IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM.

DEPARTMENT OF MATHEMATICS

CLASS	**: II B.Sc., MATHS**
SUBJECT NAME	**: VECTOR CALCULUS & FOURIER SERIES**
SUBJECT CODE	**: 16SCCMM7**
SEMESTER	**: IV**
UNIT	**: V (Half Range Fourier Series, Change of Interval, Combination of Series)**
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UNIT V

1. Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi + b_n \sin n\pi\ x\right)$$

This representation of $f(x)$ is called a Fourier series or Fourier expansion.

Here a_0, a_n, b_n are called Fourier coefficients.

Where,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos n\pi\ dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin n\pi\ dx$$

2. Odd and Even functions with examples:

If $f(x) = f(-x)$ then $f(x)$ is said to be even. **Ex.** $x^2, \cos x,\ \sin^2 x, |x|, x\sin x$.

If $f(x) = -f(-x)$ then $f(x)$ is said to be even. **Ex.** $x, x^3, x\cos x,\ \sin x, \sin 2x, \tan^3 x$.

3. Properties of odd and even functions:

(i) $\int_{-a}^{a} f(x)\, dx = 0$ if $f(x)$ is an odd.

(ii) $\int_{-a}^{a} f(x)\, dx = 2\int_0^{2\pi} f(x)dx$ if $f(x)$ is an even.

4. Half range Fourier series:

Let $f(x)$ be a periodic function with period 2π. We have considered the expression of $f(x)$ as a Fourier series in the interval $(-\pi,\pi)$ or $(0,2\pi)$. Sometimes it may be required to expand as Fourier series in a range with equal to half the period. It may further be required that series should obtain only cosine terms or only sine terms such series are known as half range series.

5. Write the development in cosines series.

Let $f(x)$ be expanded as a series containing cosines only and

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi\right)$ ------------ (1)

If we integrate both sides of (1) between limits 0 to π.

Then $\int_0^{\pi} f(x)\, dx = \int_0^{\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} a_n \int_0^{\pi} \cos nx\, dx = \frac{a_0\pi}{2}$.

$$a_0 = \frac{\pi}{2}\int_0^{\pi} f(x)\, dx.$$

If we multiply both sides of the above equation (1) by $\cos n\pi$ and integrate from 0 to π.

Then $\int_0^{\pi} f(x) \cos nx\, dx = a_n \frac{\pi}{2}$.

Since all the terms except the term containing b_n vanish.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx .$$

6. Write the development in sine series.

Let $f(x)$ be expanded as a series containing sines only and

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx .$$

Multiply both sides of the above equation by $\sin nx$ and integrate from 0 to π .

$$\text{Then } \int_0^{\pi} f(x)\sin nx\, dx = b_n \frac{\pi}{2} .$$

Since all the terms except the term containing b_n vanish.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx .$$

7. Write the change of interval.

Suppose we have to expand $f(x)$ in the interval $-l$ to l as a Fourier series.

Let $X = \frac{\pi}{l}$, ie., $x = \frac{lX}{\pi}$.

When $x = -l,\ \ X = -\pi$ and

When $x = l,\ \ X = \pi$

Hence the function becomes $f\left(\frac{lX}{\pi}\right)$ can be expanded as Fourier series of the form

$$f\left(\frac{lX}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nX + b_n \sin nX\right)$$

Where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right)\cos nX\ dX\ (n = 0,1,2,.....)$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right)\sin nX\ dX\ (n = 1,2,.....)$$

Reverting back to the original variable x .

$$x = \frac{lX}{\pi} .$$

$$dx = \frac{ldX}{\pi} .$$

When $X = \pi, x = l$;

$X = -\pi, x = -l$

$$a_n = \frac{1}{\pi}\int_{-l}^{l} f(x)\cos\frac{n\pi X}{l} . \frac{ndX}{l}$$

$$= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx .$$

Similarly, $b_n = \frac{1}{\pi}\int_{-l}^{l} f(x)\sin\frac{n\pi X}{l} . \frac{ndX}{l}$

$$= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx .$$

Therefore $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi\, x}{l}+b_n \sin\frac{n\pi\, x}{l}\right).$

Where,

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi\, x}{l}\,dx$$

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi\, x}{l}\,dx$$

8. Combination of series:

Some of the known Fourier expansions may be easily combined to yield Fourier expansions of any linear and quadratic functions of x over the half range interval $0<x<l$.

