IDHAYA COLLEGE FOR WOMEN , KUMBAKONAM

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Steady state heat conduction: Laplace equation

Laplace equation in two or three dimensions

usually arises in two types of physical problems:

1. As steady state heat conduction.
2. As equation of continuity for incompressible potential flow.

However, here we will emphasize only on the first type.

Steady state solution here means

1. the solution for large time.
2. the solution does not depend anymore on time.

Laplace equation

Laplace equation in two dimensions and three dimensions in Cartesian coordinates are respectively given by

$$u_{xx} + u_{yy} = 0, \quad (1)$$

$$u_{xx} + u_{yy} + u_{zz} = 0. \quad (2)$$

The above equations can be obtained from the two-dimensional and three-dimensional transient heat conduction equations when u does not depend on t.

Hence Laplace equation models

steady heat flow in a region where the temperature is fixed on the boundary.

Maximum Principle

Theorem: Let $u(x, y)$ satisfy Laplace's equation in D, an open, bounded, connected region in the plane; and let u be continuous on the closed domain $D \cup \partial D$ consisting of D and its boundary. If u is not a constant function, then the maximum and minimum values of u are attained on the boundary of D and nowhere inside D.

This is called maximum principle theorem for Laplace equation.

Steady state heat conduction in two dimensions

We consider steady state heat conduction

in a two-dimensional rectangular region.

To be specific,

consider the equilibrium temperature inside a rectangle $0 \le x \le a,\ 0 \le y \le b$.

Here

the temperature is a prescribed function of position on the boundary.

In general the Dirichlet BVP will be like

$$u_{xx} + u_{yy} = 0,\ 0 \le x \le a,\ 0 \le y \le b$$

$$u(0, y) = g_1(y),\ u(a, y) = g_2(y),\ 0 \le y \le b$$

$$u(x, 0) = f_1(x),\ u(x, b) = f_2(x),\ 0 \le x \le a.$$

Steady state heat conduction in two dimensions

where

$f_1(x), f_2(x), g_1(y), g_2(y)$ are given functions.

Though the equation is linear and homogenous,
the BCs are not homogenous.

Hence
the BVP is needed to be split into four BVPs with each containing
one non-homogenous BC.

Take

$$u = u_1 + u_2 + u_3 + u_4, \ 0 \leq x \leq a, \ 0 \leq y \leq b.$$

Steady state heat conduction in two dimensions

BVP I and BVP II:

$u_{1,xx} + u_{1,yy} = 0;$ $u_{2,xx} + u_{2,yy} = 0,\ 0 \le x \le a,\ 0 \le y \le b;$

$u_1(0, y) = 0,\ 0 \le y \le b;$ $u_2(0, y) = 0,\ 0 \le y \le b;$

$u_1(a, y) = 0,\ 0 \le y \le b;$ $u_2(a, y) = 0,\ 0 \le y \le b;$

$u_1(x, 0) = f_1(x),\ 0 \le x \le a;$ $u_2(x, 0) = 0,\ 0 \le x \le a;$

$u_1(x, b) = 0,\ 0 \le x \le a;$ $u_2(x, b) = f_2(x),\ 0 \le x \le a;$

BVP III and BVP IV:

$u_{3,xx} + u_{3,yy} = 0;$ $u_{4,xx} + u_{4,yy} = 0,\ 0 \le x \le a,\ 0 \le y \le b;$ $u_4(0,$

$u_3(0, y) = g_1(y),\ 0 \le y \le b;$ $y) = 0,\ 0 \le y \le b;$

$u_3(a, y) = 0,\ 0 \le y \le b;$ $u_4(a, y) = g_2(y),\ 0 \le y \le b;$

$u_3(x, 0) = 0,\ 0 \le x \le a;$ $u_4(x, 0) = 0,\ 0 \le x \le a;$

$u_3(x, b) = 0,\ 0 \le x \le a;$ $u_4(x, b) = 0,\ 0 \le x \le a.$

We will consider only one of them.......take $u_1 = u$ for

Steady state heat conduction in two dimensions

Consider the steady state heat conduction in a rectangular region $0 \leq x \leq a,\ 0 \leq y \leq b$

where three boundaries along $x = 0, x = a, y = b$ are kept at 0^0C

While the temperature along the boundary $y = 0$ is $f(x)$.

To find the temperature at any point (x, y).

Steady state heat conduction in two dimensions

BVP will consist of the following:

The governing equation is two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0,\ 0 \le x \le a,\ 0 \le y \le b. \quad (3)$$

The boundary conditions

$$\begin{aligned} u(0,y) &= 0,\ 0 \le y \le b, && (4a)\\ u(a,y) &= 0,\ 0 \le y \le b, && (4b)\\ u(x,0) &= f(x),\ 0 \le x \le a, && (4c)\\ u(x,b) &= 0,\ 0 \le x \le a. && (4d) \end{aligned}$$

It being a pure BVP and the solution being a function of x and y, obviously we will not have any initial conditions.

Hence
This problem is called a steady-state problem.

Steady state heat conduction in two dimensions

Assume a solution of the

$$u(x, y) = X(x)Y(y). \tag{5}$$

On separating the variables x and y,

$$\frac{X''}{X} = -\frac{Y''}{Y} = k(\text{say}).$$

Giving

$$X'' - kX = 0, \tag{6}$$

$$Y'' + kY = 0. \tag{7}$$

The zero and positive values of k will not give rise to solutions conforming to the boundary conditions.

