IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM.

DEPARTMENT OF MATHEMATICS

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UNIT – V HARMONIC FUNCTIONS

6.1 Definition and Basic properties

1. Define Laplace equation

Definition: Laplace equation in Cartesian co-ordinates.

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Laplace equation in Polar co-ordinates.

$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

2. Define Harmonic function (or) Potential function.

Definition: A real valued function $u(z)$ or $u(x, y)$ defined and single valued in a region Ω is said to be harmonic in Ω if it is continuous together with its partial derivatives of the first two orders and satisfies the Laplace equation.

3. Define conjugate Harmonic.

Definition: Let $f = u + iv$ be an analytic function in a region D then v is said to be conjugate harmonic function of u.

4. Define conjugate differential.

Definition:The conjugate differential of du is $*du = dv$

(i.e.,) $dv = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy.$

5. Theorem : The real and imaginary part of an analytic functions are harmonic.

Proof: Let $f = u(x, y) + iv(x, y)$ be an analytic function.

Then u and v have continuous partial derivatives of first order which satisfy C-R equations given by $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Further $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial v}{\partial x}\right)$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

Thus u is a Harmonic function.

Similarly , we can prove v is a Harmonic function.

6. Show that log r is harmonic.

Proof: Let $u = log\, r$

$\frac{\partial u}{\partial r} = \frac{1}{r}$, $\frac{\partial u}{\partial \theta} = 0$

$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}$, $\frac{\partial^2 u}{\partial \theta^2} = 0$

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Therefore log r is harmonic.

7. If u_1 and u_2 are harmonic in a region Ω show that $\int_\gamma u_1 * du_2 - u_1 * du_2 = 0$ for every cycle γ which is homologous to zero in Ω.

Proof: Let u_1, u_2 be the harmonic in a region Ω , then u_1, u_2 have harmonic conjugates $* du_1, * du_2$ are harmonic conjugate differential of u_1 and u_2.

(i.e.,) $* du_1 = dv_1$, $* du_2 = dv_2$

$u_1 * du_2 - u_2 * du_1 = u_1 dv_2 - u_2 dv_1$

$$= v_1 du_2 + u_1 dv_2 - u_2 dv_1 - v_1 du_2$$

$$= u_1 dv_2 + v_1 du_2 - d(u_2 v_1) \qquad \rightarrow \textbf{(1)}$$

$$u_1 dv_2 + v_1 du_2 = I.P\ (u_1 + iv_1)(du_2 + idv_2)$$

. $F_1(z) = u_1 + iv_1$, $F_2(z) = u_2 + iv_2$

$F_2{}'(z) = du_2 + idv_2$

(1) becomes,

$$\int_\gamma u_1 * du_2 - u_2 * du_1 = \int_\gamma Im\,(F_1(z)F_2'{}'(z)dz - \int_\gamma d(u_2, v_1)$$

Since Im part of $F_1(z)$ and $F_2'{}'(z)$ is analytic.

By Cauchy's theorem, $\int_\gamma Im\,(F_1(z)F_2'{}'(z)dz) = 0$,S

Similarly, $\int_\gamma d(u_2, v_1) = 0$

Hence , $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0.$

6.2 The Mean –value property

8. Theorem: Mean value theorem for Harmonic function.

Statement: The arithmetic mean of a harmonic function over a concentric circles $|z| = r$ is a linear function of log r $\frac{1}{2\pi}\int_{|z|=r} u\, d\theta = \alpha \log r + \beta$ and if u is harmonic in a disk $\alpha = 0$ and arithmetic mean is constant.

Proof: Case (i) WKT, if u_1 and u_2 are harmonic in a region, then $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$,for all cycle γ which is homologous to zero in Ω.

Also we know that $* du = r\left(\frac{\partial u}{\partial r}\right) d\theta.$

Let us apply this with Ω, let the disk $0 < |z| < \rho$.

Take $u_1 = log\, r$ & $u_2 = u$ is harmonic in Ω.

