IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

PG & RESEARCH DEPARTMENT OF MATHEMATICS

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UNIT – V

Differential Geometry Of Surfaces In The Large :

Different geometry of surfaces in the large is the study of relations between the local and global proportions of surfaces.

Compact surfaces whose points are umbilics.

For the first few theorems we will the definitionof surface given and assume that each point has a neighborhoodhomeomorphism to an open 2-cell which can be described by parametric equation $\bar{r} = \bar{r}(u, v)$

THEOREM :

The only compact surfaces of class$\geq 2$for which every point is an umbilic are spheres.

We shall prove that in the neighborhoodof any point the surface is either spherical or plane.

We use the property of compacts surface of class ≥ 2 for which every point is an umbilic.

Let p be any pt on s, and let v be a coordinate neighborhood of s containing p, in which part of s is represented parametrically by $\bar{r} = \bar{r}(u, v)$.

Since point of v is an umbilic it follows that every curve lying in v, must be a line of curvature.

Hence from rodrigues formula at all points of v

$$dN + kdr = 0 \qquad \rightarrow (1)$$

Where k is the normal curvature of s in the direction dr.

$$dN = -kdr \;\; , \quad \int dr = \int \frac{dN}{-k} \Rightarrow r$$

$\bar{N}_1 = -kr_1$and$\bar{N}_2 = -kr_2$

Using the identify $N_{12} = N_{21}$ with the abve equations we get

Since, $k_2 r_1 - k_1 r_2 = 0$

$r_1$andr_2 are linearly independent. We obtain

$k_1 = k_2 = 0$, so that k is a compact surface.

Integrating (1) we get

$$r = a - K^{-1} N \;\rightarrow (2)$$

For $k \neq 0$ showing that v lies n the surface of a sphere of centre$\bar{a}$ and radius k^{-1}.

When k=0, (1) gives $\bar{N} = \bar{b} \rightarrow (3)$

Showing that v lies on the Plane. This completes the local part of the theorem. i,e., so far all we have proved is that in the neighborhood f any point the surface is spherical or plane.

Associate with each point p on the surface a neighborhood v, having the above property.

The set f all neighborhoods V_p covers S and from the compactness we conclude that S is covered by a finite subcover formed by V_i (i =1,2,…N)

Consider two over lapping neighborhoodv_i,v_j, from the previous local agreement it follows that k is constant in v_i and also in v_j. By considering the value of k at the points in v_i n v_j we find that k has the same value over the whole of the surface would of the and would not be compact.

Hence the surface must be a sphere.

HILBERT'S LEMMA:

In a closed region R of a surface of constant positive mansion curvature with outumblies, the principle curvature take their extreme values of the boundary.

Let us restate the lemma in a slightly different form suggested by withnewm.

If at a point p_o of any surface, the principle curvatures Ka, K_b are such that either(1), Ka>K_b, Ka has a minimum at p_o and Kb has a minimum point at p_o[or] K_a< K_b, Ka has minimum at p_o, K_b has a maximum at p_o then the gaussian curvature K cannot be positive at p_o.

PROOF:

We shall prove the lemma by the method f contradiction.

Suppose the lemma a is false then is a point p_o at which the principle curvature have distinct extreme values, one maximum and the others minimum.

Taking the lines of curvature as parametric curves the principle curvature are

$$K_a = \frac{L}{a}, K_b = \frac{N}{G} \quad \rightarrow \quad (1)$$

The codazzi equations are

$$L_2 = \frac{1}{2} E_2 \left(\frac{L}{E} + \frac{N}{G} \right)$$

$$N_1 = \frac{1}{2} G_1 \left(\frac{L}{E} + \frac{N}{G} \right)$$

$$\frac{\partial K_a}{\partial v} = \frac{E L_2 - L E_2}{E^2}$$

$$= \frac{E \frac{1}{2} E_2 \left(\frac{L}{E} + \frac{N}{G} \right) - L E_2}{E^2}$$

$$= \frac{\frac{1}{2} E E_2 \left(\frac{N}{G} \right) - \frac{1}{2} L E_2}{E^2}$$

$$= \frac{1}{2} \frac{E_2}{E} (K_b - K_a)$$

Similarly ,

$$\frac{\partial K_a}{\partial u} = \frac{1}{2} \frac{G_1}{G} (K_a - K_b)$$

Since the principle curvatures have extema , the L.H.S members vanish at P_0. It follows that P_0.

