IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

DEPARTMENT OF MATHEMATICS

SUBJECT NAME	**:**	**FUNCTIONAL ANALYSIS**
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IMPORTANT QUESTIONS FROM UNIT I - UNIT V

Definition 1 Let N and $N^{'}$ be normed linear space and let $T : N \rightarrow N^{'}$ be a mapping with domain N and the range in $N^{'}$. The graph of T is defined to be a subset of $NXN^{'}$ which consist of all ordered pairs $(x, T(x))$. If is denoted by GT

$GT = \{(x, T(x))/x \in N\}$

Theorem 2

Statement

If B and $B^{'}$ are Banach spaces and if T is a linear transformation of B into $B^{'}$, then T is continuous iff its graph is closed.

That is, GT is closed.

Proof

Given that B and $B^{'}$ are Banach space and $T : B \rightarrow B^{'}$ is linear transformation.

Suppose GT is closed.

We have to prove T is continuous.

Let B_1 be given Banach space renamed by $\|\|$.

$$\begin{aligned} \| x \| &= \| x \| + \| T(x) \| \\ \| T(x) \| &\leq \| x \| + \| T(x) \| \\ &= \| x \| \\ \Rightarrow \| T(x) &\leq \| x \| \end{aligned}$$

Therefore T is bounded from B_1 to $B^{'}$.

Therefore It is continuous from B_1 to $B^{'}$.

Now we have to prove $T : B \to B^{'}$ is continuous.

It is sufficient to prove B_1 and B have the same topology.

It is enough if we prove B_1 and B are homeomorphic.

Let $I : B_1 \to B$ defined by

$\parallel I(x) \parallel = \parallel x \parallel, \forall x \in B_1$

Therefore I is always 1-1 and onto.

$$\begin{aligned}
\parallel I(x) \parallel &= \parallel x \parallel \\
&\leq \parallel x \parallel + \parallel T(x) \parallel \\
&= \parallel x \parallel \\
\Rightarrow \parallel I(x) \parallel &= \parallel x \parallel
\end{aligned}$$

$\Rightarrow I$ is bounded and it is continuous .

Therefore B_1 and B are homeomorphic.

$\Rightarrow B_1, B$ are having same topology.

Therefore B_1, B are homeomorphic.

T is continuous linear transformation from $B \to B^{'}$.

Now we have to show that B_1 or B is complete under $\parallel\parallel$.

Let $\parallel \{x_n\}$ be a cauchy sequence in B.

Given $\epsilon \geq 0$, there exists a positive integer n_0 such that

$$\begin{aligned}
\parallel x_n - x_m \parallel + \parallel T(x_n - x_m) \parallel &< \epsilon, \forall n, m \geq n_0 \\
\Rightarrow \parallel x_n - x_m \parallel + \parallel T(x_n - x_m) \parallel &< \epsilon \\
\Rightarrow \parallel x_n - x_m \parallel &< \epsilon, \forall\ n, m \geq n_0\ or \\
\parallel T(x_n - x_m) \parallel &< \epsilon
\end{aligned}$$

$\{x_n\}$ is a cauchy's sequence in B. $x_n \to x$ as $n \to \infty$

Therefore $x_n \to x$ and $T(x_n) \to T(x)$. $\parallel T(x_n) - T(x) \parallel < \epsilon$

$$\begin{aligned} Therefore \parallel x_n - x \parallel + \parallel T(x_n) - T(x) \parallel &< 2\epsilon, n \geq n_0 \\ \parallel x_n - x \parallel + \parallel T(x_n - T(x) \parallel &< 2\epsilon \\ \parallel x_n - x \parallel &< 2\epsilon, \forall\ n \geq n_0 \end{aligned}$$

That is B_1 is complete.

Therefore T is continuous linear transformation from $B \to B^{'}$

Conversely, $T : B \to B^{'}$ is continuous.

We have to prove GT is closed.

It is sufficient to prove that $GT = \overline{GT}$.

$GT \subset \overline{GT}$ is always true ————(1)

We have to prove $\overline{GT} \subset GT$.