Note that $log\, r$ is harmonic for γ

We take the cycle $C_1 - C_2$ where C_1 is the circle $|z| = r_1$ and C_2 is the circle $|z| = r_2$ with $0 < r_1 < r_2 < \rho$ described in the positive sense.

$$\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$$

$$\int_\gamma \log r * du - u * d(\log r) = 0$$

$$\int_{C_1 - C_2} \log r * du - u * d(\log r) = 0$$

$\int_{C_1} \log r * du - u * d(\log r) - \int_{C_2} \log r * du - u * d(\log r) = 0$

on the circle $|z| = r$.

$* du = r\left(\frac{\partial u}{\partial r}\right) d\theta$

$\int_{C_1} \log r\, r\left(\frac{\partial u}{\partial r}\right) d\theta - ur\left(\frac{\partial \log r}{\partial r}\right) d\theta - \int_{C_2} \log r\, r\left(\frac{\partial u}{\partial r}\right) d\theta - ur\left(\frac{\partial \log r}{\partial r}\right) d\theta = 0$

$\int_{C_1} \log r\, r\left(\frac{\partial u}{\partial r}\right) d\theta - \int_{C_1} u\, d\theta - \int_{C_2} \log r\, r\left(\frac{\partial u}{\partial r}\right) d\theta - \int_{C_2} u\, d\theta = 0$

In otherwords, it implies that the value of the integral is same.(i.e.,) It's constant on different path C_1 *and* C_2. Let it be β'

$\int_{C_1 - C_2} \log r\,.\, r\left(\frac{\partial u}{\partial r}\right) d\theta - \int_{C_2} u\, d\theta = \beta'$

$\int_{|z|=r} \log r\, r\, \frac{\partial u}{\partial r} d\theta - u\, d\theta = \beta'$ $\rightarrow$ **(1)**

Also , $\int_{\gamma} * du = 0.$

For any harmonic function u in Ω and for every cycle γ which is homologous to zero in Ω

$\int_{\gamma} * du = 0 \Longrightarrow \int_{C_1 - C_2} * du = 0$

$\int_{C_1} * du - \int_{C_2} * du = 0$

$\int_{C_1} r\left(\frac{\partial u}{\partial r}\right) d\theta - \int_{C_2} r\left(\frac{\partial u}{\partial r}\right) d\theta = 0$

$\int_{C_1} r\left(\frac{\partial u}{\partial r}\right) d\theta = \int_{C_2} r\left(\frac{\partial u}{\partial r}\right) d\theta$

Hence $\int_{|z|=r} r \frac{\partial u}{\partial r} \; d\theta$ is a constant say α'.

Thus $\int_{|z|=r} r \frac{\partial u}{\partial r} \; d\theta = \alpha'$ $\rightarrow$ **(2)**

(1) $\Rightarrow \int_{|z|=r} \log r\, r\, \frac{\partial u}{\partial r} d\theta - u\, d\theta = \beta'$

$logr\, \alpha' - \int_{|z|=r} u \; d\theta = \beta'$

$\int_{|z|=r} u \quad d\theta = logr\ \alpha' - \beta'$

Divide both sides by 2π

$\frac{1}{2\pi}\int u\, d\theta = \log r + \beta$ $\quad\rightarrow (\boldsymbol{A})$

where $\alpha = \frac{\alpha'}{2\pi}$ and $\beta = -\frac{\beta'}{2\pi}$

Thus the arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of log r.

Case (ii) If u is harmonic in whole disc then $\int_\gamma * du = 0$

$\Longrightarrow \int_\gamma r\frac{\partial u}{\partial r} \quad d\theta = 0$

$\int_{|z|=r} r\frac{\partial u}{\partial r} \quad d\theta = 0$

$\alpha' = 0 \Longrightarrow \alpha = 0$

Sub in (A),

$\frac{1}{2\pi}\int u\, d\theta = \beta$.

Hence if u is harmonic in a disc then the arithmetic mean is constant. The arithmetic mean is constant.

