IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

DEPARTMENT OF MATHEMATICS

CLASS	**:**	**II B.Sc., MATHEMATICS**
SUBJECT NAME	**:**	**LINEAR ALGEBRA**
SUBJECT CODE	**:**	**16SCCMM8**
SEMESTER	**:**	**IV**
UNIT	**:**	**V (EIGEN VALUE AND EIGEN VECTOR)**
FACULTY NAME	**:**	**Mrs. S.ABIRAMI**

UNIT V

CHARACTERISTIC EQUATION

Characteristic Matrix:

Let A be any square matrix of order n and let I be the identity matrix of order n. Then the matrix polynomial given by A-I is called the Characteristic matrix of A.

Characteristic equation:

The equation $|A-\lambda I|=0$ is called the characteristic equation of A.

Characteristic Polynomial:

The determinant $|A-\lambda I|$ which is an ordinary polynomial in λ of degree n is called the characteristic polynomial of A.

EIGEN VALUE AND EIGEN VECTOR

Eigen value:

Solving the Characteristic equation $|A-\lambda I| = 0$, we get n values of λ and these n roots are called the eigen values or latent root or characteristic value of A.

Eigen vector:

Corresponding to each value of λ, the equation $(A-\lambda I)X = 0$ gives a non zero solution vector X. X is called the eigen vector or latent vector or characteristic vector of A corresponding to the eigen value.

Eigen value and Eigen vector based problems

Problem 1 :

Find eigen values and eigen vectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 1 + 5 + 1 = 7$

$s_2 = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$ =0

$s_3 = |A|$ = 4+2+3(-14) = -36

The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$

$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$

$(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$

$\lambda = -2, 3, 6$

The eigen values of A are -2,3,6

To find eigen vector:

(A- λI) X = 0

$$\begin{pmatrix} -\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ ---------- (1)}$$

Case 1: $\lambda = -2$ sub in (1)

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Here 1 and 3 equations are same So we take only one equation.

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$, \frac{x}{1-21} = \frac{y}{0} = \frac{z}{21-1}$$

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20}, \quad \frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

The eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Case 2: $\lambda = 3$ sub in (1)

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Select 1 and 2 equations..

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$, \frac{x}{1-6} = \frac{-y}{-2-3} = \frac{z}{-5}$$

$$\frac{x}{-5} = \frac{y}{5} = \frac{z}{-5}, \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

The eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Case 3: $\lambda = 6$ sub in (1)

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Select 1 and 2 equations..

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$, \frac{x}{1+3} = \frac{-y}{-5-3} = \frac{z}{5-1}$$

$$\frac{x}{4} = \frac{y}{8} = \frac{z}{4}, \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

The eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Answer:

Eigen values	Eigen vectors
$\lambda = -2$	$X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
$\lambda = 3$	$X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
$\lambda = 6$	$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Problem 2 :

Find eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 2 + 3 + 2 = 7$

$s_2 = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$ = 4+3+4 = 11

$s_3 = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$ = 8-2-1 = 5

The characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$(\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0$

$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$

$\lambda = 1,1,5$

The eigen values of A are 1, 1, 5

To find eigen vector:

(A- λI) X = 0

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{---------- (1)}$$

Case 1: $\lambda = 5$ sub in (1)

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Here three equations are different, select the first two Eqaution.

$$\frac{x}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{-y}{-1} = \frac{z}{4}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

The eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Case 2: $\lambda = 1$ sub in (1)

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

In the above matrix to get 3 equation. These equations are same, From this we take only one equation, x+2y+z=0,

Put x=0, 2y+z=0

$$\frac{y}{1} = \frac{z}{-2}$$

The eigen vector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

Put y=0, x+z=0

$$\frac{x}{1} = \frac{z}{-1}$$

The eigen vectors are $X_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Answer:

Eigen values	Eigen vectors
$\lambda = 5$	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\lambda = 1$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$
$\lambda = 1$	$X_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Properties of Eigen values:

1. A square matrix and its transpose A^T have the same eigen values
2. The sum of the eigen values of a matrix A is equal to the sum of the principal diagonal elements of A i.e trace of A
3. The product of eigen values of a matrix A is equal to the determinant of A.
4. If $\lambda_1 \lambda_2, \lambda_3, ...\lambda_n$ are eigen value of A, then

   i) $k\lambda_1, k\lambda_2, k\lambda_3, ...k\lambda_n$ are the eigen values of KA, where K is non zero scalar.