Let $(x, y) \in \overline{GT}$

Then there exists a sequence $(x_n, T(x_n))$ in GT such that $(x_n, T(x_n)) \to (x, y)$

$\Rightarrow x_n \to x$ and $T(x_n) \to y$

T is continuous.

Therefore$x_n \to x$ and$T(x_n) \to T(x)$ as $n \to \infty$

$\Rightarrow y = T(x)$

That is, $(x, y) = (x, T(x))$

$(x, y) \in GT$

Therefore $\overline{GT} \subset GT$ ————(1)

Therefore from (1) and (2)$GT = \overline{GT}$.

Therefore GT is closed.

Hence the theorem.

Theorem 3

State and prove Parallelogram law in a Hilbert space.

Statement

If x and y are any two vectors in a Hilbert space, then $\| x+y \|^2 + \| x-y \|^2 = 2(\| x \|^2 + \| y \|^2)$.(OR)

The sum of the squares of the sides equals the sum of the squares of its diagonals.

Proof

$$\begin{aligned}
\| x+y \|^2 + \| x-y \|^2 &= (x+y, x+y) + (x-y, x-y) \\
&= (x,x) + (x,y) + (y,x) + (y,y) + (x,x) - (x,y) - (y,x) + (y,y) \\
&= 2 \| x \|^2 + 2 \| y \|^2 \\
&= 2(\| x \|^2 + \| y \|^2) \\
\| x+y \|^2 + \| x-y \|^2 &= 2(\| x \|^2 + \| y \|^2)
\end{aligned}$$

Theorem 4

Prove that $M_i^{'}$s span H.

Proof

We Know that $M_i^{'}$s are closed linear subspaces of H and since there are pair wise orthogonal by a Known theorem.

$M = M_1 + M_2 + M_3 + ... + M_m$ is a closed linear space of H.

Let $P_1 + P_2 + P_3 + ... + P_m$ be its associated projections.

Since M_i reduces T, by a known theorem

$TP_i = P_iT$, for each P_i

Now, $TP = \sum TP_i = \sum P_iT = PT$

Hence M also reduces T.

Consequently $M^{\perp}$ is invariant under T.

Let if possible $M^{\perp} \neq (0)$.

Now, since all the eigen vector of T are in M, the restriction of T to $M^{\perp}$ is an operator on a non trivial finite dimensional Hilbert which has no eigen vectors and hence no eigen values.

By a Known result, this situation is impossible.(The set if all eigen values cannot be empty).

Hence we conclude that $M^{\perp} = (0)$ and $M = H$.

Thus M_i's span H

Theorem 5

Let H be a Hilbert space. Let $y \in H$ be a fixed vector.Define a function $f_y : H \to C$ by $f_y(x) = (x, y)$, then $f_y \in H^*$. That is, f_y is a continuous linear functionals on H.

Proof

$y \in H$ is a fixed vector

$f_y : H \to C$ by

$$f_y(x) = (x, y) \forall x \in H$$

For, $\alpha, \beta \in C$ and $x_1, x_2 \in H$.

$$\begin{aligned} f_y(\alpha x_1 + \beta x_2) &= (\alpha x_1 + \beta x_2, y) \\ &= \alpha(x_1, y) + \beta(x_2, y) \\ &= \alpha f_y(x_1) + \beta f_y(x_2) \\ f_y(\alpha x_1 + \beta x_2) &= \alpha f_y(x_1) + \beta f_y(x_2) \end{aligned}$$

Therefore f_y is linear.

Further $| f_y(x) | = | (x, y) | \leq \| x \| \| y \|$

$\| f_y \| \leq \| y \|$

Therefore f_y is a bounded functional and hence continuous.

Therefore f_y is a continuous linear functional.

Therefore $f_y \in H^*$

Theorem 6

Prove that in any Banach algebra the multiplication is jointly continuous.

Proof

Given that A is Banach algebra.

Now, we have to prove the multiplication is jointly continuous.