9. Theorem: Mean value theorem for Harmonic function.

Statement : A non constant harmonic function has neither a maximum nor a minimum in its region of definition . Consequently, the maximum and the minimum on a closed bounded set E are taken on the boundary of E.

Proof: If u is harmonic throughout the circular disc $|z| \le r$ then by the mean value property its arithmetic mean is constant.

$\frac{1}{2\pi}\int_{|z|=r} u(z) \quad d\theta = const = \beta = u(0)$

By Continuity $u(0) = \frac{1}{2\pi}\int_{|z|=r} u(z) \quad d\theta$

By change of origin to z_0 we have , $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta}) \quad d\theta$ $\quad\rightarrow (\boldsymbol{A})$

This shows that the value of $u(z)$ at the centre is z_0.

That is, $u(z_0)$ = The arithmetic mean of the values of $u(z)$ on the circumference .

Suppose $|u(z)|$ is maximum at z_0 in Ω then

$\left|u(z_0 + re^{i\theta})\right| \leq |u(z_0)|, 0 \leq \theta \leq 2\pi.$

Assume that this inequality is strict for a single value θ.

By continuing it will held on whole arc . This means that

$|u(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta\right|$

$\leq \frac{1}{2\pi}\int_0^{2\uparrow\pi} \left|u(z_0 + re^{i\theta})d\theta\right|$

$= \frac{1}{2\pi}\int_0^{2\pi}|u(z_0)|\ |d\theta|$

$= \frac{1}{2\pi}|u(z_0)|[\theta]_0^{2\pi}$

= v$|u(z_0)|$

$|u(z_0)| = |u(z_0)|$

This is a contradiction. Thus equality holds through

$$\frac{1}{2\pi}\int_0^{2\pi}\left|u(z_0 + re^{i\theta})\right|\ d\theta \ = \frac{1}{2\pi}\int_0^{2\pi}|u(z_0|\ d\theta$$

$$\int_0^{2\pi} |u(z_0)| - \left|u(z_0 + re^{i\theta})\right|\ d\theta \ = 0$$

Since the integral is non negative and continuous $|u(z_0| - \left|u(z_0 + re^{i\theta})\right|$=0

This equation holds an all circle $|z - z_0| = r$ and therefore $|u(z)|$ is constant in any neighbourhood of z_0.

Hence V is constant on Ω

This is a contradiction.

Then the maximum and minimum are taken on the boundary of E.

6.3 Poisson's Formula

10. Theorem: State and prove Poisson's Formula.

Statement :Suppose that $u(z)$ is harmonic for $|z| < R$ and continuous for $|z| \leq R$, then

(a) $u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2} u(z)d\theta$ for all $|a| < R$ → ***(A)***

(b) $u(a) = \frac{1}{2\pi}\int_{|z|=R} Re\left(\frac{z+a}{z-a}\right) u(z)d\theta$ → ***(B)***

(c) If we replace $a = re^{i\varphi}$ *and* $z = Re^{i\theta}$

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2u(Re^{i\theta})}{R^2-2rRcos\,(\theta-\varphi)+r^2}\, d\theta \quad \rightarrow (C)$$

Proof: $u(z)$ is harmonic for then $|z| \leq$ R .

Consider the linear transformation $z = S(t) = \frac{R\,(Rt+a)}{R+\bar{a}t} = \frac{R(a)}{R} = a.$ → **(1)**

$S(0) = a$ at $t = 0$.

Since $u(z)$ is harmonic $u(s(t))$ is also harmonic in $|t| = 1$.

From the Mean value property , $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(z) \quad d\theta$ → (2)

$$u(s(0)) = \frac{1}{2\pi}\int_{|t|=1} u(s(t))d(\arg t)$$

As $t = 0$ corresponds to $z = a$.