The following half range expansions in the interval $0<x<l$ will be found useful.

1. $1=\frac{4}{\pi}\left(\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+.....\right)$
2. $x=\frac{2l}{\pi}\left(\sin\frac{\pi x}{l}-\frac{1}{2}\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}-.....\right)$
3. $x=\frac{l}{2}-\frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+.....\right)$
4. $x^2=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l}-\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+.....\right)$
5. $x^2=\frac{2l^2}{\pi^3}\left((\pi^2-4)\sin\frac{\pi x}{l}-\frac{\pi^2}{2}\sin\frac{2\pi x}{l}+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin\frac{3\pi x}{l}-\frac{\pi^2}{4}\sin\frac{4\pi x}{l}.....\right)$

9. Derive a cosine series for $f(x)=x$ in $0<x<2$.

Cosine series:

$$x=\frac{l}{2}-\frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+.....\right),$$

Given: $l=2$

$$x=\frac{2}{2}-\frac{4(2)}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+.....\right)$$

$$x=1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+.....\right).$$

10. Derive a sine series for $f(x)=x$ in $0<x<2$.

Sine series:

$$x=\frac{2l}{\pi}\left(\sin\frac{\pi x}{l}-\frac{1}{2}\sin\frac{2\pi x}{l}+\frac{1}{3}\cos\frac{3\pi x}{l}+.....\right),$$

Given: $l=2$

$$x=\frac{2(2)}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\cos\frac{3\pi x}{2}+.....\right)$$

$$x=\frac{4}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\cos\frac{3\pi x}{2}+.....\right).$$

11. Obtain a sine series for $f(x)=c$ in the range 0 to π .

Solution:

We have to find the half range sine series for $f(x)$.

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

Where, $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin n\pi\, dx$

To Find b_n :

$$b_n=\frac{2}{\pi}\int_0^{\pi} c\sin n\pi\, dx$$

$$=\frac{2c}{\pi}\int_0^{\pi}\sin n\pi\, dx$$

$$=\frac{2c}{\pi}\left[\frac{-\cos x}{n}\right]_0^{\pi}$$

$$=\frac{2c}{\pi}\left[\frac{-\cos x}{n}-\frac{-\cos 0}{n}\right]$$

$$=\frac{2c}{\pi}\left[\frac{-(-1)^n}{n}+\frac{1}{n}\right]$$

$$=\frac{2c}{n\pi}\left[1-(-1)^n\right]$$

When n is even, $b_n=0$.

When n is odd, $b_n=\frac{4c}{n\pi}$

$$f(x)=c=\frac{4c}{n\pi}\sin n\pi$$

$$=\frac{4c}{n\pi}\left\{\frac{\sin\pi}{1}+\frac{\sin 3\pi}{3}+\frac{\sin 5\pi}{5}+.....\right\}$$

Put $x=\frac{\pi}{2}$,

$$=\frac{4c}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}+.....\right\}. \qquad \text{Since } \left\{\sin\frac{\pi}{2}=1, \sin\pi=0\right\}$$

$$\frac{c\pi}{4c}=1-\frac{1}{3}+\frac{1}{5}+.....$$

Therefore,

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}+.....$$

12. Find a cosine series for $f(x)=(\pi-x)$ in $0<x<\pi$.

Solution:

We have to find the half range cosine series for $f(x)$.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos n\pi\right)$$

Where,

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos n\pi\,dx$$

To find a_0:

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\,dx$$

$$=\frac{2}{\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left\{\left[\pi(\pi)-\frac{(\pi)^2}{2}\right]-\left[\pi(0)-\frac{(0)^2}{2}\right]\right\}$$

$$=\frac{2}{\pi}\left\{\left[\pi^2-\frac{\pi^2}{2}\right]-[0]\right\}$$

$$=\frac{2}{\pi}\left[\frac{2\pi^2-\pi^2}{2}\right]$$

$$=\frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$$

Therefore, $a_0=\pi$

To find a_n:

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos n\pi\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\cos n\pi\,dx$$

This is in the form of $\int udv$, by using Bernoulli formula.

$$\int udv=uv-u'v_1+u''v_2-u'''v_3+\ldots\ldots$$

$u=\pi-x$; $dv=\cos nxdx$

$u'=-1$; $v=\frac{\sin nx}{n}$

$u''=0$; $v_1=\frac{-\cos nx}{n^2}$

$$a_n = \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right) + 0 \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi - \pi)\left(\frac{\sin n(\pi)}{n}\right) - (-1)\left(\frac{-\cos n(\pi)}{n^2}\right)\right] - \left[(\pi - 0)\left(\frac{\sin n(0)}{n}\right) - (-1)\left(\frac{-\cos n(0)}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[(0)\left(\frac{0}{n}\right) - (-1)\left(\frac{-(-1)^n}{n^2}\right)\right] - \left[(\pi)\left(\frac{0}{n}\right) - (-1)\left(\frac{-1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{-(-1)^n}{n^2}\right) + \left(\frac{1}{n^2}\right)$$

Therefore, $a_n = \frac{2}{\pi}\left(\frac{-(-1)^n}{n^2}\right) + \left(\frac{1}{n^2}\right)$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left\{ \frac{2}{\pi}\left(\frac{-(-1)^n}{n^2}\right) + \left(\frac{1}{n^2}\right)\right\}\cos nx .$$

13. In the range $(0, 2l)$ $f(x)$ is defined by the relations

$f(x) = \begin{cases} 0 & when \quad 0 < x < l \\ a & when \quad l < x < 2l \end{cases}$ **expand $f(x)$ as a fouries series of period $2l$.**

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi\, x}{l} + b_n \sin\frac{n\pi\, x}{l}\right).$$

Where,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi\, x}{l}\, dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi\, x}{l}\, dx$$

Given interval: $0 \ to \ 2l$

$$a_0 = \frac{1}{l}\int_{0}^{2l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\frac{n\pi\, x}{l}\, dx$$

$$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\frac{n\pi\, x}{l}\, dx$$

To find a_0:

$$a_0 = \frac{1}{l}\int_{0}^{2l} f(x)\, dx$$

$$f(x) = a, \quad l < x < 2l .$$

$$a_0 = \frac{1}{l}\int_l^{2l} a\, dx$$

$$= \frac{1}{l} a \int_l^{2l} dx$$

$$= \frac{a}{l}\left[x\right]_l^{2l}$$

$$= \frac{a}{l}\left[2l - l\right]$$

$$= \frac{a}{l}\left[l\right]$$

Therefore $a_0 = a$ **.**

To find a_n **:**

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\, dx$$

$$f(x) = a, \quad l < x < 2l.$$

$$a_n = \frac{1}{l}\int_l^{2l} a\cos\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l} a \int_l^{2l} \cos\frac{n\pi x}{l}\, dx$$

$$= \frac{a}{l}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi x}{l}}\right]_l^{2l}$$

$$= \frac{a}{n\pi}\left[\sin 2nx - \sin nx\right]$$

Therefore $a_n = 0$ **.**

To find b_n **:**

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\, dx$$

$$f(x) = a, \quad l < x < 2l.$$

$$b_n = \frac{1}{l}\int_l^{2l} a\sin\frac{n\pi x}{l}\, dx$$

$$= \frac{a}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi x}{l}}\right]_l^{2l}$$

$$= \frac{-a}{n\pi}\left[\cos 2nx - \cos nx\right]$$

$$= \frac{a}{l}\left[\frac{-\sin\frac{n\pi\, x}{l}}{\frac{n\pi\, x}{l}}\right]_{l}^{2l}$$

Therefore $b_n = \frac{-a}{n\pi}\left[(-1)^n - 1\right]$.