$E_2 = G_1 = 0$and hence at P_0.

$$\frac{\partial^2 K_a}{\partial v^2} = \frac{1}{2} \frac{E_{22}}{E} (K_b - K_a)$$

$$\frac{\partial^2 K_a}{\partial u^2} = \frac{1}{2} \frac{G_{11}}{G} (K_a - K_b) \quad \rightarrow (4)$$

These are (2) possibilities

Either (i) K_a has maximum

In this case $K_a - K_b > 0$

$$\frac{\partial^2 K_a}{\partial v^2} \leq 0 \ , \frac{\partial^2 K_a}{\partial u^2} \geq 0 \quad \rightarrow (5)$$

(i) K_a has a minimum

In this case $K_b - K_a > 0$

$$\frac{\partial^2 K_a}{\partial v^2} \geq 0 \ , \frac{\partial^2 K_a}{\partial u^2} \leq 0 \quad \rightarrow (6)$$

In either case $E_{22} \geq 0$ and $G_{11} \geq 0$

But this contradicts the fact that the Gaussian curvature k satisfies

$$k = -\frac{1}{2EG}(E_{22} + G_{11})$$

R.H.S is negative or zero while k is assumed strictly positive.

This contradiction completes the proof of the lemma.

Characterization of complete surfaces :

In this section we shall consider three properties each of which can be used characterize complete surfaces. The properties are:

(a) Every Cauchy sequence of points of S is convergent
(b) Every Geodesic can be prolonged indefinitely in either direction or else it forms closed curve.
(c) Every bounded set of points of S is relatively compact.

It is clear that condition (c) implies (a)

Let us prove that (a) implies (b)

If γ be a closed curve, then condition (b) is satisfied.

If γ is not a closed curve and if P(x) is some point on γ then there is some number .

Such that γ can be prolonged for distances (measured along) less than l , but cannot be prolonged for distances greater than l.

Consider the sequence of points $\{x_n\}$ lying on γ at distance from P along γ given by $l\left(1 - \frac{1}{n}\right)$.

Clear $\{x_n\}$ is a Cauchy sequence and by (a) converges to some point Q on γ whose distance from P is precisely l.

If $\{x_n'\}$ is another Cauchy sequence such that $(x_n, x_n') \rightarrow l$, then $\{x_n'\}$ tends to some limit Q'.

Now the sequence $x_1, x_1', x_2, x_2', x_3, x_3', ...$ is also a cauchy sequence tending to both Q and Q'. Hence $= Q'$, and there exists a unique end point Q at a distance l from P along γ.

Consider now a co-ordinate neighborhood of S which contains (1).

At Q there is uniquely determined a direction $\bar{t}$. Which is the direction of the geodesic $-\gamma$ which starts at (1).

In this coordinate neighborhood there is a unique geodesic at Q which has the direction $(-\bar{t})$, and this gives a continuation of γ beyond Q, contrary to the hypothesis.

If follows that γ must satisfy condition (b). thus we have proved that (a) implies (b), since (c) implies (a) we conclude than (c) implies (b).

Now we have only to prove that (b) implies (c) so that all the conditions equivalent.

Suppose property (b) holds for S. consider the point a of S and geodesic arc which start at a. we define the initial vector of a geodesic arc starting at a to be the tangent to this arc at a which has the same sense as the geodesic and whose length is equal to the length of the geodesic arc. Since property (b) is true for S, it follows that every tangent vector to S at a, whatever its length, is the initial vector of some geodesic arc stating at a which is uniquely determined. This arc may cut itself or if it forms part of a closed geodesic, may even cover part of itself.