It suffices to prove that, if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$

$\| x_n - x \| < \epsilon$ and $\| y_n - y \| < \epsilon$ for $n > n_0, \epsilon > 0$

$$\begin{aligned} \| x_n y_n - xy \| &= \| x_n y_n - x_n y + x_n y - xy \| \\ &= \| x_n(y_n - y) + y(x_n - x) \| \\ &\leq \| x_n \| \| y_n - y \| + \| y \| \| x_n - x \| \end{aligned}$$

But $x_n \to x$ and $y_n \to y$ as $n \to \infty$

Therefore $\| x_n y_n - xy \| \to xy$

Therefore $x_n y_n \to xy$

Definition 7

An element z in a Banach algebra A is called a topological divisors of zero if there exist a sequence $\{z_n\}$ in A such that $\| z_n \| = 1$ and either $z_n z \to 0$

Definition 8

The spectrum of X is denoted by $\sigma(x)$ and it is the subset of complex plane defined by

$\sigma(x) = \{\lambda : X \to \lambda I \text{ is singular } \}$

Definition 9

Radical of A is denoted by R and it is the intersection of all its maximal ideals. That is, $R = \cap\{M; M \in A\}$, where A is the set of all Maximal ideals of A.

Definition 10

The $r(x)$ is defined by $r(x) = sup\{| \lambda | : \lambda \in \sigma(x)\}$ is called spectral radius of

x.

Definition 11

The algebra A is called semi simple of the radical consist of the zero vector alone.

Theorem 12

If I is proper closed two sided ideal in A, then quotient algebra A/I is a Banach algebra.

Proof

Given that A is a Banach algebra and I is a proper closed ideal.

$\Rightarrow A/I$ is well defined.

A/I is a linear space, for all $x \in A$

$x + I \in A/I$

$\| x + I \|$ is defined by

$\| x + I \| = inf\{\| x + i \|; i \in I\}$

Stage (1)

In this stage, we have to prove A/I is a Normed linear space.

(i) $\| x + I \| = inf\{\| x + i \|; i \in I\}$, since $x \in A$ and $i \in I, x + i \in A$.

A is normed linear space.

$\| x + I \| \geq 0$

Which implies $\| x + I \| = inf\{\| x + i \|; i \in I\} \geq 0$

$$\begin{aligned}
(ii) \| (x + I) + (y + I) \| &= inf\{\| (x + i_1) + (y + i_2) \|; i_1, i_2 \in I\} \\
&\leq inf\{\| x + i_1 \| + \| y + i_2 \|; i_1, i_2 \in I\} \\
\therefore \| (x + I) + (y + I) \| &\leq inf\{\| (x + i_1) \|; i_1 \in I\} + inf\{\| (y + i_2) \|; i_2 \in I\} \\
\Rightarrow \| (x + I) \| + \| (y + I) \| &\leq \| x + I \| \| y + I \|
\end{aligned}$$

$$
\begin{aligned}
(iii) \parallel \alpha(x+I) \parallel &= inf\{\parallel \alpha(x+i) \parallel; i \in I\} \\
&= inf\{\parallel \alpha x + i \parallel; i \in I\} \\
&= inf\{\mid \alpha \mid\mid x+i \mid; i \in I\} \\
&= \mid \alpha \mid inf\{\parallel x+i \parallel; i \in I\} \\
&= \mid \alpha \mid\parallel x+I \parallel \\
\parallel \alpha(x+I) \parallel &= \mid \alpha \mid\parallel x+I \parallel
\end{aligned}
$$

Stage (2)

Now, in this stage, we have to prove A/I is a Banach space.