Hence $b_n = 0$ when n is even and $b_n = \frac{-2a}{n\pi}$ when n is odd.

$$f(x) = \frac{a}{2} - \frac{2a}{n\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + ...\right]$$

14. Find a sine and cosine series for $f(x) = 2x - 4$ **in** $0 < x < 4$**.**

Solution:

Combination of sine series:

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - .....\right)$$

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + ...\right)$$

Given: $f(x) = 2x - 4, \quad l = 4$

$$x = \frac{2(4)}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} - .....\right)$$

$$x = \frac{8}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} - .....\right)$$

$$2x = 2\left\{\frac{8}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} - .....\right)\right\}$$

$$2x = \frac{2(8)}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} - .....\right)$$

$$2x = \frac{16}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} - .....\right)$$

$$1(4) = \frac{4(4)}{\pi}\left(\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4} + ...\right)$$

$$4 = \frac{16}{\pi}\left(\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4} + ...\right)$$

$$2x-4=\frac{16}{\pi}\left(\sin\frac{\pi x}{4}-\frac{1}{2}\sin\frac{2\pi x}{4}+\frac{1}{3}\sin\frac{3\pi x}{4}-.....\right)-\frac{16}{\pi}\left(\sin\frac{\pi x}{4}+\frac{1}{3}\sin\frac{3\pi x}{4}+\frac{1}{5}\sin\frac{5\pi x}{4}+.....\right)$$

$$=\frac{16}{\pi}\left(\sin\frac{\pi x}{4}\right)-\frac{16}{\pi}\left(\frac{1}{2}\sin\frac{2\pi x}{4}\right)+\frac{16}{\pi}\left(\frac{1}{3}\sin\frac{3\pi x}{4}\right)-...-\frac{16}{\pi}\left(\sin\frac{\pi x}{4}\right)-\frac{16}{\pi}\left(\frac{1}{3}\sin\frac{3\pi x}{4}\right)-\frac{16}{\pi}\left(\frac{1}{5}\sin\frac{5\pi x}{4}\right)....$$

Odd terms are all cancel, so only got the even terms

$$2x-4=-\frac{16}{\pi}\left(\frac{1}{2}\sin\frac{2\pi x}{4}\right)-\frac{16}{\pi}\left(\frac{1}{4}\sin\frac{4\pi x}{4}\right)-\frac{16}{\pi}\left(\frac{1}{6}\sin\frac{6\pi x}{4}\right)-....$$

$$=-\frac{16}{\pi}.\frac{1}{2}\left(\sin\frac{2\pi x}{4}\right)-\frac{16}{\pi}.\frac{1}{2}\left(\frac{1}{2}\sin\frac{4\pi x}{4}\right)-\frac{16}{\pi}.\frac{1}{2}\left(\frac{1}{3}\sin\frac{6\pi x}{4}\right)-....$$

$$=-\frac{8}{\pi}\left(\sin\frac{2\pi x}{4}\right)-\frac{8}{\pi}\left(\frac{1}{2}\sin\frac{4\pi x}{4}\right)-\frac{8}{\pi}\left(\frac{1}{3}\sin\frac{6\pi x}{4}\right)-....$$

$$=-\frac{8}{\pi}\left(\sin\frac{\pi x}{2}\right)-\frac{8}{\pi}\left(\frac{1}{2}\sin\frac{2\pi x}{2}\right)-\frac{8}{\pi}\left(\frac{1}{3}\sin\frac{3\pi x}{2}\right)-....$$

$$2x-4=-\frac{8}{\pi}\left(\sin\frac{\pi x}{2}+\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+.....\right).$$

Combination of cosine series

$$x=\frac{l}{2}-\frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+.....\right)$$

