IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

DEPARTMENT OF MATHEMATICS

CLASS	:	**I B.Sc., MATHEMATICS**
SUBJECT NAME	:	**DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORMS**
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SEMESTER	:	**II**
UNIT	:	**V (LAPLACE TRANSFORMS)**
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UNIT V

Laplace Transform

Let f(t) be a function of t defined for t>0. Then the Laplace transform of function f (t) , denoted by L[f(t)] (or) F(s) is defined by $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(S)$

Important formula in Laplace Transforms and Inverse Laplace Transforms

S.No	L[f(t)]=F(S)	L^{-1} [F(s)]= f(t)
1	$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$
2	$L[t] = \frac{1}{s^2}$	$L\left[\frac{1}{s^2}\right] = t$
3	$L[t^n] = \frac{n!}{s^{n+1}}$	$L^{-1}[\frac{n!}{s^{n+1}}] = t^n$
4	$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}[\frac{1}{s-a}] = e^{at}$
5	$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}[\frac{1}{s+a}] = e^{-at}$
6	L[Sinat]=$\frac{a}{s^2+a^2}$	L^{-1} [$\frac{a}{s^2+a^2}$]=sinat
7	L[cosat]=$\frac{s}{s^2+a^2}$	L^{-1} [$\frac{s}{s^2+a^2}$]=cosat
8	L[sinhat]=$\frac{s}{s^2-a^2}$	L^{-1} [$\frac{a}{s^2-a^2}$]=sinhat
9	L[coshat]=$\frac{s}{s^2-a^2}$	L^{-1} [$\frac{s}{s^2-a^2}$]=coshat
10	$L[e^{at} t^n] =$ n!/(s-a)$^{n+1}$	$L^{-1}\left[\frac{1}{(s-a)^{n+1}}\right] = \frac{L[e^{at} t^n]}{n!}$

Problems in Laplace Transforms

1. Find $L[5t^3 + 6t - 7 + 2e^{-6t} + 10\sin 2t + 7\cosh 3t]$

Solution: $L[5t^3 + 6t - 7 + 2e^{-6t} + 10\sin 2t + 7\cosh 3t]$

$$= 5L[t^3] + 6L[t] - 7L[1] + 2L[e^{-6t}] + 10L[\sin 2t] + 7L[\cosh 3t]$$

$$= 5\frac{3!}{s^4} + 6\frac{1}{s^2} - 7\frac{1}{s} + 2\frac{1}{s+6} + 10\frac{2}{s^2+4} + 7\frac{s}{s^2-9}$$

$$= \frac{30}{s^4} + \frac{6}{s^2} - \frac{7}{s} + \frac{2}{s+6} + \frac{20}{s^2+4} + \frac{7s}{s^2-9}$$

2. Find L$[\sqrt{t}]$

Solution: L$[\sqrt{t}] = \frac{\Gamma_{\frac{1}{2}+1}}{s^{\frac{3}{2}}}$

$$= \frac{\frac{1}{2}\Gamma_{\frac{1}{2}}}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

3. Find L[sin^{2}3t]

Solution: L$[\sin^2 3t] = \frac{1}{2}L(1 - \cos 6t)$

$$= \frac{1}{2}[L(1) - L(\cos 6t)]$$

$$= \frac{1}{2}[\frac{1}{s} - \frac{s}{s^2+36}] = \frac{18}{s(s^2+36)}$$

4. Find L[cos^{3}2t]

Solution: L$[\cos^3 2t] = \frac{1}{4}L(3\cos 2t + \cos 6t)$

$$= \frac{1}{4}[3\frac{s}{s^2+4} + \frac{s}{s^2+36}]$$

$$= \frac{s}{4}[\frac{3}{s^2+4} + \frac{1}{s^2+36}]$$

$$= \frac{s(s^2+28)}{(s^2+4)(s^2+36)}$$

5. Find L[sintcost]

Solution: L[sintcost] = $L[\frac{\sin 2t}{2}]$

$$=\frac{1}{2}L[\sin 2t]=\frac{1}{2}\frac{2}{s^2+4}=\frac{1}{s^2+4}$$

6. Find $L[e^{-t}\sin 3t]$

Solution: $L[e^{-t}\sin 3t] = L[\sin 3t]_{s\to s+1}$

$$=\left[\frac{3}{s^2+9}\right]_{s\to s+1}=\left[\frac{3}{(s+1)^2+9}\right]=\left[\frac{3}{s^2+2s+10}\right]$$

