular to OP in the direction in which θ increases. These are respectively called the **radial and transverse** components of v.



2. Write the equation of motion in polar coordinates.

If R and S are the components of the external force acting on a particle of mass m in the radial and transverse directions, we have the equations R = m($\ddot{r} - r\dot{\theta}^2$) and S = m.$\frac{1}{r}$.$\frac{d}{dt}$($r^2\dot{\theta}$).

3. Define central force.

Suppose a particle describes a path, acted on by an attractive force F towards a fixed point O. Such a force is called a **central force** and the path described by the particle is called a central orbit. The fixed point is known as the centre of the force.

4. Write down the pedal equation of equiangular spiral.

In any curve p = r sin ∅ in the usual notation. In the equilateral spiral, ∅ = constant = α (say). Hence p = r sin α = k r is the (p,r) equation to the spiral.

5. Define areal velocity.

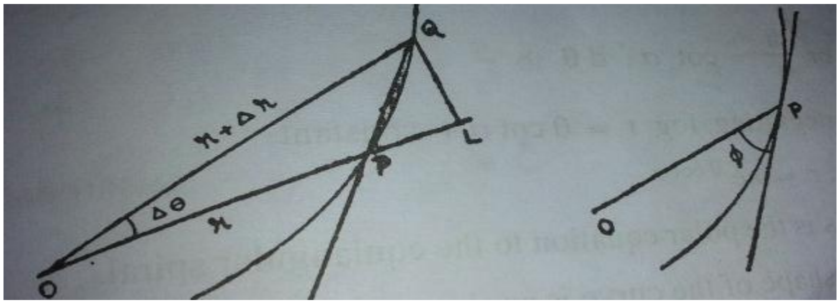
The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the **areal velocity** of the particle.

6. Define apse and apsidal distance.

If there is a point A on a central orbit at which the velocity of the particle is perpendicular to the radius OA, then the point A is called an **apse** and the length OA is the corresponding **apsidal distance**. Hence at an apse, the particle is moving at right angles to the radius vector.

PART-B

1. Describe the polar equation to the equiangular spiral.



Let OP (= r) and OQ (= r + Δr) be two consecutive radii vectors such that the included angle POQ = $\Delta\theta$.

Draw QL perpendicular to OP.

Then OL = (r+ Δr)cos. $\Delta\theta$

= r + Δr approximately.

Hence PL = OL – OP = Δ r and

LQ = (r + Δr)sin$\Delta\theta$

= (r +Δ r) $\Delta\theta$ nearly .

= r $\Delta\theta$ to the first order of smallness.

Hence tan <QPL = $\frac{QL}{PL}$

$$= r \frac{\Delta\theta}{\Delta r}$$

In the limit as Δ r and $\Delta\theta$ both tends to zero, the point Q tends to coincide with P.

The chord QP becomes in the limiting position the tangent at P.

Let $\emptyset$ be the angle made by the tangent at P with OP.

Then $\emptyset$ = Lt $<$ QPL. (Limit Q tends to P)

Hence tan $\emptyset$= Lt tan $<$QPL (limit Δ r tends to zero)

$$= \text{Lt} \quad r \frac{\Delta\theta}{\Delta r}$$

$$= r \frac{d\theta}{dr}$$

That is , tan $\emptyset$ = r $\frac{d\theta}{dr}$

This formula is an important one in dealing with curves in polar coordinates and it gives the angle between the radius vector and the tangent.

Now for the equiangular spiral, at any point P on it the angle $\emptyset$ is constant.

Let $\emptyset$ = α .

Then tan $\emptyset$ = tan α .

That is , tan α = r $\frac{d\theta}{dr}$

$\frac{dr}{r}$ = cot α . $d\theta$

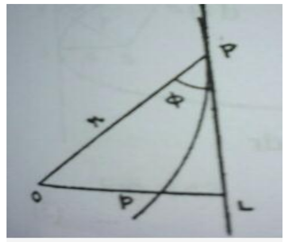
Integrating, log r = θ cot α +constant

i.e. r = a $e^{\theta \cot \alpha}$

This is the polar equation to the equiangular spiral.

2. Derive the perpendicular from the pole on the tangent formulae polar co-ordinates.

Solution:



Let $\emptyset$ be the angle made by the tangent at P with be radius vector OP.

We know that $\tan\emptyset = rd\theta/dr$ $\rightarrow$ (1)

From O draw OL perpendicular to the tangent at P and let OL = p.

Then $\sin\emptyset = OL/ OP = p / r$ or $p = r\sin\emptyset$ $\rightarrow$.(2)

Let us eliminate $\emptyset$ between (1) and (2).