Let $S_r = \{x \in s/\rho(x,a) \leq r\}$ and let E_r be the set of points x of S_r which can be joined to a by a geodesic arc whose length is equal to $\rho(x,a)$.

We claim that the set of points E_r is compact.

Let $\{x_n \rightarrow \ x_h\}_{h=1}^{\infty}$ be a sequence of points of E_r.

Let T_h be the initial vector of a geodesic arc of length $\rho(a,x_h)$ joining a to x_n. Then the sequence of vectors $\{\bar{T}_h\}$ regarded as a sequence of points in two dimensional Euclidean space, admits at least one vector of accumulation $\bar{T}$. Moreover this vector $\bar{T}$is the initial vector of a geodesic arc whose extremity belongs to E_r and is accumulation point of $\{x_n\}$.

This proves that E_r is compact.

We next claim that $E_r = S_r \ \rightarrow (1)$

Obviously (1) is true for $r = 0$ also if it is true for > 0 , then it is certainly true for $r < R$. We now prove that conversely if (1) is true $r < R$ then it is still true for $r = R$.

Now every point of S_R is the limit point of a sequence of points whose distance from a is less than R. by hypothesis these points belong to E_R , and since E_R is closed. It follows that limit belongs to E_r. Thus (1) is true for $r = R$.

In order to prove (1) completely , it is necessary show that if it holds for $r = R$ then it still holds $r = R + S$, $S > 0$.

This follows because it would then be possible to extend the range of validity of (1) to an arbitrary extent by an appropriate number of extends of the range by an amount S.

We next show that to any point y such that $\rho(a,y) > R$ there is a point x such that,

$$\rho(a,x) = R \ \rightarrow (2)$$

And $\rho(a,y) = R + \rho(y,x)$

Since $\rho(a, y)$ has been defined as the lowest bound of the lengths of arcs from a to y, it follows that we can join a to y by a curve γ whose length is less than $\rho(a, y) + h^{-1}$ for any integer h. Let x_n be the last point of this curve belonging to $E_R = S_R$

Now we have $\rho(a, y) \leq \rho(a, x_h) + \rho(x_h, y) \rightarrow (3)$

i.e., $\leq R + \rho(x_h, y)$, $\rho(a, x_h) = R$

(or) $\rho(x_h, y) \geq \rho(a, y) - R \rightarrow \quad (4)$

Since the arc length of γ from a to y is the sum of the arc lengths from a to x_h and from $x_h$to y we have,

$$\rho(x_h, y) \leq arc\ (x_h, y)$$

$$\rho(x_h, y) \leq arc\ (a, y) - arc\ (a, x_h)$$

$$\leq \rho(a, y) + h^{-1} - R$$

Let $h \rightarrow \infty$; $\{x_n\}$ will have at least are point of accumulation x with the property.

$$\rho(x, y) \leq \rho(a, y) - R \rightarrow \quad (5)$$

Comparing (4) and (5) , we find that

$$\rho(a, y) = R + \rho(y, x)$$

Thus we have proved the existence of a point x satisfying (2) and (3)

We have seen earlier that provided two points x, y are not toofor apart, then the point Y is the extremity of the one and only one geodesic arc of origin x and of length $\rho(x, y)$. More precisely there exists a continuous function $s(x) > 0$ such that if $\rho(x, y) < s(x)$, the point y is the extremity of the unique geodesic arc of length $\rho(x, y)$ joining x to y. further the continuous function s(x) attains a positive minimum value on the compact set E_R and we take S to be this minimum.

If (1) is true for $r = R$ and if $R < r(a, y) \leq R + S$ there exists an $x \in E_R$ such that $\rho(a, x) = R$ and $\rho(x, y) = \rho(a, y) - R \leq S$ consequently there exists a geodesic arc L' of length $P(a, x)$ joining a to x and a geodesic arc L'' of length $\rho(x, y)$ joining x to y. the composite arc is a geodesic formed by L' and L'' joins a to y and has its length $\rho(a, y)$. The composite arc formed by is geodesic arc and y is thus joined to a by a geodesic arc whose length is equal to the distance of y form a.

