# IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

## DEPARTMENT OF MATHEMATICS

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## UNIT V
## PART-A

1. **Define invariant subspaces with example.**

Let V be a vector space and T a linear operator on V. If W is a subspace of V, we say that W is invariant under T if for each vector α in W the vector Tα is in W, that if T(W) is contained in W.

Example: If T is any linear operator on V, then V is invariant under T, as is the zero subspace. The range of T and the null space of T are also invariant under T.

2. **Define Projection.**

If V is a vector space, a projection of V is a linear operator E on V such that $E^2 = E$.

3. **Write down the T-conductor of α into W.**

Let W be an invariant subspace for T and let α be a vector in V. The T-conductor of α into W is the set $S_T(\alpha;W)$, which consists of all polynomial g such that g(T)α is in W.

4. **When the subspaces be linearly independent ?**

Let $W_1,\dots .. W_k$ be subspaces of a vector space V. Then $W_1,\dots .. W_k$ are called linearly independent if $w_1 + \dots + w_k = 0$ for each $w_i \in W_i$ holds only for $w_1 = w_2 \dots\dots = w_k = 0$.

5. **Define direct sum.**

If $W_1,\dots ..,W_k$ are independent and if $W = W_1 + W_2 + \dots\dots + W_k$, then W is the direct sum of $W_1,\dots .. W_k$ and its denoted by $W = W_1 \oplus \dots\dots\dots \oplus W_k$

## Part -B

1. **Let V be a finite dimensional vector space V over the field F & let T be a linear operator on V. Then T is diagonalizable if and only if the minimal polynomial for T has the form p = $(x - c_1)...(x - c_k)$ where $c_1,...,c_k$ are distinct elements of F.**

**Proof**

If T is diagonalizable, its minimal polynomial is a product of distinct linear factors .

Conversely, let W be the subspace spanned by all of the characteristic vectors of T, and suppose W ≠ V. By the lemma, Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F. There is a vector α not in W and a characteristic value $c_j$ of T such that the vector β = $(T - c_jI)\alpha$ lies in W. Since β is in W, $\beta = \beta_1 + \beta_2 ........\beta_k$ where $T\beta_i = c_i\beta_i, 1 \le i \le k$ and therefore the vector $h(T)\beta = h(c_1)\beta_1 + h(c_2)\beta_2 ........ + h(c_k)\beta_k$ is in W, for every polynomial h. Now $p = (x - c_j)q$, for some polynomial $q$ and also $q - q(c_j) = (x - c_j)h$ and $q(T)\alpha - q(c_j)\alpha = h(T)(T - c_jI)\alpha = h(T)\beta$ . But h(T)β is in W and, Since $0 = p(T)\alpha = (T - c_jI)q(T)\alpha$ the vector $q(T)\alpha$ is in W. Therefore, $q(c_j)\alpha$ is in W. Since a is not in W, we have $q(c_j) = 0$. That contradicts the fact that p has distinct roots.

2. **If $V = W_1 \oplus ......... \oplus W_k$, then there exist k linear operators $E_1,.........E_k$ on V such that**

**(i) each $E_i$ is a projection $E_i^2 = E_i$**

**(ii) $E_i E_j = 0$, if $i \neq j$**

**(iii) $I = E_1 + ... + E_k$**

**(iv) the range of $E_i$ is $W_i$.**

**Conversely, if $E_1,........E_k$ are k linear operators on V which satisfy conditions (i), (ii), and (iii), and if we let $W_i$ be the range of $E_i$, then $V = W_1 \oplus .........\oplus W_k$.**

**Proof**

Suppose $E_1,........E_k$ are linear operators on V which satisfy the first three conditions, and let $W_i$ be the range of $E_i$. Then certainly $V = W_1 + . . . + W_k$ ;for, by condition (iii) and $\alpha = E_1 \alpha + . . . + E_k \alpha$ for each α in V, and $E_i \alpha$ is in $W_i$. This expression for a is unique, because if $\alpha=\alpha_1 +\ldots\ldots\alpha_k$ with $\alpha_i$ in $W_i$, say

$\alpha_i = E_i \beta_i$, then using (i) and (ii)

$$E_j\alpha = \sum_{i=1}^{k} E_j\alpha_j$$

$$E_j\alpha = \sum_{i=1}^{k} E_j E_i \beta_j$$

$$= E_j^2\beta_j$$

$$= E_j\beta_j$$

$$= \alpha_j$$

Hence V is the direct sum of the $W_i$.