$S(t) = z \Rightarrow S(0) = a$

$u(a) = \frac{1}{2\pi}\int_{|t|=1} u(z)d(\arg t)$ → **(3)**

Now, $S(t) = z = \frac{R(Rt+a)}{R+\bar{a}t}$

$z(R + \bar{a}t) = R(Rt + a)$

$zR + z\overline{at} = R^2t + aR$

$zR - Ra = R^2a - z\bar{a}t$

$R(z-a) = t(R^2 - z\bar{a})$

$$t = \frac{R(z-a)}{R^2 - z\bar{a}} \qquad \rightarrow (4)$$

Next we compute $d(arg\ t)$ by putting $t = e^{i\varphi}$

$\log t = i\varphi$.

Differentiate with respect to 't'

$\frac{1}{t}\,dt = i\,d\varphi \Rightarrow -i\,\frac{dt}{t} = d(\arg t)$

From (4) $t = \frac{R(z-a)}{R^2 - z\bar{a}}$

Taking log on both sides,

$log\ t = log\,\frac{R(z-a)}{R^2 - z\bar{a}}$

$= \log R\,(z-a) - \log(R^2 - z\bar{a})$

$= \log R + log\,(z-a) - \log(R^2 - z\bar{a})$

Differentiate with respect to 'z'

$\frac{1}{t}\,\mathrm{dt} = 0 + \frac{1}{z-a}dz - \frac{1}{R^2 - z\bar{a}}(-a)dz$

$$\frac{dt}{t} = \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - z\bar{a}}\right) dz$$

Thus $d(\arg t) = -i\,\frac{dt}{t}$.

$$= -i\left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - z\bar{a}}\right) dz \qquad \rightarrow (A)$$

Put $= e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$

$-i\,dz = z\,d\theta$.

Substitute in (A),we get

$-i\frac{dt}{t} = z\,d\theta\,(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - z\bar{a}})d\theta$

$$d\,(\arg t) = \left(\frac{z}{z-a} + \frac{z\bar{a}}{R^2 - z\bar{a}}\right) d\theta$$

On substituting $R^2 = z\bar{z}$ then the co efficient of $d\theta$ in the last expression is

$$\left[\left(\frac{z}{z-a}\right) + \left(\frac{z\bar{a}}{R^2 - z\bar{a}}\right)\right] d\theta$$

$$= \left[\left(\frac{z}{z-a}\right) + \left(\frac{z\bar{a}}{R^2 - z\bar{a}}\right)\right] d\theta$$

$$= \left[\left(\frac{z}{z-a}\right) + \left(\frac{z\bar{a}}{z\bar{z} - z\bar{a}}\right)\right] d\theta$$

$$= \left[\left(\frac{z}{z-a}\right) + \left(\frac{\bar{a}}{\bar{z} - \bar{a}}\right)\right] d\theta$$

$$= \left[\frac{z(\bar{z}-\bar{a}) + \bar{a}(z-a)}{(z-a)(\bar{z}-\bar{a})}\right] d\theta$$

$$= \left[\frac{z\bar{z} - \bar{a}a}{(z-a)(\bar{z}-\bar{a})}\right] d\theta$$

$$= \left[\frac{z\bar{z} - \bar{a}a}{(z-a)(\bar{z}-\bar{a})}\right] d\theta$$

$$= \frac{R^2 - |a|^2}{|z-a|^2}\, d\theta$$

$$d\,(\arg t) = \frac{R^2 - |a|^2}{|z-a|^2}\, d\theta \qquad \rightarrow (5)$$

Sub (5) in (1),

$$u(a) = \frac{1}{2\pi} \int_{|t|=1} u(z)\; d(\arg t)\,.$$

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\; u(z) d\theta.$$

(b) Consider $\mathrm{Re}\left(\frac{z+a}{z-a}\right) = \dfrac{\frac{z+a}{z-a} + \frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}}{2}$

$$\mathrm{Re}\left(\frac{z+a}{z-a}\right) = \frac{z\,\bar{z} - a\bar{a}}{(z-a)(\bar{z}-\bar{a})}$$