Try for: $L[\sin^3 2t], L[\sin^2 4t], L[\cos 4t\sin 2t],\ L[\sin 4t\sin 2t], L[\cos^2 4t]\ ,\ L[e^t\cosh 5t]$

7. Find $L[t\sin 2t]$

Solution: By the property $L[tf(t)]=-\frac{d}{ds}F(s)$

$$L[t\sin 2t]=-\frac{d}{ds}(\frac{2}{s^2+4})$$

$$=-(\frac{-2(2s)}{(s^2+4)^2})=\frac{4s}{(s^2+4)^2}$$

8. Find $L[\frac{\sin 3t}{t}]$

Solution: By the Property $L[\frac{f(t)}{t}]=\int_s^\infty F(s)ds$

$$L[\frac{\sin 3t}{t}]=\int_s^\infty L[\sin 3t]ds$$

$$=\int_s^\infty \frac{3}{s^2+9}ds$$

$$=3\left[\frac{1}{3}\tan^{-1}(\frac{s}{3})\right]_s^\infty=\tan^{-1}(\infty)-\tan^{-1}(\frac{s}{3})$$

$$=\frac{\pi}{2}-\tan^{-1}(\frac{s}{3})=\cot^{-1}(\frac{s}{3})=\tan^{-1}(\frac{3}{s})$$

9. Find $L[te^{-t}\sin t]$

Solution: $L[te^{-t}\sin t]=-\frac{d}{ds}L[e^{-t}\sin t]$

$$= -\frac{d}{ds}\left[L[\sin t]_{s\to s+1}\right]$$

$$= -\frac{d}{ds}\left[[\frac{1}{s^2+1}]_{s\to s+1}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right] = -\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right]$$

$$= -\left[\frac{-(2s+2)}{(s^2+2s+2)^2}\right] = \left[\frac{2(s+1)}{(s^2+2s+2)^2}\right]$$

10. Find $L[\frac{1-e^t}{t}]$

Solution: $L[\frac{1-e^t}{t}] = \int_s^\infty L[1-e^t]ds$

$$= \int_s^\infty \frac{1}{s} - \frac{1}{s-1} ds = \left[\log s - los(s-1)\right]_s^\infty$$

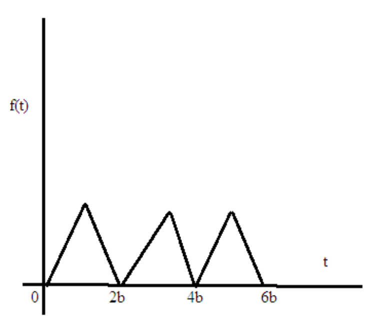
$$= \left[\log\frac{s}{s-1}\right]_s^\infty = \left[\log\frac{1}{1-\frac{1}{s}}\right]_s^\infty = \left[0 - \log\frac{1}{1-\frac{1}{s}}\right]$$

$$= \left[-\log\frac{s}{s-1}\right] = \left[\log\frac{s-1}{s}\right]$$

Laplace Transforms of a periodic function f(t) with period P given by

$$\frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$$

11. Find the Laplace transform of the triangular wave function given below:



Solution: The triangular wave function can be represented as

f(t) =t 0<t<b

=2b-t   b<t<2b

The function f(t) has a period 2b

$$L[f(t)] = \frac{1}{1-e^{-2ps}} \int_0^{2p} e^{-st} f(t)dt$$

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \left[\int_0^{2b} e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^{b} e^{-st} t \ dt + \int_b^{2b} e^{-st} (2b-t)dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\left(\frac{te^{-st}}{-s} \right)_0^b + \frac{1}{s} \int_0^b e^{-st} dt \right] + \left[\left(\frac{(2b-t)e^{-st}}{-s} \right) - \frac{1}{s} \int_b^{2b} e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{-be^{-bs}}{s} + \frac{1}{s^2} e^{-sb} + \frac{1}{s^2} + \frac{be^{-bs}}{s} + \frac{e^{-2bs}}{s^2} - \frac{e^{-bs}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{-2be^{-bs}}{s^2} + \frac{1}{s^2} + \frac{e^{-2bs}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{1-2be^{-bs} + e^{-2bs}}{s^2} \right]$$

$$= \frac{1}{s^2(1-e^{-2bs})} \left[\frac{(1-e^{-bs})^2}{(1-e^{-bs})(1+e^{-bs})} \right] = \left[\frac{(1-e^{-bs})}{s^2(1+e^{-bs})} \right]$$