Hence $Y \in E_{R+S}$ and the range of validity of (1) is thus extended from E_r to E_{R+S} we have proved incidentally that hypothesis (c) implies that any two points of S can be joined by a geodesic arc whose length is equal to their distance.

Suppose we are now given a bounded set of points of M on S. Clearly we can find some R. such that M is contained in s_R and some R such that M is contained in S_R and since $S_R(= E_R)$ is compact , it follows that M is relatively compact.

We have thus shown that (b) implies (c) and hence the equivalence of all the three conditions (a) , (b) and (c).

Hilbert's theorem :

A complete analytic surface free from singularities, with constant negative Gaussian curvature, cannot exist in three dimensional Euclidean space.

We have already seen that a compact surface with these properties cannot exist, but here the condition of compact ness is relaxed to completeness and hence the proof is much more difficult.

In the proof the theorem, the notation of universal covering space of a given space is being used.

Let P be a point on the surface S, and let a be the set of all paths of S which begin at P.

Let us divide the set Q into classes, putting into each class the totality of paths that are homologically equivalent.

Let S' denote the set of these classes, so that a point of S' is an equivalence class of paths on S.

There is a natural mapping Φ of the set S' on the space S, for if A is a point of S', then all the equivalent paths in S belonging to A must end in the same point a, and we write $a = \Phi(A)$.

It is shown that the set of points S' can be considered as forming a surface called the universal covering surface which has the following properties.

(1) The natural mapping of S' on S is a continuous open mapping. Moreover , Φ is locally homeomorphism mapping, i.e., for every point A of S' there exists a neighborhood U^* such that the mapping Φ is homeomorphism on the neighborhood U^*

(2) The universal covering of surface S' of a surface S is always simply connected.

Property (1) implies that S and S' are locally homeomorphism so that all the local properties of the space S are automatically true for S' , Moreover, the differential geometric structure on S induces a differential geometric structure on S'

Proof of Hilbert's theorem :

Let us assume that a surface S exists having the required properties and we arrive at a contradiction.

Consider an arbitrary geodesic line on the surface S and take an arbitrary point O on this geodesic as origin.

Let S denote the arc length of this geodesic measured O , Since S is complete, the geodesic can be continued in both directions from $-\infty \; to + \infty$.

It is possible that the geodesic will ultimately cross itself so that the same point of S will have two different S – values.

However, if we consider instead of S its universal covering surface S' , then different values of S will correspond to different points on S', this follows because on a surface of negative Gaussian Curvature two geodesic arcs cannot enclose a simply connected region.

At each point of parameter 'S' on the given geodesic , consider the orthogonal geodesic line and let its arc length t be chosen as parameter so the equation of geodesic is $t = 0$.

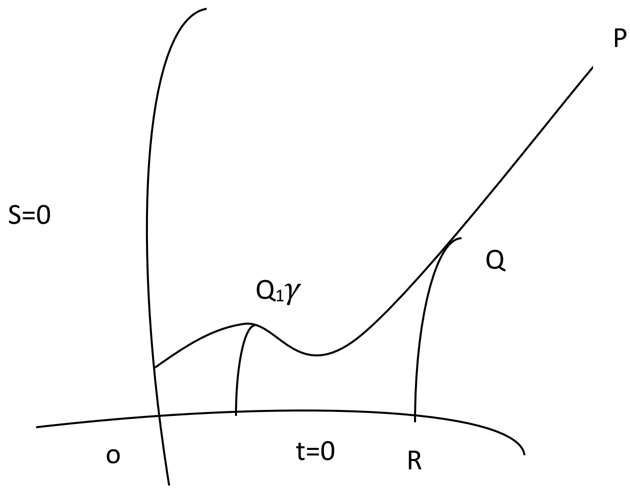
Now two of these geodesic arc at S_1, S_2 cannot meet on the surface S in order to form with the geodesic arc S_1S_2 a simply connected region. For if this were the case, then the sum of the angles of the geodesic triangle so formed would not be less than 2π, contrary to the results.