Thus the poisson formula becomes,

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} Re\left(\frac{z+a}{z-a}\right)\; u(z) d\theta.$$

(c) $u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2}\, u(z)d\theta$

Put $a = re^{i\varphi}, z = Re^{i\theta}$,

$$u\left(re^{i\varphi}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\left|re^{i\varphi}\right|^2}{\left|Re^{i\theta}-re^{i\varphi}\right|^2}\, u\left(Re^{i\theta}\right)d\theta \qquad \to \textbf{(6)}$$

$$|a|^2 = r^2 \qquad \to \textbf{(7)}$$

$Re^{i\theta} - re^{i\varphi} = R\,(cos\theta + i\, sin\theta) - r(cos\varphi + i\, sin\varphi)$

$= R\, cos\theta - r\, cos\varphi + i(R\, sin\theta - rsin\varphi)$

$Re^{-i\theta} - re^{-i\varphi} = R\,(cos\theta - i\, sin\theta) - r(cos\varphi - i\, sin\varphi)$

$= R\, cos\theta - r\, cos\varphi - i(R\, sin\theta - rsin\varphi)$

$\left|Re^{-i\theta} - re^{-i\varphi}\right|^2 = (Re^{i\theta} - re^{i\varphi})\,(Re^{-i\theta} - re^{-i\varphi})$

$= [R(cos\theta + isin\theta) - r(cos\varphi + isin\varphi)]$

$[R(cos\theta - isin\theta) - r(cos\varphi - sin\varphi)]$

$= (R\, cos\theta - rcos\varphi)^2 + (R\, sin\theta - rsin\varphi)^2$

$= R^2 + r^2 - 2Rr[cos\theta cos\varphi + sin\theta sin\varphi]$

$$|z - a|^2 = R^2 + r^2 - 2Rr[\cos(\theta - \varphi)] \qquad \to \textbf{(8)}$$

Sub (7) & (8) in (6),

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2 u(Re^{i\theta})}{R^2-2rRcos\,(\theta-\varphi)+r^2}\, d\theta$$

6.4 Schwarz's Theorem

11.Define Poisson integral of U

Definition: If U(θ) is harmonic in $0 \le \theta \le 2\pi$ and piecewise continuous in $0 \le \theta \le 2\pi$ then we defined $P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\, \frac{e^{i\theta}+z}{e^{i\theta}-z}$ U(θ) dθ and this integral is called Poisson Integral of U.

12.Theorem :State and prove Schwarz's Theorem

Statement: The function $P_U(z)$ is harmonic for $|z| < 1$ and $\lim_{z\to e^{i\theta}} U(\theta_0)$ only if U is continuous of θ_0

Proof:

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\,\frac{e^{i\theta}+z}{e^{i\theta}-z}$$

$$P_U(z) = Re\{\frac{1}{2\pi}\int_0^{2\pi} Re\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\}$$

Let $t = e^{i\theta}$ then $U(t) = U(e^{i\theta})$

$$dt = i\ e^{i\theta}\ d\theta$$

$$\frac{dt}{it} = d\theta$$

Thus,

$$P_U(z) = Re\left\{\frac{1}{2\pi}\int_{|t|=1}\frac{t+z}{t-z}\ u(t)\frac{dt}{it}\right\}$$

$$P_U(z) = Re\left\{\frac{1}{2\pi i}\int_{|t|=1}\frac{t+z}{t-z}\ u(t)\frac{dt}{t}\right\}$$

$P_U(z)$ is an real part of an analytical function.

$$P_U(z) = Re\left\{\frac{1}{2\pi i}\int_{|t|=1}\frac{t+z}{t-z}\ u(t)\frac{dt}{t}\right\}+\text{ic}$$

Therefore it's harmonic

Let c_1, c_2 be two complementary arcs of unit circles such that $u_1 = u$ on c_1 , $u_1 = 0$ on c_2 and $u_2 = 0$ on c_1, $u_2 = u$ on c_2

(i.e) $u_1 = \begin{cases} u\ on\ c_1 \\ 0\ on\ c_2 \end{cases}$

$$u_2 = \begin{cases} 0\ on\ c_1 \\ u\ on\ c_2 \end{cases}$$

Then u $= u_1 + u_2$

This implies $P_U = P_{U_1} + P_{U_2}$ $\rightarrow$ **(1)**

By continuity U$|\theta_0|$= 0.