**3. Let T be a linear operator on the space V, and let $W_1, . . . , W_k$ and $E_1, .... , E_k$ be k linear operator as in Theorem 2. Then a necessary and sufficient condition that each subspace $W_i$ be invariant under T is that T commute with each of the projections $E_i$, i.e., $TE_j = E_iT$, i = 1, . . . , k.**

**Proof.**

Suppose T commutes with each $E_i$. Let $y \in T(W_i)$.

Then $y=Tx$ where $x \in W_i$

$y=TE_ix$

$=E_iTx \in R(E_i)=W_i$

Conversely suppose that $T(W_i) \subseteq W_i$

Let $x \in V$ ; $x = E_1x + E_2x + \ldots\ldots E_kx$

$Tx \;\; = TE_1x + T\,E_2x + \ldots\ldots + TE_kx$

$= E_1y_1 + E_2y_2 + \ldots\ldots E_ky_k$

$$= \sum_{i=1}^{k} E_j y_j$$

$$E_i Tx = \sum_{j=1}^{k} E_j E_i y_j$$

$$= E_j y_j$$

$$= TE_i\, x$$

Hence $TE_i = E_iT$

**PART-C**

1. **Let T be a linear operator on a finite-dimensional space V. If T is diagonalizable and if $c_1, c_2, \ldots\ldots c_k$ are the distinct characteristic values of T, then there exist linear operators $E_1, \ldots, E_k$ on V such that**

 **(i) $T = c_1E_1 + \ldots + c_kE_k$**

 **(ii) $I = E_1 + \ldots + E_k$ ;**

 **(iii) $E_iE_j = 0,\ i \neq j$ ;**

 **(iv)** $E_i^2 = E_i \quad \forall i$

 **(v) the range of $E_i$ is the characteristic space for T associated with $c_i$ .**

 **Conversely, if there exist k distinct scalars $c_i, \ldots.., c_k$ and k non-zero linear operators $E_1, \ldots, E_k$ which satisfy conditions (i), (ii) , and (iii), then T is diagonalizable, $c_1, \ldots., c_k$ are the distinct characteristic values of T, and conditions (iv) and (v) are satisfied also.**

**Proof.**

Suppose that T is diagonalizable, with distinct characteristic values $c_1, \ldots, c_k$. Let Wi be the space of characteristic vectors associated with the characteristic value $c_i$ and $V = W_1 \oplus \ldots\ldots\ldots \oplus W_k$. Let $E_1, \ldots, E_k$ be the projections associated with this decomposition. Then (ii), (iii), (iv) and (v) are satisfied. To verify (i), proceed as follows. For each α in V,