Let us denote a point in the converging space S' by the pair of coordinates (s, t) and it can be seen that different pairs (s, t) correspond to different point on S'. Out claim is that every point of S can be represented on the covering surface S' in this manner.

The line element of the surface assumes the form $ds^2 + G(S)dt^2$

Suppose now that a point p of the surface S remained uncovered by our construction join P to O (s=0, t=0) by some rectifiable curve γ.

Then there must be some point Q on γ with the property that all points between O and Q can be covered, while points on γ arbitrarily near Q on the side if Q remote from O cannot be covered. If Q_1 lies on γ between O and Q it follows from the form of the metric that the length of the curve QQ_1 is greater than or equal to SQ_1 where SQ_1 is the S – coordinate of the corresponding point of S'



The set of values $\{SQ_1\}$ is bounded, and we define S_Q to be the last upper bound of this set.

Let R be the point on the geodesic t=0 distant S_Q from Q, and consider the orthogonal geodesic along some interval on the geodesic t=0 which contains R.

These geodesic will cover a strip of the surface which certainly contains the point Q, and the points beyond Q on the curve γ which gives a contradiction and we conclude that every point of the surface S can be covered in this way.

Thus there is a local homemorphism between points of /s and the (s-t) plane. But this correspondence may not be (1-1) in the large. However the covering space S' is homemorphic with the (s-t) plane.

Consider the asymptotic lines on the surface S.

These lines are given by the differential equation $Lds^2 + 2Mdsdt + Ndt^2 = 0$

Since $k < 0$, we conclude that $LN - M^2 < 0$ and hence that at each point of S, the asymptotic directions are real and different. Hence at each point of S' these determine two distinct directions and similarly at each point of the (s-t) plane.

Since the (s-t) plane is simply connected, the differential equation gives rise to two vector fields which can be continued over the whole plane.

The Lipchitiz condition for uniqueness of the solution of the differential equation is satisfied for we have assumed that S is of class w.

Thus throughout the (s-t) plane there are two systems of asymptotic lines with the property that a curve from each system passes through an arbitrary point.

Further since S is free from singularities, the differential equation has no Singularities.

From the theorem of Bendixon that each asymptotic line can be prolonged to an arbitrary extent in both directions and if τ denoted the arc length.

Then $\lim_{\tau\to -\infty}(S^2+t^2)=\infty$, $\lim_{\tau\to +\infty}(S^2+t^2)=\infty$

Let us next prove that two such lines cut in at most one point. Suppose this is not so. Then three would be a region of the s-t plane bounded by two asymptotic lines of different systems.

Consider the first case when the asymptotic lines meet at A and B such that the continuation of the lines does not contain any interior point of the region bounded by the two lines. Let P be the point on one of the lines lying between A and B and consider asymptotic line of the Second system which passes through P. because this second line through P cannot intersect the line AB of the opposite system in a further point Q. moreover as P moves from A towards the end B. there must be one point where P and Q coincide at that point the asymptotic directions will coincide. This contradicts the fact $k<0$.

Consider now the second case, where by continuation of the asymptotic lines at least one line penetrates the region bounded by the two asymptotic line.

Then this asymptotic line will meet the line of the opposite system at a second point C.

Then the continuation BC together with the asymptotic line BC from a system of the type discussed above and again we arrive at a contradiction.

Thus we have proved that each asymptotic line of one system cannot meet each asymptotic line of the other system is more than one point.

In order to prove such lines must meet in atleast one point, it is convenient to refer to the asymptotic lines as parametric lines.

Suppose that N is a neighborhood of S in which the lines of curvature are chosen as parametric lines.