Steady state heat conduction in two dimensions

We consider only the negative values of k, $-\lambda^2$, to write the equations (6) sayand (7) as

$$X'' + \lambda^2 X = 0, \quad (8)$$
$$Y'' - \lambda^2 Y = 0, \quad (9)$$

so that the solution $u(x, y)$ can be written

$$u(x, y) = (A \cos \lambda x + B \sin \lambda x)(C \cosh \lambda y + D \sinh \lambda y). \quad (10)$$

Using boundary condition (4a)

$A = 0$.

Using boundary condition (4b),

$$\lambda_n = \frac{n\pi}{a}$$

$n = 1, 2, 3, \ldots$

$$u_n(x, y) = \sin \frac{n\pi x}{a} \left(A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a} \right).$$

Steady state heat conduction in two dimensions

Similarly we can find the other solutions u_2, u_3 and u_4 and write the total solution as $u = u_1 + u_2 + u_3 + u_4$.

This problem with Dirichlet conditions along all boundaries is called a Dirichlet problem for a rectangle.

The problem with Neumann conditions along all boundaries is called a Neumann problem for a rectangle.

This new problem can be solved by writing the boundary conditions as

$$u_x(0, y) = 0, \quad (13a)$$

$$u_x(a, y) = 0, \quad (13b)$$

$$u_y(x, 0) = f(x), \quad (13c)$$

$$u_y(x, b) = 0. \quad (13d)$$

yourself.

Solution in different types of domains

Till now the problems that we have taken up are for bounded domains in Cartesian coordinates.

There are other types of domain which occur frequently in many physical problems.

We need to rewrite our governing equations and the related conditions.

Some domains of importance are

- circular domain
- spherical domain
- cylindrical domain.

Solution in different types of domains

For two dimensional problem with (x, y)

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

For three dimensional problem with (x, y, z)

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

For two dimensional problem with (r, θ), that is, in polar

$$\nabla^2 u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Laplace's equation in polar coordinates

For a problem involving circular disk, polar coordinates are more appropriate than rectangular coordinates.

Let us formulate the steady-state heat flow problem in polar coordinates r, θ, where $x = r\cos\theta, y = r\sin\theta$.

A circular plate of radius a can be simply represented by $r \leq a$ with $0 \leq \theta \leq 2\pi$.

The unknown temperature inside the plate is now $u = u(r, \theta)$,

The given temperature on the boundary of the plate is $u(a, \theta) = f(\theta)$, where f is a known function.

Laplace's equation in polar coordinates

What we have just solved is called

Interior Dirichlet problem for a circle since we have used Dirichlet condition find the region $r \leq a$.

If we change only the condition to Neumann

we have what is called Interior Neumann problem for a circle.

Now if we change the region to $r > a$ with Dirichlet condition
we have what is called Exterior Dirichlet problem for a circle.

If in above condition is replaced by Neumann
we have what is called Exterior Neumann problem for a circle.

Diffusion in a disk

Consider a circular, planar disk of radius a for which

- initial temperature is a function of the radial distance r alone
- boundary is held at zero degrees.

Intuition tells that

the temperature u in the disk depends only on time and the distance r from the centre.

To be precise, the initial temperature u will be the same for some $r = r_n$ irrespective of what value of θ is assigned.

That is, if the initial temperature u is, say, u_1 for some $r = r_1$, then it is imperative that the temperature is same for that specific r.

Consider now that the initial temperature u is, say, u_2 for some $r = r_2$, then it is imperative that the temperature is same for that specific r.

which, in turn means that

the heat flow will take place either from $r = r_1$ to $r = r_2$ or from $r = r_2$ to $r = r_1$ depending on which has higher temperature.

Diffusion in a disk

It then allows us to consider u simply as

$u = u(r, t)$ though the given equation initially seemed to contain r, θ, t as independent variables.

This assumption looks perfectly alright

because there is nothing in the initial condition or boundary condition to cause heat to diffuse in an angular direction

Heat will flow only along rays emanating from the origin.

Now it is obvious that

the diffusion equation looks simpler than what it was originally.

Diffusion in a disk

Diffusion equation is now as simple as follows:

$$u_t = \alpha\left(u_{rr} + \frac{1}{r}u_r\right), \quad 0 \le r \le a, \; t > 0. \tag{1}$$

Boundary

$$u(a, t) = 0, \; t > 0, \tag{2}$$

Initial

$$u(r, 0) = f(r), \quad 0 \le r < a, \tag{3}$$

f is a given initial radial temperature distribution.

Diffusion in a disk

Assume a solution in the form:

$$u(r,t) = R(r)T(t).$$

From given equation

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{T'}{\alpha T} = k.$$

$k = -\lambda^2$ gives rise to the following pair of ODEs:

$$r^2R'' + rR' + \lambda^2 r^2 R = 0, \quad (4)$$

$$T' + \alpha\lambda^2 T = 0. \quad (5)$$

Equation can be easily solved to write as

$$T(t) = Ce^{-\alpha\lambda^2 t}. \quad (6)$$

Can you recognize equation

Diffusion in a disk

Bessel's equation of order 0.

Its solution can be written as

$$R(r) = AJ_0(\lambda r) + BY_0(\lambda r), \quad (7)$$

where

J_0 and Y_0 are, respectively, Bessel's function of first kind and second kind of order zero.

We are looking for a bounded solution as $r \to 0$,

we must take $B = 0$ as $Y_0(\lambda r) \to -\infty$ as $r \to 0$.