Let $\{x_n + I\}$ is a cauchy sequence in $A/I, x_n \in \forall\forall n$

Given $\epsilon > 0$, there exist $n \geq n_0$ such that

$$
\begin{aligned}
\parallel (x_n + I) - (x_m) \parallel &< \epsilon \\
&\Rightarrow inf\{\parallel (x_n + I) - (x_m + I) \parallel; i \in I\} < \epsilon \\
&\Rightarrow inf\{\parallel (x_n - x_m) + I \parallel; i \in I\} < \epsilon \\
&\Rightarrow \parallel x_n - x_m \parallel < \epsilon \forall\ n, m \geq n_0
\end{aligned}
$$

Since A is complete, $x_n \to x$ as $n \to \infty$

$inf\{\parallel (x_n - x + i) \parallel; i \in I\} < \epsilon \forall\ n \geq n_0$

$inf\{\parallel (x_n + i) - (x + i) \parallel; i \in I\} < \epsilon$

$\parallel (x_n + I) - (x + I) \parallel < \epsilon$

$\Rightarrow \parallel (x_n + I) - (x + I) \parallel \to 0$ as $n \to \infty$

$\{x_n + I\} \to \{x + I\}$

Therefore A/I is complete.

Which implies A/I is a Banach space.

Stage (3)

In this stage, we have to prove A/I is an algebra.

Let $(x+I),(y+I) \in A/I$

Now, $(x+I),(y+I) = xy+I$

$$\begin{aligned}
\| (x+I)(y+I) \| &= \| xy+I \| \\
&= inf\{\| xy+i \|, i \in I\} \\
&= inf\{\| (x+i_1)(y+i_2) \|, i_1, i_2 \in I\} \\
&\leq inf\{\| x+i_1 \|, i_1 \in I\} inf\{\| y+i_2 \|, i_2 \in I\} \\
&\leq \| x+I \| \| y+I \| \\
\| (x+I)(y+I) \| &\leq \| x+I \| \| y+I \|
\end{aligned}$$

Note that A/I is a Banach algebra with identity $e+I$.

$$\begin{aligned}
\| e+I \| &= inf\{\| e+i \|, i \in I\} \\
&\leq \| e \| = 1 \\
\| e+I \| &\leq 1 - - - - - - - - - (1) \\
\| e+I \| &= \| ee+I \| \\
&= \| (e+I)(e+I) \| \\
&\leq \| e+I \| \| e+I \| \\
\| e+I \| &\leq \| e+I \| \| e+I \| \\
1 &\leq \| e+I \|
\end{aligned}$$

$i.e., \| e+I \| \geq 1 - - - - - - - - - - (2)$

From (1) and (2) $\| e+I \| = 1$

Hence from stage(1), stage(2), stage(3) A/I is a Banach Algebra.

Theorem 13

If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other than $N_1 + N_2$ and $N_1 N_2$ are normal.

Proof

N_1 and N_2 are normal operators.

$N_1 N_1^* = N_1^* N_1$ ——-(1)

$N_2 N_2^* = N_2^* N_2$ ——-(2)

Further it is given that

$N_1 N_2^* = N_2^* N_1$

$N_2 N_1^* = N_1^* N_2$

$$\begin{aligned}
(N_1 + N_2)(N_1 + N_2)^* &= (N_1 + N_2)(N_1^* + N_2^*) \\
&= N_1 N_1^* + N_2 N_1^* + N_1 N_2^* + N_2 N_2^* \\
&= N_1^* N_1 + N_2^* N_1 + N_1^* N_2 + N_2^* N_2 \\
&= (N_1^* + N_2^*)(N_1 + N_2) \\
&= (N_1 + N_2)^*(N_1 + N_2) \\
(N_1 + N_2)(N_1 + N_2)^* &= (N_1 + N_2)^*(N_1 + N_2)
\end{aligned}$$

Therefore $N_1 + N_2$ is a normal operator. Now,

$$\begin{aligned}
(N_1 N_2)(N_1 N_2)^* &= (N_1 N_2)(N_1^* N_2^*) \\
&= (N_1(N_2 N_2^*) N_1^*) \\
&= N_1(N_2^* N_2) N_1^* \\
&= (N_1 N_2^*)(N_2 N_1^*) \\
&= N_2^*(N_1^* N_1) N_2) \\
&= (N_2^* N_1^*)(N_1 N_2) \\
(N_1 N_2)(N_1 N_2)^* &= (N_2^* N_1^*)(N_1 N_2)
\end{aligned}$$

Therefore N_1N_2 is a normal operator.