   ii) $\lambda_1^k \lambda_2^k, \lambda_3^k, ...\lambda_n^k$ are the eigen values of A^k, where k is a positive integer.

   iii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, ... \frac{1}{\lambda_n}$ are the eigen values of A^{-1}

Properties based problems

Problem 3: Find the sum and product of all eigen values of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & 3 \end{bmatrix}$

Solution:

Sum of the eigen values = Trace of A=1+0+3=4

Product of eigen values= |A|

= 1(0+3)-2(3+6)-2(-1) = 3-18+2= -13

Problem 4: Find the eigen values A^2 and A^{-1} given the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Solution:

Since A is a triangular matrix, the eigen values of A are 3,2,5

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}$

The eigen values of A^2 are $9,4,25$

Problem 5: If 3 and 15 are the eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 0 & 0 & 5 \end{bmatrix}$, find |A|

Solution:

If X in 3rd eigen values of A, then

3+15+X=8+7+3

X=0

|A| = product of eigen values of A = 3x15x0=0

Problem 6: If 3 and 6 are the eigen values of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, write down all the eigen values of A^{-1} , A^2 and 3A

Solution: If x is 3rd eigen value,

x+3+6 = 1+5+1

x=-2

The eigen values are 3, 6, -2

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{1}{-2}$

and the eigen values of A^2 are 9,36,4

and the eigen values of 3A are 9,18,-6

Problem 7: Find the eigen values of A-2I if $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Solution:

The characteristic equation of A is

$\lambda^2 - 2\lambda + 1 = 0$,

$(\lambda - 1)(\lambda - 1) = 0$

$\lambda = 1,1$

The eigen values of A are 1,1

The eigen values of A-2I are -1,-1

CAYLEY HAMILTON THEOREM

Statement

Any square matrix A satisfy its own Characteristic equation

Cayley Hamilton based problems

Problem 8: (CHT Problems)

Show that the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies its own characteristic equation and hence find A^{-1}

Solution: The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

S_1=2+2+2=6

S_2=3+3+3=9

S_3=6-1-1=4

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

We have to prove $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Cayley Hamilton theorem is verified

To find A^{-1}

Multiplying both sides of (1) by A^{-1}

$A^2 - 6A + 9I - 4A^{-1} = 0$

$4A^{-1} = A^2 - 6A + 9I$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Problem 9 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ prove that $A^3 - 3A^2 - 9A - 5I = 0$. Hence find A^4 and A^{-1}

Solution: Characteristic equation is $A^3 - 3A^2 - 9A - 5I = 0$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$$

$$A^3 - 3A^2 - 9A - 5I = \begin{bmatrix} 41 & 42 & 41 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix} + \begin{bmatrix} -27 & -24 & -24 \\ -24 & -27 & -24 \\ -24 & -24 & -27 \end{bmatrix} + \begin{bmatrix} -9 & -18 & -18 \\ -18 & -9 & -18 \\ -18 & -18 & -9 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 3A^2 - 9A - 5I = 0$ ----(1)

To find A^4:

Multitiplying (1) by A

$A^4 - 3A^3 - 9A^2 - 5A = 0$

$A^4 = 3A^3 + 9A^2 + 5A$

$$= \begin{bmatrix} 123 & 126 & 126 \\ 126 & 123 & 126 \\ 126 & 126 & 123 \end{bmatrix} + \begin{bmatrix} 81 & 72 & 72 \\ 72 & 81 & 72 \\ 72 & 72 & 81 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 209 & 208 & 208 \\ 208 & 209 & 208 \\ 208 & 208 & 209 \end{bmatrix}$$

To find A^{-1}:

Multitiplying (1) by A^{-1}

$A^2 - 3A^2 - 9I - 5A^{-1} = 0$

$-5A^{-1} = A^2 - 3A \;\; - 9I$

$$5A^{-1} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & -6 \\ -6 & -6 & -3 \end{bmatrix} + \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Try for :

1.Verify CHT for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

2.Using CHT for the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ find i) A^{-1} ii) A^2

3. Verify the statement that the sum of the elements in the diagonal of a matrix is the sum of the eigen values of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

4. Product of two eigen values of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the 3rd eigen value.

What is the sum of the eigen values of A?

5.Find the eigen values and eigen vectors of i) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

6. Find characteristic roots of the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$

7.Find the sum of squares of the eigen values of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

8.Find the Inverse of the matrix $A = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

9.Show that the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ satisfies the equation A(A-I)(A+2I)=0

10.Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$