If K_a , K_b denote the principal curvatures at a point P on N and if $k=\frac{1}{a^2}$ is the constant negative Gaussian curvature, we can write $K_a=a^{-1}\cot P$, $K_b=a^{-1}\tan P$, $0<P<\frac{\pi}{2}$

Using an argument similar to section 3, we obtain

$$\frac{\partial K_a}{\partial v}=\frac{1}{2}\frac{E_2}{E}(K_b-K_a)$$

$$\frac{\partial K_a}{\partial u} = \frac{1}{2}\frac{G_1}{G}(K_a - K_b)$$

Using $K_a = a^{-1}\cot\rho$ and $K_b = -a^{-1}\tan\rho$ we get

$$\frac{E_2}{E} = 2\rho_2 \cot\rho$$

$$\frac{G_1}{G} = -2\rho_1 \tan\rho$$

From these equations on integration we obtain

$$E = U(u)sin^2\rho\,;G = V(u)cos^2\rho \quad \rightarrow (3)$$

Where U(u) , V(v) are certain functions of u and v respectively,

By means of a suitable parameterization, these functions may be taken as unity and the first fundamental form becomes,

$$sin^2\rho du^2 + cos^2\rho dv^2$$

In terms of the new parameters

$$L = K_a E = a^{-1}\sin\rho\cos\rho$$

$$N = K_b G = -a^{-1}\sin\rho\cos\rho$$

$M = 0$, and the asymptotic lines are given by $du^2 - dv^2 = 0$

Choose new parameter σ, τ where $\sigma = \frac{1}{2}(v + u)\,, \tau = \frac{1}{2}(u - v)$

Then , the parametric curves $\sigma = constant\,, \tau = constant$ are asymptotic lines.

Moreover, the metric assumes the form

$$d\sigma^2 + 2\cos 2\rho \;\; d\tau + d\tau^2 \qquad \rightarrow \quad (4)$$

And σ, τ measure the arc length of the asymptotic lines.

Through O of the (s-t) plane there pass two asymptotic lines.

Through each point on these two lines we draw the asymptotic lines of opposite system.

Then we prove that each point of the (s-t) plane lies on one asymptotic line of each system.

Suppose that there is a point P on the plane which cannot be reached in this way. Join P to O by a continuous curve γ then there will be a point Q on γ with the property that every point

γ between O and Q can be reached in this way, but points on γ arbitrarily near to Q on the side remote from O and possibly including Q cannot be reached.

Consider a neighborhood of Q which is covered by the asymptotic lines and has the property that each pair of lines from different systems cur in a single point in this neighborhood. Consider a point Q_0 lying in this neighborhood and let asymptotic lines through Q_0 cut the coordinate curve $t = 0, \sigma = 0$ in two points ${Q_0}^{(1)}, {Q_0}^{(2)}$ respectively.

Let Q_1 denote a typical point which lies on γ between Q_0 and Q. let the asymptotic lines through Q_1 meet the coordinate curves at ${Q_1}^{(1)}, {Q_1}^{(2)}$ and let these lines meet the lines through $Q_0\ in\ {\bar{Q}_0}^{(1)}, {\bar{Q}_0}^{(2)}$

Then $Q_0{\bar{Q}_1}^{(1)} = {\bar{Q}_0}^{(1)}{Q_i}^{(1)}$and $Q_0{\bar{Q}_i}^{(2)} = {\bar{Q}_0}^{(2)}{Q_i}^{(2)}$,

Provided Q_i lies in a neighborhood of Q_0 where the line element is of the form given by (4).

Any asymptotic line which cuts $Q_0{Q_1}^{(1)}$ between $Q_0 and\ \ {Q_1}^{(1)}$ which is sufficiently near to Q_0

Will cut equal lengths from all asymptotic lines which meet $Q_0{Q_0}^{(1)}$.