Given: $f(x)=2x-4,\quad l=4$

$$x=\frac{(4)}{2}-\frac{4(4)}{\pi^2}\left(\cos\frac{\pi x}{4}+\frac{1}{3^2}\cos\frac{3\pi x}{4}+\frac{1}{5^2}\cos\frac{5\pi x}{4}+.....\right)$$

$$x=2-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{4}+\frac{1}{3^2}\cos\frac{3\pi x}{4}+\frac{1}{5^2}\cos\frac{5\pi x}{4}+.....\right)$$

$$2x=2\left\{2-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{4}+\frac{1}{3^2}\cos\frac{3\pi x}{4}+\frac{1}{5^2}\cos\frac{5\pi x}{4}+.....\right)\right\}$$

$$=4-\frac{32}{\pi^2}\left(\cos\frac{\pi x}{4}+\frac{1}{3^2}\cos\frac{3\pi x}{4}+\frac{1}{5^2}\cos\frac{5\pi x}{4}+.....\right)$$

$$2x-4=4-\frac{32}{\pi^2}\left(\cos\frac{\pi x}{4}+\frac{1}{3^2}\cos\frac{3\pi x}{4}+\frac{1}{5^2}\cos\frac{5\pi x}{4}+.....\right)-4$$

$$2x-4=-\frac{32}{\pi^2}\left(\cos\frac{\pi x}{4}+\frac{1}{3^2}\cos\frac{3\pi x}{4}+\frac{1}{5^2}\cos\frac{5\pi x}{4}+.....\right)$$

15. Find a sine series and cosine series for the function $f(x) = x(l-x)$, $0 < x < l$.

Solution:

Given: $f(x) = x(l-x)$, $0 < x < l$.

$f(x) = xl - x^2$, $l = l$

Combination of sine series:

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - .....\right)$$

$$x^2 = \frac{2l^2}{\pi^3}\left((\pi^2-4)\sin\frac{\pi x}{l} - \frac{\pi^2}{2}\sin\frac{2\pi x}{l} + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin\frac{3\pi x}{l} - \frac{\pi^2}{4}\sin\frac{4\pi x}{l} .....\right)$$

$$xl = \frac{2l(l)}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - .....\right)$$

$$xl = \frac{2l^2}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - .....\right)$$

$$f(x) = xl - x^2 = \frac{2l^2}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - .....\right)$$

$$-\frac{2l^2}{\pi^3}\left((\pi^2-4)\sin\frac{\pi x}{l} - \frac{\pi^2}{2}\sin\frac{2\pi x}{l} + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin\frac{3\pi x}{l} - \frac{\pi^2}{4}\sin\frac{4\pi x}{l} .....\right)$$

$$= \sin\frac{\pi x}{l}\left(\frac{2l^2}{\pi} - \frac{2l^2}{\pi^3}(\pi^2-4)\right) + \sin\frac{2\pi x}{l}\left(\frac{2l^2}{\pi}\left[-\frac{1}{2}\right] - \frac{2l^2}{\pi^3}\left[-\frac{\pi^2}{2}\right]\right) + \sin\frac{3\pi x}{l}\left(\frac{2l^2}{\pi}\left[\frac{1}{3}\right] - \frac{\pi^2}{2}\left[\frac{\pi^2}{3} - \frac{4}{3^3}\right]\right) - ....$$

Therefore $f(x) = \dfrac{8l^2}{\pi^3}\left(\sin\dfrac{\pi x}{l} + \dfrac{1}{3^3}\sin\dfrac{3\pi x}{l} + \dfrac{1}{5^3}\sin\dfrac{5\pi x}{l} .....\right)$.

Combination of cosine series:

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + ...\right)$$

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + ...\right)$$

$$xl = \frac{l(l)}{2} - \frac{4l(l)}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + ...\right)$$

$$xl = \frac{l^2}{2} - \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + ...\right)$$

$$f(x) = xl - x^2 = \frac{l^2}{2} - \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + ...\right)$$

$$-\left[\frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + ...\right)\right]$$

$$f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} .....\right).$$