Problems based on Inverse Laplace Transforms

12. Find $L^{-1}[\frac{1}{s^2+2s+5}]$

Solution: $L^{-1}[\frac{1}{s^2+2s+5}] = L^{-1}\left[\frac{1}{(s+1)^2+4} \right]$

$$= e^{-t}[\frac{1}{s^2+2^2}] = [\frac{e^{-t}\sin 2t}{2}]$$

13. Use Laplace Transform to solve the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 3y = \sin t, y(0) = 0, y'(0) = 0$$

Solution:

$y'' + y' - 3y = \sin t$

Taking the Laplace Transform on both sides, we get

$$L[y''] + L[y'] - 3L[y] = L(\sin t)$$

$$[s^2 L[y] - sy(0) - y'(0)] + [sL(y) - y(0)] - 3L[y] = \frac{1}{s^2+1}$$

$$L[y][s^2 + s - 3] - s(0) - 0 - 0 = \frac{1}{s^2+1}$$

$$L[y] = \frac{1}{(s^2+1)(s^2+s-3)}$$

$$y = L^{-1}\left[\frac{1}{(s^2+1)(s^2+s-3)}\right]$$

Resolving $\frac{1}{(s^2+1)(s^2+s-3)}$ into partial fraction

$$\frac{1}{(s^2+1)(s^2+s-3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+s-3}$$

$$1 = (As+B)(s^2+s-3) + (Cs+D)(s^2+1)$$

Coefficient of s^3, A+C=0 ------(1)

Coefficient of s^2, A+B+D=0 -----(2)

Coefficient of s, -3A+B+C=0 -----(3)

Constant, -3B+D=1 -----(4)

3-1 -4A+B=0 -----(5)

4-2 -A-4B=1 -----(6)

(5)x4 -16A+4B=0 ------(7)

(6+7) -17A=1, **A= -1/17**

C=1/17

B=4A=-14/17 , **B = -14/17 D = 5/17**

$$L^{-1}\left[\frac{1}{(s^2+1)(s^2+s-3)}\right] = \frac{-1}{17}L^{-1}\left[\frac{s+4}{(s^2+1)}\right] + \frac{1}{17}L^{-1}\left[\frac{s+5}{(s^2+s-3)}\right]$$

$$= \frac{-1}{17}\left[L^{-1}\left(\frac{s}{(s^2+1)}\right) + 4L^{-1}\left(\frac{1}{(s^2+1)}\right)\right] + + \frac{1}{17}L^{-1}\left(\frac{(s+\frac{1}{2}) + 5 - \frac{1}{2}}{(s+\frac{1}{2})^2 - 3 - \frac{1}{4}}\right)$$

$$= \frac{-1}{17}\left[L^{-1}\left(\frac{s}{(s^2+1)}\right) + 4L^{-1}\left(\frac{1}{(s^2+1)}\right) - L^{-1}\left(\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 - \frac{13}{4}}\right) - \frac{9}{2}L^{-1}\left(\frac{1}{(s+\frac{1}{2})^2 - \frac{13}{4}}\right)\right]$$

$$= \frac{-1}{17}\left[\cos t + 4\sin t - e^{-\frac{t}{2}} \cosh\left(\frac{\sqrt{13}}{2} t\right) - \frac{9}{\sqrt{13}} e^{\frac{-1}{2}} \sin\left(\frac{\sqrt{13}}{2} t\right)\right]$$

Try for:

1 .Solve, using LT $\frac{d^2 y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{-t}$, $Given, y(0) = 1, y'(0) = 0$

2.Solve the differential equation $(D^2 + 4D + 8)y = 1,\ Given, y(0) = 0, y'(0) = 1$

3.Solve $L^{-1}\left[\frac{s+3}{s^2 - 4s + 13)}\right]$

4.Solve $L^{-1}\left[\frac{s}{(s+5)^2}\right]$