Now given t > 0

We can find c_1, c_2 such that $e^{i\theta_0}$ is a interior point of c_2 . $|u_2(\theta)| < \frac{\epsilon}{2}$ for $e^{i\theta} \epsilon c_2 \rightarrow (2)$

By continuity $u(\theta_0) = 0$ .Now given $t > 0$. We can find c_1, c_2 such that $e^{i\theta_0}$ is a interior point of c_2

$$P_{U_2}(z) = \frac{1}{2\pi} \int_0^{2\pi} Re \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u_2(\theta) d\theta$$

Further

$$\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = Re \left(\frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \frac{e^{-i\theta}+\bar{z}}{e^{-i\theta}-\bar{z}}\right)$$

$$= Re \left(\frac{e^{i\theta} \cdot e^{-i\theta} + ze^{-i\theta} - \bar{z}e^{i\theta} \cdot - z\bar{z}}{(e^{i\theta}-z)\ (e^{-i\theta}-\bar{z})}\right)$$

$$= \text{Re} \left(\frac{1-|z|^2+ze^{-i\theta}-\bar{z}e^{i\theta}}{|e^{i\theta}-z|^2}\right) \rightarrow (A)$$

$$ze^{-i\theta} - \bar{z}e^{i\theta} = (a+ib)\ (cos\ \theta - isin\ \theta) - (a-ib)\ (cos\ \theta + isin\ \theta)$$

$$= (a\ cos\ \theta - ia\ sin\theta + ib\ cos\ \theta + b\ sin\ \theta) - (a\ cos\ \theta - ia\ sin\theta + ib\ cos\ \theta + b\ sin\ \theta)$$

$$= 2ib\ cos\ \theta - 2ia\ sin\theta$$

$$= 2i(b\ cos\ \theta - a\ sin\theta)$$

$ze^{-i\theta} - \bar{z}e^{i\theta}$ is purely imaginary .

Equation (A) becomes

$$\text{Re} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = \frac{1-|z|^2}{|e^{i\theta}-z|^2} < 1 \qquad \rightarrow (\boldsymbol{B})$$

(i.e.,) multiplying by the conjugate both numerator and denominator has $|z| < 1$

$$|P_{U_2}(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |u_2(\theta)|\ d\theta \qquad \rightarrow (\mathbf{2})$$

$$\leq \frac{1}{2\pi} |u_2(\theta)| \int_0^{2\pi} d\theta$$

$\leq \frac{1}{2\pi}|u_2(\theta)|$

$= u_2(\theta)$

Therefore $\left|P_{U_2}(z)\right| \leq \frac{\epsilon}{2}$ $\rightarrow$ **(3)**

Since u_1 is continuous and vanishes at $e^{i\theta_0}$ there exist $\delta > 0$ such that

(2) implies

$\left|P_{U_2}(z)\right| \leq \frac{\epsilon}{2}$ for $\left|z - e^{i\theta}\right| < \delta$

$\left|P_{U_2}(z)\right| \leq \left|P_{U_1}(z)\right| + \left|P_{U_1}(z)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

(i.e.,) $\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$

6.5 Reflection Principle

13. Theorem : Let Ω^+ be the part in the upper half plane of a symmetric region Ω , and let σ be the part of the real axis in Ω. Suppose that $V(x)$ is continuous in $\Omega^+ \cup \sigma$,harmonic in Ω^+ and zero on σ , then V has a harmonic extension to which satisfies thesymmetry relation $V(\bar{z}) = -V(z)$. In the same situation , if V is the imaginary part of an analytic function f(z) in Ω^+ , then f(z) has an analytic extension which satisfies f(z) = $\overline{f(\bar{z})}$

Proof: Consider the function $V: \Omega \rightarrow C$ defined by

$$V(Z) = \begin{cases} V(z) if\ z \in \Omega^+ \\ 0\ if\ z \in \sigma \\ -V(z) if\ z \in \Omega^- \end{cases}$$

Where $\Omega^- = \{z \in \Omega: Im\ z < 0\}$

To prove that :V is harmonic in Ω.