$\alpha = E_1 \alpha + \ldots + E_k\alpha$

$T\alpha = TE_1\alpha + \ldots + TE_k\alpha$

$\quad = c_1E_1\alpha + \ldots + c_kE_k\alpha.$

In other words, $T = c_1E_1 + \ldots + c_kE_k$. Now suppose that we are given a linear operator T along with distinct scalars $c_i$ and non-zero operators $E_i$ which satisfy (i), (ii) and (iii). Since $E_iE_j = 0$ when $i \neq j$, we multiply both sides of $I = E_1 + \ldots + E_k$ by $E_i$ and obtain immediately $E^2_i = E_i$. Multiplying $T = c_1E_1 + \ldots + c_kE_k$ by $E_i$, then $TE_i = c_iE_i$, which shows that any vector in the range of $E_i$ is in the null space of $(T – c_i I)$. Since assumed that $E_i \neq 0$, this proves that there is a non-zero vector in the null space of $(T – c_iI)$, that $c_i$ is a characteristic value of T. Furthermore, the $c_i$ are all of the characteristic values of T; for, if c is any scalar, then $T - cI = (c_1 - c)E_1 + \ldots + (c_k - c)E_k$ so if $(T - cI)\alpha = 0$ and have $(c_i - c)E_i \alpha = 0$. If α is not the zero vector, then $E_i \neq 0$ for some i, that for this i ,to get $c_i - c = 0$. Certainly T is diagonalizable, since we have shown that every nonzero vector in the range of $E_i$ is a characteristic vector of T, and the fact that $I = E_1 + \ldots + E_k$ shows that these characteristic vectors span V. All that remains to be demonstrated is that the null space of $(T – c_iI)$ is exactly the range of $E_i$. But this is clear, because if $T\alpha = c_i\alpha$, then

$$\sum_{j=1}^{k}(c_j - c_i)E_j\alpha = 0$$

hence $(c_j - c_i)E_j\alpha = 0$ and then $E_j\alpha = 0;\ j \neq i$. Since $\alpha = E_1\alpha + \ldots + E_k\alpha$, and $E_j \alpha = 0$ for $j \neq i$, we have $\alpha = E_i\alpha$, which proves that a is in the range of $E_i$.

2. **State and Prove Primary decomposition theorem**.

**Let T be a linear operator on the finite-dimensional vector space V over the field F. Let p be the minimal polynomial for T, $p_1^{r_1} \dots . p_k^{r_k}$ where $p_i$ are distinct irreducible monic polynomials over F and the $r_i$ are positive integers. Let $W_i$ be the null space of $P_i(T)^{ri}$, i = 1, ...,k. Then**

$(i) V = W_1 \oplus \dots \oplus W_k$;

**(ii) each $W_i$ is invariant under T;**

**(iii) if $T_i$ is the operator induced on $W_i$ by T, then the minimal polynomial for $T_i$ is $P_i^{r1}$.**

**Proof**

For each I, $f_i = \prod_{j \neq i} p_j^{r_i}$ Since $p_i'$ s are distinct prime polynomials, the polynomial $f_i^{'}$ s are relatively prime. Thus there are polynomials $g_1, \dots g_k$ to show that the polynomials $h_i = f_i g_i$; $E_l + \dots + E_k = I$ and $E_i E_j =: 0$, if i # j. Thus the $E_i$ are projections which correspond to some direct-sum decomposition of the space V. To show that the range of $E_i$ is exactly the subspace $W_i$. It is clear that each vector in the range of $E_i$ is in $W_i$ , for if α is in the range of $E_i$, then α = Eα and so $p_i(T)^{ri}\alpha = p_i(T)^{ri} E_i \alpha = P_i(T)' f_i,(T) g_i(T)\alpha = 0$. Because $p^r f_i g_i$ is divisible by the minimal polynomial p.

Conversely, suppose that a is in the null space of $p_i(T)^{ri}$. If $j \neq i$, then $f_j g_j$ is divisible by $p_i$, i.e., $E_j \alpha = 0$ for $j \neq i$. But then it is immediate that Eα= α, i.e., that α is in the range of $E_i$. This completes the proof of statement (i). It is certainly clear that the subspaces $W_i$ are invariant under T. If $T_i$ is the operator induced on $W_i$ by T, then evidently $p_i(T_i)^{ri} = 0$, because by definition $p_i(T)$ is 0 on the subspace $W_i$. This shows that the minimal polynomial for $T_i$ divides $p_i$.

Conversely, let g be any polynomial such that $g(T_i) = 0$. Then $g(T) f_i(T) = 0$. Thus $gf_i$ is divisible by the minimal polynomial p of T:, i.e., $p_i^{ri} f_i$ divides $gf_i$.

It is easily seen that $p_i$ divides g. Hence the minimal polynomial for $T_i$ is $p_i^{ri}$.