From (2), $1/p^2 = 1/ r^2\sin^2\emptyset$

$= 1/ r^2.\text{cosec}^2\emptyset$

$= 1/r^2.(1+\cot^2\emptyset)$

$=1/r^2[1+1/r^2(dr/d\emptyset)^2]$

i.e. $1/p= 1/r^2+ 1/r^4 (dr/d\theta)$ $\rightarrow$ (3)

Using $r = 1/u$, $dr/d\theta$

$= dr/du.\ du/d\theta$

$= -1/u^2.\ du/d\theta$.

Hence (3) becomes $1/p^2 = u^2+(du/d\theta)^2$ $\rightarrow$ (4)

3. Derive the pedal equation of the central orbit.

Solution:

In certain curves the relation between p(the perpendicular from the pole on the tangent) and r (the radius vector) is very simple.

Such a relation is called the pedal equation or the (p,r) equation to the curve.

We can get the(p,r) equation to a central orbit as follows:

In the usual notation, we have from, $1/p^2 = u^2+(du/d\theta)^2$ $\rightarrow$ (1)

Differentating both sides of (1) with respect to .,

$-2/p^3..dp/d\theta = 2u.\ du/d\theta+2du/d\theta.d^2u/d\theta^2 = 2du/d\theta\ (u+d^2u/d\theta^2)$ $\rightarrow$ (2)

But the differential equation in polars is $u+d^2u/d\theta^2 = p/h^2u^2$.

Hence (2) becomes, $-1/p^3.dp/d\theta = p/h^2u^2.du/d\theta$

i.e. $-\frac{1}{p^3}.dp = \frac{p}{h^2u^2}.du$

$= \frac{p}{h^2}. r^2 d(1/r)$

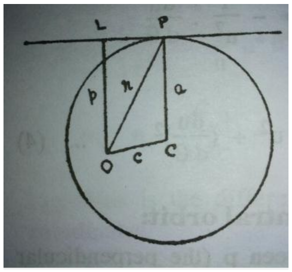
$= -\frac{p}{h^2} dr$ (or) $\frac{h^2}{p^3}.\frac{dp}{dr} = p$ → (3)

(3) is the (p,r) equation or the pedal equation to the central orbit.

4. Derive the pedal equation of circle pole at any point and parabola pole at focus.

Solution:

(1) circle – pole at any point:



Let C be the centre, 'a' the radius, O the pole where OC = c.

Let p be any point on the circle and OL be the perpendicular from O on the tangent at P.

OP = r and

OL= p.

From ΔOPC,

$c^2 = r^2+a^2-2ra \cos\angle OPC$

$= r^2+a^2-2ra \cos\angle POL$

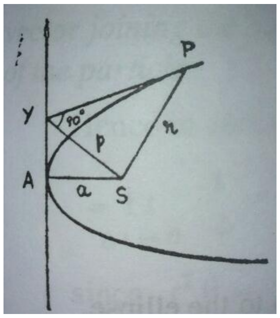
$= r^2+a^2-2ra.p/r$

$= r^2+a^2-2ap.$

Hence the pedal equation of the circle for a general position of the pole is $c^2 = r^2+a^2-2ap$.

When c = a, the pole is on the circumference and the equation is $r^2 = 2ap$.

(2) Parabola – pole at focus:



To get the (p,r) equation to a parabola, we assume the geometrical property that if the tangent at P meets the tangent at the vertex A in Y and S is the focus, then SY is perpendicular to PY and the triangles SAY and SYP are similar.

Hence SA/ SY = SY/SP

i.e.a/p = p/r or $p^2 = ar$.

5. Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described.

Solution :

Given , $r^n = a^n \cos n\theta$.

since r = 1/u, the equation $u^n a^n \cos n\theta = 1$ → (1)

Taking logarithms,

$n\log u + n\log a + \log \cos n\theta = 0$ → (2)

Differentiating (2) with respect to θ .

$n.1/u \, du/d\theta - n\sin n\theta/\cos n\theta = 0$

i.e. $du/d\theta = u \tan n\theta$ → (3)

Differentiating (3) with respect to θ

$d^2u / d\theta^2 = un\sec^2 n\theta + \tan n\theta \, du/d\theta$

$= nu\sec^2 n\theta + u\tan^2 n\theta$ using (3)