Theorem 14

The mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homomorphism of G onto itself.

Proof

Given that $x \to x^{-1}$ is a mapping from G into G.

Let $x_0 \in G$ and $\{x_n\}$ be a sequence in G such that $x \to x^{-1}$ as $n \to \infty$.

Now, we have to prove that $x \to x^{-1}$ is continuous.

It is enough if we prove $x \to x^{-1}$

$$\begin{aligned}
\| x_n^{-1} - x_0^{-1} \| &= \| x_n^{-1}x_0x_0^{-1} - x_n^{-1}x_nx_0^{-1} \| \\
&= \| x_n^{-1}(x_0 - x_n)x_0^{-1} \| \\
&\leq \| x_n^{-1} \| \| x_0 - x_n \| \| x_0^{-1} \| \\
\| x_n^{-1} - x_0^{-1} \| &\leq \| x_n^{-1} \| \| x_0 - x_n \| \| x_0^{-1} \| - - - - (1)
\end{aligned}$$

Since $x_n \to x$.

Therefore given $\epsilon > 0$, there exists n_0 such that $\| x_n - x_0 \| < \epsilon, \forall n > n_0$

We take $\epsilon = \frac{1}{2\|x_0^{-1}} \|$

$\| x_n - x_0 \| \leq \frac{1}{2\|x_0^{-1}} \|$ ——(2)

Consider,

$$\begin{aligned}
\| e - x_0^{-1}x_n \| &= \| x_0^{-1}x_0 - x_0^{-1}x_n \| \\
&= \| x_0^{-1}(x_0 - x_n)) \| \\
&\leq \| x_0^{-1} \| \| x_0 - x_n \| \\
\| e - x_0^{-1}x_n \| &\leq \| x_0^{-1} \| \| x_0 - x_n \| - - - - - - - - - (3)
\end{aligned}$$

Substitute (2)in (3), we get

$\| e - x_0^{-1}x_n \| \leq \| x_0^{-1} \| \frac{1}{2\|x_0^{-1}} \| \leq \frac{1}{2}$

That is, $\| e - x_0^{-1}x_n \| \leq \frac{1}{2}$

$(x_0^{-1}x_n)^{-1} = e + \sum(e - x_0^{-1}x_n)^n$

$x_n^{-1}x_0 = e + \sum(e - x_0^{-1}x_n)^n$

$$\begin{aligned}
\| x_n^{-1}x_0 \| &= \| e + \sum(e - x_0^{-1}x_n)^n \| \\
&\leq \| e \| + \sum \| e - x_0^{-1}x_n) \|^n \\
&\leq 1 + (e - x_0^{-1}x_n) + (e - x_0^{-1}x_n)^2 + ... \\
&= \frac{1}{1 - (e - x_0^{-1}x_n)} \\
&= \frac{1}{1 - \frac{1}{2}} \\
&= \frac{1}{\frac{1}{2}} \\
&= 2 \\
\text{Therefore } \| x_n^{-1}x_0 \| &\leq 2 \\
\| x_n^{-1} \| &= \| x_n^{-1}x_0x_0^{-1} \| \\
&\leq \| x_n^{-1}x_0 \| \| x_0 \| \\
\| x_n^{-1} &\leq 2 \| x_0^{-1} \| - - - - - - - - - - (4)
\end{aligned}$$

Sub(2)and (4) in 1, we get

$$\begin{aligned}
\| x_n^{-1} - x_0^{-1} \| &\leq \| x_n^{-1} \| \| x_0 - x_n \| \| x_0^{-1} \| \\
&\leq 2 \| x_0^{-1} \| \frac{1}{\frac{1}{2\|x_0^{-1}\|}}
\end{aligned}$$

AS $n \to \infty, \| x_n - x_0 \| \to 0.$

So $\| x_n^{-1} - x_0^{-1} \to 0$

That is, $x_n^{-1} \to x_0^{-1}$

Therefore mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homomorphism of G onto itself.