For a point x_0 in σ. Consider a disk with centre x_0 contained in Ω and let P_v denote the poisson Integral with respect to this disk formed with bounded values v.

$$P_v(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) V(\theta) d\theta$$

From Schwarz theorem, the function $P_v(z)$ is harmonic for $|z| < 1$ and $\lim_{z \to e^{i\theta_0}} P_v(z) = v(\theta_0)$

Provided that V is continuous at $\theta_{0.}$

Consider the difference $V - P_{V.}$

$V - P_{V.}$ is harmonic in the upper half of the disc. $V - P_{V.}$ vanishes on the real diameter and consequently $P_{V.}$ is zero on the real diameter.

This together $V - P_{V.}$ is zero on the boundary of upper half circle. The maximum and minimum principle implies that $V - P_{V.} = 0$ in upper half disk. (i.e.,) $V = P_{V.}$ in the upper half disk.

Therefore $V(z)$ is harmonic in whole disk . Hence $V(z)$ is harmonic in Ω.

Also, $V(\bar{z}) = -V(z)$. Now assume that V is the imaginary part of analytic function $f(z)$ in Ω^+.

We have already extended V in the whole disk. Let $-u_0$ be the conjugate harmonic function of v in the same disk. We normalized so that $u_0 = Re(f(z))$ in upper half.

Consider $u_0(z) = \overline{u_0(z)} - u_0(\bar{z})$ on the real diameter. It's clear that $\frac{\partial u_0}{\partial x} = 0$ also $\frac{\partial v_0}{\partial y} = 2\left(\frac{\partial u_0}{\partial y}\right)$.

$$\Rightarrow \frac{\partial v_0}{\partial y} = 2\left(\frac{-\partial v}{\partial x}\right) = -2\left(\frac{\partial v}{\partial x}\right) = 0$$

It follows that the analytic function $\frac{\partial u_0}{\partial x} - i\frac{\partial u_0}{\partial y}$ vanishes on the real axis and hence identical.

Therefore u_0 is a constant and this constant is evidently zero.

$\therefore\ u_0(z) = u_0(\bar{z})$.

14. Theorem : State and prove the reflection principle (or) Symmetry principle.

Statement:

(i) If $u(z)$ is harmonic function in Ω then $u(\bar{z})$ is a harmonic function in Ω^*.

(ii) If $f(z)$ is analytic function in Ω then $\overline{f(\bar{z})}$ is an analytic function in obtained by reflecting Ω in real axis (i.e.,) $z \in \Omega^*$.if and only if $\bar{z} \in \Omega$.

Proof: consider the case of symmetric region $\Omega = \Omega^*$.

Since Ω is connected it must intersect the real axis along atleast one open interval.

Assume that f(z) is analytic in Ω and real on atleast one interval of the real axis.

Since $f(z) - \overline{f(\bar{z})}$ is analytic and vanishes on an interval it must be identically zero.

$f(z) - \overline{f(\bar{z})} = 0$

$f(z) = \overline{f(\bar{z})}$ in Ω.

We know that, $f = u + iv$

$u(z) = u(\bar{z})$, $v(z) = -v(\bar{z})$.

Ω^+ =The intersection of Ω in the upper half plane.

$\sigma =$ The intersection of Ω with real axis.

Suppose that $f(z)$ is defined on $\Omega^+ \cup \sigma$ analytic in Ω^+,continuous and real on σ

By symmetry condition $f(z) = \overline{f(\bar{z})}$.

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