If this were not true for all the asymptotic lines meeting $Q_0{Q_1}^{(1)}$, we could choose a point R on the asymptotic line joining $Q_0 to\ \overline{Q_1}$ such that all points between Q_0 and R posses this property, but there are points arbititarily close to R (may be R itself) which does not possess this property the asymptotic line through R will cut the coordinate line $\tau = 0$ in the point $R^{(1)}$ such that the lengths $Q_0R, {Q_0}^{(1)}R^{(1)}$ are equal and further all the asymptotic lines between ${Q_0}^{(1)}$ and Q_0 will have equal lengths intercepted by the asymptotic lines through R. let us measure off from all these asymptotic lines the length Q_0R in the direction of increasing σ.

We assert that the end points of these segments form an asymptotic line. This is clearly the case when we consider neighborhoods of points on the line $RR^{(1)}$ and make use of the net of asymptotic lines in this neighborhood.

It is true for all asymptotic lines which meet Q_0Q_1 in a neighborhoods of points on the line $RR^{(1)}$ and make use of the net of asymptotic lines in this neighborhood of R and in particular it is true for the asymptotic lines through $\overline{Q_1}$ and for those in a certain neighborhood of Q_1 which contradicts the hypothesis.

Thus the two asymptotic lines through O will cut an arbitrary asymptotic line in the planer, and sine the point O has been chosen arbitrarily, it follows that each asymptotic line of one system meets every asymptotic line of the other system in exactly one point. We can take (σ, τ) as coordinates for points in the whole plane and the metric is of the form $d\sigma^2 + 2\cos 2\rho\ d\sigma d\tau + d\tau^2$

Let w be the angle between the parametric curve.

Thus $\cos w = \frac{F}{\sqrt{EG}} = \cos 2\rho$

Here $F = \cos 2\rho \; , E = 1, G = 1$

$w = 2\rho$and hence $0 < w < \pi$

Now using $k = -\frac{1}{2H}\left\{\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right\}$ for the Gaussian curvature we obtain $k = -\frac{1}{a^2}$

Also $\frac{\partial^2 w}{\partial s \partial t} = -k \sin w$

Consider now the quadrilateral formed by the asymptotic lines $\sigma = \pm\infty, \tau = \pm\infty$

Total curvature

$$= \iint k \, ds = \iint k \sin w \, d\sigma \, d\tau$$

$$= w_1 - w_2 + w_3 - w_4$$

$$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\pi$$

It follows that the absolute magnitude of the total curvature of an arbitrarily large region cannot exceed 2π.Consider the first form of metric

$$ds^2 + G(s)dt^2$$

We have $k = -\frac{1}{2G}\frac{\partial}{\partial s}\left(\frac{G_s}{\sqrt{G}}\right)$and $\sqrt{G} = \cos h\left(\frac{s}{a}\right)$

The total curvature over a region bounded by parametric lines $s = \pm l \, , t = \pm l$ is

$$\iint k \, ds = \iint k\sqrt{G} ds \, dt$$

$$= -\frac{1}{2}\iint \frac{\partial}{\partial s}\left(\frac{G_s}{\sqrt{G}}\right) ds \, dt$$

$$= -\frac{4l}{a} \sin h \frac{l}{a}$$

But in magnitude this tends to $\infty \; as \; l \rightarrow \infty$ which contradicts the earlier assertion that the absolute magnitude of the total curvature cannot exceed 2π.

This complete the proof of Hilbert’s theorem.

Definition : Field of Geodesics

A field of geodesics is meant a one parameter set of geodesics defined over a region R of a surface such that through each point of R passes one and only one curve of the set.

Bonnet theorem Statement :

If along a geodesic the Gaussian curvature exceeds a positive constant $\frac{1}{a^2}$, then the curve cannot be the shortest distance between its extremities along an arc length exceeding πa.

Strum's theorem Statement :

Consider the two distinct differential equations $\frac{d^2V}{dx^2} = HV$, $\frac{d^2V}{dx^2} = H'V$ where for all values of x in the range considered ,$H'(x) \geq H(x)$.

Then if $\Phi(x)$ is a solution of the first equation having to consecutive zero at x_0 and x_1 a solution of the second equation which has a zero at x_0 cannot have another zero in the closed interval $[x_0, x_1]$.