$u + d^2u / d\theta^2 = u + nu\sec^2 n\theta + u\tan^2 n\theta$

$= nu \sec^2 n\theta + u (1+ \tan^2 n\theta)$

$= nu\sec^2 n\theta + u\sec^2 n\theta$

$= (n+1)\ u \sec^2 n\theta$

$=(n+1)\ u.u^{2n} a^{2n}$ using (1) to substitute for $\sec^2 n\theta$

$=(n+1)a^{2n}\, u^{2n+1}$

$P = h^2u^2(u + d^2u / d\theta^2)$

$= h^2u^2(n+1)\ a^{2n}\, u^{2n+1}$

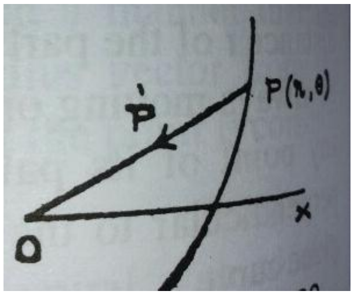
$= (n+1)\ a^{2n}\ \ u^{2n+3}\, h^2$

$= (n+1)\ a^{2n}\ \ h^2.1/r^{2n+3} \qquad \rightarrow \qquad (4)$

i.e $P \propto 1/r^{2n+3}$ which means that the central acceleration varies inversely as the (2n+3) rd power of the distance.

PART-C

1. Derive the differential equation of central orbit.



A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane .

Take O as the pole and a fixed line through O as the initial line. Let $P(r,\theta)$ be the polar coordinates of the particle at time t and m be its mass.

Also let P be the magnitude of the central acceleration along PO.

The equation of motion of the particle are

$m(\ddot{r} - r\dot{\theta}^2)$ = - mP

i.e $(\ddot{r} - r\dot{\theta}^2)$ = - P $\qquad \rightarrow (1)$

and $m/r.d/dt(r^2\dot{\theta}) = 0$

i.e. $1/r.d/dt(r^2\dot{\theta}) = 0 \qquad \rightarrow(2)$

Equation (2) follows from the fact that as there is no force at right angles to OP, the transverse component of the acceleration zero throughout the motion.

From (2), $r^2\dot{\theta}$ = constant = h . $\rightarrow$ (3)

To get the polar equation of the path ,we have to eliminate the element of time between equations (1) and (3).

For this purpose, it is found convenient to put u = 1/r and work with u as the dependent variable.

From (3),. $\dot{\theta} = h/r^2$

$= hu^2$

Also $\dot{r} = dr/dt$

$= d/dt(1/u)$

$= -1/u^2.du/dt$

$= -1/u^2.du/d\theta.\ d\theta/dt$

$= -1/u^2.(du/d\theta).hu^2$

$= -h(du/d\theta)$

$\ddot{r} = d/dt(-hdu/d\theta)$

$= -h^2u^2d^2u/d\theta^2$

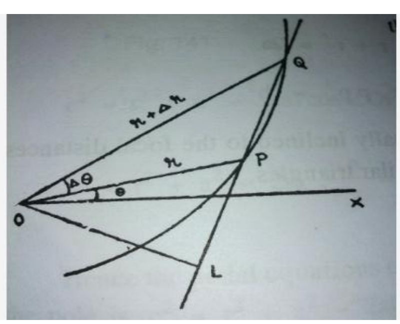
substituting r and $\dot{\theta}$ in (1), we get

$-h^2u^2(d^2u/d\theta^2) - 1/u\ .h^2u^4 = -P$ (or)

$u + d^2u/d\theta^2 = P/\ h^2u^2$.

This is the differential equation of the central orbit, in polar coordinates.

2. Describe about the velocities in a central orbit.



In every central orbit the areal velocity is constant and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

Let at time t the particle be at P(r, θ) and at time t + Δt, let it be at Q(r + Δr, θ + Δθ).

sectorial area OPQ described by the radius vector OP.

= Area of ΔOPQ nearly

=1/2 OP.OQ sin<POQ

=1/2.r(r + Δr)sin Δθ

=1/2 r^2 Δ θ, to the first order of smallness.

The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle.

Hence in the central orbit, areal velocity of P

= Lt ½ (Δθ/Δt)

= ½ dθ/dt r^2

= ½ h →(1)

since $r^2\dot{\theta}$ = constant = h.

Hence h = twice the areal velocity and as h is a constant, the areal velocity is constant.

In other words, equal areas are described by the radius vector in equal times.

We can get another expression for the areal velocity.

Let Δs be the length of the arc PQ. Draw OL perpendicular to PQ.

Sectorial area POQ = ΔPOQ nearly

= 1/2PQ.OL.

As Δt tends to 0, Q tends to coincide with P along the curve and the chord QP becomes the tangent at P.

length PQ = Δs nearly and OL becomes the perpendicular from O on the tangent at P. let OL = p.

Hence areal velocity

$= Lt \ 1/2 \ (\Delta s/\Delta t).p$

$= \frac{1}{2} p \ (ds / dt)$

$= \frac{1}{2} pv$ →(2)

as ds/dt is the rate of describing s and so is the linear velocity of P.

Hence combining (1) and (2), areal velocity $= \frac{1}{2} h = \frac{1}{2} pv$.

(or) $h = pv$

(i.e) $v = h/ p$.

Hence linear velocity varies inversely as OP.